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Computing the Solution of the m-Korteweg-de Vries Equation on Turing Machines

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**Abstract**

In this paper, we study the initial value problem of the mKdV equation, and convert the constant coefficients mKdV equation into its standard norm by linear transformation. Then, we define a nonlinear map *KR* : *Hs*(R) *C*(R*,Hs*(R)) (*s* ≥ 1 )*,* from the initial data to the solution of the equation, and prove *KR* is Turing computable for any integer *s* ≥ 3. Therefore, the solution of the mKdV equation with arbitrary

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*→*

precision on Turing machines can be satisfied.

*Keywords:* mKdV equation, conservation equation, computable, Turing machines.

# 1 Introduction

In engineering and other scientific computation there are a lot of practical applica- tions which are related to finding solutions of some kind of differential equations. It is always a great challenge for mathematicians to determine whether the equa- tions of certain type have solution and, if it is the case, how these solutions can be computed. Unfortunately, this task cannot always be satisfactorily fulfilled for all equations. However, there are a lot of special equations whose solutions do exist and can be exactly and precisely specified. Those equations are usually called ex- actly solvable equations. The Korteweg-de Vries equation (KdV equation for short) *ut* + *uux* + *uxxx* = 0 is a good example of such kind of equations. Actually, earlier, Gay, Zhang and Zhong[4] studied the Cauchy of the Korteweg-de Vries equation posed on a periodic domain, defined the nonlinear operator and proved that the op- erator is computable. Later, Klaus Weihrauch and Ning Zhong [12]have shown that

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the solution operator of the KdV equation is computable on real line in the frame- work of Type-2 computability theory. In this paper, we extend the investigations of

[12] to the modified Korteweg-de Vries (mKdV) equations *ut* + *αu*2*ux* + *βuxxx* = 0

for *α, β ∈* R.

The mKdV equations are used to describe the acoustic spread of non-harmonic Lattice and the Alfen wave sport of non-collision plasma in plasma physics, solid physics, atomic physics, hydrodynamics and the theory of quantum, etc. The mKdV equations have been intensively studied from various aspects of both physics and mathematics. Anne Boutet de Monvel and Dmitry Shepelsky [9] have analyzed an initial-boundary value problem for the mKdV equation in a finite interval. F. Gesztesy and B. Simon [5]have constructed solutions of the mKdV equation. Jing Yu and Ruguang Zhou[15] have presented two kinds of integrable decompositions of the mKdV equation. Turabi Geyikliand and Dogan Kaya [6]have obtained a numerical solution to mKdV equation. In[1,2,3,14], more new kinds of solutions such as periodic wave solutions and solitary wave solutions have been obtained. The mKdV equation has widespread application. Therefore, it is very significant to study the solution operator of the equation.

Our goal is to show that the solution operator of mKdV equations is also com- putable in the same framework as in [[12](#_bookmark8)]. Although a similar approach to that of

[[12](#_bookmark8)] is used in this paper, the construction is more complicated because the non- linearity of the mKdV equation is stronger than the KdV equation. Especially, we need conservation equations of the mKdV equation and use properties of Banach algebra to prove our main result. It seems that this method can be further applied to study the operator of the generalized KdV equations.

The paper is organized as follows. In Section 2, we recall the Turing machine, Type 2 theory of effectivity, basic spaces, and representations. In Section 3, we prove the main result and put the tedious inferential process of the proof to Section 4 which contains also three estimates and their proofs for the purpose of effectively determining a computable subsequence.

# 2 Preliminaries

The computability of subsets and functions on the discrete (countable) sets is usually defined by means of Turing machines. Both inputs and outputs of a Turing machine are finite words. In order to investigate the computability on uncountable sets, the Turing machines have been extended by Klaus Weihrauch [11] so that their inputs and outputs can be infinite sequences as well. These machines are usually called *Type 2 Turing machines* and they can be used to define the computability on the set Σ*ω* of infinite sequences in an analogous way while the (classic) Turing machines introduce the computability to the set Σ*∗* of finite sequences on the alphabet Σ. If we want to introduce the computability to other set *D* of a cardinality up to continuum, we can choose a representation *δ* : Σ*ω D* of *D* which is simply a surjective function. That is, the representation *δ* assigns (possibly infinite) names (*δ*-names) to each element *x ∈ D* and transfers the computability on Σ*ω* straightforwardly

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to the set *D*. For example, an element *x D* is called *δ*-computable if it has a computable *δ*-name.

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In order to investigate the computability of the solution operation of various dif- ferential equations, we have to introduce the corresponding computability notion to the function spaces at first. In this section, we recall the definitions of computability on several function spaces which are necessary for our discussions. They essentially belong to Klaus Weihrauch and Ning Zhong [12].

Usually, we are interested in the computability on some metric spaces. If a metric space (*M, d*) has a countable dense subset, then we can define its effectivization as a *computable quadruple metric space M* = (*M, d, A, ν*) in which (1) *A* is a dense subset of *M* , (2) *ν* : Σ*∗ A* is a surjective function ( so-called *notation* of *A*); and (3) the set *u, v, w, x* Σ*∗* : *νQ*(*w*) *< d*(*ν*(*u*)*, ν*(*v*)) *< νQ*(*x*) is a recursively enumerable set, where *νQ* : Σ*∗* Q is the standard notation of the rational numbers. In a computable metric space (*M, d, A, ν*) we can introduce the computability to the following Cauchy representation *δC* :*⊆* Σ*ω → M* which is a surjective function such that *δC* (*p*) = *x* if and only if *p* = *w*0#*w*1#*w*2# *···* for *wi ∈ dom*(*ν*) and the sequence *{ν*(*wi*)*}* converges effectively to *x* in the sense that *d*(*x, ν*(*wi*)) *≤* 2*−i* for all *i ∈* N.

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For example, let *L*2(R) be the set of all *L*2-functions, i.e., the functions *f* meet the condition that ( *f* (*x*) 2*dx*)1*/*2 *<* and let *d* be the standard *L*2-norm defined by *dL*2 (*f, g*) = *||f − g||* = ( R *| f* (*x*) *− g*(*x*) *|*2 *dx*)1*/*2. Then the *L*2-function space (*L*2(R)*, dL*2 ) has a countable dense subset *σ* consisting of all rational finite step

∫R *| | ∞ L*2

∫

functions. Let *vL*2 be a canonical notation of *σ*. Thus we achieve the computable metric space (*L*2(R)*, dL*2 *, σ, vL*2 ). On this computable metric space we can define the Cauchy representation *δL*2 as follows: a sequence *p ∈* Σ*ω* is a *δL*2 -name of *g ∈ L*2(R) (i.e., *δL*2 (*p*) = *g*) if and only if *p* = *w*0#*w*1#*w*2# *···* with *wi ∈ dom*(*vL*2 ) and *||vL*2 (*wi*) *− g|| ≤* 2*−i* for all *i ∈* N.

In order to prove our main theorem, we need to introduce some representations of the Schwardtz space. Let the Schwardtz space *S*(R) be the set of all functions *φ ∈ C∞* (R), such that sup *|xαφ*(*β*) (*x*) *| < ∞* for all *α, β ∈* N,where *C∞* (R) is the

*x* R

*∈*

space of complex-valued functions of class *C∞*

topology. Let *ds* be the metric defined by

equipped with the compact open

*∞*

Σ

*ds*(*φ, ϕ*) =

*α,β*=0

2*−<α,β>*  *φ − ϕ α,β*

1+ *φ − ϕ α,β*

*∀α, β ∈* N

where *⟨α, β⟩* is the bijective Cantor pairing function, defined by *< α, β >*:= *β* + (*α* + *β*)(*α* + *β* + 1)*/*2, and *φ α,β*:= sup*x∈*R *| xαφ*(*β*)(*x*) *|*. Then the space (*S*(R)*, ds*) has a dense subset *P∗* consisting of the set of truncated polynomials with

rational coefficients. Let *νp* be the canonical notation of *P∗* . Thus we obtain the

*∞*

computable metric space (*S*(R)*, ds, P*

*∞ sc*

*∗, νp* ). Let *δ* :*⊆* Σ*ω*

*→ S*(R) be the Cauchy

representation. However, we have to introduce another representation of *S*(R) for

the proof, denoted as *δs*, defined by

⎧⎪

*δP* (*p*) = *φ and*

*∞*

*δs* (*⟨q, p⟩*) = *φ ⇔* ⎪⎨ *q* = *u*0#*u*1#*u*2 *··· , where uk ∈ dom* (*v*N)

sup

⎪⎩ *and*

*|x|*≥*v*N(*u<i,j,n>*)

*|xiφ*(*j*) (*x*) *|* ≤ 2*−n*

**Remark 2.1** *δP* is the Cauchy representation of the computable metric space

(*C∞*(R) *c ∞p*), where *dc* is the metric defined as follows:

*,d , P,ν*

*∞*

Σ

*dc*(*φ, ϕ*) =

*α,β*=0

2*−<α,β>*  *φ − ϕ α,β ,*

1+ *φ − ϕ α,β*

*P* is the set of polynomials with rational coefficients, and *νp* is a canonical notation of *P* . *ν*N is a canonical notation of the set N.

The representations on *L*2(R) and *S*(R) lead straightforwardly to a representa- tion of the Sobolev space *Hs*(R). By definition, the Sobolev space consisits of all functions *f ∈ L*2(R) such that *Ts*(*f* ) *∈ L*2(R), where

*Ts*(*f* )(*ξ*) := (1 + *|ξ|*2)*s/*2*F*(*f* )(*ξ*)

is a weighted Fourier transform of *f* with weight (1 + *|ξ|*2)*s/*2 and *F*(*f* ) denotes the Fourier transform of *f* . An infinite word *p ∈* Σ*ω* is a *δHs* -name of *f ∈ Hs*(R), iff *p* is a *δL*2 -name of the weighted transform *Ts*(*f* ) *L*2(R)(i.e *δHs* (*p*) = *Ts−*1 *δL*2 (*p*)). During the proof, we also need another representation of *Hs*(R). When *s* ≥ 0 is an integer, *Hs*(R) is the same as the set: *{f ∈ L*2(R) : the *kth*order derivative *fk* of *f* is in *L*2(R) for all 0 *≤ k ≤ s}*, define the norm as following:

*∈ ◦*

 *f* (*x, t*)  *s*= (  *f* (*x, t*)  2 +  *f'*(*x, t*)  2 + *···* +  *f* (*s*)(*x, t*)  )1*/*2

An infinite word *p ∈* Σ*ω* is *δ*˜*Hs* -name of *f* , iff *p* = *⟨p*0*, p*1*, p*2*, ·· ·⟩* with *pi ∈ dom*(*δsc*) and *δsc*(*pi*) *− f s*≤ 2*−i.*

**Remark 2.2** *S*(R) is dense in *Hj*(R) for all *j ∈* N. If *f* (*t*) *∈ S*(R) ,we can see that

*f* (*t*) *∈ Hj*(R), in particular , *Dj f* (*t*) *∈ L*2 (R) for all *j ∈* N and *t ∈* R

*x*

Finally, we consider the representation of functions. Suppose that we have two sets *M* and *M'* with the representations *δ* : Σ*ω M* and *δ'* : Σ*ω M'*, respectively. We say that a function *f* : *M M'* is (*δ, δ'*)-computable if there is a type-2 Turing machine which transfers any *δ*-name of an *x dom*(*f* ) to a *δ'*-name of *f* (*x*) for any *x dom*(*x*). It is well known that, any (*δ, δ'*)-computable function is (*δ, δ'*)-continuous. Here a function *f* : *M M'* is (*δ, δ'*)-continuous if there is a continuous function *g* : Σ*ω* Σ*ω* which transfers any *δ*-name of *x dom*(*f* ) to a *δ'*-name of *f* (*x*). Denote by *C*(*M, M'*) the set of all (*δ, δ'*)-continuous functions from *M → M'*. For the set *C*(*M, M'*), there is a canonical representation (see [[11](#_bookmark9)])

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[*δ → δ'*] : Σ*ω → C*(*M, M'*) such that a function *f* is (*δ, δ'*)-computable iff it has a computable [*δ → δ'*]-name.

# 3 Main result

In this section, we will prove our main result, that is, the solution operator of the mKdV equations is computable. We explain in this section the essential idea of how this result can be proved. Some technical details will be given in Section [4](#_bookmark2).

Precisely, we are interested in the following initial value problem (IVP, for short) of mKdV equation on the real line R,

⎧⎨ *ut* + *αu*2*ux* + *βuxxx* = 0*,* (*t, x ∈* R)

1. ​

*u* (*x,* 0) = *ϕ* (*x*) *∈ Hs*(R)

⎩

By a computable linear transformation *u* = 1 *u'*, *x* = *β*1*/*3*x'* and *t* = *t'*,

*√*

*αβ−*1*/*3

the IVP ([1](#_bookmark0)) can be transformed computably to the following

⎧⎨ *ut* + *u*2*ux* + *uxxx* = 0*,* (*t, x ∈* R)

(2)

⎩ *u* (*x,* 0) = *ϕ* (*x*) *∈ Hs*(R)

Thus, it suffices to consider only the IVP ([2](#_bookmark0)) because any computable solution of

1. leads straightforwardly by a computable reverse transformation to a computable solution of ([1](#_bookmark0)). Originally, the IVP ([2](#_bookmark0)) looks for a function *u* of two arguments which satisfies both equations from any given function *ϕ*. However, if we write the function *u* as a functional *u*(*x, t*) := *u*(*t*)(*x*), then the IVP ([2](#_bookmark0)) can be regarded equivalently as a solution operator mapping a function *ϕ* to the functional *u* : *t u*(*t*) where *u*(*t*) : R R is a real function of one argument. Furthermore, the initial function *ϕ* is in the Sobolev space *Hs*(R), then ([2](#_bookmark0)) does have a solution *u* which, as a functional,

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is a continuous function from R to *Hs*(R) for any *s* 1 (see[[7](#_bookmark7)]). In other words, the

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*≥*

IVP ([2](#_bookmark0)) has a solution operator *K*R : *Hs*(R) *C*(R; *Hs*(R)).The special solidary wave solution possessed by the mKdV equation in [[2](#_bookmark6)] is

*→*

*α*

*β*

*u*(*x, t*) = 6*csech*( *c* (*x − ct*))

where c is wave speed. The solution is obviously computable when *c* is computable real number. Our main result asserts that this solution operator is actually com- putable in the framework of type-2 computability theory.

**Theorem 3.1** *The solution operator K*R : *Hs*(R) *→ C* (R; *Hs*(R)) *of the initial value problem* ([2](#_bookmark0)) *is* (*δHs ,* [*ρ → δHs* ])*-computable for any integer s* ≥ 3*.*

During the proof, firstly we will give the equivalent integral equation of the IVP(2), and use the iterative approach to solute this integral equation. Secondly, we prove that this iterative sequence is computable, and converges uniformly in *t ∈* [0*,T* ]. Thus, the limit of the converging sequence will also be computable, and

satisfy the equivalent integral equation of the IVP(2). In other words, the limit will be the solution of the IVP(2), and then the solution will be computable.

The following is the equivalent integral equation of the initial value problem ([2](#_bookmark0))

1. ​

1 *t*

*u* (*t*) = *F−*1 (*E* (*t*) *·F* (*ϕ*)) *−* 3

∫

0

*F−*1

*E* (*t − τ* ) *·F*

*d* (*u* (*τ* ))3

*dx*

d*τ*

Where *u* (*t*) (*x*) := *u* (*x, t*) *,E* (*t*) (*x*) := *eix*3*t*, and (*ϕ*) (*x*) = 1

*F √*

2*π*

*e−ixξϕ* (*ξ*) d*ξ*.

R

∫

1. ​

We show that the following iterative sequence with the initial data *ϕ* as the seed:

⎧⎨ *v*0 (*t*) = *F−*1 (*E* (*t*) *·F* (*ϕ*))

⎩ *vj*+1 (*t*) = *v*0 (*t*) *−* 1 [∫](#_bookmark1) *t F−*1 *E* (*t − τ* ) *·F* d (*vj* (*τ* ))3 d*τ.*

3

0

dx

The iterative sequence ([4](#_bookmark1)) is contracting near *t* = 0*,* thus the sequence converges to a unique limit. Since the limit satisfies the integral equation ([3](#_bookmark1)), it is the solution of the initial value problem ([2](#_bookmark0)) near *t* = 0*.* To prove that the solution operator is computable, we need to construct a type-2 Turing machine which computes fast approximations to the solution *u* (*t*) when given enough information on the initial data *ϕ*.The machine will be designed in such a way that is capable of computing the iterative sequence in ([4](#_bookmark1)) when inputting Schwartz function. Thus for any given initial data *ϕ Hs* (R) and any *δ*˜*Hs* -name *p*0*, p*1 *.. .* of *ϕ*, the machine is able to compute the iterative sequence for each seed *δsc* (*pi*).We recall that *δsc* (*pi*) is an approximation to *ϕ* in *Hs* (R) with accuracy 2*−i.*

*∈ ⟨ ⟩*

Firstly, we define the operator

1 *t*

∫

*S*(*u, ϕ, t*) = *F−*1(*E*(*t*) *· F*(*ϕ*)) *−* 3

0

*F−*1

*E*(*t − τ* ) *·F*

*d* (*u*(*τ* ))3 d*τ*

which is ([*ρ → δs*]*, δs, ρ, δs*)-computable. This follows from Lemma 3.2 in

*dx*

[[12](#_bookmark8)] straightforwardly. Therefore, the function *S*¯(*u, ϕ*)(*t*) := *S*(*u, ϕ, t*) is

([*ρ δs*]*, δs,* [*ρ δs*])-computable. Then we define the function *v* : *S*(R) N

*→ → × →*

*C*(R : *S*(R)) by

*v*(*ψ,* 0) = *S*¯(0*, ψ*)

*v*(*ψ, j* + 1) = *S*¯(*v*(*ψ, j*)*, ψ*)*.*

It is easy to verify that *v* is (*δS, γ*N*,* [*ρ → δS*])-computable.

**Proof.** (of Theorem [3.1](#_bookmark1)) For a given initial value *ϕ Hs*(R) and a rational number

*∈*

*T*¯ *>* 0 we will show how to compute the solution *u*(*t*) of the initial value problem

([2](#_bookmark0)) at the time interval 0 ≤ *t* ≤ *T*¯. For this purpose, we first find some appropriate rational number *T* such that 0 *< T < T*¯, and show how to compute *u*(*t*) from *t'* and *ψ* := *u*(*t'*) at the time interval [*t', t'* + *T* ], 0 ≤ *t'* ≤ *T*¯, by a fixed point iteration.Using this method, we can compute the values *u* (*T /*2*m*) successively for *m* = 1*,* 2*,* and finally *u* (*t*) for any 0 ≤ *t* ≤ *T*¯.

*···*

If *ut*+*u*2*ux*+*uxxx* = 0*,u* (*x, t'*) = *ψ* (*x*), and *v* is defined by *v* (*x, t*) := *u* (*x, t* + *t'*) *,*

then

⎨

1. ​

⎧ *vt* + *v*2*vx* + *vxxx* = 0 *x ∈* R*,t* ≥ 0*,*

⎩ *v*(*x,* 0) = *ψ*(*x*)*.*

We assume that the initial value *ψ ∈ Hs*(R) is given by a *δ*˜*Hs* -name, i.e., by a sequence *ψ*0*, ψ*1*, ···* of Schwartz functions such that *ψ − ψn s*≤ 2*−n*. For any *n ∈* N, we define function *v*0*, v*1*, ··· ∈ C*(R : *S*(R)) by

*n*

*n*

1. *v*0 := *S*¯(0*, ψn*)*, vj*+1 := *S*¯(*vj , ψn*)*.*

*n n n*

We note that the sequence *{vj }* can be computed from *ψ* . If the iterative sequence

*n*

*n*

*v*0*, v*1*, ···* converges to some *vn*, then *vn* is the fixed point of the iteration *S*¯ and

*n*

*n*

satisfies the following internal equation:

1. ​

*vn* (*t*)= *S*¯ (*vn,ψn*)

1 ∫*t*

d

1. ​

= *F−*1(*E*(*t*) *· F*(*ψn*)) *−* 3

0

*F−*1

*E* (*t − τ* ) *·F*

dx (*vn* (*τ* ))3 d*τ*

hence *vn* solves the initial value problem:

1. ​

*∂vn* + *v*2 *∂vn* + *∂vn* = 0*,* and *v*

(*x,* 0) = *ψ*

(*x*)*.*

*∂t n ∂x*

*∂x*3 *n n*

We will show that, by a contraction argument, for some sufficiently small com-

putable real number *T >* 0 (depending only on *ϕ* and *T*¯), *vj* (*t*) *→ v* (*t*) as *j →∞*

*n*

*n*

for all *n*, and *vn* (*t*) *v* (*t*) as *n ,* sufficiently fast and uniformly in *t* [0*,T* ]. We recall that *v* is the solution of the initial value problem (5). Then we can effec- tively determine a computable subsequence of the double sequence *vj* which will converge fast to *v* uniformly in *t* [0*,T* ].The inferential process is tedious,we will finish it in next section.

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*∈*

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*→ → ∞ ∈*

Since *v* is the limit of a fast convergent computable sequence, *v* itself is com- putable. So *K*R : (*ϕ, t*) *'→ u* (*t*) is (*δHs , ρ, δHs* )-computable for *t* ≥ 0, the re- flection *R* : *S* (R) *→ S*(R)*, R*(*ψ*)(*x*) := *ψ*(*−x*), is (*δHs , δHs* )-computable. Define *u'* (*t*) (*x*) := *u* ( *t*) ( *x*). Then *u't* + *u'*2*u'x* + *u'xxx* = 0 and for *t* ≥ 0*,u* ( *t*) =

*− − −*

*R u'* (*t*) = *R K*R (*u'* (0) *, t*) = *R K*R (*R* (*ϕ*) *, t*), i.e., *u* (*t*) = *R K*R (*R* (*ϕ*) *, t*) for

*◦ ◦ ◦ ◦ −*

*t* ≤ 0. Therefore, as the two computable functions join at 0, *K*R is computable for

*t ∈* R. (see e.g. Lemma 4.35 in [12]).

# 4 Three estimates

From the above section, we obtain the result that *vn* is the solution of the IVP(9), and  *ψ ψn * *s* 2*−n*. If the sequence  *v vn * *s* is controlled by  *ψ ψn * *s*, we can obtain the result that the sequence *vn* converges uniformly. For the purpose of effectively determining a computable subsequence *vj* that is convergent fast and uniformly, we need three estimates listed below as Propositions 4.1,4.6,4.9. In the first proposition, we prove the solution of IVP(2) can be controlled by the initial function *ϕ*(*x*).

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**Proposition 4.1** *If u* (*x, t*) *is the solution of the IV P (*[*2*](#_bookmark0)*), Then there is a com- putable function e* : N R R R *which is non-decreasing in the second and third argument such that*

*× × →*

*T*

sup

0≤*t*≤*T*

 *u* (*x, t*)  *s*≤ *es* (  *ϕ * *s*) *,*

*where es* (*r*) := *e* (*s, T, r*)*, s is an integer and s* ≥ 3*.*

*T*

In order to prove the proposition, we need to introduce the conservation equa- tion. As we know, there are three significant conservation laws in the area of physics, which are conservation of mass, of energy and of momentum. In the area of math- ematics, when a physical problem can be described by a differential equation, like *ut* = *k*(*u*), the conservations law of this equation can be described as the following form:

(10)

*∂T* + *∂X* = 0*.*

*∂t ∂x*

Where *T* and *X* are relative to *u*(*x, t*)*, X* = 0 on the boundary of the field of definition. From (10) we know that *I* = *T* dx is irrelative to *t*. From the mKdV equation, we have the following:

∫

[*u*2

1

2 4 + 2*uuxx − u*2]*x* = 0

]*t* +[ *u*

*x*

[*−* 3 *u*2 + 1 *u*4]

2

*x*

4

*t*

+ [*−*3*u*2*u*2 *−* 3*u u*

+ 3 *u*2

+ 1 *u*6 + *u*3*u*

] = 0

[ 5 *u*2*u*2 *− u*2

*x*

*x*

*xxx*

2

*xx*

6

*xx x*

1 6 5 4 2 10

*— u u u* + *u*2*u*

*u xx −* 8 *u*2*u x −* 10 *uu*2*u*

3 *x* *xx*

*t*

1 4

] +[

18 2 *x*

2 1 8

3 *x x* 3 *x* 3

1 5

*x xx*

*−* 6 *ux −* 2*uxxuxxxx* + *uxxx −* 24 *u*

Define

∫

*—* 3 *u uxx*]*x* = 0

Φ0(*u*)=

R

Φ (*u*)= ∫

*u*2(*x, t*)d*x*

*−* 3 *u*2 + 1 *u*4 d*x*

1 2 *x* 4

R

Φ (*u*)= ∫ [ 5 *u*2(*x, t*)*u*2(*x, t*) *− u*2

(*x, t*) *−*  1 *u*6(*x, t*)]d*x.*

2 3 *x*

R

*xx* 18

It is easy to verify that

d Φ dt *j*

(*u*) = 0 for *j* = 0*,* 1*,* 2 and *t ∈* R*.*

Consequently, for any *t ∈* R,

∫

∫

*u*2(*x, t*)*dx* = *ϕ*2(*x*)d*x* thus *u*(*·, t*)  =  *ϕ *

R R

∫ *−* 3 *u*2(*x, t*)+ 1 *u*4(*x, t*) d*x* = *−* 3 ∫

*ϕ*2(*x*)d*x* + 1 ∫

*ϕ*4(*x*)d*x*

1. *x* 4

R

2 *x* 4

R R

∫ [ 5 *u*2(*x, t*)*u*2(*x, t*) *− u*2 (*x, t*) *−*  1 *u*6(*x, t*)]d*x*

3 *x xx* 18

R

= ∫ [ 5 *ϕ*2(*x*)*ϕ*2(*x*) *− ϕ*2 (*x*) *−*  1 *ϕ*6(*x*)]d*x*

1. *x xx* 18

R

During the proof, we need the following inequalities, which are in common use and whose proofs are seen in [8].

**Lemma 4.2 (Young’s inequality)** *For p, q >* 1*, p−*1 + *q−*1 = 1 *and a, b* ≥ 0*,*

*ab* ≤ *εap* + *c*(*ε*)*bq,*

*where c*(*ε*) = (*p −* 1) */* *pqε* 1 *.*

*p−*1

**Lemma 4.3** *For any f ∈ H*1 (R)*,*

 *f L∞* ≤  *f *1 *and * *f L∞* ≤ *√*2  *f *1*/*2  *f' *1*/*2

**Lemma 4.4 (Gronwall’s inequality)** *If, for any t* ≥ 0 *and a, b >* 0*, Dtx* (*t*) ≤

*ax* (*t*)+ *b, then*

*x* (*t*) ≤ *x* (0) *eat* + *b* *eat −* 1 *,*

*a*

*for any t* ≥ 0*, in particular,*

*x* (*t*) ≤ *x* (0) *eaT* + *b* *eaT −* 1

*a*

*for any* 0 ≤ *t* ≤ *T.*

Since  *u*(*x, t*)  *s*= ( *u*(*x, t*)  2 +  *u'*(*x, t*)  2 + +  *u*(*s*)(*x, t*)  )1*/*2, we give the following lemma 4.5 firstly.

*···*

**Lemma 4.5** *There is a computable function C* : N *×* R *×* R *→* R*, such that for any*

*ϕ ∈ Hs* (R)*, any integer s* ≥ 1 *and any T >* 0*,*

¨*u*(*s*) (*·, t*)¨ ≤ *Cs*  *ϕ *

*T*

*s−*1  *ϕ * *s*

*,* (0 ≤ *t* ≤ *T* )*,*

*where u is the solution of the IV P (2), u*(*s*) := *∂su and Cs* (*r*) := *C* (*s, T, r*)*.*

*x T*

**Proof.** In the following let *t* R and abbreviate *u * := *u* (*t*)  *L∞* . From the second conservation, we obtain

*∈ ∞*

 *u *2 = ∫ *u*2(*x, t*)*dx* = ∫ *ϕ*2*dx −* 1 ∫ *ϕ*4(*x*)*dx* + 1 ∫ *u*4*dx*

*x x x* 6 6

R R R R

≤ *ϕx*

2 + 1

6

*u* 2 *u* 2

*∞*

≤ *ϕx*

 2 + 1 *·* 2 *· u *  *u * *u * 2

1

6

*x*

≤ 2 *ux*

2 + 1

18

 *u * 6 + *ϕx*

2 (Young*'*s inequality)

Subtracting both sides by 1

2

 *ux * 2 we have

*u* 2 ≤ 1

*x* 9

 *ϕ * 6 +2 *ϕx*

 2 ≤ 2  *ϕ * 4 + 1 *· ϕ * 2 *.*

  *√*   1  

1

2

Thus *u*

*x*

≤

2

*ϕ* 4 +1

*· ϕ*

1

≤ *C*1(*ϕ*)*. ϕ*

1

*√* 1

When *s* = 1, we prove the result.

we observe that when *s >* 1 , *Hs*(R) is a Banach algebra. Therefore,  *um * *s*≤ 

2

, where *C*1

(*r*) =

2

*r*4 +1 2 *.*

*u m*, from the third conservation, we have

*s*

 *uxx * 2 =

∫

R

*u*

*xx*

2 (*x, t*)*dx*

= ∫ *ϕ*2 (*x, t*)*dx −* 5 ∫ *ϕ*2(*x*)*ϕ*2(*x*)d*x* + 1 ∫

*ϕ*6(*x*)d*x*

*xx* 3 *x* 18

R R R

+ 5 ∫ *u*2(*x, t*)*u*2(*x, t*)d*x −*  1 ∫ *u*6(*x, t*)d*x*

3

R

≤ *ϕ* 2 + 1

*xx*

18

*x*

*ϕ* 6 + 5

1

3

18

R

*u* 2 *u* 2

*∞*

*x*

≤ *ϕ* 2 + 1 *ϕ* 6 + 10 *u u* 3



*xx*

18

1

3

*x*

≤ *ϕ* 2 + 1

*xx*

18

*ϕ* 6 + 5

*u* 2 + 5 *u* 6

≤ [2 + 2(1 + *c*6(  *ϕ * ))*· ϕ * 4] *·* (  *ϕ * 2 + *ϕx * 2 + *ϕxx * 2)

1

3

3

*x*

1 1

≤ *C*2(  *ϕ * 1)  *ϕ * 2

where *C*2 (*r*) = 2 + 2[1 + *c*6(*r*)*r*4], *c*6(*r*) = [*c*1(*r*)]6*.* When *s* = 2, we obtain the result.

1 1

We prove the lemma by induction on *s* . The inequality holds for *s* = 1*,* 2

*C*1 := *C*1*, C*2 := *C*2 . Assume *s* ≥ 3 and that the inequality holds for all *sj < s*.

*T*

*T*

*x*

*x*

Differentiating (2) *s* times with respect to *x* , we obtain

⎧⎨ *u*(*s*) + *u*(*s*)

*t*

*xxx*

= *−Ds* *u*2*u* *,*

Since

*Ds* *u*2*ux*

*x*

⎩ *u*(*s*) (*x,* 0) = *ϕ*(*s*) (*x*) *.*

*s*

= Σ *u* *u −*

*s* 2 (*j*) (*s j*+1) *j*

*j*=0

## Σ Σ

*s*

*j*

=

*s j*

*j*

*i*

*u*(*i*)*u*(*j−i*)*u*(*s−j*+1)

*j*=0

*i*=0

Σ Σ

*s−*2

*s j*

*s−*2 *j*

=

*j*=2

*i*=0

*i*

*j u*(*i*)*u*(*j−i*)*u*(*s−j*+1) +

*i*=1

(*s*)*u*(*i*)*u*(*s−i*)*ux* + *u*2*u*(*s*+1)

*s−*2

Σ

*i*

+2 (*s* + 1) *uuxu*(*s*) + *su*(*s−*1)*uxux* + 2*suu*(*s−*1)*uxx* + *s* Σ *s−*1 *u*(*i*)*u*(*s−i−*1)*uxx*

*i*

we have

∫

*d dt*

R

*u*(*s*) 2*dx* = *−*2 ∫

(*s*)

*u*

*xxx*

∫

* *u*(*s*)*dx −* 2

R

∫

*x*

∫

∫

*Ds* *u*2*u*

*i*=1

*x* *u*(*s*)*dx*

= *−*2

R

*u*2*u*(*s*+1)*u*(*s*)*dx −* 4 (*s* + 1)

R

*uux*

*u*(*s*) 2 *dx −* 2*s* ∫

R

*u*(*s−*1)*uxux*

R

*u*(*s*)*dx*

*−*4*s*

∫

R

*uu*(*s−*1)*uxxu*(*s*)*dx −* 2

*s−*2

*i*=1

Σ

(*s*)

R

*i*

*u*(*i*)*u*(*s−i*)*uxu*(*s*)*dx*

*s−*2 *j*

Σ *s* Σ

*−*2 *j*

*j* *·* ∫

*u*(*i*)*u*(*j−i*)*u*(*s−j*+1)*u*(*s*)*dx*

where

*j*=2

*s−*2

*i*

Σ

*—* 2*s*

*i*=1

*i*=0

*s* 1

*i*

R

R

*u*(*i*)*u*(*s−i−*1)*uxxu*(*s*)*dx* = *A* + *B*

*—* ∫

*A* = *−*2

∫

R

*u*2*u*(*s*+1)*u*(*s*)*dx −* 4 (*s* + 1)

R

∫

∫

*uux*

*u*(*s*) 2*dx*

*—* 2*s*

R

*u*(*s−*1)*uxuxu*(*s*)*dx −* 4*s*

R

∫

*uu*(*s−*1)*uxxu*(*s*)*dx*

*B* = *−*2

*s−*2

*i*=1

Σ

(*s*)

R

*i*

∫

*u*(*i*)*u*(*s−i*)*uxu*(*s*)*dx*

*s−*2 *j*

Σ *s* Σ

*—* 2 *j*

*j* *·* ∫

*u*(*i*)*u*(*j−i*)*u*(*s−j*+1)*u*(*s*)*dx*

*j*=2 *s−*2

*i*

Σ

*—* ∫

*—* 2*s*

*i*=1

*i*=0

*s* 1

*i*

R

R

*u*(*i*)*u*(*s−i−*1)*uxxu*(*s*)*dx*

First , consider

∫

*A* = *−*

R

*u*2*d* *u*(*s*) 2 *−* 4 (*s* + 1) *·* ∫

R

*uux*

*u*(*s*) 2 *dx*

*−*2*s*

∫

R

*u*(*s−*1)*uxuxu*(*s*)*dx −* 4*s*

R

∫

∫

*uu*(*s−*1)*uxxu*(*s*)*dx*

= *−*2 (2*s* + 1) *·*

R

∫

*uux*

*u*(*s*) 2 *dx −* 2*s* ∫

R

*u*(*s−*1)*uxux*

*u*(*s*)*dx*

*—* 4*s*

R

2

(*s*)

*uu*(*s−*1)*uxxu*(*s*)*dx*

≤ 2 (2*s* + 1) *u*

(*s*)

 *· u *

*·* ¨*u*(*s*)¨ + 2*s* ¨*u*(*s−*1)¨ *· u *2

*·* ¨*u* ¨

+4*s u*(*s−*1)

¨ ¨

*x*

*∞*

*x*

*∞*

*∞*

*·* ¨*u * ¨ *·u ∞ · uxx *

≤ 2 (2*s* + 1) *u * *·* ( *u* + *u* ) *·* ¨*u*(*s*)¨ + 2*s*¨*u*(*s−*1)¨ *·* (  *u * + *u *)2

2

¨

¨

¨

¨

*x x* ¨ ¨   ¨   ¨  *x* *xx*

2

*·* ¨*u*(*s*)¨ + 4*s* ¨*u*(*s−*1)¨ + ¨*u*(*s*)¨ *·* ( *u* + *ux* ) *· uxx * *·* ¨*u*(*s*)¨

¨

¨

≤ 2 (2*s* + 1) *u * *·* (  *u * + *u * ) *·* ¨*u*(*s*)¨ + 4*s* (  *u * + *u*

2

 ) *· u*

*·* ¨*u*(*s*)¨

 *x * *x * ¨ ¨ ¨ ¨ ¨ *x* *xx*

+ 2*s* ( *ux* + *uxx* )2 *·* ¨*u*(*s−*1)¨ *·* ¨*u*(*s*)¨

+ 4*s* ¨*u*(*s−*1)¨ (  *u * + *ux* ) *· uxx * *·* ¨*u*(*s*)¨

≤ 2(4*s* + 1)(  *u * + *u * + *u *)2 *·* ¨*u*(*s*)¨ + 6*s*(  *u * + *u*

2

+ *u* )2

*x* *x*

¨ ¨ ¨ ¨

¨ ¨ *x* *x*

Then

*·* ¨*u*(*s−*1)¨ *·* ¨*u*(*s*)¨

*A*

2 ¨*u*(*s*)

2 (4*s* + 1) ( *ϕ* + *C*1

¨ 

¨ ≤

(  *ϕ * ) *· ϕ * 1

+ *C*2

(  *ϕ * 1

) *· ϕ *

)2 *u*(*s*)

 

¨ ¨

*T*

2

2

+3*s* ( *ϕ * + *C*1 (  *ϕ * ) *· ϕ * 1

*· ϕ * *s−*1

+ *C*2 (  *ϕ *

1) *· ϕ *

)2 *· Cs−*1 *ϕ*

*s−*2

Secondly, consider

*s* 2

*—*

¨

¨ ¨ ¨ ¨ ¨  

*B* ≤ 2

(*s*) ¨*u*(*s−i*)¨ *·* ¨*u*(*s*)¨ *·* ¨*u*(*i*)¨

*· ux*

*i*=1

Σ

*i*

## Σ Σ

*s−*2 *j*

(*i*)

+2

*j*=2

*s j*

*j*

*i*

*·* ¨*u* ¨*∞*

*∞ ∞*

¨ ¨ ¨ ¨ ¨

*·* ¨*u*(*j−i*)¨

*·* ¨*u*(*s−j*+1)¨ *·* ¨*u*(*s*)¨

*∞*

¨ ¨ ¨

*s−*2

Σ

*—*

*i*=0

¨ ¨ ¨ ¨   ¨ ¨

Thus

+ 2*s*

*i*=1

*s* 1

*i* ¨*u*(*i*)¨ *·* ¨*u*(*s*)¨ *· uxx ∞ ·* ¨*u*(*s−i−*1)¨

*∞*

*B s−*2

Σ

*i*

2

≤

¨

(*s*) ¨*u*(*s−i*)¨ *·* ( *ux* + *uxx* )

¨

¨     ¨

¨ ¨ ¨

+ *s*Σ*−*2 *s* Σ*j*

¨*u*(*s*)

*j*

*i*=0

*j*=2

*j* ¨

(*j−i*+1)¨ + ¨

¨ ¨ ¨ ¨

*u*(*i*+1) + *u*(*i*)

*u*

*u*

*·*

*u*

*u*

(*j−i*)¨

*· uxx ·*

¨ (*s−j*+1)¨

*·*

*u*

*u*

*u*

¨ (*i*+1)¨ + ¨

*u*

(*i*)¨

+ *s*Σ*−*2 *s−*1 ¨

*i*=1

*i*

*s*

*i*

*u*

*i*=1

(*i*+1)¨ + ¨

(*i*)¨  

(¨ (*s−i−*1)¨ + ¨

(*s−i*)¨)

≤ Σ (*s*) *Ci*+1 (  *ϕ * ) *ϕ * + *Ci*  *ϕ *  *ϕ *

*s−*2

*i*

*T*

*i*

*i*+1

*T*

*i−*1

*i*

*i*=1

*·* (*C*1 (  *ϕ * ) *ϕ * 1 + *C*2 (  *ϕ * 1) *ϕ * 2) *· Cs−i*  *ϕ * *s−i−*1  *ϕ * *s−i*

## Σ Σ

*T*

*s−*1 *j*

+

*s j*

*j*

*i*

*Cj−i*+1

+ *Cj−i*

       

*T*

*ϕ*

*j−i*

*ϕ*

*j−i*+1

*T*

*ϕ*

*j−i−*1

*ϕ*

*j−i*

*i*+1 *T*

*·* *C* (  *ϕ*

*j*=2

*i*=0

*s−*2

*i*) *ϕ *

*i*+1

+ *Ci*  *ϕ *

*i−*1  *ϕ*

 *· Cs−j*+1  *ϕ *

*s−j*  *ϕ *

*s−j*+1

+ 2*s* Σ *s−*1 *Ci*+1 (  *ϕ * ) *ϕ * + *Ci*  *ϕ *  *ϕ *

*T*

*T*

*i*

*T*

*i*

*i*+1

*T*

*i−*1

*i*

*· C*2 ( *ϕ* 1) *ϕ* 2 *·* [*Cs−i−*1  *ϕ * *s−i−*2  *ϕ * *s−i−*1 + *Cs−i*  *ϕ * *s−i−*1  *ϕ * *s−i*]

*i*

*i*=1

   

Therefore

*d u*(*s*) = 1

¨ ¨

*dt*

2 ¨*u*(*s*)¨

*T*

* *d* ¨*u*(*s*)¨ 2 = 1

*dt*

2 ¨*u*(*s*)¨

*· d* ∫

R

*T*

*u*(*s*) 2 *dx* = *A* + *B*

*dt*

2 ¨*u*(*s*)¨

where

≤ *a*(  *ϕ * *s−*1)  *u*(*s*)  + *b*(  *ϕ * *s−*1)  *ϕ * *s*

*a* (*r*) := (4*s* + 1)[*r* + *C*1(*r*) *· r* + *C*2(*r*) *· r*]2 +1

*s−*2

*T T T*

*i*=1

*b* (*r*) := Σ *Ci*+1 (*r*)+ *Ci* (*r*) (*C*1(*r*)+ *C*2(*r*)) *· Cs−i* (*r*) *· r*3

## Σ Σ

*s−*2 *j*

*s−*2

+

*s*

*j*

*Cj−i*+1 (*r*)+ *Cj−i* (*r*)

*·*

*Ci*+1 (*r*)+ *Ci* (*r*)

*· Cs−j*+1 (*r*) *· r*3

*i*=0

*T*

*T*

*T*

*i*

*T*

*T*

*T*

*T*

*j*

*i*

*T*

*T*

*i*=2

+ *s* Σ *s−*1 *Ci*+1 (*r*)+ *Ci* (*r*) *· C*2 (*r*) *·* (*Cs−i−*1 (*r*)+ *Cs−i* (*r*)) *· r*3

*i*=1

+ 3*s*[*r* + *C*1(*r*) *· r* + *C*2(*r*) *· r*]2 *· C*(*s−*1)(*r*) *· r*

Applying the Gronwall inequality, we have for any 0 ≤ *t* ≤ *T*

*T*

¨ (*s*)(*·, t*)¨ ≤ *ea*(  *ϕ * *s−*1)*T* ¨*u*(*s*)(*·,* 0)¨ + *b* *ϕ s−*1 *ϕ s ea*(  *ϕ * *s−*1)*T*



*u*

¨

¨

¨

*a* *ϕ*

*s−*1

= *ea*( *ϕ s−*1)*T* ¨*ϕ*(*s*)¨ + *b*  *ϕ * *s−*1  *ϕ * *s ea*( *ϕ * *s−*1)*T*

≤ *ea*(  *ϕ * *s−*1)*T* 1+ *b*  *ϕ *

*s−*1

 *ϕ * *s*

≤ *Cs* ( *ϕ*

*s−*1) *ϕ * *s*

Where *Cs* (*r*) := *ea*(*r*)*T* (1 + *b* (*r*)).

*T*

*T*

Now we can prove the Proposition [4.1](#_bookmark3)

**Proof.** (of Proposition [4.1](#_bookmark3)) Let *es* (*y*) = (1 + *C*1 + *C*2 + *···* + *Cs* ) *· y*. By Lemma

*T*

*T*

*T*

*T*

[4.5](#_bookmark4), the function *d* (*s, T, r*) := *ds* (*r*) *,* is computable and for any 0 ≤ *t* ≤ *T ,*

*T*

 *u* (*·, t*)  *s*= *{ u * 2 +  *u*(1) 2 + *···* +  *u*(*s*) 2*}*1*/*2

*≤{ * *ϕ * 2 + *C*1 (  *ϕ * )  *ϕ*

*T*

 2 + *···* + (*Cs* (  *ϕ*

 *s−*1)  *ϕ*

 *s*)2*}*1*/*2

*≤* [ 1+ *C*1 (  *ϕ *

*T*

*s−*1

)+ *···* + *Cs* (  *ϕ *

*s−*1

*T*

*T*

1

) 2]1*/*2  *ϕ *

*≤ ds* (  *ϕ * *s*)  *ϕ * *s*

*T*

*s*

This proves Proposition 4.1

We will prove the convergence of the iterative sequence *{vj }* about *j*.

*n*

**Proposition 4.6** *Let v*0 := *S*¯(0*, ϕ*)*, and vj*+1 := *S*¯(*vj, ϕ*)*. If αs T* 1*/*2(3 + *T* )3*/*2

*T*

 *ϕ * 2 +8(3 + *T* )*αs T* 1*/*2  *ϕ * *s*≤ 1*, then we have*

*s T*

 *vj*+1 (*t*) *− vj* (*t*)  *s*≤ 2*−j* (3 + *T* )1*/*2  *ϕ * *s,*

*where αs* = (*es* (  *ϕ * *s*)+ 1)*·√s·*2*s·*(2*s* + 1)*·T* 1*/*2 +1 *, for all* 0 ≤ *t* ≤ *T , ϕ ∈ Hs* (R)*.*

*T*

*T*

Firstly, we will construct the space to obtain the estimate of the nonlinearity of the mKdV equation, and then prove the Proposition 4.6.

**Definition 4.7** Let *T >* 0, and continuous functions *u* : *Y Hs* (R) with [0 *,T* ] *Y* define

*→ ⊆*

*s*

Λ

1*,T*

(*u*) = sup

0≤*t*≤*T*

⎛

⎝

 *u* (*· , t*)  *s*

∫*T*

*s*+1 2

⎞1*/*2

*s*

Λ

2*,T*

(*u*) = sup

*x∈*R 0

⎛∫

*Dx u* (*x, t*) *dt*⎠

⎞1*/*2

( )*|*2*dx*

and

⎝

2*,T*

3*,T*

*s*

3*,T*

Λ

(*u*) =

sup

0≤*t*≤*T*

R

*|u x, t* ⎠

 *u * := Λ*s* (*u*) := Λ*s*

*X*

*s T*

*T*

1*,T*

(*u*) 2 + Λ*s*

(*u*) 2 + Λ*s*

(*u*) 2 1*/*2

Then *Xs*

*T*

norm  *u * *s T*

*X*

= *{u ∈ C* ([0*,T* ]; *Hs* (R); Λ*s* (*u*) *< ∞*)*}* is a Banach space with the

*.*

*T*

*s T*

**Lemma 4.8** *If T >* 0 *,u ∈ X* (R) *,* sup

0≤*t*≤*T*

 *u* (*x, t*) *s*≤ *es* (  *ϕ * *s*) *, then we*

*have*

*T*

∫*T* ¨ ¨

¨*u*2*ux*¨ *dt* ≤ *αs T* 1*/*2  *u * *s * *u * *s*

*s*

*T*

*XT*

*XT*

0

*where αs* = (*es* (  *ϕ * *s*)+ 1) *· √s·* 2*s ·* (2*s* + 1) *· T* 1*/*2 +1 *,and es* (  *ϕ * *s*) *is the same*

*T*

*T*

*T*

*form as Proposition* 4*.*1*.*

**Proof.** For *s* ≥ 3,

*s−*1 *j*

*x*

+

*s j*

*j*

*i*

*u*(*i*+1)*u*(*j−i*)*u*(*s−j*)

*s−*1

Σ

(*s*)

*u*(*i*+1)*u*(*s−i−*1)*u*

*D*

*s x*

*u*2*u*

*x*

≤

*i*

*i*=1

## Σ Σ

*j*=2

*i*=0

+*|u*2*us*+1*|* +2 (*s* + 1) *|uuxu*(*s*)*|*

¨ ¨

*s* 1

¨

Σ

¨ ¨

*i*

*—*

¨   ¨ ¨

¨*Ds*

*x*

*u*2*ux* ¨ ≤

(*s*) *u*(*s−i−*1) *ux*

*i*=1

*∞*

*u*(*i*+1)

## Σ Σ

¨ ¨

*s−*1 *j*

¨ ¨

*∞*

+

*s j*

*i*=0

*j*

*i*

¨*u*(*i*+1)¨ *·* ¨*u*(*j−i*)¨

*u*(*s−j*)

*∞*

*j*=2

¨

¨ ¨ ¨ ¨ ¨

*∞*

+ ¨*u*2*us*+1¨ +2 (*s* + 1) *u * *∞ ux * *∞* ¨*u*(*s*)¨

Since ¨*f* ¨ =¨*Ff* ¨ = *|ξ| Ff ,* 1+ *|ξ|* + *...* + *|ξ| ≤ s* (1 + *|ξ|* )

(*k*) (*k*) *k* 2 2*s s* 2

¨*u*2*ux*¨

≤ *√s* ¨*u*2*ux*¨ + ¨*Ds* *u*2*ux* ¨

*s*

≤ *√s*

*x*

*i*

*j*

*i*

*s*Σ*−*1

(*s*) +

*s*Σ*−*1 *s* Σ*j*

*j*

+2 (*s* + 1)+ 1 

*u * *s · u * *s · u * *s*

*i*=1

*j*=2

*i*=0

+ *√s* ¨*u*2*us*+1¨

Since ∫ *T f* (*t*)*dt* 2 ≤ *T* ∫ *T* *f* (*t*)2 *dt,* we have

0

0

0

∫ *T* ¨*u*2*ux*¨ *dt*

≤ *√s · T ·*

*s*

*s*Σ*−*1

(*s*) +

*i*

*s*Σ*−*1 *s* Σ*j*

*j*

+2 (*s* + 1)+ 1

* sup

 *u * *s ·* sup *u * *s*

*i*=1

*j*=2

⎛

*j*

⎝

*i*=0

∫*T*

*i*

⎞1*/*2 ⎛

0*≤t≤T*

∫

0*≤t≤T*

⎞1*/*2

sup *u * *s*

*·*

0*≤t≤T*

+ *√sT* 1*/*2 sup

*x∈*R 0

⎝

2

*u x, t* ⎠

(*s*+1) ( )

⎜⎝

Σ *s* Σ

sup

0≤*t*≤*T*

R

*u*2 (*x, t*) 2*dt*⎟⎠

≤ *√s*

sup

0≤*t*≤*T*

 *u * *s* [*T*

⎛Σ*s−*1

(*s*) +

*i*

*s−*1 *j*

*j*

*i*=0

*j*

*i*

+2 (*s* + 1)+ 1⎞⎠

* sup *u * *s*

0≤*t*≤*T*

∫ *T*

*·*

*i*=1

sup

0≤*t*≤*T*

*u * *s* + *T* 1*/*2

sup

*x∈*R

0

2 1*/*2 ∫

2 1*/*2

The Lemma follows straightforwardly.

*j*=2

*u*(*s*+1) (*x, t*)

sup

0≤*t*≤*T*

R *|u* (*x, t*)*| dt*

]

1

∫

**Proof.** (of Proposition [4.6](#_bookmark5)) Let *W* (*t*) *ϕ* = *√*2*π*

R

*eixξ*

*eiξ*3*t*

*ϕ*ˆ(*ξ*)d*ξ.*

Since *v*0 := *S*¯ (0*, ϕ*) *, vj*+1 := *S*¯ *vj, ϕ* , by Lemma 4.8 and Lemma 4.8,4.9,4.10 in [8], for *j* ≥ 1*,*

1 *t*

∫

*vj Xs* = *W* (*t*) *ϕ −* 3

*T*

0

∫

*W* (*t − τ* ) (*vj−*1)3

*dτ * *s*

*X*

*x T*

≤ (3 + *T* )1*/*2 *ϕ*

*T * *s* + (3 + *T* )1*/*2 

0

*vj−*1

2

*x s dτ*

*v −*

*j* 1

≤ (3 + *T* )1*/*2  *ϕ * *s* + (3 + *T* )1*/*2 *αs T* 1*/*2  *vj−*1 2 *s*

Let *T >* 0 *,* such that

*T XT*

24*αs T* 1*/*2 (3 + *T* )3*/*2  *ϕ * 2 +8 (3 + *T* ) *αs T* 1*/*2  *ϕ * *s*≤ 1

*T*

*s*

*T*

From  *v*0 *Xs* =  *W* (*t*) *ϕ * *Xs ≤* (3 + *T* )1*/*2  *ϕ * *s* we obtain by induction

*T*

*T*

 *vj * *Xs* ≤ 2 (3 + *T* )1*/*2  *ϕ * *s* (*for all j ∈* N)

*T*

For *j* ≥ 2

 *vj − vj−*1 

*t*

1

∫

*s* = 3 *W* (*t − τ* )

*X*

*T*

0

(*vj−*1)3

(*vj−*2)3 *dτ * *s*

*X*

*x −*

*x T*

≤ (3 + *T* )1*/*2 *αs T* 1*/*2  *vj−*1 2 + *vj−*1*vj−*2 + *vj−*2 2 *·* *vj−*1 *− vj−*2  *s*

*T*

*X*

*T*

≤ (3 + *T* )1*/*2 12*αs T* 1*/*2 *ϕ* 2

*vj−*1 *− vj−*2  *Xs*

*T s T*

1

≤ 2  *vj−*1 *− vj−*2 *Xs*

*T*

Therefore, for 0 ≤ *t* ≤ *T ,*

 *vj*+1 (*t*) *− vj* (*t*)  *s*≤  *vj*+1 (*t*) *− vj* (*t*)  *Xs ≤* 2*−j* (3 + *T* )1*/*2  *ϕ * *s*

*T*

if 24*αs T* 1*/*2 (3 + *T* )3*/*2  *ϕ * 2 +8 (3 + *T* ) *αs T* 1*/*2  *ϕ * *s*≤ 1. This proves Proposi-

*T*

*s*

*T*

tion 4.6.

From the above proof, if we use *vj* instead of *vn*, when *j → ∞*, we can obtain

*n*

the result *vj → vn*. Then we will prove the uniform convergence of the sequence

*n*

*j*

*{vn}* or *{vn}*.

**Proposition 4.9** *v* (*t*) = *S*¯ (*v, ψ*) (*t*) *, vn* (*t*) = *S*¯ (*vn, ψn*) (*t*) *,*

 *v* (*t*) *− vn* (*t*)  ≤ 2 (3 + *T* )1*/*2  *ψ − ψn *

*s s*

*for all* 0 ≤ *t* ≤ *T, if* 24*αs T* 1*/*2 (3 + *T* )3*/*2 (  *ψ * *s* +1)2 +

8 *s* 1*/*2

*s s T √ s s*

1*/*2

*αT T*

(3 + *T* )(  *ψ * *s* +1) ≤ 1*, where αT* = (*eT ϕ * *s* + 1)*·*

*s·*2 *·*(2

+ 1)*·T*

+1*.*

**Proof.** Since *v* (*t*) = *S*¯ (*v, ψ*) (*t*) *, vn* (*t*) = *S*¯ (*vn, ψn*) (*t*) *,* by Lemma 4.8 in [8],we obtain the result as following:

1 *t*

∫

*v − vn Xs* = *W* (*t*) (*ψ − ψn*) *−* 3

*W* (*t − τ* )

*v*3 *− v*3

*dτ Xs*

*T n x T*

0

≤ (3 + *T* )1*/*2  *ψ − ψn * *s* + (3 + *T* )1*/*2 *as T* 1*/*2  *v*2 + *vvn* + *v*2 *Xs * *v − vn * *Xs*

*T*

*n*

*T*

*T*

where *as* = *√s ·* 2*s · T* 1*/*2 + 1(*see*[8])*.* By Proposition 4.6 (notice that  *ψn s*≤ 

*T*

*ψ * *s* +1*,* ) if 24*αs T* 1*/*2 (3 + *T* )3*/*2 (  *ψ * *s* +1)2 + 8*αs T* 1*/*2 (3 + *T* )(  *ψ * *s* +1) ≤ 1,

then

*T T*

 *v − vn * *Xs*

*T*

≤ (3 + *T* )1*/*2  *ψ − ψn * *s* + (3 + *T* )1*/*2 *αs T* 1*/*2  *v*2 + *vvn* + *v*2

*T*

*n*

 *Xs * *v − vn * *Xs*

≤ (3 + *T* )1*/*2  *ψ − ψn * *s* +12 (3 + *T* )3*/*2 *αs T* 1*/*2 (  *ψ * *s* +1)2  *v − vn * *Xs*

*T*

*T*

## ≤ (3 + *T* )

1*/*2

*T T*

1

*ψ − ψn s* + 2 *v − vn Xs*

*T*

Therefore  *v − vn Xs* ≤ 2 (3 + *T* )1*/*2  *ψ − ψn s,* the sequence *{vn}* is uniform

*T*

convergence.

Thus, by Proposition 4.6 and 4.9, the sequence *vj* converges fast to *v* uni- formly in *t* [0 *, T* ]*.* We can see that the machine searches fast approximations to *u*(*x, t*), and computes the solutions of mKdV equation with arbitrary precision. This approach can be extended to other nonlinear equations.

*n*

*∈*

*{ }*

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# References

1. Chen, Y. and B. Li. *The stochastic soliton-like solutions of stochastic mKdV equations.* Czechoslovak

J. Phys.**55**(2005),1–8.

1. Fu, Z. and S. Liu, S. Liu. *New solutions to mKdVequation.* Phys. Lett. A. **326**(2004),364–374.
2. Fu, Z. and L. Zhang, S. Liu, S. Liu. Fractional transformation and *new solutions to mKdV equation.*

Phys. Lett. A. **325**(2004),363–369.

1. Gay, W. and B.Y. Zhang, N. Zhong. *Computability of solution of the Korteweg-de Vries equation.*MLQ Math. Log.Q. **47**(2001),93-110.
2. Gesztesy, F. and B. Simon. *Constructing solutions of the mKdV-equation.* J. Funct. Anal. **89**(1990),53– 60.
3. Geyikli, T. and D. Kaya. *An application for a modified KdV equation by the decomposition method and finite element method.* Appl. Math. Comput. **169**(2005),971–981.
4. Kenig, C. and G. Ponce, L. Vega. *On the ill-posedness of some canonical dispersive equations.* Duke Math. **106**(2001),617-633.
5. Kenig, C. and G. Ponce, L. Vega. *A bilinear estimate with applications to the KdV equation.* Amer. Math. Soc. **9**(1996),573-603.
6. Boutet de Monvel, A. and D. Shepelsky. *The modified KdV equation on a finite interval.* C. R. Math. Acad. Sci. Paris. **337**(2003),517–522.
7. Pour-El, M.B. and I. Richards. *Computability in Analysis and Physics.* Springer, Berlin,1989.
8. Weihrauch, K. *Computable analysis*. Springer-Verlag, Berlin, 2000.
9. Weihrauch, K. and N. Zhong. *Computing the solution of the Korteweg-de Vries equation with arbitrary precision on Turing machines.* Theoret. Comput. Sci. **332**(2005),337–366.
10. Weihrauch, K. and N.Zhong. *Computing Schrodinger propagator on type-2 Turing machines.*Complexity. **22**(2006),918-935.
11. Yan, Z. *Approximate Jacobi elliptic function solutions of the modified KdV equation via the decomposition method.* Appl. Math. Comput. **166**(2005),571–583.
12. Yu, J. and R. Zhou. *Two kinds of new integrable decompositions of the mKdV equation.* Phys. Lett.

A. **349**(2006),452–461.