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Continuous Prequantale Models of *T*1

Topological Semigroups [4](#_bookmark0)

Hui Li[1](#_bookmark0) Xiangnan Zhou[2](#_bookmark0) Qingguo Li[3](#_bookmark0)

*College of Mathematics and Econometrics Hunan University*

*Changsha, Hunan, 410082, China*

**Abstract**

In this paper, we show that every *T*1 topological semigroup satisfying condition (Δ) can be embedded into a topological semigroup (*D, σ, ⊙*), where (*D, ±*) is a domain. Furthermore, by considering the maximal point topological semigroup of a continuous prequantale, it is proven that every *T*1 topological semigroup satisfying condition (Δ) has a continuous prequantale model, which may not be bounded complete.

*Keywords:* topological semigroup, prequantale, stable ordered semigroup, Scott topology, prequantale model.

# Introduction

Dana Scott introduced the domain theory in order to provide the mathematical foundation for denotational semantics of programming languages [[12](#_bookmark19)]. After that, the domain theory has been a major impetus in the development of topological spaces. Thanks to the topological tools, the domain theory has developed very quickly. Indeed there are deep ties and interactions between the topological theory of topological spaces and the order theory of partially ordered sets. One aspect that many scholars are interested in is that the classical topological spaces are embedded into the set of the maximal points of the appropriate posets so as to use the domain theory to solve the problems about the topology.

Now the research for the maximal point space max(*P* ) of the poset *P* has become one of the most central fields in domain theory. A poset model of a topological

1 Email: [lihui871869166@163.com](mailto:lihui871869166@163.com)

2 Corresponding author, Email: [xnzhou81026@163.com](mailto:xnzhou81026@163.com)

3 Corresponding author, Email: [liqingguoli@aliyun.com](mailto:liqingguoli@aliyun.com)

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space *X* is a poset *P* such that the maximal point space max(*P* ) is homeomorphic to *X* [[7](#_bookmark14)]. There have been many results which give necessary or sufficient conditions for topological spaces to have a poset model. Please refer to ([[8,](#_bookmark15)[9,](#_bookmark16)[10,](#_bookmark17)[11](#_bookmark18)]) if the reader wants to know more about poset models.

In [[4](#_bookmark10)], Kamimura and Tang characterized the spaces that have a bounded com- plete algebraic dcpo model with a countable base. Edalat and Heckmann [[2](#_bookmark9)] proved that every complete metric space has a domain model. Later, in [[5](#_bookmark12)], it was proven that every complete metric space has a bounded complete domain model. Lawson proved that a topological space is a Polish space if and only if it has an *ω*-continuous dcpo model satisfying the *Lawson condition* [[7](#_bookmark14)]. In [[1](#_bookmark8)], it was shown that any *T*1 space has a continuous poset model. In ([[3,](#_bookmark11)[14](#_bookmark21)]), Ern´e and Zhao proved that every *T*1 space has a bounded complete algebraic poset model, respectively. In [[15](#_bookmark22)], Zhao and Xi revealed that every *T*1 space has a directed complete poset model (in fact, a locally quasialgebraic dcpo model), and verified that the *T*1 space is sober if and only if the dcpo model is sober with respect to the Scott topology. In [[13](#_bookmark20)], Bin Zhao et al. proposed the notions of maximal point topological semigroups and prequantale models of topological semigroups. In that paper, they introduced a condition (Δ) on topological semigroups and obtained a result that every *T*1 topological semigroup satisfying condition (Δ) has a bounded complete algebraic prequantale model.

In this paper, we take advantage of this fact that every *T*1 topological space has a continuous poset model, and define a new binary operation on the continuous poset to prove that every *T*1 topological semigroup satisfying condition (Δ) has a continuous prequantale model. In particular, the continuous poset may not be bounded complete. Thus the bounded completeness is not a necessary condition for a prequantale to be the model of a *T*1 topological semigroup. In addition, in this paper, the way to find the model of a *T*1 topological semigroup is different from the method used by Bin Zhao et al..

# Preliminaries

Now we will recall some basic notions on domain theory and topological theory to be used in the sequel.

A poset *P* is called *bounded complete* if every subset that is bounded above has a least upper bound. In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set. A nonempty subset *E* of a poset *P* is *directed* if every two elements of *E* have an upper bound in *E*. A poset is called a *directed complete poset*, or *dcpo* for short, if every directed subset has a supremum. For any subset *A* of an ordered set *P* , we denote *↓A* =*{x ∈ P | x ≤ y* for some *y ∈ A}* and *↑A* =*{x ∈ P | x ≥ y* for some *y ∈ A}*. For any element *a ∈ P* , one simply writes *↓a* for *↓{a}* and *↑a* for *↑{a}*. A subset *X* is called a *lower set (upper set)* if

*X* = *↓X* (*X* = *↑X*, respectively).

**Definition 2.1** A subset *U* of a poset *P* is *Scott open* if

* 1. it is an upper set, that is, *U* = *↑U* ;
  2. for any directed subset *E* of *P* , W *E ∈ U* implies *E ∩ U /*= *∅* , whenever W *E*

exists.

All Scott open sets of a poset *P* form a topology on *P* , denoted by *σ*(*P* ) and called Scott topology on *P* . The space (*P, σ*(*P* )) is written as Σ*P* , called Scott space of *P* .

Let *P* be a poset. We say that *x* is *way below y*, in notation *x y*, if for all directed subsets *E ⊆ P* for which W *E* exists, the relation *y ≤* W *E* always

implies the existence of *e ∈ E* with *x ≤ e*. Denote *⇓x* = *{y ∈ P | y x}*,

*⇑x* = *{y ∈ P | x y}*. A poset *P* is called *continuous* if for any *x ∈ P* , *x* = W*↑ ⇓x*,

that means for all *x ∈ P* , the set *⇓x* is directed and its supremum is *x*. For any continuous poset *P* , the family *{⇑x | x ∈ P}* form a base for the Scott topology on

*P* . A dcpo which is continuous as a poset will be called a *domain*.

A *basis B* of a poset *P* is a subset of *P* such that for each *x ∈ P* , *x* = W*↑*(*B∩⇓x*).

From this, we obtain an equivalent condition for continuous posets. A poset is continuous if and only if it has a basis.

**Definition 2.2** A triple (*S, ≤, ·*) is called an *ordered semigroup* if it satisfies:

1. (*S, ≤*) is a poset;
2. (*S, ·*) is a semigroup;
3. For all elements *x*, *y*, *z* in *S*, *x ≤ y* implies *x · z ≤ y · z* and *z · x ≤ z · y*.

**Definition 2.3** ([[13](#_bookmark20)]) A triple (*P, ≤, ·*) is called a *prequantale* if it satisfies:

1. (*P, ≤*) is a poset;
2. (*P, ·*) is a semigroup;

W (3) For all directed subsets *E* of *P* with W *E* existing, *a ·* (W *E*)= W(*a · E*) and

( *E*) *· a* = W(*E · a*), where *a · E* = *{a · e | e ∈ E}* and *E · a* = *{e · a | e ∈ E}*. Note that in Definition [2.3,](#_bookmark1) we do not ask that a prequantale is a dcpo.

A prequantale (*P, ≤, ·*) is called *continuous (algebraic)*, if (*P, ≤*) is continuous

(algebraic).

**Remark 2.4** It is easy to see that a prequantale is an ordered semigroup. Con- versely, it is not true. For example, let *P* be the subset of the square [0*,* 1]2 consisting of its interior ]0*,* 1[2 and the points (0*,* 0) = *⊥,* (1*,* 1) = *T*. Then, obviously, (*P, ≤, ·*) is an ordered semigroup, where *≤* is the pointwise order and *a· b* = *a∧b* for *a, b ∈ P* .

Pick *E* = *{* 1 *}×*]0*,* 1[, then *E* is directed and W *E* = (1*,* 1) = *T*. For *x* = ( 2 *,* 1 ),

*x ·* (W 2 2 1 *· E* = *{* 1 *}×*]0*,* 1 ]. So W(*x · E*) = ( 1 *,* 1 ) */*= *x ·* (W3 2

*E*) = *x* = ( 3 *,* 2 ), but *x* 2 2

Hence (*P, ≤, ·*) is not a prequantale.

2 2 *E*).

# Continuous Prequantale Models of Topological Semi- groups

In this section, we will discuss the prequantale models of topological semigroups. Now we give some concepts which will be used afterwards.

**Definition 3.1** A *topological semigroup* (*S, τ, ·*) consists of a semigroup (*S, ·*) and a topology *τ* on the set *S* such that the mapping *f* : *S×S → S* defined by *f* (*x, y*)= *x·y*

is continuous when *S × S* is endowed with the product topology, that is, for each *x* and *y* in *S* and each open neighborhood *W* of *x · y*, there exist open neighborhoods *U* of *x* and *V* of *y* such that *U · V ⊆ W* , where *U · V* = *{u · v | u ∈ U, v ∈ V }*.

**Example 3.2** (1) Let *L* = *{T, a, b}* be a poset, where *a, b* are incomparable and *T*

is above *a, b*. Define a binary operation *∗* on *L* as follows:

|  |  |  |  |
| --- | --- | --- | --- |
| *∗* | a | b | *T* |
| a | a | a | a |
| b | b | b | b |
| *T* | a | b | *T* |

It is easy to verify that (*L, ∗*) is a semigroup. The topology on *L* is the Scott topology and *σ*(*L*) = *{∅, L, {a, T}, {b, T}, {T}}*. For *a, b ∈ L*, *a ∗ b* = *a ∈ {a, T}*, but for all open neighborhoods *U* of *a* and *V* of *b*, *U ∗ V ⊆ {a, T}* never happens. Hence (*L, σ, ∗*) is not a topological semigroup.

(2) Let *M* = *{T, a, ⊥}* be a poset with *⊥ < a < T*. Define a binary operation *·*

on *M* as follows:

|  |  |  |  |
| --- | --- | --- | --- |
| *·* | *⊥* | a | *T* |
| *⊥* | *⊥* | *⊥* | *⊥* |
| a | *⊥* | a | a |
| *T* | *⊥* | *T* | *T* |

Then (*M, ·*) is a semigroup and *σ*(*M* )= *{∅, M, {a, T}, {T}}*. So one can easily see that (*M, σ, ·*) is a topological semigroup.

Let (*S, τ*1*, ·*) and (*T, τ*2*, •*) be topological semigroups. A mapping *f* : (*S, τ*1*, ·*) *→* (*T, τ*2*, •*) is called an *isomorphism (embedding)*, if it is both a topological homeo- morphism (embedding) and a semigroup homomorphism.

**Definition 3.3** ([[13](#_bookmark20)]) Let (*P, ≤, ∗*) be an ordered semigroup. The triple (max(*P* )*,σ |*max(*P* )*, ∗*) is called a *maximal point topological semigroup* of *P* , if it satisfies the following conditions:

1. (max(*P* )*, ∗*) is a subsemigroup of *P* ;
2. (max(*P* )*,σ |*max(*P* )*, ∗*) is a topological semigroup.

**Definition 3.4** ([[13](#_bookmark20)]) Let (*S, τ, ·*) be a topological semigroup. A *prequantale model* of the topological semigroup (*S, τ, ·*) is a prequantale (*P, ≤, ∗*) together with an isomorphism

*φ* : (*S, τ, ·*) *→* (max(*P* )*,σ |*max(*P* )*, ∗*)*,*

where (max(*P* )*,σ |*max(*P* )*, ∗*) is the maximal point topological semigroup of *P* . We will use (*P, φ*) to denote a prequantale model of *S*.

Note that if (*P, ≤*) is a continuous poset, then (*P, φ*) is called a

*continuous prequantale model* of *S*.

**Definition 3.5** ([[13](#_bookmark20)]) A topological semigroup (*S, τ, ·*) is said to satisfy *condition*

(Δ), if *U · V ∈ τ* holds for all *U, V* in *τ* .

Topological semigroups may not satisfy the condition (Δ) as is shown by the following example.

**Example 3.6** (1) (N*, σ, ×*) is a topological semigroup, where *×* is the multiplication on the set of natural number N, but it does not satisfy condition (Δ). For Scott open sets *U*1 = *{a | a ≥* 2*,a ∈* N*}*, and *U*2 = *{b | b ≥* 3*,b ∈* N*}*, *U*1 *× U*2 = *{a × b | a ∈ U*1*,b ∈ U*2*}* implying 6 *∈ U*1 *× U*2 but 7 *∈/ U*1 *× U*2. So *U*1 *× U*2 is not an upper set. Hence *U*1 *× U*2 is not Scott open, which is equivalent to say that (N*, σ, ×*) does not satisfy condition (Δ).

* 1. Let *T* = *{⊥, a, b, T}* be a poset, where *a, b* are incomparable, *T* is above *a, b*

and *⊥* is below *a, b*. The binary operationon *T* is defined by :

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | *⊥* | a | b | *T* |
| *⊥* | *⊥* | *⊥* | *⊥* | *⊥* |
| a | *⊥* | a | a | a |
| b | *⊥* | b | b | b |
| *T* | *⊥* | *T* | *T* | *T* |

Obviously, (*T,* ) is a semigroup and (*T, σ,* ) is a topological semigroup. Clearly,

*{T}* and *T* are Scott open sets, but *{T} T* = *{⊥, T}* is not Scott open. Thus (*T, σ,* ) does not satisfy condition (Δ).

* 1. We will show that it may change the result as the binary operation on a poset changes. Let *T* be the poset described in (2). The binary operation on *T* is defined by *∧*, in other words, the meet of each pair of elements in *T* . It is easy to see that (*T, ∧*) is a semigroup and (*T, σ, ∧*) is a topological semigroup. It is not hard to verify that for any *U, V ∈ σ*(*T* ), *U ∧V ∈ σ*(*T* ), where *U ∧V* = *{u∧v | u ∈ U, v ∈ V }*. Hence (*T, σ, ∧*) is a topological semigroup satisfying condition (Δ).

These examples presented in Example [3.6](#_bookmark2) (2), (3) show that even for the same poset, things can be quite different with different binary operation.

**Remark 3.7** Let (*X, τ, ·*) be a topological semigroup. If (*X, τ, ·*) satisfies condition (Δ), then (*O∗*(*X*)*, ⊆, ·*) is an ordered semigroup, where *O∗*(*X*)= *{U ∈ τ | U /*= *∅}* and for any *U, V ∈ τ* , *U · V* = *{u · v | u ∈ U, v ∈ V }*.

The following theorem shows that for any *T*1 topological space (*X, τ* ) there exist a domain (*D, ±*) and a topological embedding *f* : *X → D*, but we do not know that what exactly the set max(*D*) is.

**Theorem 3.8** ([[1](#_bookmark8)]) *Let* (*X, τ* ) *be a T*1 *topological space. Then there exist a domain* (*D, ±*) *and a topological embedding f* : *X → D, where D is equipped with the Scott topology.*

Let (*X, τ* ) be a *T*1 topological space. Set *D* = *D*0 *∪ D*1, where

*D*0 = *{*(*U, n*) *| U ∈ τ, U /*= *∅* and *n ∈* N*}*

and where

*D*1 = *{F | F ∈ Filt*(*O∗*(*X*))*},*

*Filt*(*O∗*(*X*)) = *{F ⊆ O∗*(*X*) *|F* is a filter of *O∗*(*X*)*}.*

For any *x ∈ X*, let *N* (*x*)= *{U ∈ τ | x ∈ U}*. Obviously, *N* (*x*) is in *D*1.

Define the binary relation *±* on *D* as follows: for each *p*1*, p*2 *∈ D p*1 *± p*2 if and only if one of the following holds:

1. *p*1 = *p*2;
2. *pi* = (*Ui, ni*) *∈ D*0 for *i* = 1*,* 2, *n*1 *< n*2 and *U*2 *⊆ U*1;
3. *p*1 = (*U*1*, n*1) *∈ D*0, *p*2 = *F ∈ D*1 and *U*1 *∈ F*;
4. *p*1*, p*2 *∈ D*1 and *p*1 *⊆ p*2.

It is easy to verify that the relation *±* is a partial order on *D*. Note that for

*p*1 *∈ D*1 and *p*2 *∈ D*0, the relation *p*1 *± p*2 never occurs.

Then we have the following conclusions:

1. *D* is a dcpo. For any directed subset *E* of *D* with *E* containing no maximal element of itself. If *E ⊆ D*0, then W *E* = *F* generated by *π*1(*E*), *π*1(*E*)= *{U ∈ τ |* (*U, n*) *∈ E, n ∈* N*}*. If *E ∩ D*1 */*= *∅*, then W *E* = *{R | R ∈ E ∩ D*1*}*.
2. *D* is continuous. For all *p*0 *∈ D*0, *p*0 *p*0. For *p*1 *∈ D*1,

*⇓p*1 = *{*(*U, n*) *∈ D*0 *| U ∈ p*1*, n ∈* N*}*

is directed and its supremum is *p*1. For any *q*1*, q*2 *∈ D*1, *q*1 *q*2 never occurs.

1. Define the function *f* : (*X, τ* ) *→* (*D, σ*) by *f* (*x*) = *N* (*x*). Then *f* is a topological embedding.

Unless explicitly stated otherwise, *D* always stands for the domain constructed in Theorem [3.8](#_bookmark3) in the following.

In order to discuss the embedding between topological semigroups, we will define a binary operation *≥* on *D*.

If (*X, τ, ·*) is a topological semigroup satisfying condition (Δ), then we can define a binary operation *≥* on *D*, for any *p*1*, p*2 *∈ D*:

1. If *pi* = (*Ui, ni*) *∈ D*0 for *i* = 1*,* 2, then *p*1 *≥ p*2 = (*U*1 *· U*2*, n*1 + *n*2).
2. If *p*1 = (*U*1*, n*1) *∈ D*0, *p*2 = *F ∈ D*1, then *p*1 *≥ p*2 = *{W ∈ O∗*(*X*) *| U*1 *· V ⊆*

*W, V ∈ F}* (*↑{U*1 *· V | V ∈ F}* for short).

1. If *p*1 = *F ∈ D*1, *p*2 = (*U*2*, n*2) *∈ D*0, then *p*1 *≥ p*2 = *{W ∈ O∗*(*X*) *| V · U*2 *⊆*

*W, V ∈ F}* (*↑{V · U*2 *| V ∈ F}* for short).

1. If *p*1*, p*2 *∈ D*1, then *p*1 *≥ p*2 = *{W ∈ O∗*(*X*) *| U · V ⊆ W, U ∈ p*1*,V ∈ p*2*}*

(*↑{U · V | U ∈ p*1*,V ∈ p*2*}* for short).

The following proposition about the binary operation *≥* holds.

**Proposition 3.9** *Let* (*X, τ, ·*) *be a topological semigroup and it satisﬁes condition*

(Δ)*, then* (*D, ±, ≥*) *is a prequantale.*

**Proof.** Since (*X, τ, ·*) satisfies condition (Δ), one can easily see that (*D, ≥*) is a

semigroup. For any directed subset *E* in *D*, and *d ∈ D*, if *E* has a maximal element *e*ˆ, then W *E* = *e*ˆ. We can easily verify that *d ≥* (W *E*) = W(*d ≥ E*) and (W *E*) *≥d* = W(*E ≥d*). Now we consider the case where *E* has no maximal element.

* 1. If *E ⊆ D*0, then W *E* = *F* generated by *π*1(*E*).

If *d* = (*V, n*) *∈ D*0, then (W *E*) *≥ d* = *↑{U · V | U ∈ F}*. *E ≥ d* = *{*(*W ·*

*V, m* + *n*) *|* (*W, m*) *∈ E}*. *E ≥ d* is directed with no maximal element since *E* is directed. So W(*E ≥ d*) = *Fj* generated by *π*1(*E ≥ d*). One can easily show that

*↑{U · V | U ∈ F}* = *Fj i.e.* (W *E*) *≥ d* = W(*E ≥ d*).

If *d* = *E ∈ D*1, then (W *E*) *≥ d* = *F ≥ E* = *↑{U · V | U ∈ F,V ∈ E}*.

*E ≥ d* = *E ≥E* = *{↑{W · V | V ∈ E}| W ∈ π*1(*E*)*}⊆ D*1

and *E ≥ d* is directed from the fact that *E* is directed. So

(*E ≥ d*)= *{↑{W · V | V ∈ E}| W ∈ π*1(*E*)*}.*

We claim that (W *E*) *≥ d* = W(*E ≥ d*). For any element *A ∈* (W *E*) *≥ d*, there exist *U*1 *∈ F*, *V*1 *∈ E* such that *U*1 *· V*1 *⊆ A*. From *U*1 *∈ F*, we can achieve that there exists *W*1 *∈ π*1(*E*) such that *W*1 *⊆ U*1, implying *W*1 *· V*1 *⊆ U*1 *· V*1 *⊆ A*. Then *A ∈* (*E ≥ d*). Conversely, for any *B ∈* (*E ≥ d*), there exists *W*0 *∈ π*1(*E*) such that *B ∈ ↑{W · V | V ∈ E}*, which implies that there is an element *V* of *E* such that

W W

0 0

*W*0 *· V*0 *⊆ B*. Also *B ∈* (W *E*) *≥d* due to *W*0 *∈ F*. Therefore, (W *E*) *≥d* = W(*E ≥d*).

* 1. If *E ∩ D*1 */*= *∅*, then W *E* = *{R | R ∈ E ∩ D*1*}*.

If *d* = (*V, n*) *∈ D*0, then (W *E*) *≥ d* = *↑{U · V | U ∈* W *E}* and

*E ≥ d* = *{*(*W · V, m* + *n*) *|* (*W, m*) *∈ E ∩ D*0*} ∪ {↑{U · V | U ∈ R}|R∈ E ∩ D*1*}.*

Then (*E ≥ d*) *∩ D*1 */*= *∅* and *E ≥ d* is directed since *E* is directed. Let

*{↑{U · V | U ∈ R}|R∈ E ∩ D*1*}* = *A.*

Then W(*E ≥ d*)= *{Rj | Rj ∈ A}*. We should verify that (W *E*) *≥ d* = W(*E ≥ d*). For any *W ∈* (W *E*) *≥ d* there exists *U ∈* W *E* such that *U · V ⊆ W* . Then there is

an element *R* in *E ∩ D* such that *U ∈ R*. So *W ∈ ↑*(*U · V* ) for *U ∈R∈ E ∩ D* . Hence *W ∈* W(*E ≥ d*). For the reverse, for any element *W* in W(*E ≥ d*), there exists

1 1

*Rj ∈ A* such that *W ∈ Rj*. Then there is an element *U ∈ R ∈ E ∩ D*1 such that *Rj* = *↑*(*U · V* ). That is, *W ∈ ↑*(*U · V* ) for *U ∈* W *E*, implying *W ∈* (W *E*) *≥ d*. It thus follows that (W *E*) *≥ d* = W(*E ≥ d*).

If *d* = *F ∈ D*1, then (W *E*) *≥ d* = *↑{U · V | U ∈* W *E, V ∈ F}*. Let *E ∈ E ∩ D*1. Since *E* is directed, for any (*W, n*) *∈ E ∩ D*0, there is an *E*1 in *E* such that *E,* (*W, n*) *± E*1. Obviously, *E*1 is in *D*1, so *W ∈ E*1, which implies that

*E ≥ d* = *{↑{W · V | V ∈ F}|* (*W, n*) *∈ E ∩ D*0*} ∪ {↑{U · V | U ∈ R,V ∈ F}|R∈ E ∩ D*1*}*

= *{↑{U · V | U ∈ R,V ∈ F}|R∈ E ∩ D*1*}.*

So W(*E ≥d*)= *{Fj | Fj ∈ E ≥d}*. We can easily prove that (W *E*) *≥d* = W(*E ≥d*). Using a similar argument, we can deduce that *d ≥* (W *E*) = W(*d ≥ E*) for any directed subset *E*. It thus follows that (*D, ±, ≥*) is a prequantale. *2*

The following corollary can be obtained easily.

**Corollary 3.10** *For any topological semigroup* (*X, τ, ·*) *satisfying condition* (Δ)*, then* (*D, ±, ≥*) *is an ordered semigroup.*

Our main aim is to show that (*D, σ, ≥*) is a topological semigroup. In [[13](#_bookmark20)], the authors provided an approach using stable ordered semigroups to prove that ordered semigroups with the Scott topology are topological semigroups. Before we move on to prove the topological semigroup, we recall some knowledge of the auxiliary relation and stable ordered semigroups.

**Definition 3.11** ([[6](#_bookmark13)]) A binary relation *≺* on a poset (*L, ≤*) is called an

*auxiliary relation* if it satisfies the following conditions for all *p, q, x, y ∈ L*:

1. *x ≺ y* implies *x ≤ y*;
2. *p ≤ x ≺ y ≤ q* implies *p ≺ q*;
3. If *F* is a finite subset of *L*, *F ≺ y* (which we take to abbreviate the fact that

*a ≺ y* for all *a ∈ F* ) implies that there exists *r ∈ L* such that *F ≺ r ≺ y*.

For the sake of convenience, let ***↑****p* = *{x ∈ L | p ≺ x}*, and ***↓****p* = *{x ∈ L | x ≺ p}*. We call an auxiliary relation *≺* on a poset *L approximating* if for all *p, q ∈ L*,

***↓****p ⊆* ***↓****q* if and only if *p ≤ q*. We can easily see that if *L* is a continuous poset, then the way-below relationis an approximating auxiliary relation on *L*.

We call the topology generated by the set *{****↑****p | p ∈ L}* the *pseudo-Scott topology*

and denote it by **P***σ*.

**Definition 3.12** ([[13](#_bookmark20)]) Let (*S, ≤, ·*) be an ordered semigroup. An auxiliary relation

*≺* on *S* is called *stable*, if it satisfies the following conditions for all *x*1*, x*2*, y*1*, y*2 *∈ S*:

1. *x*1 *≺ y*1 and *x*2 *≺ y*2 imply *x*1 *· x*2 *≺ y*1 *· y*2;
2. *x ≺ y*1 *· y*2 implies that there exist *x*1 *≺ y*1, *x*2 *≺ y*2 such that *x ≤ x*1 *· x*2.

We call the quadruple (*S, ≤, ·, ≺*) a *stable ordered semigroup*, if the auxiliary relation *≺* on *S* is stable.

**Example 3.13** (1) Clearly, ((0*,* 1]*, ≤, ×*) is a continuous prequantale and the way- below relationon (0*,* 1] is stable, where *×* is the multiplication on the set (0*,* 1] and *a b* if and only if *a < b* for *a, b ∈* (0*,* 1].

* 1. Let *L* = *{ai | i ∈* N+*}∪ {a, b*1*, b*2*, c*1*, c*2*, T}*. The partial order *≤* on *L* is given by:

*a*1 *< a*2 *< a*3 *··· < a < bi < ci < T* (*i* = 1*,* 2)*,*

(See Fig. [1](#_bookmark4)) Then (*L, ≤*) is a continuous lattice. It is easy to prove that (*L, ≤, ∧*) is an ordered semigroup, where the binary operation *∧* on *L* means that the meet of each pair of elements in *L*. Obviously, *b*1 *c*1, *b*2 *c*2, and *b*1 *∧b*2 = *a*, *c*1 *∧c*2 = *a*, but *a a* does not occur, implying that the way below relationas an auxiliary relation on *L* is not stable. So (*L, ≤, ∧,* ) is not a stable ordered semigroup.

*c*1 *b*1

*T*

*c*2 *b*2

*a*

*a*3 *a*2 *a*1

Fig. 1. continuous lattice *L*

**Proposition 3.14** ([[13](#_bookmark20)]) *The stable ordered semigroup* (*S, ≤, ·, ≺*) *endowed with the pseudo-Scott topology* ***P****σ is a topological semigroup.*

**Proposition 3.15** *For any topological semigroup* (*X, τ, ·*) *satisfying condition* (Δ)*,* (*D, ±, ≥,* ) *is a stable ordered semigroup.*

**Proof.** For *p*1*, p*2*, q*1*, q*2 *∈ D*, if *p*1 *q*1, *p*2 *q*2, by the way-below relation on *D*, *p*1 is of the form (*U*1*, n*1) and *p*2 is of the form (*U*2*, n*2). So *p*1*≥p*2 = (*U*1*·U*2*, n*1+*n*2). For *q*1*, q*2 we consider the following cases.

* + 1. *q*1 = (*V*1*, m*1)*, q*2 = (*V*2*, m*2) *∈ D*0. Then *q*1*≥q*2 = (*V*1*·V*2*, m*1+*m*2). From the way-below relation on *D*0, we know that *V*1 *⊆ U*1*, n*1 *< m*1 and *V*2 *⊆ U*2*, n*2 *< m*2. Since (*O∗*(*X*)*, ⊆, ·*) is an ordered semigroup, *V*1 *· V*2 *⊆ U*1 *· U*2 and in addition, *n*1 + *n*2 *< m*1 + *m*2, implying

(*U*1 *· U*2*, n*1 + *n*2) *±* (*V*1 *· V*2*, m*1 + *m*2)*.*

Because (*V*1 *· V*2*, m*1 + *m*2)(*V*1 *· V*2*, m*1 + *m*2) then (*U*1 *· U*2*, n*1 + *n*2)(*V*1 *·*

*V*2*, m*1 + *m*2) *i.e. p*1 *≥ p*2 *q*1 *≥ q*2.

* + 1. One of *q*1 and *q*2 is in *D*1, without loss of generality, let *q*2 = *F ∈ D*1. Let *q*1 = (*V*1*, m*1), then *q*1 *≥ q*2 = *↑{V*1 *· W | W ∈ F}*. From the way-below relation on *D*, it is known that *V*1 *⊆ U*1, and *U*2 *∈ F*. So *V*1 *· U*2 *⊆ U*1 *· U*2, implying *U*1 *· U*2 *∈ q*1 *≥ q*2, that is, *p*1 *≥ p*2 *q*1 *≥ q*2.
    2. *q*1 = *F*1*, q*2 = *F*2 *∈ D*1, then

*q*1 *≥ q*2 = *↑{V*1 *· V*2 *| V*1 *∈ F*1*, V*2 *∈ F*2*}.*

It is known that *U*1 *∈ F*1, *U*2 *∈ F*2, so *U*1 *· U*2 *∈ q*1 *≥ q*2, implying *p*1 *≥ p*2 *q*1 *≥ q*2. Let *q p*1 *≥p*2. Since *D* is continuous, *p*1 = W*↑ ⇓p*1, *p*2 = W*↑ ⇓p*2, where W*↑ ⇓p*1 means the supremum of the directed set *⇓p*1 and W*↑ ⇓p*2 means the supremum of

the directed set *⇓p*2. So

*q p*1 *≥ p*2 = (W*↑ ⇓p*1) *≥* (W*↑ ⇓p*2)

= W*↑*(*⇓p*1 *≥ ⇓p*2)

= W*↑{r ≥ s | r p*1*,s p*2*}.*

Then there exist *r*1 *p*1*, s*1 *p*2 such that *q ± r*1 *≥ s*1. Hence the way-below relationon *D* is a stable approximating auxiliary relation, that is, (*D, ±, ≥,* ) is a stable ordered semigroup. *2*

By combining Proposition [3.14](#_bookmark5) and Proposition [3.15](#_bookmark6), the following corollary is immediate.

**Corollary 3.16** *For any topological semigroup* (*X, τ, ·*) *satisfying condition* (Δ)*, then* (*D, σ, ≥*) *is a topological semigroup.*

From the above discussion, we can derive the topological embedding between *X*

and *D* easily.

**Theorem 3.17** *Let* (*X, τ, ·*) *be a T*1 *topological semigroup satisfying condition* (Δ)*, then the function f* : (*X, τ, ·*) *→* (*D, σ, ≥*) *deﬁned by f* (*x*)= *N* (*x*) *for x ∈ X is an embedding.*

**Proof.** It follows from Theorem [3.8](#_bookmark3) that *f* is a topological embedding. We shall show that *f* is a semigroup homomorphism, that is, for any *x, y ∈ X*, *f* (*x · y*) = *f* (*x*) *≥ f* (*y*).

We know that

*f* (*x*) *≥ f* (*y*)= *N* (*x*) *≥N* (*y*)= *↑{U · V | U ∈ N* (*x*)*,V ∈ N* (*y*)*}.*

For any element *W* in *f* (*x*) *≥ f* (*y*), there exist *U*1 *∈ N* (*x*)*, V*1 *∈ N* (*y*) such that *U*1 *· V*1 *⊆ W* . We achieve that *x · y ∈ W* , *i.e. W ∈ N* (*x · y*) = *f* (*x · y*), since *x · y ∈ U*1 *· V*1 *⊆ W* . So *f* (*x*) *≥ f* (*y*) *⊆ f* (*x · y*). Conversely, for each *U* belonging to *f* (*x · y*), that is, *x · y ∈ U* , there exist open neighborhoods *U*0 of *x* and *V*0 of *y* such that *x · y ∈ U*0 *· V*0 *⊆ U* . Then *U ∈ N* (*x*) *≥N* (*y*), implying

*f* (*x · y*) *⊆ N* (*x*) *≥N* (*y*)*, i.e. f* (*x · y*) *⊆ f* (*x*) *≥ f* (*y*)*.*

Therefore, *f* (*x · y*)= *f* (*x*) *≥ f* (*y*), completes the proof. *2*

In order to achieve the most significant result of this paper, we complete it with the help of Theorem 2*.*5 in [[1](#_bookmark8)]. We will rewrite the theorem and give a brief description for the proof.

**Theorem 3.18** ([[1](#_bookmark8)]) *Let* (*M, ±*) *be a domain and let X be a T*1 *subspace of* Σ*M. Then there is a continuous poset* (*P, ≤*) *such that the function f* : *X →* max(*P* ) *is a topological homeomorphism, where the topologies on X and* max(*P* ) *are the relative Scott topology.*

Set *P* = *P*0 *∪ P*1, where *P*0 = (*⇓X*) *×* N and *P*1 = (*↓X*) *× {∞}*. For any (*y, m*)*,* (*z, n*) *∈ P* ,

(*y, m*) *≤* (*z, n*) if and only if one of the following cases holds:

1. (*y, m*)= (*z, n*);
2. *y z* and *m < n*;
3. *y ± z* and *m* = *n* = *∞*.

Note that whenever (*y, ∞*) *≤* (*z, n*) then *n* = *∞*. Then we have the following conclusions:

W (1) For any directed subset *E* of *P* , if *E* has a supremum in *P* , then W *E* =

( *E*1*,* W *E*2), where *Ei* = *πi*(*E*) and *πi* is the projection on the *i*th coordinate,

*i* = 1*,* 2.

1. (*P, ≤*) is a continuous poset. For any (*y, m*) *∈ P* , (*y, m*)(*y, m*). For any (*y∗, ∞*) *∈ P*1, we have (*y∗, ∞*)= W(*P*0 *∩ ⇓*(*y∗, ∞*)), where

0

*P*0 *∩ ⇓*(*y∗, ∞*)= *{*(*y, m*) *∈ P*0 *| y y∗,m ∈* N*}*

is directed. In addition, (*y*1*, ∞*)(*y*2*, ∞*) never happens.

1. By the partial order on *P* , it follows that max(*P* )= *{*(*x, ∞*) *| x ∈ X}*.
2. Define the function *f* : *X →* max(*P* ) by *f* (*x*)= (*x, ∞*), then it is a topolog- ical homeomorphism.

**Remark 3.19** The continuous poset *P* constructed in Theorem [3.18](#_bookmark7) may not be bounded complete. Here is a simple example. Let *M* = *{a*1*, a*2*, b*1*, b*2*}*. The partial order on *M* is defined by:

*b*1 *≤ ai* (*i* = 1*,* 2)*,*

*b*2 *≤ ai* (*i* = 1*,* 2)*,*

Pick *X* = *{a*1*, a*2*}*, which is a *T*1 subspace of Σ*M* . Then it is known that *P*0 = *M ×* N*, P*1 = *M × {∞}* and *P* = *P*0 *∪ P*1. Then *P* is a continuous poset. By the order relation *≤* on *P* , it is clear that (*b*1*, ∞*) and (*b*2*, ∞*) have the common upper bounds *{*(*a*1*, ∞*)*,* (*a*2*, ∞*)*}*, but they do not have a supremum. So *P* is not bounded complete.

We now define a binary operation *⊗* on *P* .

Let (*M, ±*) be a domain and (*M, ±, •*) be a prequantale. We define a binary operation *⊗* on *P* as follows for any (*y, m*)*,* (*z, n*) *∈ P* :

1. (*y, m*) *⊗* (*z, n*)= (*y • z, m* + *n*) for *m, n ∈* N;
2. (*y, m*) *⊗* (*z, n*)= (*y • z, ∞*) for *m* = *∞* or *n* = *∞*.

**Theorem 3.20** *Let* (*M, ±*) *be a domain,* (*M, ±, •*) *a prequantale and* (*M, ±, •,* ) *a stable ordered semigroup. Let X be a T*1 *subspace of* Σ*M and* (*X, •*) *a subsemi- group of* (*M, •*)*. Then*

* + - 1. (*P, ≤, ⊗*) *is a continuous prequantale, where P stands for the continuous poset constructed in* Theorem [3.18](#_bookmark7)*.*
      2. *The function f* : (*X, σ |X, •*) *→* (max(*P* )*,σ |*max(*P* )*, ⊗*) *deﬁned by f* (*x*) = (*x, ∞*) *is an isomorphism.*

**Proof.** (1) It follows from Theorem [3.18](#_bookmark7) (*P, ≤*) is a continuous poset. For any (*y, m*)*,* (*z, n*) *∈ P* , one case is that (*y, m*) and (*z, n*) are both in *P*0, then *m, n ∈* N and there exist *y*1*, z*1 *∈ X* such that *y y*1*,z z*1. Since the way-below relation

on *M* is stable, *y • z y*1 *• z*1. We know that *y*1 *• z*1 *∈ X* because (*X, •*) is a subsemigroup of (*M, •*). In addition, (*y, m*) *⊗* (*z, n*) = (*y • z, m* + *n*). So (*y • z, m* + *n*) *∈ P*0, *i.e.* (*y, m*) *⊗* (*z, n*) *∈ P* . The other case is that at least one of (*y, m*) and (*z, n*) is in *P*1. Without loss of generality, let (*z, n*) *∈ P*1 *i.e. n* = *∞*. Then (*y, m*) *⊗* (*z, ∞*)= (*y • z, ∞*). Pick *y*1*, z*1 *∈ X* such that *y y*1, and *z ± z*1. Then *y • z ± y*1 *• z*1 and *y*1 *• z*1 *∈ X*, which imply that (*y • z, ∞*) *∈ P*1, that is, (*y, m*) *⊗* (*z, ∞*) *∈ P* . For any *p*1*, p*2*, p*3 *∈ P* , it is easy to prove that (*p*1 *⊗ p*2) *⊗ p*3 = *p*1 *⊗* (*p*2 *⊗ p*3). So (*P, ⊗*) is a semigroup. For any element (*y, m*) of *P* and any directed subset *E* of *P* with *E* exists. It is obvious

W

(*y, m*) *⊗* ( *E*)= ((*y, m*) *⊗ E*)*,*

( *E*) *⊗* (*y, m*)= (*E ⊗* (*y, m*))*.*

Therefore, (*P, ≤, ⊗*) is a continuous prequantale.

(2) We know that (*M, ±, •,* ) is a stable ordered semigroup, by Proposi- tion [3.14](#_bookmark5), (*M, σ, •*) is a topological semigroup. Then (*X, σ |X, •*) is also a topo- logical semigroup. It should be claimed that (max(*P* )*,σ |*max(*P* )*, ⊗*) is a topological semigroup. By the definition of the binary operation *⊗*, we know that max(*P* ) is closed under the binary operation *⊗*. For any *x, y, z ∈ X*, (*x, ∞*)*,* (*y, ∞*)*,* (*z, ∞*) are all in max(*P* ), since (*X, •*) is a subsemigroup of (*M, •*), ((*x, ∞*) *⊗* (*y, ∞*)) *⊗* (*z, ∞*)= (*x, ∞*) *⊗* ((*y, ∞*) *⊗* (*z, ∞*)). Thus (max(*P* )*, ⊗*) is a subsemigroup of *P* . Define a function *g* : max(*P* ) *×* max(*P* ) *→* max(*P* ) by

*g*((*x, ∞*)*,* (*y, ∞*)) = (*x, ∞*) *⊗* (*y, ∞*)

for all *x, y ∈ X*. We prove that *g* is continuous. For any *x, y ∈ X*, (*x, ∞*)*,* (*y, ∞*) *∈* max(*P* ) satisfying (*x, ∞*) *⊗* (*y, ∞*) *∈ ⇑r ∩* max(*P* ) for some *r ∈ P* . Since *P* is continuous,

*r* (*x, ∞*) *⊗* (*y, ∞*)= (W*↑ ⇓*(*x, ∞*)) *⊗* (W*↑ ⇓*(*y, ∞*))

= W*↑*(*⇓*(*x, ∞*) *⊗ ⇓*(*y, ∞*))

= W*↑{*(*a, m*)*⊗*(*b, n*) *|* (*a, m*)(*x, ∞*)*,* (*b, n*)(*y, ∞*)*, m,n ∈* N*}.*

Then there exist (*a*1*, m*1)(*x, ∞*), (*b*1*, n*1)(*y, ∞*), *m*1*, n*1 *∈* N

such that *r ≤* (*a*1*, m*1) *⊗* (*b*1*, n*1)= (*a*1 *• b*1*, m*1 + *n*1). We need to prove (*⇑*(*a*1*, m*1) *∩* max(*P* )) *⊗* (*⇑*(*b*1*, n*1) *∩* max(*P* )) *⊆ ⇑r ∩* max(*P* )*.*

For any element *p* in (*⇑*(*a*1*, m*1)*∩*max(*P* ))*⊗*(*⇑*(*b*1*, n*1)*∩*max(*P* )), there are elements

(*x*1*, ∞*)*,* (*y*1*, ∞*) in max(*P* ) such that (*a*1*, m*1)(*x*1*, ∞*)*,* (*b*1*, n*1)(*y*1*, ∞*) and

*p* = (*x*1*, ∞*) *⊗* (*y*1*, ∞*)= (*x*1 *• y*1*, ∞*)*.*

By the definition of the way-below relation on *P* , it is known that *a*1 *x*1*, b*1 *y*1. So *a*1 *• b*1 *x*1 *• y*1, implying (*a*1 *• b*1*, m*1 + *n*1)(*x*1 *• y*1*, ∞*). Therefore *r p*, that is, *p ∈ ⇑r ∩* max(*P* ), as desired. Thus (max(*P* )*,σ |*max(*P* )*, ⊗*) is a topological semigroup.

It follows from Theorem [3.18](#_bookmark7) that *f* is a topological homeomorphism. Obviously,

*f* is a semigroup homomorphism. Therefore, *f* is an isomorphism. *2*

From the above results, we can now derive our major result in this paper.

**Theorem 3.21** *Every T*1 *topological semigroup satisfying condition* (Δ) *has a con- tinuous prequantale model.*

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