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Comparing Control Constructs by Double-barrelled CPS Transforms

# Hayo Thielecke

[H.Thielecke@cs.bham.ac.uk](mailto:H.Thielecke@cs.bham.ac.uk) School of Computer Science University of Birmingham Birmingham B15 2TT

United Kingdom

Abstract

We investigate continuation-passing style transforms that pass two continuations. Altering a single variable in the translation of λ-abstraction gives rise to different control operators: first-class continuations; dynamic control; and (depending on a further choice of a variable) either the return statement of C; or Landin’s J- operator. In each case there is an associated simple typing. For those constructs that allow upward continuations, the typing is classical, for the others it remains intuitionistic, giving a clean distinction independent of syntactic details.

# Introduction

Control operators come in bewildering variety. Sometimes the same term is used for distinct constructs, as with catch in early Scheme or throw in Standard ML of New Jersey, which are very unlike the catch and throw in Lisp whose names they borrow. On the other hand, this Lisp catch is fundamentally similar to exceptions despite their dissimilar and much more ornate appearance.

Fortunately it is sometimes possible to glean some high-level “logical” view of a programming language construct by looking only at its type. Specifically for control operations, Griffin’s discovery [3] that call/cc and related op- erators can be ascribed classical types gives us the fundamental distinction between languages that have such classical types and those that do not, even though they may still enjoy some form of control. This approach complements comparisons based on contextual equivalences [10,14].

Such a comparison would be difficult unless we blot out complication. In particular, exceptions are typically tied in with other, fairly complicated features of the language which are not relevant to control as such: in ml with the datatype mechanism, in Java with object-orientation. In order to

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simplify, we first strip down control operators to the bare essentials of labelling and jumping, so that there are no longer any distracting syntactic differences between them. The grammar of our toy language is uniformly this:

*M* ::= *x* | *λx.M* | *MM* | here *M* | go *M.*

The intended meaning of here is that it labels a “program point” or expression without actually naming any particular label—just uttering the demonstrative “here”, as it were. Correspondingly, go jumps to a place specified by a here, without naming the “to” of a goto.

Despite the simplicity of the language, there is still scope for variation: not by adding bells and whistles to here and go, but by varying the meaning of *λ*-abstraction. Its impact can be seen quite clearly in the distinction be- tween exceptions and first-class continuations. The difference between them is as much due to the meaning of *λ*-abstraction as due to the control operators themselves, since *λ*-abstraction determines what is statically put into a closure and what is passed dynamically. Readers familiar with, say, Scheme imple- mentations will perhaps not be surprised about the impact of what becomes part of a closure. But the point of this paper is twofold:

* small variations in the meaning of *λ* completely change the meaning of our control operators;
* we can see these differences at an abstract, logical level, without delving into the innards of interpreters.

We give meaning to the *λ*-calculus enriched with here and go by means of continuations in Section 2, examining in Sections 3–5 how variations on *λ*- abstraction determine what kind of control operations here and go represent. For each of these variations we present a simple typing, which agrees with the transform (Section 6). We conclude by explaining the significance of these typings in terms of classical and intuitionistic logic (Section 7).

# Double-barrelled CPS

Our starting point is a continuation-passing style (cps) transform. This trans- form is double-barrelled in the sense that it always passes *two* continuations. Hence the clauses start with *λkq....* instead of *λk* Other than that, this

cps transform is in fact a very mild variation on the usual call-by-value one [8]. As indicated by the ? , we leave one variable, the extra continuation passed to the body of a *λ*-abstraction, unspecified.

[[*x*]] = *λkq.kx*

[[*λ*?*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r* ? )

[[*MN* ]] = *λkq.*[[*M* ]](*λm.*[[*N* ]](*λn.mnkq*)*q*)*q* [[here *M* ]] = *λkq.*[[*M* ]]*kk*

[[go *M* ]] = *λkq.*[[*M* ]]*qq*

The extra continuation may be seen as a jump continuation, in that its

manipulation accounts for the labelling and jumping. This is done symmet- rically: here makes the jump continuation the same as the current one *k*, whereas go sets the current continuation of its argument to the jump contin- uation *q*. The clauses for variables and applications do not interact with the additional jump continuation: the former ignores it, while the latter merely distributes it into the operator, the operand and the function call.

Only in the clause for *λ*-abstraction do we face a design decision. Depend- ing on which continuation (static *s*, dynamic *d*, or the return continuation *r*) we fill in for “?” in the clause for *λ*, there are three different flavours of *λ*-abstraction.

[[*λ*s*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r s* )

[[*λ*d*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r d* )

[[*λ*r*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r r* )

The lambdas are subscripted to distinguish them, and the box around the last variable is meant to highlight that this is the crucial difference between the transforms. Formally there is also a fourth possibility, the outer continuation *k*, but this seems less meaningful and would not fit into simple typing.

For all choices of *λ*, the operation go is always a jump to a place specified by a here. For example, for any *M* , the term here ((*λx.M* )(go *N* )) should be equivalent to *N* , as the go jumps past the *M* . But in more involved examples than this, there may be different choices *where* go can go to among several occurrences of here. In particular, if *s* is passed as the second continuation argument to *M* in the transform of *λx.M* , then a go in *M* will refer to the here that was in scope at the point of definition (unless there is an intervening here, just as one binding of a variable *x* can shadow another). By contrast, if *d* is passed to *M* in *λx.M* , then the here that is in scope at the point of definition is forgotten; instead go in *M* will refer to the here that is in scope at the point of call when *λx.M* is applied to an argument. In fact, depending upon the choice of variable in the clause for *λ* as above, here and go give rise to different control operations:

* first-class continuations like those given by call/cc in Scheme [4];
* dynamic control in the sense of Lisp, and typeable in a way reminiscent of checked exceptions;
* a return-operation, which can be refined into the J-operator invented by Landin in 1965 and ancestral to call/cc [4,6,7,13].

We examine these constructs in turn, giving a simple type system in each case. An unusual feature of these type judgements is that, because we have two continuations, there are two types in the succedent on the right of the turnstile, as in

Γ ▶ *M* : *A, B.*

The first type on the right accounts for the case that the term returns a value; it corresponds to the current continuation. The second type accounts for the

Fig. 1. Typing for static here and go

Γ*,x* : *A,* Γ' ▶s *x* : *A, C*

Γ ▶s *M* : *B, B*

Γ ▶s here *M* : *B, C*

Γ ▶s *M* : *B, B*

Γ ▶s go *M* : *C, B*

Γ*,x* : *A* ▶s *M* : *B, C*

Γ ▶s *λ*s*x.M* : *A* → *B, C*

Γ ▶s *M* : *A* → *B, C*

Γ ▶s *N* : *A, C*

Γ ▶s *MN* : *B, C*

jump continuation. In logical terms, the comma on the right may be read as a disjunction. It makes a big difference whether this disjunction is classical or intuitionistic. That is our main criterion of comparing and contrasting the control constructs.

# First-class continuations

The first choice of which continuation to pass to the body of a function is arguably the cleanest. Passing the static continuation *s* gives control the same static binding as ordinary *λ*-calculus variables. In the static case, the transform is this:

[[*x*]] = *λkq.kx*

[[*λ*s*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r s* )

[[*MN* ]] = *λkq.*[[*M* ]](*λm.*[[*N* ]](*λn.mnkq*)*q*)*q* [[here *M* ]] = *λkq.*[[*M* ]]*kk*

[[go *M* ]] = *λkq.*[[*M* ]]*qq*

We type our source language with here and go as in Figure 1.

In logical terms, both here and go are a combined right weakening and contraction. By themselves, weakening and contraction do not amount to much; but it is the combination with the rule for →-introduction that makes the calculus “classical”, in the sense that there are terms whose types are propositions of classical, but not of intuitionistic, minimal logic.

To see how →-introduction gives classical types, consider *λ*-abstracting over go.

*x* : *A* ▶s go *x* : *B, A*

▶s *λ*s*x.*go *x* : *A* → *B, A*

If we read the comma as “or”, and *A* → *B* for arbitrary *B* as “not *A*”, then this judgement asserts the classical excluded middle, “not *A* or *A*”. We build on the classical type of *λ*s*x.*go *x* for another canonical example: Scheme’s

call-with-current-continuation (call/cc for short) operator [4]. It is syntactic sugar in terms of static here and go:

call/cc = *λ*s*f.*(here (*f* (*λ*s*x.*go *x*)))*.*

As one would expect [3], the type of call/cc is Peirce’s law “if not *A* implies

*A*, then *A*”. We derive the judgement

▶s *λ*s*f.*(here (*f* (*λ*s*x.*go *x*))) : ((*A* → *B*) → *A*) → *A, C* as follows. Let Γ be the context *f* : (*A* → *B*) → *A*. Then we derive:

Γ ▶s *f* : (*A* → *B*) → *A, A*

Γ*,x* : *A* ▶s *x* : *A, A*

Γ*,x* : *A* ▶s go *x* : *B, A*

Γ ▶s *λ*s*x.*go *x* : *A* → *B, A*

Γ ▶s (*f* (*λ*s*x.*go *x*)) : *A, A*

Γ ▶s here (*f* (*λ*s*x.*go *x*)) : *A, C*

▶s *λ*s*f.*(here (*f* (*λ*s*x.*go *x*))) : ((*A* → *B*) → *A*) → *A, C*

As another example, let Γ be any context, and assume we have Γ ▶s *M* : *A, B*. Right exchange is derivable in that we can also derive Γ ▶s *M* ' : *B, A* for some *M* '.

In the typing of call/cc, a go is (at least potentially, depending on *f* ) exported from its enclosing here. Conversely, in the derivation of right ex- change, a go is imported into a here from without. What makes everything work is static binding.

# Dynamic control

Next we consider the dynamic version of here and go. The word “dynamic” is used here in the sense of dynamic binding and dynamic control in Lisp. Another way of phrasing it is that with a dynamic semantics, the here that is in scope at the point where a function is *called* will be used, as opposed to the here that was in scope at the point where the function was *deﬁned* —the latter being used for the static semantics.

In the dynamic case, the transform is this: [[*x*]] = *λkq.kx*

[[*λ*d*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r d* )

[[*MN* ]] = *λkq.*[[*M* ]](*λm.*[[*N* ]](*λn.mnkq*)*q*)*q* [[here *M* ]] = *λkq.*[[*M* ]]*kk*

[[go *M* ]] = *λkq.*[[*M* ]]*qq*

In this transform, the jump continuation acts as a handler continuation; since it is passed as an extra argument on each call, the dynamically enclosing handler is chosen. Hence under the dynamic semantics, here and go become a stripped-down version of Lisp’s catch and throw with only a single catch

Fig. 2. Typing for dynamic here and go

Γ*,x* : *A,* Γ' ▶d *x* : *A, C*

Γ ▶d *M* : *B, B*

Γ ▶d here *M* : *B, C*

Γ ▶d *M* : *B, B*

Γ ▶d go *M* : *C, B*

Γ*,x* : *A* ▶d *M* : *B, C*

Γ ▶d *λ*d*x.M* : *A* → *B* ∨ *C, D*

Γ ▶d *M* : *A* → *B* ∨ *C, C*

Γ ▶d *N* : *A, C*

Γ ▶d *MN* : *B, C*

tag. These catch and throw operation are themselves a no-frills version of exceptions with only identity handlers. We can think of here and go as a special case of these more elaborate constructs:

here *M* ≡ (catch ’e *M* ) go *M* ≡ (throw ’e *M* )

Because the additional continuation is administered dynamically, we can- not fit it into our simple typing without annotating the function type. So for dynamic control, we write the function type as *A* → *B* ∨ *C*. Syntactically, this

should be read as a single operator with the three arguments in mixfix. We

regard the type system as a variant of intuitionistic logic in which → and ∨

always have to be introduced or eliminated together.

This annotated arrow can be seen as an idealization of the Java throws

clause in method definitions, in that *A* → *B* ∨ *C* could be written as

*B*(*A*) throws *C*

in a more Java-like syntax. A function of type *A* → *B* ∨ *C* may throw things of type *C*, so it may only be called inside a here with the same type. Our typing for the language with dynamic here and go is presented in Figure 2.

We do not attempt to idealize the ML way of typing exceptions because ML uses a universal type exn for exceptions, in effect allowing a carefully delimited area of untypedness into the language. The typing of ML exceptions is therefore much less informative than that of checked exceptions.

Note that here and go are still the same weakening and contraction hybrid as in the static setting. But here their significance is a completely different one because the →-introduction is coupled with a sort of ∨-introduction. To see the difference, recall that in the static setting *λ*-abstracting over a go reifies the jump continuation and thereby, at the type level, gives rise to classical disjunction. This is not possible with the version of *λ* that gives go the dynamic semantics. Consider the following inference:

*x* : *A* ▶d go *x* : *B, A*

▶d *λ*d*x.*go *x* : *A* → *B* ∨ *A, C*

The *C*-accepting continuation at the point of definition is not accessible to the go inside the *λ*d. Instead, the go refers only to the *A*-accepting continuation that will be available at the point of call. Far from the excluded middle, the

type of *λ*d*x.*go *x* is thus “*A* implies *A* or *B*; or anything”.

In the same vein, as a further illustration how fundamentally different the

dynamic here and go are from the static variety, we revisit the term that, in the static setting, gave rise to call/cc with its classical type:

*λf.*here (*f* (*λx.*go *x*))*.*

Now in the dynamic case, we can only derive the intuitionistic formula

((*A* → *B* ∨ *A*) → *A* ∨ *A*) → *A* ∨ *C*

as the type of this term.

Let Γ be the context *f* : (*A* → *B* ∨ *A*) → *A* ∨ *A*. Then we have:

Γ ▶d *f* : (*A* → *B* ∨ *A*) → *A* ∨ *A, A*

Γ*,x* : *A* ▶d *x* : *A, A*

Γ*,x* : *A* ▶d go *x* : *B, A*

Γ ▶d *λ*d*x.*go *x* : *A* → *B* ∨ *A, A*

Γ ▶d (*f* (*λ*d*x.*go *x*)) : *A, A*

Γ ▶d here (*f* (*λ*d*x.*go *x*)) : *A, C*

▶d *λ*d*f.*here (*f* (*λ*d*x.*go *x*)) : ((*A* → *B* ∨ *A*) → *A* ∨ *A*) → *A* ∨ *C, D*

# Return continuation

Our last choice is passing the return continuation as the extra continuation to the body of a *λ*-abstraction. So the cps transform is this:

[[*x*]] = *λkq.kx*

[[*λ*r*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r r* )

[[*MN* ]] = *λkq.*[[*M* ]](*λm.*[[*N* ]](*λn.mnkq*)*q*)*q* [[here *M* ]] = *λkq.*[[*M* ]]*kk*

[[go *M* ]] = *λkq.*[[*M* ]]*qq*

This transform grants *λ*r the additional role of a continuation binder. The original operator for this purpose, here, is rendered redundant, since here *M* is now equivalent to (*λ*r*x.M* )(*λ*r*y.y*) where *x* is not free in *M* . At first sight, binding continuations seems an unusual job for a *λ*; but it becomes less so if

we think of go as the return statement of C or Java.

* 1. *Non-ﬁrst class* return

Because the enclosing *λ* determines which continuation go jumps to with its argument, the go-operator has the same effect as a return statement. The

Fig. 3. Typing for go as a return-operation

Γ ▶r *M* : *B, B*

Γ*,x* : *A,* Γ' ▶r *x* : *A, C*

Γ ▶ go *M* : *C, B*

r

Γ*,x* : *A* ▶r *M* : *B, B*

Γ ▶r *λ*r*x.M* : *A* → *B, C*

Γ ▶r *M* : *A* → *B, C* Γ ▶r *N* : *A, C*

Γ ▶r *MN* : *B, C*

type of extra continuation assumed by go needs to agree with the return type of the nearest enclosing *λ*:

Γ*,x* : *A* ▶r *M* : *B, B*

Γ ▶r *λ*r*x.M* : *A* → *B, C*

The whole type system for the calculus with *λ*r is in Figure 3.

The agreement between go and the enclosing *λ*r is comparable with the typing in C, where the expression featuring in a return statement must have the return type declared by the enclosing function. For instance, *M* needs to

have type int in the definition:

int f(){ *...* return *M* ; *...* }

With *λ*r, the special form go cannot be made into a first-class function. If we try to *λ*-abstract over go *x* by writing *λ*r*x.*go *x* then go will refer to that *λ*r.

The failure of *λ*r to give first-class returning can be seen logically as follows. In order for *λ*r to be introduced, both types on the right have to be the same:

*x* : *A* ▶r go *x* : *A, A*

▶r *λ*r*x.*go *x* : *A* → *A, C*

Rather than the classical “not *A* or *A*” this asserts merely the intuitionistic “*A* implies *A*; or anything”.

One has a similar situation in Gnu C, which has both the return statement and nested functions, without the ability to refer to the return address of another function. If we admit go as a first-class function, it becomes a much more powerful form of control, Landin’s JI-operator.

* 1. *The* JI-operator

Keeping the meaning of *λ*r as a continuation binder, we now consider a control operator JI that always refers to the statically enclosing *λ*r, but which, unlike the special form go, is a first-class expression, so that we can pass the return continuation to some other function *f* by writing *f* (JI). The cps of this

operator is this:

[[JI]] = *λks.k*(*λxrd. s x*)

That is almost, but not quite, the same as if we tried to define JI as *λ*r*x.*go *x*:

Fig. 4. Typing for JI

Γ*,x* : *A,* Γ' ▶j *x* : *A, C*

Γ ▶j JI : *B* → *C, B*

Γ*,x* : *A* ▶j *M* : *B, B*

Γ ▶j *λ*r*x.M* : *A* → *B, C*

Γ ▶j *M* : *A* → *B, C* Γ ▶j *N* : *A, C*

Γ ▶j *MN* : *B, C*

[[JI]] = [[*λ*r*x.*go *x*]]

= *λks.k*(*λxrd. r x*)

We can, however, define JI in terms of go if we use the static *λ*s, that is JI = *λ*s*x.*go *x*, as this does not inadvertently shadow the continuation *s* that we want JI to refer to.

The whole transform for the calculus with JI is this:

[[*x*]] = *λkq.qx*

[[*λ*r*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r r* )

[[*MN* ]] = *λkq.*[[*M* ]](*λm.*[[*N* ]](*λn.mnkq*)*q*)*q* [[JI]] = *λks.k*(*λxrd. s x*)

Recall that the role of here has been usurped by *λ*r, and we replaced go by its first-class cousin JI.

In the transform for JI, the jump continuation is the current “dump” in the sense of the secd-machine. The dump in the secd-machine is a sort of call stack, which holds the return continuation for the procedure whose body is currently being evaluated. Making the dump into a first-class object was precisely how Landin invented first-class control, embodied by the J-operator. The typing for the language with JI is given in Figure 4. In particular,

the type of JI is the classical disjunction

Γ ▶j JI : *B* → *C, B*

As an example of the type system for the calculus with the JI-operator, we see that Reynolds’s [9] definition of call/cc in terms of JI typechecks. (Strictly speaking, Reynolds used escape, the binding-form cousin of call/cc, but call/cc and escape are syntactic sugar for each other.) We infer the type of call/cc ≡ *λ*r*f.*((*λ*r*k.f k*)(JI)) to be:

((*A* → *B*) → *A*) → *A*)

To write the derivation, we abbreviate some contexts as follows: Γ*fk* ≡ *f* : (*A* → *B*) → *A, k* : (*A* → *B*)

Γ*f* ≡ *f* : (*A* → *B*) → *A*

Then we can derive:

Γ*fk* ▶j *f* : (*A* → *B*) → *A, A* Γ*fk* ▶j *k* : (*A* → *B*)*,A* Γ*fk* ▶j *f k* : *A, A*

Γ*f* ▶j *λ*r*k.fk* : (*A* → *B*) → *A, A* Γ*f* ▶j JI : *A* → *B, A*

Γ*f* ▶j (*λ*r*k.f k*)(JI) : *A, A*

▶j *λ*r*f.*((*λ*r*k.f k*)(JI)) : ((*A* → *B*) → *A*) → *A*)*,C* Because JI has such evident logical meaning as classical disjunction, we

have considered it as basic. Landin [6] took another operator, called J, as

primitive, while JI was derived as the special case of J applied to the identity combinator:

JI = J (*λx.x*)

This explains the name “JI”, as “J” stands for “jump” and I for “identity”. We were able to start with JI, since (as noted by Landin) the J-operator is syntactic sugar for JI by virtue of:

J = (*λ*r*r.λ*r*f.λ*r*x.r*(*fx*)) (JI)*.*

To accommodate J in our typing, we use this definition in terms of JI to derive the following type for J:

▶j J : (*A* → *B*) → (*A* → *C*)*,B*

Let Γ be the context *x* : *A, r* : *B* → *C, f* : *A* → *B*. We derive:

Γ ▶j *r* : *B* → *C, C*

Γ ▶j *f* : *A* → *B, C* Γ ▶j *x* : *A, C*

Γ ▶j *fx* : *B, C*

Γ ▶j *r*(*fx*) : *C, C*

*r* : *B* → *C, f* : *A* → *B* ▶j *λ*r*x.r*(*fx*) : *A* → *C, A* → *C*

*r* : *B* → *C* ▶j *λ*r*f.λ*r*x.r*(*fx*) : (*A* → *B*) → (*A* → *C*)*,* (*A* → *B*) → (*A* → *C*)

▶j *λ*r*r.λ*r*f.λ*r*x.r*(*fx*) : (*B* → *C*) → (*A* → *B*) → (*A* → *C*)*,B*

▶j (*λ*r*r.λ*r*f.λ*r*x.r*(*fx*)) (JI) : (*A* → *B*) → (*A* → *C*)*,B*

This type reflects the behaviour of the J-operator in the secd machine. When J is evaluated, it captures the *B*-accepting current dump continuation; it can then be applied to a function of type *A* → *B*. This function is composed with the captured dump, yielding a non-returning function of type *A* → *C*, for arbitrary *C*. By analogy with call-with-current-continuation, we may read the J-operator as “compose-with-current-dump” [13].

The logical significance, if any, of the extra function types in the general J seems unclear. There is a curious, though vague, resemblance to exception handlers in dynamic control, since they too are functions only to be applied on jumping. This feature of J may be historical, as it arose in a context where

greater emphasis was given to attaching dumps to functions than to dumps as first-class continuations in their own right.

# Type preservation

The typings agree with the transforms in that they are preserved in the usual way for cps transforms: we have a “double-negation” transform for types, contexts and judgements. The only (slight) complication is in typing the dynamic continuation in those transforms that ignore it.

The function type of the form *A* → *B* ∨ *C* for the dynamic semantics is translated as follows:

[[*A* → *B* ∨ *C*]] = [[*A*]] → ([[*B*]] → Ans) → ([[*C*]] → Ans) → Ans

Each call expects not only the *B*-accepting return continuation, but also the

*C*-accepting continuation determined by the here that encloses the call.

Because we have not varied the transform of application, functions defined with *λ*s and *λ*r are also passed this dynamic continuation, even though they ignore it:

[[*λ*s*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r s* )

[[*λ*r*x.M* ]] = *λks.k*(*λxrd.*[[*M* ]]*r r* )

In both of these cases, the dynamic jump continuation *d* is fed to each function call, but never needed. Each function definition must expect this argument to be of certain type. Because different calls of the same function may have dynamically enclosing here operators with different types, the type ascribed to *d* should be polymorphic.

So the function type of the form *A* → *B* is transformed so as to accept this unwanted argument polymorphically:

[[*A* → *B*]] = ∀*β.*[[*A*]] → ([[*B*]] → Ans) → *β* → Ans

That is, a function of type *A* → *B* accepts an argument of type *A*, a *B*- accepting return continuation, and the continuation determined by the here dynamically enclosing the call.

For all the transforms we have preservation of the respective typing: if Γ ▶? *M* : *A, B*, then

[[Γ]] ▶ [[*M* ]] : ([[*A*]] → Ans) → ([[*B*]] → Ans) → Ans*.*

The proof is a straightforward induction over the derivation.

As a typical example, consider how the classical axiom of excluded middle

▶j JI : *A* → *B, A*

is translated to the *λ*-term **[**JI]] = *λks.k*(*λxrd.rx*) with the type

((∀*β.*[[*A*]] → ([[*B*]] → Ans) → *β* → Ans) → Ans) → ([[*A*]] → Ans) → Ans*.*

Fig. 5. Comparison of the type systems as logics

Static here and go, implies call/cc

Γ ▶s *B, B* Γ ▶s *B, B*

s

Γ*,A* ▶s *B, C*

Γ ▶s *A* → *B, C*

Γ ▶ *B, C*

Γ ▶ *C, B*

s

Γ*, A,* Γ' ▶s *A, C*

Γ ▶s *A* → *B, C* Γ ▶s *A, C*

Γ ▶s *B, C*

Dynamic here and go, like checked exceptions

Γ ▶d *B, B* Γ ▶d *B, B*

d

Γ*,A* ▶d *B, C*

Γ ▶d *A* → *B* ∨ *C, D*

Γ ▶ *B, C*

Γ ▶ *C, B*

d

Γ*, A,* Γ' ▶d *A, C*

Γ ▶d *A* → *B* ∨ *C, C* Γ ▶d *A, C*

Γ ▶d *B, C*

Non-first class return-operation

Γ ▶r *B, B*

r

Γ*,A* ▶r *B, B*

Γ ▶r *A* → *B, C*

Γ ▶ *C, B*

Γ*, A,* Γ' ▶r *A, C*

Γ ▶r *A* → *B, C*

Γ ▶r *A, C*

Γ ▶r *B, C*

Landin’s JI-operator

Γ ▶j *B* → *C, B*

Γ*,A* ▶j *B, B*

Γ ▶j *A* → *B, C*

Γ*, A,* Γ' ▶j *A, C*

Γ ▶j *A* → *B, C*

Γ ▶j *A, C*

Γ ▶j *B, C*

# Conclusions

As a summary of the four control constructs we have considered, we present their typings in Figure 5, omitting the terms for conciseness. As logical sys- tems, these toy logics may seem a little eccentric, with two succedents that can only be manipulated in a slightly roundabout way. But they are sufficient for our purposes here, which is to illustrate the correspondence of first-class con- tinuations with classical logic and weaker control operation with intuitionistic logic, and the central role of the arrow type in this dichotomy.

Recall the following fact from proof theory (see for example [15]). Suppose

one starts from a presentation of intuitionistic logic with sequents of the form Γ ▶ ∆. If a rule like the following is added that allows →-introduction even if there are multiple succedents, the logic becomes classical.

Γ*,A* ▶ *B,* ∆

Γ ▶ *A* → *B,* ∆

In continuation terms, the significance of this rule is that the function clo- sure of type *A* → *B* may contain any of the continuations that appear in ∆; to use the jargon, these continuations become “reified”. The fact that the logic becomes classical means that once we can have continuations in func- tion closures, we gain first-class continuations and thereby the same power as call/cc. We have this form of rule for static here and go; though not for JI, since JI as the excluded middle is already blatantly classical by itself.

But the logic remains intuitionistic if the →-introduction is restricted. The rule for this case typically admits only a single formula on the right:

Γ*,A* ▶ *B*

Γ ▶ *A* → *B,* ∆

Considered as a restriction on control operators, this rule prohibits *λ*-abstraction for terms that contain free continuation variables. There are clearly other pos- sibilities how we can prevent assumptions from ∆ to become hidden (in that they can be used in the derivation of *A* → *B* without showing up in this type itself). We could require these assumptions to remain explicit in the arrow type, by making ∆ a singleton that either coincides with the *B* on the right of the arrow, or is added to it:

Γ*,A* ▶r *B, B*

Γ ▶r *A* → *B, C*

Γ*,A* ▶d *B, C*

Γ ▶d *A* → *B* ∨ *C, D*

These are the rules for →-introduction in connection with the return-operation, and dynamic here and go, respectively. Neither of which gives rise to first- class continuations, corresponding to the fact that with these restrictions on

→-introduction the logics remain intuitionistic.

The distinction between static and dynamic control in logical terms ap- pears to be new, as is the logical explanation of Landin’s JI-operator.

* 1. *Related work*

Following Griffin [3], there has been a great deal of work on classical types for control operators, mainly on call/cc or minor variants thereof. A similar cps transform for dynamic control (exceptions) has appeared in [5], albeit for a very different purpose. Felleisen describes the J-operator by way of cps, but since his transform is not double-barrelled, J means something different in each *λ* [2]. Variants of the here and go operators are even older than the notion of continuation itself: the operations valof and resultis from cpl later appeared in Strachey and Wadsworth’s report on continuations [11,12].

These operators led to the modern return in C. As we have shown here, they lead to much else besides if combined with different flavours of *λ*.

* 1. *Further work*

In this paper, control constructs were compared by cps transforms and typing of the *source*. A different, but related approach compares them by typing in the *target* of the cps [1]. On the source, we have the dichotomy between intuitionistic and classical typing, whereas on the target, the distinction is between linear and intuitionistic. We hope to relate these in further work.

# References

1. Berdine, J., P. W. O’Hearn, U. Reddy and H. Thielecke, *Linearly used continuations*, in: A. Sabry, editor, *Proceedings of the 3rd ACM SIGPLAN Workshop on Continuations*, 2001.
2. Felleisen, M., *Reflections on Landin’s J operator: a partly historical note.*, Computer Languages 12 (1987), pp. 197–207.
3. Griffin, T. G., *A formulae-as-types notion of control*, in: *Proc. 17th ACM Symposium on Principles of Programming Languages*, San Francisco, CA USA, 1990, pp. 47–58.
4. Kelsey, R., W. Clinger and J. Rees, editors, *Revised*5 *report on the algorithmic language Scheme*, Higher-Order and Symbolic Computation 11 (1998), pp. 7– 105.
5. Kim, J., K. Yi and O. Danvy, *Assessing the overhead of ML exceptions by selective CPS transformation*, in: *Proceedings of the 1998 ACM SIGPLAN Workshop on ML*, 1998.
6. Landin, P. J., *A generalization of jumps and labels*, Report, UNIVAC Systems Programming Research (1965).
7. Landin, P. J., *A generalization of jumps and labels*, Higher-Order and Symbolic Computation 11 (1998), reprint of [6].
8. Plotkin, G., *Call-by-name, call-by-value, and the* λ*-calculus*, Theoretical Computer Science 1 (1975), pp. 125–159.
9. Reynolds, J. C., *Deﬁnitional interpreters for higher-order programming languages*, in: *Proceedings of the 25th ACM National Conference* (1972), pp. 717–740.
10. Riecke, J. G. and H. Thielecke, *Typed exceptions and continuations cannot macro-express each other*, in: J.Wiedermann, P.van Emde Boas and M.Nielsen, editors, *Proceedings 26th International Colloquium on Automata, Languages and Programming (ICALP)*, LNCS 1644 (1999), pp. 635–644.
11. Strachey, C. and C. P. Wadsworth, *Continuations: A mathematical semantics for handling full jumps*, Monograph PRG-11, Oxford University Computing Laboratory, Programming Research Group, Oxford, UK (1974).
12. Strachey, C. and C. P. Wadsworth, *Continuations: A mathematical semantics for handling full jumps*, Higher-Order and Symbolic Computation 13 (2000), pp. 135–152, reprint of [11].
13. Thielecke, H., *An introduction to Landin’s “A generalization of jumps and labels”*, Higher-Order and Symbolic Computation 11 (1998), pp. 117–124.
14. Thielecke, H., *On exceptions versus continuations in the presence of state*, in: G. Smolka, editor, *Programming Languages and Systems, 9th European Symposium on Programming, ESOP 2000,*, number 1782 in LNCS (2000), pp. 397–411.
15. Troelstra, A. S. and H. Schwichtenberg, “Basic Proof Theory,” Cambridge University Press, 1996.