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Coalgebraic Logic over Measurable Spaces: Behavioral and Logical Equivalence

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**Abstract**

We study the relationship between logical and behavioral equivalence for coalgebras on general mea- surable spaces. Modal logics are interpreted in these coalgebras using predicate liftings. Prominent examples include stochastic relations and labelled Markov transition systems and corresponding Hennessy–Milner type logics. Local versions of logical and behavioral equivalence are introduced and it is shown that these notions coincide for a wide class of functors. We relate these notions to the corresponding global ones common in model checking. Throughout, we work in general measurable spaces. In contrast to previous work, no topological assumptions on the state spaces are needed.

*Keywords:* coalgebraic logic, measurable space

# Introduction

Coalgebras for an endofunctor provide a uniform framework for the study of reactive systems. In this article, we will study coalgebras for functors on the category **Meas** of measurables spaces and maps. The subprobability functor *S* will play a role similar to the powerset functor in that it allows us to treat stochastic aspects instead of non-deterministic ones.

Prominent examples include stochastic relations as coalgebras for the sub- probability functor *S* and so-called Markov transition systems [[4](#_bookmark19)], which arise as coalgebras for the functor *X '−→* (*SX*)*A* for some set *A* of actions.

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Modal logic, on the other hand, seems to be an appropriate language to talk about properties of coalgebras, and hence of reactive systems.

Modal operators for coalgebras are introduced using predicate liftings for the functor *T* [[10](#_bookmark25)], that is, natural transformations *Bn −→ B · T* . Here, *B* is the functor **Meas**op *−→* **Set** which sends every measurable space to its set of measurable subsets.

Two states in two coalgebras are called logically equivalent (with respect to a given family of predicate liftings) if they satisfy exactly the same formu- las. As usual, the two states are called behaviorally equivalent if we can find two morphisms whose domains are the given coalgebras, whose codomains coincide, and which map the two given states to the same state.

We show that under condition on the set of predicate liftings, the two no- tions of equivalence of states mentioned above coincide. This generalizes pre- vious results in this area in that we do not require surjectivity of the coalgebra morphisms, which leads us to local versions of the two notions of equivalence. This seems to be more akin to usual coalgebraic modal logic (over **Set**). We show how to deduce the previous, global, results from ours.

The main technical difficulty is in proving that the logic is *expressive*, that is, that logical equivalence implies behavioral equivalence. This is shown using quite a simple factoring technique. To ensure applicability of this technique, we have to rely on a somewhat technical concept which was—in lack of a better name—called *admissibility*.

In contrast to previous work, we do not impose any topological assumptions on the underlying spaces of the coalgebras. We believe that this makes the exposition more accessible.

## Related Work

Expressivity of modal logics for coalgebras over suitable measurable spaces have been studied quite extensively in recent years. The works can be divided into two groups. One deals with coalgebras over measurable spaces which satisfy some topological properties [[3](#_bookmark20),[4](#_bookmark19),[6](#_bookmark22),[12](#_bookmark26)].

Results for general measurable spaces have been established in [[2](#_bookmark18),[8](#_bookmark24),[5](#_bookmark21)], but only for coalgebras for the subprobability functor *S* or for *SA*. We try to reunite this two groups by establishing expressivity results for general endo- functors on **Meas**.

The basic factoring technique used here stems from [[6](#_bookmark22)] and was used in [[12](#_bookmark26)] for coalgebras for a general endofunctor on the category of analytic spaces. In

[[13](#_bookmark27)] the existence of final coalgebras based on Standard Borel spaces was used to extend the expressivity results from [[12](#_bookmark26)] to general measurable spaces.

The insight that it might be useful to consider a quotient-structure custom-

made to the logic at hand (and not the canonical one as used in [[6](#_bookmark22),[12](#_bookmark26)]) was first used in [[5](#_bookmark21)] and is somewhat hinted at in [[2](#_bookmark18)].

# Preliminaries

We collect here some basic definitions and results from measure theory for the reader’s convenience and for easier reference. Nearly all results are well-known.

## Measurable spaces

Recall that a measurable space *X* consists of a set *|X|* and a *σ*-algebra *BX* on *X*, that is: a family of subsets of *|X|* which is closed under comple- mentation, countable intersections, and countable unions. For each family *A* of subsets of a set *M* there is a smallest *σ*-algebra on *M* containing *A*, which we denote by *σ*(*A*). If *B*(*X*) = *σ*(*A*) then *A* is called a *generator* of *B*(*X*). A *measurable function X → Y* is given by a function *f* : *|X| → |Y |* such that *f−*1[*B*] *∈ BX* for all *B ∈ B*(*Y* ). In case *BY* = *σ*(*A*), measurability of *f* is guaranteed by *f−*1[*A*] *∈ BX* for all *A ∈ A*. The category of measurable spaces with measurable functions as morphisms is denoted by **Meas**. Observe that we do not notationally distinguish between a **Meas**-morphisms and its underlying function. Often we will just write *X* in place of *|X|*.

The assignment *X '→ XB* defines a functor **Meas**op *−→* **Set** with its action on morphisms given by the restriction of the inverse-image function, thus *Bf* = *f−*1 : *BY −→ BX* for *f* : *X −→ Y* . Observe that *B* is naturally isomorphic to the hom-functor **Meas**(*−,* **2**), with **2** the two-point space in which every subset is measurable. In particular, we obtain:

**Lemma 2.1** *B* : **Meas**op *−→* **Set** *preserves limits and Bf is injective pro- vided f is surjective.*

**Lemma 2.2** *We have σ*(*f−*1(*A*)) = *f−*1(*σ*(*A*))*.*

## Special morphisms

Given a family (*Yi*)*I* of measurable spaces and a family (*fi* : *A → |Yi|*)*I* of functions with common domain, we define the *initial σ-algebra* with respect

this data to be *A* = *σ*( *I f* [*B*(*Yi*)]). It is characterized by the following

*−*1

*i*

property: a function *g* : *|X| → A* is measurable with respect to *BX* and *A* if and only if all *fi · g* : *X → Yi* are measurable. In case *I* = *{∗}*, we have *A* = *f−*1[*B*(*Y∗*)].

*∗*

**Lemma 2.3** *Let* (*A −f→i*

*|Yi|*)*I be a family of functions and assume that each*

*Yi has a generator Gi. Then the initial σ-algebra with respect to the fi is*

*σ* *I*

*i*

*f−*1[*Gi*] *.*

**Proof.** See [[7](#_bookmark23), Satz 5.2].

Dually, we define the *ﬁnal σ-algebra* for (*|X | →gi*

*i*

*B*)*I*

by *{ E ⊆ B | ∀i ∈*

*I* : *g−*1[*E*] *∈ B*(*Xi*) *}*. It is characterized by the property that measurability of *h* : *B → |Y |* is guaranteed by the measurability of all *h · gi*.

*i*

This can be used to show that **Meas** is complete and cocomplete. Lim- its are constructed as follows: construct the limit (*L,* (*li*)) of the underlying diagram in **Set** and equip *L* with the initial *σ*-algebra with respect to the projections *li*, thus, with the *σ*-algebra generated by *l−*1[*BXi*]. Colimits are

*i*

formed dually using final *σ*-algebras.

## Equivalence relations and invariant sets

Let *α* be an equivalence relation on a set *X*. Let *ηα* : *X −→ X/α* denote the projection. We obtain an adjunction:

*ηα*[*−*]

*PX* ¸, *P* (*X/α*) *,*

*η−*1[*−*]

*α*

where *PX* denotes the powerset of *X*, ordered by inclusion. Since *ηα* is sur- jective, every *B ∈ P* (*X/α*) is a fixpoint of the above adjunction. On the other hand, the fixpoints of *η−*1 *· ηα*[*−*] are easily characterized as the *α-invariant* subsets of *X*. Here, *A ⊂ X* is *α*-invariant provided *x ∈ A* and *xα x'* imply *x' ∈ A* for each *x*, *x'* in *X*. Write Inv(*α*) for the *α*-invariant subsets of *X*. The adjunction restricts to a pair of mutually inverse functions

*α*

*ηα*[*−*]

Inv(*α*) ¸, *P* (*X/α*)*.*

*η−*1[*−*]

*α*

(1)

Fix a family *A* of subsets of a set *X*. Define an equivalence relation Eq(*A*) by:

(*x, x'*) *∈* Eq(*A*) *⇐⇒ ∀A ∈A* : [*x ∈ A ⇐⇒ x' ∈ A*]*.*

Obviously, every *A ∈A* is Eq(*A*)-invariant.

**Lemma 2.4** *We have* Eq(*A*)= Eq(*σA*)*.*

## Separable Measurable spaces

We say that a family *A* of subsets of a set *X separates points* if whenever

*x /*= *x'* then there exists *A ∈A* with *x ∈ A*, *x' ∈/ A*, or vice versa.

**Lemma 2.5** *A separates points if, and only if, σ*(*A*) *separates points.*

**Definition 2.6** A measurable space *X* is called *separable* if *BX* separates points. The full subcategory of **Meas** spanned by the separable objects is denoted by **Sep**.

**Proposition 2.7 Sep** *is closed under mono-sources and coproducts in* **Meas Proof.** Let (*fi* : *X −→ Yi*)*I* be a mono-source in **Meas** and assume *x /*= *x'*

in *X*. There exists *i ∈ I* with *fi*(*x*) */*= *fi*(*x'*), hence *B ∈ BYi* with (say)

*fi*(*x*) *∈ B*, *fi*(*x'*) *∈/*

*B*. Hence *x ∈ f−*1[*B*], *x' ∈/ f−*1[*B*], and *f−*1[*B*] *∈ BX*

*i i* *i*

since *fi* is measurable. Closure under coproducts is obvious.

## Subprobability measures

A *subprobability measure* on a measurable space *X* is a *σ*-additive func- tion *BX →* [0*,* 1]. The set of all subprobability measures on *X* becomes a measurable space *SX* when equipped with the initial *σ*-algebra with respect to (ev*A*)*A∈BX* with ev*A* : *SX →* [0*,* 1], *μ '→μ*(*A*). Here, [0*,* 1] is equipped with the *σ*-algebra generated by the open subsets. Another generator of this *σ*-algebra is *{* [*r,* 1] *| r ∈* Q *∩* [0*,* 1] *}*.

This subprobability construction gives rise to a functor *S* : **Meas** *→* **Meas**

by setting

*Sf* (*μ*)(*B*)= *μ*(*f−*1[*B*])

for *f* : *X → Y* in **Meas**, *μ ∈ SX*, *B ∈ BY* .

**Lemma 2.8** *Each SX is separable.*

**Proof.** Observe that (ev*A*)*A∈BX* is a mono-source. Thus the claim follows from Proposition [2.7](#_bookmark4)

**Lemma 2.9** *Let A ⊂ BX be closed under ﬁnite intersections with σ*(*A*) =

*BX. Then B*(*SX*) *is generated by the set {* ev*−*1[*r,* 1] *| A ∈ A,r ∈* Q *∩* [0*,* 1] *}.*

*A*

**Proof.** Combine Lemma 3.6 from [[14](#_bookmark30)] with Lemma [2.3](#_bookmark1).

# Coalgebras and Models

Let *T* : **Meas** *−→* **Meas** be a functor. A *T-coalgebra* A = (*A, d*) is given by measurable space *A* and a **Meas**-morphism *d* : *A → T A*, called the *dynamics*. A morphism of coalgebras (*A, d*) *→* (*A', d'*) is given by a **Meas**-morphism

*f* : *A → A'* such that

*f*  *'*

*A A*

*d d'*

J J

*TA Tf* *T* *A'*

commutes. This leads to the category **Coalg** *T* of *T* -coalgebras and mor- phisms.

**Examples 1** (i) *Coalgebras for the subprobability functor are stochastic re- lations.*

(ii) *Coalgebras for the functor on* **Meas** *given by X '→*(*SX*)Act *for some set*

Act *are labelled Markov processes.*

We fix a set Var of *variables*, and define *models* with respect to *T* and Var

as follows:

**Definition 3.1** A (*T,* Var)*-model* consists of a *T* -coalgebra (*A, d*) together with a *valuation*; that is, a function Var *−V→ B*(*A*). A morphism (*A, d, V* ) *−f→* (*A', d',V '*) is given by a coalgebra morphism (*A, d*) *−f→* (*A', d'*) which satisfies *B*(*f* ) *· V '* = *V* .

This leads us to a category **Mod**(*T,* Var) of models. Observe that we have an isomorphism **Mod**(*T, ∅*) *∼*= **Coalg** *T* .

From now on we fix functor *T* , and the set of variables; **Mod**(*T,* Var) will simply be denoted by **Mod**.

It is well-known that the obvious forgetful functor **Coalg** *T −→* **Meas** creates colimits; see [[1](#_bookmark17)]. Thus, **Coalg** *T* is cocomplete since **Meas** is so. This generalizes to models:

**Proposition 3.2** *The obvious forgetful functor* **Mod** *−→* **Coalg** *T creates colimits. Hence also the forgetful functor* **Mod** *−→* **Meas** *creates colimits.*

**Proof.** Let *D* : **D** *−→* **Mod** be a (small) diagram. Write *D*(*i*)= (*Xi, di, Vi*) and let ((*X, d*)*, ci*) denote the colimit of the (*Xi, di*) in **Coalg** *T* . Observe that (*BX, Bci*) is a limit in **Set**. Since *Dd* is a model morphism for each *d* : *i → j* in **D** the collection (*Vi*) of valuations forms a natural cone, hence there is a unique *V* : Var *−→ BX* with *Bci · V* = *Vi* for each *i*. We claim that (*X, d, V* ) is a colimit of *D* in **Mod**. Let (*fi* : *Di −→* (*Y, e, W* )) be a natural cocone, and let *f* : *X −→ Y* be the induced **Coalg** *T* -morphism. Thus, *f · ci* = *fi* holds. We have

*Bci · Bf · W* = *Bfi · W* = *Vi* = *Bci · V,*

where the first equation holds since *fi* is a model morphism. Thus, *Bf · W* =

*f*

*V* follows from the fact that (*Bci*) is a limit-source, that is: (*X, d.V* ) *−→*

(*Y, e, W* ) is a model-morphism.

**Corollary 3.3 Mod** *is cocomplete.*

# Predicate Liftings

Let *T* : **Meas** *→* **Meas** be a functor. An *n-ary predicate lifting* for *T* is a natural transformation *λ* : *Bn →B · T* . We write *n* = ar(*λ*).

**Examples 2** (i) *Let r be a rational number and deﬁne λr* : *BX −→ BSX*

*X*

*via:*

*λr* (*B*)= *{ μ | μ*(*B*) *≥ r }* = ev*−*1[*r,* 1]*.*

*X B*

(*λr*) *is easily seen to be natural. We write* Λ1 = *{ λr | r ∈* Q *∩* [0*,* 1] *}.*

1. *Let* Act *be a countable set of “actions”. For each rational r and each*

*a ∈* Act*, we deﬁne a predicate lifting λr,a for S*Act *via*

*λr,a*(*B*)= *{* (*μi*)*i∈*Act *| μa*(*B*) *≥ r }.*

*X*

*We write* ΛAct = *{ λr,a | a ∈* Act*,r ∈* Q *∩* [0*,* 1] *}.*

1. *Let T denote the functor given by X '→S*(*X ×X*) *with its obvious action on morphisms. We deﬁne a family* (*κq*)*q∈*Q*∩*[0*,*1] *via*

*κq* (*A, B*)= *{ μ ∈ S*(*X × X*) *| μ*(*A × B*) *≥ q }* = ev*−*1

[*q,* 1]*.* (2)

*X A×B*

## The Logic Induced by a Family of Predicate Liftings

Fix a set Λ of predicate liftings. We define the logic induced by Λ (and

Var) by the following grammar:

*φ* ::= *T| φ*1 *∧ φ*2 *| v | ⟨λ⟩*(*φ*1*,..., φ*ar(*λ*))

for *v ∈* Var, *λ ∈* Λ.

**Remark 4.1** It is possible to enrich the logic by using further Boolean connec- tives (modelled by natural transformations *Bn −→ B*) and fixpoint-operators (modelled by natural transformations *Bω −→ B*); see [[6](#_bookmark22)] for a discussion. We refrain from doing so, the result go through verbatim.

## Interpreting the Logic in a Model

Given a model *M* = (*A, d, V* ), we may interpret *L*(Λ) by assigning a set

*φ*) *∈ BA* to every formula *φ* as follows:

*T*)*M* = *A*

*φ*1 *∧ φ*2)*M* = *φ*1)*M ∩* *φ*2)*M*

*v*)*M* = *V* (*v*)

*⟨λ⟩*(*φ*1*,..., φ*ar(*λ*)))*M* = *Bd · λA*( *φ*1)*M,...,* *φ*ar(*λ*))*M*) For a state *x* of *M* we write Th*M*(*x*)= *{ φ ∈L| x ∈* *φ*)*M }*.

**Proposition 4.2** *If f* : *M −→ N is a model morphism, then we have*

*Bf* ( *φ*)*N* )= *φ*)*M for each φ.*

**Proof.** The proof proceeds by structural induction on formula *φ* and makes use of the naturality of the predicate liftings. For details see [[6](#_bookmark22)], where the unimodal case was treated.

**Corollary 4.3** *If M −→ N, then* Th*M*(*x*)= Th*N* (*f* (*x*))*.*

*f*

## An equivalence relation

Fix a model *M* = (*A, d, V* ). Let *l* denote the equivalence on *A* which is determined by the extensions of the formulas. Thus,

*xlx' ⇐⇒ ∀φ* : [*x ∈* *φ*) *⇐⇒ x' ∈* *φ*)]*,*

or, more compressed, *l* = Eq(*{* *φ*) *| φ ∈ L}*). Observe that we have *xlx' ⇐⇒*

Th*M*(*x*)= Th*M*(*x'*).

We write *EM* = *σ*(*{ ηl*[ *φ*)*M*] *| φ* formula *}*).

**Lemma 4.4** *We have { ηl*[ *φ*)] *| φ formula }* = *{ A ⊂ X/l | η−*1[*A*]= *φ*) *for*

*l*

*some φ }. Both sets are closed under ﬁnite intersections.*

**Proof.** Just use the fact that each *φ*) is *l*-invariant and apply the bijec- tions from ([1](#_bookmark2)). The second claim follows since the set of validity-sets is closed under finite intersection and the inverse image function preserves finite inter- sections.

**Lemma 4.5** *ηl* : *X −→* (*X/l, EM*) *is measurable.*

**Proof.** Obvious by Lemma [4.4](#_bookmark8).

In general, *EM will not be the ﬁnal σ-algebra* with respect to *BX* and *ηl*. In the following, *ηM* will always denote the measurable function whose domain is (*X/l, EM*).

**Lemma 4.6** *Fix a measurable space Y and a generator A of BY . A function f* : *X/l −→ |Y | is measurable if for each A ∈ A there exists φ such that η−*1 *· f−*1[*A*]= *φ*)*.*

**Proof.** Obvious since it suffices to check measurability on a generator.

We will introduce two technical conditions on sets of predicate liftings: separability and admissibility. Our notion of separable predicate liftings is a specialization of the one used in coalgebraic logic over **Set**, cf. [[11](#_bookmark28)]. It was first introduced in [[6](#_bookmark22)] for sets of unary predicate liftings.

Admissibility is another technical condition which we need to impose in order to prove measurability of an induced function. It is introduced below and related to separability.

## Separating Predicate Liftings

Let *λ* : *Bn −→ B· T* be an *n*-ary predicate lifting. We define an equivalence

relation *≡λ*

*M*

on *TX* by setting

*λ* = Eq (*{ λX*( *φ*1)*M,...,* *φn*)*M*) *| φ*1*,..., φn ∈ L}*) *.*

*≡*

*M*

**Definition 4.7** A set Λ of predicate liftings is said to *separate* a model *M* if we have

*≡λ*

*M*

*⊆* ker *T* (*ηM*)*.*

*λ∈*Λ

Λ is said to be *separating* if it separates every model *M*.

**Examples 3** *The sets* Λ1 *and* ΛAct *are separating. This follows from Lemma*

[*4.9*](#_bookmark10) *(below) and Proposition* [*4.11*](#_bookmark11)*.*

## Admissible predicate liftings

**Definition 4.8** We call a set Λ of predicate liftings *admissible* if the set

*{ λX/l*(*η* *φ*1)*,..., η* *φ*ar(*λ*))) *| λ ∈* Λ*, φ*1*,..., φ*ar(*λ*) *∈ L}*

generates *B*(*T* (*X/l, EM*)) for every model *M*.

**Lemma 4.9** *For each set* Act*, the family* ΛAct *is admissible.*

**Proof.** Write *Q* = (*X/l, EM*). First consider the case Act = *{∗}*. We know from Lemma [2.9](#_bookmark5) and Lemma [4.4](#_bookmark8) that *BSQ* is generated by

*D* = *{* ev*−*1

*η* *φ*)

[*r,* 1] *| φ ∈ L,r ∈* Q *∩* [0*,* 1] *}.*

Since ev*−*1 [*r,* 1] = *λr* (*η* *φ*)) holds, this is just the condition for admissibility.

*y* *φ*)

*Q*

For a general set Act, observe that we have

*C* = *{ λr,α*(*η* *φ*)) *| a ∈* Act*,r ∈* Q *∩* [0*,* 1]*,φ ∈ L}*

*Q*

= *{ λr,α*(*η* *φ*)) *| r ∈* Q *∩* [0*,* 1]*,φ ∈ L}*

*Q*

*α∈*Act

=

*π−*1 *{ λr* (*η* *φ*)) *| r ∈* Q *∩* [0*,* 1]*,φ ∈ L}*

*α∈*Act

=

*α∈*Act

*α Q*

*π−*1(*D*)

*α*

holds, where *πα* : *SQ*Act *−→ SQ* is the *a*th projection. Since *D* generates

*BSQ*, *C* generates *B*(*SQ*Act) by Lemma [2.3](#_bookmark1).

**Lemma 4.10** *The set* (*κr*)*r∈*Q*∩*[0*,*1] *is admissible.*

**Proof.** For measurable spaces *X* and *Y* the set *{ A × B | A ∈ BX, B ∈ BY }* is a closed under finite intersections and a generator for *B*(*X × Y* ). Thus, the claim follows from Lemma [2.9](#_bookmark5).

**Proposition 4.11** *If each TX is separated then every admissible set of pred- icate liftings is separated.*

**Proof.** Fix a model *M* = (*X, d, V* ), *λ ∈* Λ of arity *n* and *φ*1*,..., φn ∈ L*. We have, for each *t ∈ TX*:

*Tη*(*t*) *∈ λX/l*(*η* *φ*1)*,..., η* *φn*)) *⇐⇒ t ∈ BTη · λX/l*(*η* *φ*1)*,..., η* *φn*))

*⇐⇒ t ∈ λX ·* (*Bη*)*n*(*η* *φ*1)*,..., η* *φn*))

*⇐⇒ t ∈ λX*( *φ*1)*,...,* *φn*))

where the last equality holds since each *φi*) is invariant.

Take *t, t' ∈ TX* with *t ≡λ t'* for all *λ ∈* Λ. We need to show *Tη*(*t*) =

*M*

*Tη*(*t'*). By separability of *T* (*X/l, EM*) and Lemma [2.5](#_bookmark3), it suffices to show

(*Tη*(*t*)*,Tη*(*t'*)) *∈* Eq *{ λX/l*(*η* *φ*1)*,..., η* *φ*ar(*λ*))) *| λ ∈* Λ*, φ*1*,..., φ*ar(*λ*) *∈ L}* *.*

From the above calculation it follows that this last condition is equivalent to

(*t, t'*) *∈ ≡λ* .

*M*

## The congruence theorem

**Theorem 4.12** *Let* Λ *be a separating and admissible set of predicate liftings. For every model M there exists a model structure M/l over X/lM such that ηM* : *M −→ M/lM is a model morphism.*

**Proof.** We write *M* = (*X, d, V* ), *l* for *lM*, and *Q* for (*X/lM, EM*). We claim that *xlx'* implies (*d*(*x*)*, d*(*x'*)) *∈* ker *Tη*. By separatedness of Λ it suffices to

show that we have *d*(*x*) *≡λ d*(*x'*) for each *λ ∈* Λ.

*M*

Take *λ ∈* Λ, write *n* = ar(*λ*) and fix *φ*1*,..., φn ∈ L*. We obtain:

*d*(*x*) *∈ λX*( *φ*1)*,...,* *φn*)) *⇐⇒ x ∈ Bd · λX*( *φ*1)*,...,* *φn*))

*⇐⇒ x ∈* *⟨λ⟩*(*φ*1*,..., φ*1))

hence (*d*(*x*)*, d*(*x'*)) *∈ ≡λ* follows from *xlx'*. Hence, there exists a unique

*M*

function *q* : *Q −→ TQ* for which

*X y*  *Q*

*d q*

J J

*TX Ty* *T* *Q*

commutes. We need to show that *q* is measurable.

Fix any *λ* of arity *n* and *φ*1*,..., φn ∈L* and observe:

*η−*1 *· q−*1 *· λQ*(*η* *φ*1)*,..., η* *φn*))= *Bd · BTη · λQ*(*η* *φ*1)*,..., η* *φn*))

= *Bd · λX ·* (*Bη*)*n*(*η* *φ*1)*,..., η* *φn*))

= *Bd · λX*(*η−*1[*η* *φ*1)]*,... η−*1[*η* *φn*)])

*†*

= *Bd · λX*( *φ*1)*,...,* *φn*))

= *⟨λ⟩*(*φ*1*,..., φn*))*,*

where (*†*) holds by invariance of *φi*). Measurability of *q* follows from admis- sibility of Λ and Lemma [4.6](#_bookmark9).

We are left to define a valuation *W* on *Q*. For *v ∈* Var we set *W* (*v*) = *η* *v*)*M ∈ EM*. By *l*-invariance of *v*)*M* we obtain *Bη · W* (*v*) = *η−*1[*ηV* (*v*)] = *V* (*v*), thus *W* makes *η* a model morphism. Uniqueness of *W* with this property holds by injectivity of *Bη*.

**Definition 4.13** The *reduct* Red(*M*) of *M* is the model (*Q, q, W* ) constructed in Theorem [4.12](#_bookmark12).

Thus, the underlying set of *Q* is given by *X/lM* and we have *BQ* = *EM* =

*σ*(*{ η*[ *φ*)*M*] *| φ ∈ L}*).

## Over analytic spaces

Recall that a measurable space is called *analytic* if its measurable subsets arise as the Borel sets of a topological space which is the continuous image of a metrizable topological space with a countable base.

In case the underlying measurable space of the model *M* is analytic— which is the blanket assumption in [[4](#_bookmark19),[6](#_bookmark22),[12](#_bookmark26)]—we can form the reduct Red(*M*) without relying on admissibility of Λ.

**Lemma 4.14** *Let M* = (*A, d, V* ) *be a model with A analytic and assume that L is countable. Then EM is the ﬁnal σ-algebra with respect to the projection ηlM .*

**Proof.** See [[5](#_bookmark21), Corollary 3].

Observe that *L* is countable provided both Λ and Var are countable.

**Theorem 4.15** *Let* Λ *be a countable, separating set of predicate liftings and let* Var *be countable. For every model M* = (*A, d, V* ) *with A analytic there* *exists a model structure M/l over A/l such that η* : *M −→ M/l is a model morphism.*

**Proof.** We proceed as in the proof of Theorem [4.12](#_bookmark12) to define the quotient dynamics *q*. Observe that well-definedness of *q* just makes use of separatedness of Λ. Measurability of *q* follows immediately from Lemma [4.14](#_bookmark13) since we have *q · η* = *Tη · d* and the latter function is measurable.

# Logical and Behavioral Equivalence

**Definition 5.1** We fix models *M*, *M'* and states *a* in *M*, *a'* in *M'*.

* 1. We say that the states *a* and *a'* are *logically equivalent* if Th*M*(*a*) =

Th*M'* (*a* ).

*'*

* 1. We say that the models *M* and *M'* are *logically equivalent* if *{* Th*M*(*a*) *|*

*a ∈ A }* = *{* Th*M'* (*a* ) *| a ∈ A }*.

*' ' '*

* 1. We say that the states *a* and *a' behaviorally equivalent* if there exists a model *N* and a cospan

*M −f→ N ←g− M'* (3)

of model morphisms such that *f* (*a*)= *g*(*a'*) holds;

* 1. We say that the models *M* and *M'* are *behaviorally equivalent* if there exists a cospan ([3](#_bookmark14)) with *f* and *g surjective* model morphisms.

Observe that the notions of equivalence introduced above naturally divide themselves into two groups: *local* or state-based notions (1,3), and *global* notions (2,4). Proposition [4.2](#_bookmark6) and Corollary [4.3](#_bookmark7) entail that:

* + - (local) behavioral equivalence of states implies their (local) logical equiva- lence.
    - (global) behavioral equivalence of models implies their (global) logical equiv- alence;

We will now show that under suitable conditions, all four implications can be reversed. So far, work on modal logics for stochastic relations was somewhat concentrated on the global aspects. We think that the local notions (as cus- tomary for coalgebras on **Set**) are more in the spirit of classical modal logic, which is intrinsically local.

**Theorem 5.2** *Assume that* Λ *is separating and admissible. Fix models M and N. If states of x of M and y of N are logically equivalent, then they are behaviorally equivalent.*

**Proof.** We form first the coproduct *M* + *N* of *M* and *N* and then the reduct Red(*M* + *N* ) according to Theorem [4.12](#_bookmark12). Thus, we obtain the following diagram of model morphisms

*M*  *iM*  *M* + *N* ¸*iN*, *N*

*y*

J

Red(*M* + *N* )

(4)

By Corollary [4.3](#_bookmark7), we have Th*M*+*N* (*iM*(*x*)) = Th*M*(*x*)= Th*N* (*y*)= Th*M*+*N* (*iN* (*b*)). Therefore, *η · iM*(*x*)= *η · iN* (*y*); that is, Red(*M* + *N* ) witnesses that *a* and *b* are behaviorally equivalent.

The following result on the equivalence of the global properties is in fact an easy consequence of Theorem [5.2](#_bookmark15):

**Theorem 5.3** *Assume that* Λ *is separating and admissible. Then logical equivalence of models implies behavioral equivalence of models.*

**Proof.** Let *M* and *N* be logically equivalent and form the diagram ([4](#_bookmark16)). We claim that *η · iM* and *η · iN* are surjective. Consider *η · iM* and take any state

[*z*] in Red(*M* + *N* ). In case *z* is in the image of *iM*, we are done. Otherwise, *z* = *iN* (*y*) for some state of *N* . We find a state *x* of *M* with Th*M*(*x*)= Th*N* (*y*), hence Th*M*+*N* (*iM*(*x*)) = Th*M*+*N* (*iN* (*y*)), that is *η · iM*(*x*)= *η · iN* (*y*)= *η*(*z*)= [*z*].

# Conclusion and Further Work

We have established expressivity-results for coalgebraic logic over general mea- surable spaces. In doing so, we have improved over previously published work in this area in three aspects:

* 1. we were able to work without any topological assumptions on the state space;
  2. we work with a general functor on the category of measurable spaces without relying on what was called left or right coalgebras [[6](#_bookmark22)];
  3. we considered local versions of behavioral and logical equivalence which deal with single states as opposed to the whole models. This seems to be in the spirit of coalgebraic modal logic over the category of sets. In fact, the global properties are simple consequences of the local ones.

One obvious extension of the work presented here would be to include a discussion of bisimilarity as well. This is hindered by the fact that the subprobability functor does not preserve weak pullbacks.

Another possible extension of the results presented here stems from the observation that a measurable space can be seen as a set equipped with a basis for a zero-dimensional topology on it. In particular, the technical notion of admissibility can be rephrased as a continuity condition for these topolo- gies. This hints at a possible unification of our work and the work on Stone coalgebras [[9](#_bookmark29)].

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