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Characterizations of Various Continuities of Posets Via Approximated Elements

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**Abstract**

In this paper, various continuities of posets which may not be dcpos are considered. The concepts of approximated elements and hyper-approximated elements on posets are introduced. New characterizations of continuous posets and hypercontinuous posets are given. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is introduced and some characterizations of quasicontinuous domains are also obtained. It is proved that under some reasonable conditions, the set B(*L*) (resp., QB(*L*)) of approximated elements (resp., quasi-approximated elements) in the induced order of a dcpo *L* is a continuous domain (resp., a quasicontinuous domain).

*Keywords:* continuous poset; hypercontinuous poset; approximated element; interpolation property; quasicontinuous domain

# Introduction

The notion of continuous lattices as a model for the semantics of programming lan- guages was introduced by Scott in [[10](#_bookmark39)]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied (see [[1](#_bookmark30)], [[5](#_bookmark34)], [[6](#_bookmark35)]). Since some naturally arising

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posets are important but fail to be directed complete, there are more and more occa- sions to study posets which fail to possess all directed suprema (see [[7](#_bookmark36)]-[[9](#_bookmark38)], [[11](#_bookmark40)]-[[14](#_bookmark42)]). Lawson in [[6](#_bookmark35)] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. By the technique of embedded bases and sobrification via the Scott topology, Xu in [[11](#_bookmark40)] successfully embedded continuous posets into continuous domains and proved that a poset is continuous iff its Scott topology is completely distributive. Quasicontinuous domains were introduced by Gierz, Lawson and Stralka (see [[3](#_bookmark32)]) as a common generalization of both generalized continuous lattices (see [[2](#_bookmark31)]) and continuous domains. It was proved that quasi- continuous domains have many properties similar to that of continuous domains and quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous lattices. In terms of intrinsic topologies on posets, Mao and Xu in [[8](#_bookmark37)] introduced the concept of hypercontinuous posets and quasicontinuous posets and proved that a poset is quasicontinuous iff its Scott topology is a hypercontinuous lattice.

In [[15](#_bookmark43)], Zhao introduced the concept of weakly approximated elements on com- plete lattices and gave several characterizations of continuous lattices and com- pletely distributive lattices. According to Zhao, an element *x* of a complete lattice

*L* is said to be weakly approximated if it holds that *x* = W *x*. Zhao derived a novel

←

characterization [[15](#_bookmark43), Theorem 3] of continuous lattice that a complete lattice *L* is

continuous iff *L* satisfies (i) the interpolation property (INT) for the way-below re-

←

lationon *L* and (ii) for any *x, y ∈ L*, *x /*= *y* implies

←

*x /*=

*y*. He also constructed

two counterexamples (see [[15](#_bookmark43), p.163]) to show that none of the conditions (i) and

(ii) may be omitted.

In this paper, we introduce the concepts of approximated elements and hyper- approximated elements on posets and discuss some properties and relations of ap- proximated elements and hyper-approximated elements. With these new concepts, we give several characterizations of continuous posets and hypercontinuous posets, generalizing relevant results in [[15](#_bookmark43)]. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is also introduced and some new characterizations of quasicontinuous domains are obtained. We will prove that under some reasonable conditions, the set B(*L*) (resp., QB(*L*)) of ap- proximated elements (resp., quasi-approximated elements) in the induced order of a dcpo *L* is a continuous domain (resp., a quasicontinuous domain).

# Preliminaries

We quickly recall some basic notions and results (see [[1](#_bookmark30)], [[3](#_bookmark32)], [[8](#_bookmark37)] and [[11](#_bookmark40)]). Let (*L*, *≤*) be a poset. A *principal ideal* (resp., *principal ﬁlter* ) is a set of the form

*↓x* = *{y ∈ L | y* ≤ *x}* (resp., *↑x* = *{y ∈ L | x* ≤ *y}*). For *A ⊆ L*, we write

*↓A* = *{y ∈ L |∃ x ∈ A, y* ≤ *x}*, *↑A* = *{y ∈ L |∃ x ∈ A, x ≤ y}*. A subset *A* is a(n) *lower set* (resp., *upper set* ) if *A* = *↓A* (resp., *A* = *↑A*). We say that *z* is a(n) *lower bound* (resp., *upper bound* ) of *A* if *A ⊆ ↑z* (resp., *A ⊆ ↓z*). The set of lower bounds

of *A* is denoted by lb(*A*). The supremum of *A* is denoted by W *A* or sup *A*. The

infimum of *A* is denoted by *A* or inf *A*. A nonempty subset *D* of *L* is *directed* if every finite subset of *D* has an upper bound in *D*. A poset *L* is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of *L* has a supremum. A *complete lattice* is a poset in which every subset has a supremum.

In a poset *L*, we say that *x approximates y*, written *x y* if whenever *D* is a directed set that has a supremum sup *D* ≥ *y*, then *x* ≤ *d* for some *d ∈ D*. For

→

*x ∈ L*, we write

←

*x* = *{z ∈ L | z x}* and

*x* = *{z ∈ L | x z}*. The poset *L* is

said to be *continuous* if every element is the directed supremum of elements that

approximate it, i.e., for all *x ∈ L*, the set *x* is directed and *x* = W *x*. A continuous

←

←

poset which is also a dcpo is called a *continuous domain* or *domain*. A continuous poset which is also a complete lattice is called a *continuous lattice*.

A subset *A* of a poset *L* is *Scott closed* if *↓A* = *A* and for any directed set *D ⊆ A*, sup *D ∈ A* whenever sup *D* exists. The complements of the Scott closed sets form a topology, called the *Scott topology* and denoted by *σ*(*L*). It is well-known that for

a continuous poset the Scott topology has a base of all sets of the form *x*. The

→

topology generated by the complements of all principal filters *↑x* (resp., principal ideals *↓x*) is called the *lower topology* (resp., *upper topology* ) and denoted by *ω*(*L*) (resp., *ν*(*L*)).

**Proposition 2.1** (see [[1,](#_bookmark30)[13](#_bookmark41)]) *Let L be a continuous poset. Then the following are true:*

1. The relationhas the interpolation property: *x z* =*⇒ ∃y ∈ L* such that

*x y z*;

1. The Scott open filters form a topological basis of *σ*(*L*).

Let *L* be a poset. We define a binary relation *≺ν*(*L*) on *L* such that *x ≺ν*(*L*)

*y ⇐⇒ y ∈ intν*(*L*)*↑x*, where the interior is taken in the upper topology *ν*(*L*).

**Proposition 2.2** (see [[8](#_bookmark37), Proposition 3.1]) *Let L be a poset. Then for all u, x, y, z ∈ L:*

1. *x ≺**ν*(*L*) *y implies x y;*
2. *u* ≤ *x ≺ν*(*L*) *y* ≤ *z implies u ≺ν*(*L*) *z;*
3. *x ≺ν*(*L*) *z and y ≺ν*(*L*) *z imply x ∨ y ≺ν*(*L*) *z whenever the join x ∨ y exists in L;*
4. *⊥ ≺ν*(*L*) *x whenever L has a smallest element ⊥.*

**Definition 2.3** (see [[8](#_bookmark37)]) *A poset L is called a hypercontinuous poset if for all x ∈ L,*

*the set ⇓**≺*

*ν*(*L*)

*x* = *{y ∈ L | y ≺ν*(*L*) *x} is directed and x* = sup *⇓≺*

*x.*

*ν*(*L*)

**Lemma 2.4** (see [[8](#_bookmark37), Proposition 3.5]) *Let L be a poset. Then the following state- ments are equivalent:*

1. *L is a hypercontinuous poset;*
2. *L is a continuous poset and for all x, y ∈ L, x y implies x ≺ν*(*L*) *y;*
3. *L is a continuous poset and the Scott topology σ*(*L*) *is the upper topology ν*(*L*)*.*

**Remark 2.5** *Applying Proposition* [*2.2*](#_bookmark2)*(1) and Lemma* [*2.4*](#_bookmark4)*, one can easily see that the relation ≺ν*(*L*) *on a hypercontinuous poset L is precisely the approximation relation of L. By Proposition* [*2.1*](#_bookmark1)*(1), the relation ≺ν*(*L*) *on a hypercontinu- ous poset L satisﬁes the interpolation property: x ≺ν*(*L*) *z* =*⇒ ∃y ∈ L such that*

1. *≺ν*(*L*) *y ≺ν*(*L*) *z.*

For a set *X*, we use *P*(*X*) to denote the powerset of *X* and *Pfin*(*X*) to denote the set of all nonempty finite subsets of *X*. For a dcpo *L*, we define a preorder

*≤* (sometimes called *Smyth preorder*) on *P*(*L*) *\{∅}* by *G ≤ H* iff *↑H ⊆ ↑G* for all *G, H ⊆ L*. That is, *G ≤ H* iff for each *y ∈ H* there is an element *x ∈ G* with *x ≤ y*. We say that a nonempty family *F* of subsets of *L* is *directed* if it is directed in the Smyth preorder. More precisely, *F* is directed if for all *F*1, *F*2 *∈ F*, there exists *F*3 *∈ F* such that *F*1, *F*2 *≤ F*3, i.e., *F*3 *⊆ ↑F*1 *∩ ↑F*2. We say that *G* is *way below H* or *G approximates H* and write *G H* if for every directed set *D ⊆ L*, sup *D ∈ ↑H* implies *d ∈ ↑G* for some *d ∈ D*. We write *G x* for *G {x}* and

1. *H* for *{y} H*. For all *G ⊆ L*, we write

←

*G* = *{y ∈ L | G y}*.

→

**Proposition 2.6** *(see [*[*1*](#_bookmark30)*,*[*3*](#_bookmark32)*]) Let L be a dcpo. Then*

1. *∀ G, H ⊆ L*, *G H* =*⇒ G ≤ H*;
2. *∀ G, H ⊆ L*, *G H ⇐⇒ ∀ h ∈ H, G h*;
3. *∀ E, F, G, H ⊆ L*, *E ≤ F G ≤ H* =*⇒ E H*;
4. *∀ x, y ∈ L*, *{x} {y} ⇐⇒ x y*.

*G* = *{y ∈ L | y G}* and

**Lemma 2.7** (see [[1](#_bookmark30), Corollary III-3.4]) *Let F be a directed family of nonempty ﬁnite sets in a dcpo. If G H and* *F∈F ↑F ⊆ ↑H, then F ⊆ ↑G for some F ∈ F.*

As a generalization of continuous domains, the following definition gives the concept of quasicontinuous domains.

**Definition 2.8** (see [[1,](#_bookmark30)[3](#_bookmark32)]) *A dcpo L is called a quasicontinuous domain if it satisﬁes the following two conditions:*

1. *For each x ∈ L,* fin(*x*)= *{F ⊆ L | F is ﬁnite and F x} is a directed family;*
2. *For all x, y ∈ L, if x /*≤ *y, then there exists F ∈* fin(*x*) *such that y /∈ ↑F .*

One can also derive the interpolation property for quasicontinuous domains.

**Proposition 2.9** (see [[1](#_bookmark30), Proposition III-3.5]) *Let L be a quasicontinuous domain,*

*H ⊆ L and x ∈ L. If H x, then there exists a ﬁnite set F such that H F x.*

# Continuous posets and approximated elements

In this section, we introduce the concept of approximated elements and give several characterizations of continuous posets.

**Definition 3.1** *Let L be a poset and x ∈ L. If there is a directed set Dx ⊆ x*

←

*such that* W *Dx* = *x, then x is called an approximated element. The set of all*

*approximated elements of L is denoted by* B(*L*)*.*

**Proposition 3.2** *Let L be a poset and x ∈ L. Then x is an approximated element*

*iff the set x is directed and* W *x* = *x.*

←

←

**Proof.** *⇐*=: Straightforward.

=*⇒*: If *x* is an approximated element, then there is a directed set *Dx ⊆*

←

*x* such

that W *Dx* = *x*. It is easy to prove that W *x* = *x*. We next show that the set *x* is

←

←

directed. For all *a*, *b ∈ x*, since *a x* = W *Dx* and *Dx* is directed, there is *da ∈ Dx*

←

such that *a* ≤ *da*. Similarly, there is *db ∈ Dx* such that *b* ≤ *db*. By the directedness

of *Dx*, there is some *d ∈ Dx ⊆*

←

*x* such that *da*, *db* ≤ *d*, showing that the set

*x* is

directed. *2*

←

**Remark 3.3** *In view of the fact that in a complete lattice L, for any x ∈ L one*

*has that x is directed. So, by Proposition* [*3.2*](#_bookmark11)*, an element x of L is a weakly*

←

*approximated element (in the sense of [*[*15*](#_bookmark43)*]) iff it is an approximated element in the sense of Deﬁnition* [*3.1*](#_bookmark10)*, revealing that the notions of approximated elements and weakly approximated elements coincide in a complete lattice. However, for a poset L, we will see in Proposition* [*3.6*](#_bookmark13) *that the requirement for the directedness of*

*Dx ⊆*

←

*x is indeed important. This is the main reason for us to give the concept of*

*approximated elements.*

**Proposition 3.4** *Let L be a poset. Then L is a continuous poset iff every element of L is an approximated element.*

**Proof.** It follows from the definition of continuous poset and Proposition [3.2](#_bookmark11). *2*

Recall that a basis of a poset *L* is a set *B ⊆ L* such that for every *x ∈ L*, the

subset *B ∩ x* is directed and *x* = W(*B ∩ x*). It is well-known that a poset is

←

←

continuous iff it has a basis. By Proposition [3.4](#_bookmark12), we have immediately the following

**Corollary 3.5** *A poset L has a basis if and only if every element of L is an ap-* *proximated element.*

One will be interested in giving a direct proof of the above corollary. Clearly, the existence of a basis for *L* implies that every element of *L* is approximated.

Conversely, for all *x ∈ L*, choose a directed set *Dx ⊆ x* such that W *Dx* = *x*. Let

← ←

*B* = *x∈L Dx*. Then for all *x ∈ L*, clearly W(*B ∩ x*) = *x*. To show *B ∩ x* is

←

←

←

directed, let *s, t ∈ B ∩ x*. Then there exists an element *d ∈ Dx ⊆ B ∩ x* such that

*s, t* ≤ *d*, revealing that *B ∩ x* is directed. Thus *B* is a basis of *L*.

←

**Proposition 3.6** *Let L be a dcpo. If* B(*L*) *is nonempty, then* B(*L*) *is closed under the directed sups of L and thus a dcpo. Moreover, for all x, y ∈* B(*L*)*, x y implies x* B(*L*) *y, where* B(*L*) *means the approximation relation on* B(*L*)*.*

**Proof.** To show that B(*L*) is closed under the directed sups, let *S ⊆* B(*L*) be a

directed set. For each *s ∈ S*, by Definition [3.1](#_bookmark10), there is a directed set *Ds ⊆ s* such

←

that W *Ds* = *s*. Since *s* ≤ W *S*, we have *Ds ⊆ s ⊆* (W *S*). Let *D*W *S* = *s∈S Ds*.

←

←

Clearly, *D*W *S ⊆* (W *S*). For all *xa*, *xb ∈ D*W *S* , there exist *a*, *b ∈ S* such that

←

*xa ∈ Da ⊆ a* and *xb ∈ Db ⊆ b*. By the directedness of *S*, there is *c ∈ S* such

←

←

that *a*, *b* ≤ *c*. So, *xa c* and *xb c*. Since *Dc* is directed and *c* = W *Dc*, there

exist *d*, *e ∈ Dc* such that *xa* ≤ *d* and *xb* ≤ *e*. By the directedness of *Dc*, there is *w ∈ Dc ⊆ D*W *S* such that *d*, *e* ≤ *w*. Thus there is *w ∈ D*W *S* such that *xa*,

*xb* ≤ *w*, showing the directedness of *D*W *S* . Clearly, W *S* is an upper bound of the

set *D*W *S* = *s∈S Ds*. Let *t* be any upper bound of *D*W *S* . Then for each *s ∈ S*, *s* = W *Ds* ≤ *t*. So, W *S* ≤ *t*, showing that W *S* is the least upper bound of the

set *D*W *S*. By Definition [3.1](#_bookmark10), W *S ∈* B(*L*), showing that B(*L*) is closed under the directed sups. With the above result, the second part of the proposition is clear. *2*

**Proposition 3.7** *Let L be a poset and the relation on L has the interpolation*

*property. If for all x ∈ L, the set x is directed and* W *x exists, then there exists*

←

← ←

*an approximated element B*(*x*) *satisfying:* (i) *B*(*x*) = *x;* (ii) *∀ y ∈ L, y* = *x*

←

←

←

*implies y* ≥ *B*(*x*)*.*

**Proof.** Let *L* be a poset. If for all *x ∈ L*, the set *x* is directed and W *x* exists, then

←

←

let *B*(*x*) = W *x*. We next show that *B*(*x*) is an approximated element satisfying

←

condition (i) and (ii). Since *B*(*x*) ≤ *x*, we have *B*(*x*) *⊆ x*. Let *a ∈ x*. By the

←

←

←

interpolation property of the relation, there is *b ∈ L* such that *a b x*. Thus

←

←

we have *a b* ≤ *B*(*x*) and *a ∈*

←

*B*(*x*). This shows that *x ⊆*

*B*(*x*) and thus

*B*(*x*) = *x*. So, the element *B*(*x*) satisfies condition (i). By condition (i), it is

←

←

easy to prove that *B*(*x*) is an approximated element. For all *y ∈ L*, if *y* = *x*, then

←

←

*y* ≥ W *x* = *B*(*x*). Thus, *B*(*x*) also satisfies condition (ii). *2*

←

**Theorem 3.8** *Let L be a dcpo and the relation on L has the interpolation prop-*

*erty. If for all x ∈ L, the set continuous domain.*

←

*x is directed, then* B(*L*) *in the induced order is a*

**Proof.** For each *u ∈ L*, by Proposition [3.7](#_bookmark14), there exists an approximated element

←

*B*(*u*) such that

←

*B*(*u*)=

*u* and *B*(*u*) ≤ *u*. So, B(*L*) is nonempty. By Proposition

[3.6](#_bookmark13), B(*L*) in the induced order is a dcpo. For each *x ∈* B(*L*), let *Dx* = *x ∩* B(*L*).

←

←

It follows from Proposition [3.6](#_bookmark13) that *Dx ⊆*

← ←

B(*L*)*x*, where the set

B(*L*)*x* = *{y ∈*

B(*L*) *| y B*(*L*) *x}*. By the directedness of

*x*, there is *v ∈*

*x* such that *v x*.

By Proposition [3.7](#_bookmark14) again, there exists an approximated element *B*(*v*) such that

←

*B*(*v*) ≤ *v x*. This shows that *B*(*v*) *∈ x ∩ B*(*L*)= *Dx* and thus *Dx* is nonempty.

← ←

For all *a*, *b ∈ Dx*, by the directedness of

←

*x*, there is *c ∈*

*x* such that *a*, *b* ≤ *c x*.

Since the relationon *L* has the interpolation property, there is *e*(*x*) such that

*a*, *b* ≤ *c e*(*x*) *x*. By Proposition [3.7](#_bookmark14), there exists an approximated element

←

←

*B*(*e*(*x*)) such that

←

*B*(*e*(*x*)) =

*e*(*x*) and *B*(*e*(*x*)) ≤ *e*(*x*). So, *B*(*e*(*x*)) *∈*

*x∩*B(*L*)=

*Dx* and *a*, *b* ≤ *c B*(*e*(*x*)). This shows the directedness of the set *Dx*.

We next show that WB(*L*) *Dx* = *x*, where the subscript B(*L*) means to take

relevant operations in poset B(*L*). Clearly, *x* is an upper bound of *Dx*. Let *t* be

any upper bound of *Dx* in *B*(*L*). For all *z ∈ x*, since the relationon *L* has

←

the interpolation property, there is *h*(*x*) such that *z h*(*x*) *x*. By Proposition

←

[3.7](#_bookmark14), there exists an approximated element *B*(*h*(*x*)) such that

←

*B*(*h*(*x*)) =

*h*(*x*) and

*B*(*h*(*x*)) ≤ *h*(*x*). Thus *z B*(*h*(*x*)) ≤ *h*(*x*) *x* and *B*(*h*(*x*)) *∈ x ∩* B(*L*) = *Dx*.

←

This shows that *z* ≤ *t* and hence W *x* = *x* ≤ *t*. So, *x* is the least upper bound of

←

*Dx* in B(*L*), i.e., *x* = WB(*L*) *Dx*. By Definition [3.1](#_bookmark10), *x* is an approximated element of

*B*(*L*). It follows from the arbitrariness of *x* and Proposition [3.4](#_bookmark12) that *B*(*L*) in the

induced order is a continuous domain. *2*

**Remark 3.9** *Note that in a complete lattice L, for each x ∈ L, the set x is*

←

*automatically directed. So, Theorem* [*3.8*](#_bookmark15) *is a generalization of [*[*15*](#_bookmark43)*, Theorem 2] for continuous lattices.*

The following example shows that the requirement for the set not be omitted from Theorem [3.8](#_bookmark15).

←

*x* is directed can

**Example 3.10** *In the dcpo L obtained by pasting the two top elements in the two*

*paralleled closed intervals* [0*,* 1]*, x is empty for each x ∈ L and thus not directed.*

←

*The set* B(*L*) *of all approximated elements is empty and is not a continuous domain.*

Zhao constructed [[15](#_bookmark43), Example 2] a complete lattice *L* = *{a} ∪* [0*,* 1] with *a /∈* [0*,* 1]. He defined the partial order on *L* such that 1*/*2 *< a <* 1 and that the order of elements in [0*,* 1] is in the ordinary order of [0*,* 1]. For this *L*, one sees that B(*L*)= [0*,* 1] is continuous but the relationdoes not have the interpolation property (INT). Considering Zhao’s example, we pose the following problem.

**Problem** Does Theorem [3.8](#_bookmark15) still hold if the condition “the relationon *L* has the interpolation property” is removed?

**Theorem 3.11** *Let L be a poset. If for all x ∈ L, the set x is directed and* W *x*

←

←

*exists, then L is continuous iff L satisﬁes the following two conditions:* (1) *the*

*relation on L has the interpolation property;* (2) *6 x, y ∈ L, x /*= *y* =*⇒ x /*= *y.*

←

←

**Proof.** *⇐*=: Let *x ∈ L*. By Proposition [3.7](#_bookmark14), there exists an approximated element

←

*B*(*x*) ≤ *x* such that

←

*B*(*x*)=

*x*. Applying condition (2), we have *x* = *B*(*x*). Thus

*L* = B(*L*). By Proposition [3.4](#_bookmark12), *L* is a continuous poset.

=*⇒*: By the definition of continuous posets and Proposition [2.1](#_bookmark1)(1). *2*

**Remark 3.12** *Theorem* [*3.11*](#_bookmark16) *deals with general posets. So, Theorem* [*3.11*](#_bookmark16) *general- izes the result [*[*15*](#_bookmark43)*, Theorem 3] mentioned in the introduction section.*

**Definition 3.13** *Let L be a poset and* Φ *⊆ P*(*L*) *be a family of subsets of L. We say that* Φ *strongly separates points of L if for all x, y ∈ L with x /*≤ *y, there exist A ∈* Φ *and u ∈* lb(*A*) *such that x ∈ A and u /*≤ *y.*

**Lemma 3.14** *Let L be a poset and x, y ∈ L with x /*≤ *y. If there is a Scott open set U and z ∈* lb(*U* ) *such that x ∈ U and z /*≤ *y, then z x.*

**Proof.** Let *D ⊆ L* be directed with existing sup *D* ≥ *x ∈ U* . By the Scott openness of *U* , we have sup *D ∈ U* and hence there is *d ∈ U ∩ D* such that *z* ≤ *d*, showing that *z x*. *2*

**Theorem 3.15** *Let L be a poset. If for all x ∈ L, the set x is directed, then L is*

←

*continuous iff the family of all Scott open ﬁlters strongly separates points of L.*

**Proof.** *⇐*=: For each *x ∈ L*, we only need to show that *x* = W *x*. Clearly, *x* is an

←

upper bound of the set *x*. Let *t* be any upper bound of *x*. Suppose that *x /*≤ *t*.

←

←

Then there is a Scott open filter *U* and *z ∈* lb(*U* ) such that *x ∈ U* and *z /*≤ *t*. By Lemma [3.14](#_bookmark18), *z x* but *z /*≤ *t*, a contradiction to the assumption that *t* is an upper

←

bound of as desired.

←

*x*. Therefore, *x* ≤ *t*. This shows that *x* is the least upper bound of *x*,

=*⇒*: Let *L* be a continuous poset. For all *x*, *y ∈ L* with *x /*≤ *y*, there is *z x*

such that *z /*≤ *y*. It follows from *x ∈ z ∈ σ*(*L*) and Proposition [2.1](#_bookmark1) (2) that there

→

is a Scott open filter *U* such that *x ∈ U ⊆ z ⊆ ↑z*. So, *z ∈* lb(*U* ) and *z /*≤ *y*. By

→

Definition [3.13](#_bookmark17), the family of all Scott open filters strongly separates points of *L*.*2*

**Corollary 3.16** (see [[15](#_bookmark43), Theorem 1]) *A complete lattice L is continuous iff for all*

*x, y ∈ L with x /*≤ *y, there exists a Scott open ﬁlter U such that x ∈ U and*  *U /*≤ *y.*

**Proof.** Note that in a complete lattice *L*, the set *x* is automatically directed for

←

each *x ∈ L*. Then the corollary follows immediately from Theorem [3.15](#_bookmark19). *2*

By this corollary, one sees that for complete lattices, the concept of strongly separation in Definition [3.13](#_bookmark17) coincides with Zhao’s [[15](#_bookmark43), Definition 1]).

# Hypercontinuity of posets and hyper-approximated elements

In this section, we introduce the concept of hyper-approximated elements and give several characterizations of hypercontinuous posets.

**Definition 4.1** *Let L be a poset and x ∈ L. If there is a directed set Dx ⊆ ⇓≺*

W

*such that*

*Dx* = *x, then x is called a hyper-approximated element. The set of all*

*x*

*ν*(*L*)

*hyper-approximated elements of L is denoted by* HB(*L*)*.*

**Proposition 4.2** *Let L be a poset and x ∈ L. If x is a hyper-approximated element, then x is an approximated element.*

**Proof.** Follows from Proposition [2.2](#_bookmark2)(1), Definition [3.1](#_bookmark10) and Definition [4.1](#_bookmark20). *2*

**Proposition 4.3** *Let L be a poset and x ∈ L. Then x is a hyper-approximated*

W *≺*

*element iff the set ⇓≺*

*x is directed and ⇓*

*ν*(*L*)

*x* = *x.*

*ν*(*L*)

**Proof.** *⇐*=: Straightforward.

=*⇒*: If *x* is a hyper-approximated element, then there exists a directed set

W *x* W *≺*

*Dx ⊆ ⇓≺*

*x* such that *D* = *x*. It is easy to prove that *⇓*

*ν*(*L*)

*x* = *x*. We next

*ν*(*L*)

show that the set *⇓≺ x* is directed. For all *a*, *b ∈ ⇓≺*

W

*x*, by Proposition [2.2](#_bookmark2)(1),

*ν*(*L*) *ν*(*L*)

we have *a x* and *b x*. Since *a x* = *Dx* and *Dx* is directed, there is *da ∈ Dx*

such that *a* ≤ *da*. Similarly, there is *db ∈ Dx* such that *b* ≤ *db*. By the directedness

of *Dx*, there is some *d ∈ Dx ⊆ ⇓≺*

*x* such that *d* , *d* ≤ *d*. This shows that there

*ν*(*L*)

*a b*

is *d ∈ ⇓≺*

*x* such that *a*, *b* ≤ *d*. Thus the set *⇓*

*ν*(*L*)

*≺*

*x* is directed. *2*

*ν*(*L*)

**Proposition 4.4** *Let L be a poset. Then L is a hypercontinuous poset iff every element of L is a hyper-approximated element.*

**Proof.** Follows from Definition [2.3](#_bookmark3) and Proposition [4.3](#_bookmark21). *2*

**Proposition 4.5** *Let L be a poset and the relation ≺ν*(*L*) *on L has the interpolation*

W *≺*

*property. If for all x ∈ L, the set ⇓≺*

*x is directed and ⇓*

*ν*(*L*)

*x exists, then there*

*ν*(*L*)

*exists a hyper-approximated element* HB(*x*) *satisfying:* (i) *⇓≺ν*(*L*) HB(*x*) = *⇓≺*

*x;*

*ν*(*L*)

(ii) *6 y ∈ L, ⇓≺*

*y* = *⇓*

*ν*(*L*)

*≺*

*x implies y* ≥ HB(*x*)*.*

*ν*(*L*)

**Proof.** For all *x ∈ L*, let HB(*x*) = W *⇓≺*

*x*. We next show that HB(*x*) is a

*ν*(*L*)

hyper-approximated element satisfying condition (i) and (ii). Since HB(*x*) ≤ *x*,

we have *⇓≺ν*(*L*) HB(*x*) *⊆ ⇓≺*

*x*. Let *a ∈ ⇓*

*ν*(*L*)

*≺*

*x*. By the interpolation property

*ν*(*L*)

of the relation *≺ν*(*L*), there is *b ∈ L* such that *a ≺ν*(*L*) *b ≺ν*(*L*) *x*. Therefore,

*a ≺ν*(*L*) *b* ≤ HB(*x*) and *a ∈ ⇓≺ν*(*L*) HB(*x*). This shows that *⇓≺*

*x ⊆ ⇓*

*ν*(*L*)

*≺*

*ν*(*L*)

HB(*x*)

and thus *⇓≺ν*(*L*) HB(*x*)= *⇓≺*

*x*. So, the element HB(*x*) satisfies condition (i). By

*ν*(*L*)

condition (i), it is easy to prove that HB(*x*) is a hyper-approximated element. For

W *≺*

all *y ∈ L*, if *⇓≺*

*y* = *⇓*

*ν*(*L*)

*≺*

*x*, then *y* ≥ *⇓*

*ν*(*L*)

*x* = HB(*x*). Thus, HB(*x*) satisfies

*ν*(*L*)

condition (ii), as desired. *2*

**Theorem 4.6** *Let L be a poset. If for all x ∈ L, the set ⇓≺*

W

*⇓≺*

*x exists, then L is a hypercontinuous poset iff L satisﬁes the following two*

*ν*(*L*)

*x is directed and*

*ν*(*L*)

*conditions:* (1) *the relation ≺ν*(*L*) *on L has the interpolation property;* (2) *6 x, y ∈ L,*

*x /*= *y* =*⇒ ⇓≺*

*x /*= *⇓*

*ν*(*L*)

*≺*

*y.*

*ν*(*L*)

**Proof.** *⇐*=: Let *x ∈ L*. By Proposition [4.5](#_bookmark23), there is a hyper-approximated element

HB(*x*) ≤ *x* such that *⇓≺ν*(*L*) HB(*x*) = *⇓≺*

*x*. Applying condition (2), we have

*ν*(*L*)

*x* = HB(*x*). Thus *L* = HB(*L*). By Proposition [4.4](#_bookmark22), *L* is a hypercontinuous poset.

=*⇒*: By Definition [2.3](#_bookmark3) and Remark [2.5](#_bookmark5). *2*

**Theorem 4.7** *Let L be a poset and the relation ≺ν*(*L*) *on L has the interpolation*

*property. If for all x ∈ L, the set ⇓≺*

*x is directed, then L is a hypercontinuous*

*ν*(*L*)

*poset iff the family of all ν*(*L*)*-open ﬁlters strongly separates points of L.*

**Proof.** *⇐*=: For each *x ∈ L*, we only need to show that *x* = W *⇓≺*

*x*. Clearly, *x*

*ν*(*L*)

is an upper bound of the set *⇓≺*

*x*. Let *t* be any upper bound of *⇓*

*ν*(*L*)

*x*. Suppose

*ν*(*L*)

*≺*

that *x /*≤ *t*. Then there is a *ν*(*L*)-open filter *U* and *z ∈* lb(*U* ) such that *x ∈ U ⊆ ↑z*

and *z /*≤ *t*. Since *U* is *ν*(*L*)-open, we have *x ∈ intν*(*L*)*↑z*. So, *z ≺ν*(*L*) *x* but *z /*≤ *t*,

a contradiction to the assumption that *t* is an upper bound of *⇓≺*

*x*. Therefore,

*ν*(*L*)

*x* ≤ *t*. This shows that *x* is the least upper bound of *⇓≺*

*x*, as desired.

*ν*(*L*)

=*⇒*: By Lemma [2.4](#_bookmark4) and Theorem [3.15](#_bookmark19). *2*

# Quasicontinuous domains and quasi-approximated el- ements

In this section, we introduce the concept of quasi-approximated elements and give several characterizations of quasicontinuous domains.

**Definition 5.1** *Let L be a dcpo and x ∈ L. If there is a directed family D ⊆* fin(*x*) *such that ↑x* = *{↑F | F ∈ Dx}, then x is called a quasi-approximated element. The set of all quasi-approximated elements of L is denoted by* QB(*L*)*.*

*x*

**Proposition 5.2** *Let L be a dcpo and x ∈ L. If x is an approximated element, then x is a quasi-approximated element.*

**Proof.** Let *L* bea dcpo and *x ∈ L*. If *x* is an approximated element, then there is a

directed set *Sx ⊆ x* such that W *Sx* = *x*. Let *Dx* = *{{d}| d ∈ Sx}*. By Proposition

←

[2.6](#_bookmark6)(4), *Dx ⊆* fin(*x*). It is straightforward to show that *Dx* is a directed family and

*↑x* = *{↑d | {d}∈ Dx}*. So, *x* is a quasi-approximated element. *2*

**Proposition 5.3** *Let L be a dcpo and x ∈ L. Then x is a quasi-approximated element iff the family* fin(*x*) *is directed and ↑x* = *{↑F | F ∈* fin(*x*)*}.*

**Proof.** *⇐*=: Straightforward.

=*⇒*: If *x* is a quasi-approximated element, then there exists a directed family *Dx ⊆* fin(*x*) such that *↑x* =  *{↑H | H ∈ Dx}*. It is easy to verify that *↑x* = *{↑F | F ∈* fin(*x*)*}*. We next show that the family fin(*x*) is directed. For all *F*1, *F*2 *∈* fin(*x*), since *F*1 *x* and *↑x* = *{↑H | H ∈ Dx}*. By Lemma [2.7](#_bookmark7), there is *H*1 *∈ Dx* such that *H*1 *⊆ ↑F*1. Similarly, there is *H*2 *∈ Dx* such that *H*2 *⊆ ↑F*2. By the directedness of *Dx*, there is some *H ∈ Dx* such that *H*1, *H*2 ≤ *H*. Thus *↑H ⊆ ↑H*1 *∩ ↑H*2 *⊆ ↑F*1 *∩ ↑F*2. This shows that there is *H ∈ Dx ⊆* fin(*x*) such that *F*1, *F*2 ≤ *H*. Therefore, the family fin(*x*) is directed, as desired. *2*

**Proposition 5.4** *Let L be a dcpo. Then L is a quasicontinuous domain iff every element of L is a quasi-approximated element.*

**Proof.** By Definition [2.8](#_bookmark8) and Proposition [5.3](#_bookmark25). *2*

**Proposition 5.5** *Let L be a dcpo. If* QB(*L*) *is nonempty, then* QB(*L*) *is closed under the directed sups of L and thus a dcpo. Moreover, for all x ∈* QB(*L*) *and F ∈ Pfin*(QB(*L*))*, F x implies F* QB(*L*) *x, where* QB(*L*) *means the approximation relation on* QB(*L*)*.*

**Proof.** We show that QB(*L*) is closed under the directed sups of *L*. Let *S ⊆*

QB(*L*) be a directed set. For each *s ∈ S*, by Definition [5.1](#_bookmark24), there is a directed family *Ds ⊆* fin(*s*) such that *↑s* = *{↑F | F ∈ Ds}*. Since *s* ≤ W *S*, we have *Ds ⊆* fin(*s*) *⊆* fin(W *S*). Let *D*W *S* = *s∈S Ds*. Clearly, *D*W *S ⊆* fin(W *S*). For all *Fa*,

*Fb ∈ D*W *S*, there exist *a*, *b ∈ S* such that *Fa ∈ Da ⊆* fin(*a*) and *Fb ∈ Db ⊆* fin(*b*).

By the directedness of *S*, there is *c ∈ S* such that *a*, *b* ≤ *c*. So, *Fa c* and *Fb c*. Since *Dc* is a directed family and *↑c* = *{↑F | F ∈ Dc}*, by Lemma [2.7](#_bookmark7), there exist *F*1, *F*2 *∈ Dc* such that *F*1 *⊆ ↑Fa* and *F*2 *⊆ ↑Fb*. This shows that *Fa* ≤ *F*1 and *Fb* ≤ *F*2. By the directedness of *Dc*, there is *F*3 *∈ Dc ⊆ D*W *S* such that *F*1, *F*2 ≤ *F*3. Thus there is *F*3 *∈ D*W *S* such that *Fa*, *Fb* ≤ *F*3. This shows the directedness of

*D*W *S*. Clearly, *↑*(W *S*) *⊆* *{↑F | F ∈ D*W *S}*. Let *t ∈* *{↑F | F ∈ D*W *S}*. Then

for all *s ∈ S*, we have *t ∈* *{↑F | F ∈ Ds}* = *↑s*. Thus, W *S* ≤ *t* and *t ∈ ↑*(W *S*). This shows that *{↑F | F ∈ D*W *S} ⊆ ↑*(W *S*). So, *↑*(W *S*) = *{↑F | F ∈ D*W *S}*. By Definition [5.1](#_bookmark24), W *S ∈* QB(*L*), showing that QB(*L*) is closed under the directed

sups.

Moreover, let *x ∈* QB(*L*), *F ∈ Pfin*(QB(*L*)). Suppose that *F x*. Let

W W

*S ⊆* QB(*L*) be a directed set such that QB(*L*) *S* ≥ *x*, where QB(*L*) *S* means the supremum of *S* in QB(*L*). Since QB(*L*) is closed under the directed sups in *L*, we have W *S* = WQB(*L*) *S* ≥ *x*. It follows from *F x* that *S ∩ ↑F /*= *∅*. This implies

that *S ∩↑*QB(*L*)*F /*= *∅*, where the subscript QB(*L*) means to take relevant operations

in poset QB(*L*). Thus, *F* QB(*L*) *x*. *2*

**Proposition 5.6** *Let L be a dcpo and the relation on L has the interpolation property as stated in Proposition* [*2.9*](#_bookmark9)*. If for all x ∈ L, the family* fin(*x*) *is directed and* *{↑F | F ∈* fin(*x*)*}* = *↑t for some t, then there exists a quasi-approximated element* QB(*x*) *satisfying:* (i) fin(QB(*x*)) = fin(*x*)*;* (ii) *6 y ∈ L,* fin(*y*) = fin(*x*) *implies y* ≥ QB(*x*)*.*

**Proof.** Let *L* be a dcpo. If for all *x ∈ L*, fin(*x*) is directed and *{↑F | F ∈* fin(*x*)*}* = *↑t* for some *t*, then let QB(*x*)= *t*. We next show that QB(*x*) is a quasi- approximated element satisfying condition (i) and (ii). It follows from QB(*x*) ≤ *x* and Proposition [2.6](#_bookmark6) that fin(QB(*x*)) *⊆* fin(*x*). Let *F ∈* fin(*x*). By the interpolation property of the relationas stated in Proposition [2.9](#_bookmark9), there is a finite set *E* such that *F E x*. Thus we have *F E* ≤ QB(*x*) and *F ∈* fin(QB(*x*)). This shows that fin(*x*) *⊆* fin(QB(*x*)) and thus fin(*x*) = fin(QB(*x*)). So, the element QB(*x*) satisfies condition (i). By condition (i), it is easy to prove that QB(*x*) is a quasi-approximated element. For all *y ∈ L*, if fin(*y*) = fin(*x*), then *y ∈* *{↑F | F ∈* fin(*x*)*}* = *↑t* = *↑*QB(*x*). Thus *y* ≥ QB(*x*) and QB(*x*) satisfies condition (ii), as desired. *2*

**Lemma 5.7** *Let L be a dcpo and the relation on L has the interpolation property as stated in Proposition* [*2.9*](#_bookmark9)*. If for all x ∈ L, the family* fin(*x*) *is directed and*

*{↑F | F ∈* fin(*x*)*}* = *↑t for some t, then for each F ∈* fin(*x*) *there exists* QB(*F* ) *∈ Pfin*(QB(*L*)) *such that F* QB(*F* ) *x.*

**Proof.** Let *x ∈ L*. For each *F ∈* fin(*x*), by the interpolation property as stated in Proposition [2.9](#_bookmark9) of the relationon *L*, there is nonempty finite set *EF* such that *F EF x*. It follows from Proposition [2.6](#_bookmark6)(2) that for all *e ∈ EF* , we have *F e*. By Proposition [5.6](#_bookmark28), there exists a quasi-approximated element QB(*e*) such that fin(QB(*e*)) = fin(*e*) and QB(*e*) ≤ *e*. Let QB(*F* ) = *{*QB(*e*) *| e ∈ EF } ∈ Pfin*(QB(*L*)). It is straightforward to prove that *F* QB(*F* ) *x*. *2*

**Theorem 5.8** *Let L be a dcpo and the relation on L has the interpolation prop- erty as stated in Proposition* [*2.9*](#_bookmark9)*. If for all x ∈ L, the family* fin(*x*) *is directed and* *{↑F | F ∈ fin*(*x*)*}* = *↑t for some t, then* QB(*L*) *in the induced order is a quasicontinuous domain.*

**Proof.** For each *u ∈ L*, by Proposition [5.6](#_bookmark28), there exists a quasi-approximated element QB(*u*) such that fin(QB(*u*)) = fin(*u*) and QB(*u*) ≤ *u*. Thus QB(*L*) is nonempty. By Proposition [5.5](#_bookmark26), QB(*L*) in the induced order is a dcpo. For all *x ∈* QB(*L*) and *F ∈* fin(*x*), by Lemma [5.7](#_bookmark29), there exists QB(*F* ) *∈ Pfin*(QB(*L*)) such that *F* QB(*F* ) *x*. Let *Dx* = *{*QB(*F* ) *| F ∈* fin(*x*)*}*. It follows from Proposition [5.5](#_bookmark26) that *Dx ⊆* finQB(*L*)(*x*), where the set finQB(*L*)(*x*)= *{H ∈ Pfin*(QB(*L*)) *| H* QB(*L*) *x}*. Since the family fin(*x*) is nonempty, the family *Dx* is nonempty. For all QB(*F*1), QB(*F*2) *∈ Dx*, we have QB(*F*1) *x* and QB(*F*2) *x*. By the directedness of fin(*x*) and the interpolation property as stated in Proposition [2.9](#_bookmark9) of the relationon *L*, there is *F*3 *∈* fin(*x*) such that QB(*F*1), QB(*F*2) *F*3 *x*. By Lemma [5.7](#_bookmark29), there exists QB(*F*3) *∈ Pfin*(QB(*L*)) such that *F*3QB(*F*3) *x*. This shows that there exists QB(*F*3) *∈ Dx* such that QB(*F*1), QB(*F*2) ≤ QB(*F*3). Thus the family *Dx* is

directed.

We next show that *↑*QB(*L*)*x* = *{↑*QB(*L*)QB(*F* ) *|* QB(*F* ) *∈ Dx}*, where the subscript QB(*L*) means to take relevant operations in poset QB(*L*). Since *Dx ⊆* finQB(*L*)(*x*), we have *↑*QB(*L*)*x ⊆* *{↑*QB(*L*)QB(*F* ) *|* QB(*F* ) *∈ Dx}*. Let *z ∈* *{↑*QB(*L*)QB(*F* ) *|* QB(*F* ) *∈ Dx}*. By the construction of *Dx*, for all *F ∈* fin(*x*), we have *F* QB(*F* ) ≤ *z* and thus *z ∈ ↑F* . This shows that *z ∈* *{↑F | F ∈* fin(*x*)*}*. Since *x ∈* QB(*L*) is a quasi-approximated element, we have *z ∈* *{↑F | F ∈* fin(*x*)*}* = *↑x*. Thus *z ∈ ↑x ∩* QB(*L*) = *↑*QB(*L*)*x*. This shows that

*{↑*QB(*L*)QB(*F* ) *|* QB(*F* ) *∈ Dx}⊆ ↑*QB(*L*)*x.*

Therefore, *↑*QB(*L*)*x* = *{↑*QB(*L*)QB(*F* ) *|* QB(*F* ) *∈ Dx}*. By Definition [5.1](#_bookmark24), *x* is a quasi-approximated element of QB(*L*). It follows from the arbitrariness of *x* and Proposition [5.4](#_bookmark27) that QB(*L*) in the induced order is a quasicontinuous domain. *2*

**Theorem 5.9** *Let L be a dcpo. If for all x ∈ L, the family* fin(*x*) *is directed and*

*{↑F | F ∈* fin(*x*)*}* = *↑t for some t, then L is a quasicontinuous domain iff L satisﬁes the following two conditions:* (1) *the relation on L has the interpolation* *property as stated in Proposition* [*2.9*](#_bookmark9)*;* (2) *6 x, y ∈ L, x /*= *y* =*⇒* fin(*x*) */*= fin(*y*)*.*

**Proof.** *⇐*=: Let *x ∈ L*. By Proposition [5.6](#_bookmark28), there exists a quasi-approximated element QB(*x*) ≤ *x* such that fin(QB(*x*)) = fin(*x*). Applying condition (2), we have *x* = QB(*x*). Thus *L* = QB(*L*). By Proposition [5.4](#_bookmark27), *L* is a quasicontinuous domain.

=*⇒*: By Definition [2.8](#_bookmark8) and Proposition [2.9](#_bookmark9). *2*

# Conclusion remarks and future development

In this paper, in order to characterize (1) continuous posets, (2) hypercontinuous posets, and (3) quasicontinuous domains, we introduce notions of (hyper) approxi- mated elements for posets and quasi-approximated elements for dcpos by modifying Zhao’s concept of weakly approximated elements in [[15](#_bookmark43)]. We extend Zhao’s results in [[15](#_bookmark43)] for complete lattices in each of the above settings of posets in a systematic manner.

It should be stated that quasi-approximated elements are now only defined for dcpos. How to define quasi-approximated elements in general posets remains open. A recent result established in [[4](#_bookmark33)] that a dcpo *L* is quasicontinuous iff the poset *Qf L* of nonempty finitely generated upper sets ordered by reverse inclusion

*⊇* is continuous. It is not clear, though, whether this result can be generalized to posets with the concept of quasicontinuous posets in [[8](#_bookmark37)]. However, using the above result, it is often convenient to pass from quasicontinuous domains to continuous posets via the poset *Qf L*. By this trick one may view the single element *x* as its principal filter *↑x* in the poset *Qf L*. Then it may be true that an element *x ∈ L* is quasi-approximated iff *↑x* is approximated in *Qf L*. In this way, on one hand, for a general poset *L*, an element *x ∈ L* is quasi-approximated can be defined by that *↑x* is approximated in the poset *Qf L*; on the other hand, the results reported in Section 5 can be viewed as direct corollaries of those reported in Section 3 and

generalized to the setting of posets. For the details, we leave them as future work.

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