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Decomposing Split Graphs into Locally Irregular Graphs [1](#_bookmark0)*,*[2](#_bookmark0)

C. N. Lintzmayer G. O. Mota

*Center for Mathematics, Computing and Cognition, Federal University of ABC. Santo Andr´e, S˜ao Paulo, Brazil*

M. Sambinelli

*Institute of Mathematics and Statistics, University of Sa˜o Paulo. S˜ao Paulo, Brazil*

Abstract

A graph is locally irregular if any pair of adjacent vertices have distinct degrees. A locally irregular de- composition of a graph *G* is a decomposition of *G* into locally irregular subgraphs. A graph is said to be decomposable if it admits a locally irregular decomposition. In this paper we prove that any decomposable split graph whose clique has at least 10 vertices can be decomposed into at most three locally irregular sub- graphs. Furthermore, we characterize those whose decomposition can be into one or two locally irregular subgraphs.

*Keywords:* Locally irregular, graph decomposition, split graphs.

# Introduction

All graphs considered in this text are simple and finite, i.e., contain neither loops nor parallel edges and their vertex and edge sets are finite, and we follow standard notation and terminology (see [[6,](#_bookmark29)[7](#_bookmark30)]).

Given a graph *G*, a collection *D* = *{H*1*, H*2*,..., Hk}* of subgraphs of *G* is a *decomposition* of *G* if *{E*(*H*1)*, E*(*H*2)*,..., E*(*Hk*)*}* is a partition of *E*(*G*). A graph is *locally irregular* if any pair of adjacent vertices have distinct degrees. A *locally irregular decomposition* of a graph *G* is a decomposition *D* of *G* such that every

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2 Emails: [carla.negri@ufabc.edu.br](mailto:carla.negri@ufabc.edu.br), [g.mota@ufabc.edu.br](mailto:g.mota@ufabc.edu.br), [sambinelli@ime.usp.br](mailto:sambinelli@ime.usp.br)

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subgraph *H ∈ D* is locally irregular. Not all graphs admit a locally irregular decomposition; take for example the *K*3 (complete graph with three vertices). We say that a graph is *decomposable* if it admits a locally irregular decomposition. Given a decomposable graph *G*, the *irregular chromatic index* of *G*, denoted by

*j*

*χ*

irr

(*G*), is the smallest size of a locally irregular decomposition of *G*. Alternatively,

one can see a locally irregular decomposition of a graph *G* as being an edge coloring

of *G* such that each color induces a locally irregular graph. We call such coloring of a *locally irregular edge coloring*.

This problem arose from the question “what is the antonym of regular graphs?”. The natural answer, which is “graphs for which all vertices have distinct degrees” is immediately infeasible, since it is well-known that all graphs have at least two vertices with same degree. The notion of locality is thus considered. It is worth mentioning that there are other attempts to answer the question, such as the class of highly irregular graphs [[1](#_bookmark24)]. Also, the locally irregular edge coloring is closely related to the 1-2-3 Conjecture, the 1-2 Conjecture, and to the concepts of distinguishing colorings and irregularity strength [[5,](#_bookmark28)[2](#_bookmark25)].

Locally irregular decomposition as defined above was introduced by Baudon, Bensmail, Przyby-lo, and Wo´zniak [[2](#_bookmark25)]. They characterized all graphs that are de- composable and another main result is that *d*-regular graphs with *d ≥* 107 have a locally irregular 3-edge coloring. They also showed a locally irregular 3-edge color- ing for trees that are not an odd-length path, and for *Kn* with *n ≥* 4, and showed a locally irregular 2-edge coloring for regular bipartite graphs with minimum degree at least 3, and hypercubes *Qn* with *n ≥* 2. Furthermore, the authors showed that

*j*

*χ*

irr

(*G*) *≤ [|E*(*G*)*|/*2*♩* for all decomposable graphs *G* and proposed the following

conjecture.

**Conjecture 1.1 (Baudon, Bensmail, Przyby~lo, and Wo´zniak, 2015 [**[**2**](#_bookmark25)**])** *If*

*G is a decomposable graph, then χj*

irr

(*G*) *≤* 3*.*

Although we can check in polynomial time whether a graph *G* is locally irregular, deciding if there exists a locally irregular 2-edge coloring of *G* is NP-complete, even when restricted to planar graphs with maximum degree at most 6 [[3](#_bookmark26)]. Note that proving Conjecture [1.1](#_bookmark1) would show that deciding whether there exists a locally irregular *k*-edge coloring for *k ≥* 3 is in P.

A result from Przyby-lo [[9](#_bookmark32)] shows that every graph with minimum degree at least 1010 has a locally irregular 3-edge coloring. Bensmail, Merker, and Thomassen [[4]](#_bookmark27)

irr

gave the first constant upper bound on *χj*

irr

(*G*), showing that *χj*

(*G*) *≤* 328 for any

decomposable graph *G*. They also showed that every 16-edge-connected bipartite

irr

graph *G* has *χj*

irr

(*G*) *≤* 2 and every bipartite graph *G* has *χj*

(*G*) *≤* 10. Luˇzar,

Przyby-lo, and Sot´ak [[8](#_bookmark31)] then improved the previous results showing that *χj*

irr

(*G*) *≤*

220 for any decomposable graph *G* and that *χj* (*G*) *≤* 7 for any bipartite graph *G*.

irr

They also showed that if *G* is subcubic, then *χj* (*G*) *≤* 4.

irr

Our focus is to characterize decomposable split graphs. A graph *G* is *split* if there exists a partition *{X, Y }* of *V* (*G*) such that *G*[*X*] is a complete graph and *Y* is a stable set. When defining a split graph *G* for the first time, we usually write

*G*(*X, Y* ) to also define a partition *{X, Y }* of *V* (*G*), where *X* is a maximal clique and *Y* = *V* (*G*) *\ X* is a stable set.

Let *G*(*X, Y* ) be a split graph with *X* = *{v*1*,..., vn}*. For any *vi ∈ X*, we denote by *dG*(*vi,Y* ) the number of neighbors of *vi* in *Y* , i.e., *dG*(*vi,Y* )= *|NG*(*vi*) *∩ Y |*. For simplicity we just write *di* for *dG*(*vi,Y* ) whenever the graph *G* and the stable set *Y* are clear from the context.

It is easy to verify that for split graphs *G*(*X, Y* ) with *X* = *{v*1*,..., vn}*, we have

*j*

*χ*

irr

(*G*) = 1 if and only if *d*1 *> ··· > dn*. In fact, since *X* is a maximal clique,

*dG*(*y*) *≤ n −* 1 for all *y ∈ Y* , which implies that *dG*(*y*) is smaller than the degree of

all its neighbors in *X*. Therefore, *G* is locally irregular if and only if the vertices in *X* have distinct degrees in *G*, which is possible if and only if *d*1 *> ··· > dn*, and hence the result follows. We state this in the following fact.

**Fact 1.2** *Let G*(*X, Y* ) *be a split graph with X* = *{v*1*,..., vn} where d*1 *≥ ··· ≥ dn*

*and n ≥* 2*. We have χj* (*G*)=1 *if and only if d*1 *> ··· > dn.*

irr

Our main result is Theorem [1.3](#_bookmark3), which describes *χj G*(*X, Y* ) such that *X* is a clique with at least 10 vertices.

irr

(*G*) for all split graphs

**Theorem 1.3** *Let G*(*X, Y* ) *be a split graph with X* = *{v*1*,..., vn} where d*1 *≥ · · ·≥*

*dn. If n ≥* 10*, then the following holds.*

(*i* ) *χj*

irr

(*ii* ) *χj*

irr

(*G*) *≤* 2 *if and only if d*1 *≥ [n/*2*♩ or d*2 *≥* 1*;* (*G*)=3 *if and only if d*1 *< [n/*2*♩ and d*2 = 0*.*

Theorem [1.3](#_bookmark3) is proved in Section [2.](#_bookmark4)

In the remainder of the paper, given a graph *G* and a coloring *ϕ*: *E*(*G*) *→*

*{*red*,* blue*}*, we denote the two edge-disjoint spanning monochromatic subgraphs under *ϕ* by *G*red*,ϕ* and *G*blue*,ϕ*. Formally,

*G*red*,ϕ* = (*V* (*G*)*, ϕ—*1(red)) and *G*blue*,ϕ* = (*V* (*G*)*, ϕ—*1(blue)) *.*

We may omit the term *ϕ* from *G*red*,ϕ* and *G*blue*,ϕ* whenever *ϕ* is clear from the context. This notation naturally extends to colorings that use more than two colors.

# Decomposing split graphs with a large maximal clique

In this section we give a characterization of the irregular chromatic index of all split graphs with a maximal clique that has at least 10 vertices.

Let *G*(*X, Y* ) with *X* = *{v*1*,..., vn}* with *n ≥* 10. In Lemma [2.3](#_bookmark16) we prove that

*d*1 *< [n/*2*♩* and *d*2 = 0 implies *χj*

irr

(*G*) = 3. We also prove that if *d*1 *≥ [n/*2*♩*

or *d*2 *≥* 1, then *χj*

irr

(*G*) *≤* 2, which follows directly from Lemmas [2.4](#_bookmark17) and [2.7](#_bookmark23).

Therefore, note that Theorem [1.3](#_bookmark3) follows from Lemmas [2.3](#_bookmark16), [2.4](#_bookmark17) and [2.7](#_bookmark23).

In Section [2.1](#_bookmark5) we prove Lemmas [2.3](#_bookmark16) and [2.4](#_bookmark17). The starting point for proving these lemmas is a specific coloring of *E*(*Kn*), which we call *normal*, given in Definition [2.1](#_bookmark6). In Section [2.2](#_bookmark18) we prove Lemma [2.7](#_bookmark23). For proving this result, we start with an intricate

coloring of *E*(*Kn*), which we call *strange* (see Definition [2.5](#_bookmark19)).

Given a graph *G*, we say that the edge *uv ∈ E*(*G*) is a *conflicting edge* if

*dG*(*u*)= *dG*(*v*).

* 1. *Normal colorings of complete graphs*

We start this section by defining *normal colorings* of complete graphs. See Figure [1](#_bookmark15) for example.

**Definition 2.1 (*Normal colorings*)** *Given a complete graph G with n vertices and a sequence* ***V*** = (*v*1*,..., vn*) *of V* (*G*)*, the* normal coloring for ***V*** *is the* 2*-edge coloring ϕ*: *E*(*G*) *→ {*red*,* blue*} deﬁned as follows, where X*1 = *{v*1*,..., v[n/*2*|} and X*2 = *V* (*G*) *\ X*1*:*

(*i* ) *G*red[*X*1] *is a complete graph;*

(*ii* ) *G*red[*X*2] *contains no edges;*

(*iii* ) *NG*red (*vi*)= *{v*1*,..., vn—i*+1*} for [n/*2*|* +1 *≤ i ≤ n.*

Note that in a normal coloring of a complete graph *G* for a sequence ***V*** = (*v*1*,..., vn*), we have

* *dG*red*,ϕ* (*vi*)= *n − i*, for 1 *≤ i ≤ [n/*2*|*;
* *dG*red*,ϕ* (*vi*)= *n − i* + 1, for *[n/*2*|* +1 *≤ i ≤ n*. Therefore, we know that, for a normal coloring *ϕ* of *G*,

the only vertices with same degree in *G*red*,ϕ* (and *G*blue*,ϕ*) are *v[n/*2*|* and *v[n/*2*|*+1 *.*

(1)

From the definition of normal colorings and ([1](#_bookmark7)), since there is a (unique) con- flicting edge *v[n/*2*|v[n/*2*|*+1, we know that if *n* is even (resp. odd), then *G*blue (resp. *G*red) is locally irregular and *G*red (resp. *G*blue) is not locally irregular.

The following proposition will be useful for proving Lemma [2.3](#_bookmark16).

**Proposition 2.2** *Let G be a connected graph with V* (*G*) = *{u*1*, u*2*,..., un} and* *dG*(*u*1) *≥ ··· ≥ dG*(*un*)*. If G contains only one pair of vertices u, v with dG*(*u*) = *dG*(*v*)*, then the following holds:*

(*i* ) *dG*(*u*)= *dG*(*v*)= *[n/*2*♩;*

(*ii* ) *u* = *u[n/*2*| and v* = *u[n/*2*|*+1*;*

(*iii* ) *X* = *{u*1*,..., u[n/*2*|—*1*} is a clique and Y* = *{u[n/*2*|*+2*,..., un} is a stable set;*

(*iv* ) *X ⊆ NG*(*u*) *∩ NG*(*v*)*;*

(*v* ) (*NG*(*u*) *∪ NG*(*v*)) *∩ Y* = *∅;*

(*vi* ) *uv ∈ E*(*G*) *if and only if n is even.*

**Proof** The proof follows by induction on *n*. If *n* = 2, then *G K*2, and if *n* = 3, then *G P*3. In both cases, [(*i* )](#_bookmark9)-[(*vi* )](#_bookmark14) hold. Thus, we may assume that *n ≥* 4.

Since *G* is a connected graph with *n* vertices and *u* and *v* are the only vertices of

*G* with the same degree, there are *n −* 1 distinct values of degrees in *G*. Moreover,

*v*6 (5, 4) *v*5 (5, 4)



*v*7 (4, 5)

*v*8 (3, 6)

*v*9 (2, 7)

*v*4 (6, 3)

*v*3 (7, 2)

*v*2 (8, 1)

*v*10 (1, 8) *v*1 (9, 0)

*v*7 (5, 5)

*v*8 (4, 6)

*v*9 (3, 7)

*v*6 (5, 5)

*v*5 (6, 4)

*v*4 (7, 3)

*v*3 (8, 2)

*v*10 (2, 8)

*v*2 (9, 1)

*v*11 (1, 9) *v*1 (10, 0)

Figure 1. Normal colorings of *E*(*Kn*) with sequence (*v*1*,..., vn*), for *n* = 10*,* 11. The vertices of the conflicting edge are highlighted. For better visualization, we omit all blue edges.



since *G* is connected, for any vertex *w* of *G* we have 1 *≤ dG*(*w*) *≤ n −* 1, and as a result of this, we know that the set of degrees of all vertices in *G* is *{*1*,* 2*,...,n−* 1*}*. Therefore, *dG*(*u*1) = *n −* 1 and *dG*(*un*) = 1. Let *Gj* = *G − {u*1*, un}*. Note that *dG′* (*w*)= *dG*(*w*) *−* 1 for all *w ∈ V* (*G*) *\ {u*1*, un}*.

We will show that *Gj* is a connected graph with only one pair of vertices with the same degree. If *dG*(*u*) *∈ {*1*,n −* 1*}*, then the vertices of *Gj* have distinct degrees. In particular, there exists a non-trivial component of *Gj* where all vertices have distinct degrees, which is an absurd. Thus, we may assume that *dG*(*u*) *∈/ {*1*,n −* 1*}*, and hence *Gj* has precisely two vertices with the same degree. Now note that graph *Gj* has no trivial components, since *un* is the only vertex of *G* with degree 1. Also, if *Gj* had more than one component, then it would contain a component where all vertices have distinct degrees, which is an absurd. Therefore, the graph *Gj* is connected and contains precisely two vertices with the same degree (*u* and *v*), and hence, by induction hypothesis, [(*i* )](#_bookmark9)-[(*vi* )](#_bookmark14) hold for *Gj*.

For clarity, let *V* (*Gj*) = *{uj , uj ,..., uj ′ }* = *{u*2*, u*3*,..., un—*1*}*. Since [(*i* )](#_bookmark9) holds

1 2 *n*

for *Gj*, and *dG′* (*w*)= *dG*(*w*)*−*1 for all *w ∈ V* (*G*)*\{u*1*, un}*, we have *dG*(*u*)= *dG*(*v*)=

*dG′* (*u*)+1 = *[nj/*2*♩* = *[*(*n −* 2)*/*2*♩* +1 = *[n/*2*♩*. Thus, [(*i* )](#_bookmark9) holds for *G*. Since [(*ii* )](#_bookmark10)

holds for *Gj*, the vertices *u* = *uj ′*

= *u[*(*n—*2)*/*2*|*+1 = *u[n/*2*|* and *v* = *uj ′* =

*[n /*2*| [n /*2*|*+1

*u*(*[*(*n—*2)*/*2*|*+1)+1 = *u[n/*2*|*+1 have the same degree in *Gj*, and consequently in *G*, and hence [(*ii* )](#_bookmark10) holds for *G*. Since [(*iii* )](#_bookmark11) holds for *Gj*, the set *Xj* = *{u*2*,..., u[n/*2*|—*1*}* is a clique of *Gj*, the set *Y j* = *{u[n/*2*|*+2*,..., un—*1*}* is a stable set of *Gj*, and hence *X* = *{u*1*}∪Xj* is a clique of *G* and *Y* = *{un}∪Y j* is a stable set of *G*, from where we conclude that [(*iii* )](#_bookmark11) holds for *G*. Since [(*iv* )](#_bookmark12) holds for *Gj*, and *Xj ⊆ NG′* (*u*) *∩ NG′* (*v*),

and since *dG*(*u*1) = *n −* 1, we have *X ⊆ NG*(*u*) and *X ⊆ NG*(*v*). Therefore, [(*iv* )](#_bookmark12) holds for *G*. Since [(*v* )](#_bookmark13) holds for *Gj*, and (*NG′* (*u*) *∪ NG′* (*v*)) *∩ Y j* = *∅*, and since *un* has degree 1 in *G* and *u*1*un ∈ E*(*G*), we have *NG*(*u*) *∩ Y* = *∅* and *NG*(*v*) *∩ Y* = *∅*. Therefore, [(*v* )](#_bookmark13) holds for *G*. Finally, since [(*vi* )](#_bookmark14) holds for *Gj*, and *Gj* has *n −* 2 vertices, [(*vi* )](#_bookmark14) holds for *G*, which finishes the proof. *2*

**Lemma 2.3** *Let G*(*X, Y* ) *be a split graph with X* = *{v*1*,..., vn} where d*1 *≥* *≥*

*dn and n ≥* 4*. If d*1 *< [n/*2*♩ and d*2 = 0*, then χj*

irr

(*G*) = 3*.*

**Proof** *Sketch.* Let *G*(*X, Y* ) be a split graph with *X* = *{v*1*,..., vn}*, *d*1 *≥ · · ·≥ dn*,

*n ≥* 4, *d*1 *< [n/*2*♩*, and *d*2 = 0. We start by proving that *χj*

irr

(*G*) *≥* 3. This

is shown by supposing that there exists a locally irregular 2-edge coloring of *G*, *ϕ*: *E*(*G*) *→ {*red*,* blue*}*, which allows us to show that *G*red satisfies Proposition [2.2](#_bookmark8). Then we find a contradiction by showing that the degree of *v*1 in *G*red is the same as one of its neighbors, contradicting the fact that *G*red was locally irregular in the first place. To finish the proof, we exhibit a locally irregular 3-edge coloring of *G*.*2*

In the proof of Lemma [2.4](#_bookmark17) below the reader may find it useful to refer to Figure [1.](#_bookmark15)

**Lemma 2.4** *Let G*(*X, Y* ) *be a split graph with X* = *{v*1*,..., vn} where d*1 *≥ ··· ≥*

*dn and d[n/*2*♩ ≥* 1*. If n ≥* 3*, then, χj*

irr

(*G*) *≤* 2*.*

**Proof** There are two cases to consider depending on the parity of *n*, but the only difference in the proofs is the coloring we give to *E*(*G*[*X*]), which are symmetric. For *n* even, we start with a normal coloring *ϕ*: *E*(*G*[*X*]) *→ {*red*,* blue*}* for the sequence (*v*1*,..., vn/*2*, vn, vn—*1*,..., vn/*2+1). In case *n* is odd we consider a normal coloring of *E*(*G*[*X*]) for sequence (*v[n/*2*|,..., vn, v[n/*2*♩,..., v*1). Thus, for the rest of this proof, we may assume, without loss of generality, that *n* is even.

Let *X*1 = *{v*1*, v*2*,..., vn/*2*}* and *X*2 = *X \ X*1. From ([1](#_bookmark7)) we know that the only vertices with the same degree in *G*red[*X*] or *G*blue[*X*] are *vn/*2 and *vn*.

We will obtain a locally irregular 2-edge coloring of *G* from *ϕ*. We start by extending *ϕ* to a coloring *ϕj* of *E*(*G*) with colors red and blue in the following way. For all edges *xy* between *X*1 and *Y* let *ϕj*(*xy*) = red, and for all edges *xy* between *X*2 and *Y* let *ϕj*(*xy*) = blue.

Let us first analyze the graph *G*red*,ϕ′* . Since *dn/*2 *≥* 1, for every vertex *x ∈ X*1 we have *dG*red*,ϕ′* (*x*) *> dG*red*,ϕ* (*x*) *≥ n/*2, and since *d*1 *≥ · · · ≥ dn/*2, the degree of any two vertices of *X*1 remain different in *G*red*,ϕ′* . Also, since there are no red edges between *vn* and *Y* , we have *dG*red*,ϕ′* (*vn*)= *n/*2 *< dG*red*,ϕ′* (*x*) for every *x ∈ X*1. The red degree of vertices *vn/*2+1*,..., vn* are the same in *ϕ* and *ϕj*, and the red degree

of any vertex *y ∈ Y* is at most *n/*2, since there are no red edges between *X*2 and *Y* . Therefore, since the degrees of the vertices of *X*1 in *G*red*,ϕ′* are at least *n/*2+ 1, we conclude that *G*red*,ϕ′* is locally irregular.

It remains to show that *G*blue*,ϕ′* is locally irregular. Since *ϕ* is a normal coloring and *n* is even, we know that *G*blue*,ϕ′* [*X*] is locally irregular. If there is no *y ∈ Y*

that has a neighbor *x ∈ X*2 with *dG*blue*,ϕ′* (*y*)= *dG*blue*,ϕ′* (*x*), then the result follows. Thus we may assume the opposite, i.e., there exist *y ∈ Y* and *x ∈ X*2 with the same degree in *G*blue*,ϕ′* . Since the maximum possible degree of a vertex of *Y* in *G*blue*,ϕ′* is

*n/*2 and the minimum degree of a vertex of *X*2 in *G*blue*,ϕ′* [*X*] is *n/*2 *−* 1, we conclude that

*dG*blue*,ϕ′* (*y*)= *dG*blue*,ϕ′* (*vn*)= *n/*2*.*

Therefore, because of the pair *y, vn*, the graph *G*blue*,ϕ′* is not locally irregular. In this case, we can change the color of one or two edges in *ϕj* to obtain a locally irregular 2-edge coloring *ϕjj* for *G*, as we explain next.

If *dn/*2 *≥* 2, then let *ϕjj* be the coloring obtained from *ϕj* by changing the color of *yvn* from blue to red. We claim that the graphs *G*red*,ϕ′′* and *G*blue*,ϕ′′* are locally irregular. In fact, this holds since *vn* has degree *n/*2+1 in *G*red*,ϕ′′* and every vertex in *X*1 has degree at least *n/*2+2 in *G*red*,ϕ′′* .

Now assume that *dn/*2 =1 and let *z ∈ Y* be the only neighbor of *vn/*2 in *Y* . In this case consider the coloring *ϕjj* obtained from *ϕj* by changing the color of *yvn* from blue to red and the color of *zvn/*2 from red to blue. Although *dG*blue*,ϕ′′* (*y*)= *dG*blue*,ϕ′′* (*vn*) = (*n/*2) *−* 1, they are not neighbors in *G*blue*,ϕ′′* . Also, any vertex in *X*2 *\ {vn}* has degree at least (*n/*2) + 1 in *G*blue*,ϕ′′* , so there are no conflicts in *G*blue*,ϕ′′* involving *y* or *vn*. Note that we have *dG*red*,ϕ′′* (*vn*) = (*n/*2) + 1, but since *dG*red*,ϕ′′* (*vn/*2) = *n/*2 and every vertex in *X*1 *\ {vn/*2*}* has red degree at least (*n/*2) + 2 in *G*red*,ϕ′′* , there are no conflicts in *G*red*,ϕ′′* involving *vn*. This also implies that, since *dG*red*,ϕ′′* (*vn/*2) = *n/*2, there are no conflicts involving *vn/*2 in *G*red*,ϕ′′* . Furthermore, since *dG*blue*,ϕ′′* (*vn/*2)= *n/*2 and any vertex in *X*2 *\ {vn}* has degree at least (*n/*2) + 1 in *G*blue*,ϕ′′* , we conclude that there are no conflicts involving *vn/*2 in *G*blue*,ϕ′′* . Therefore, *G*red*,ϕ′′* and *G*blue*,ϕ′′* are locally irregular, and the result follows. *2*

*2.2 Strange colorings of complete graphs*

As in Section [2](#_bookmark4), we start by defining the colorings of complete graphs that are the starting point for proving the results in this section. The following definition is technical, so we refer the reader to Figure [2](#_bookmark21) for a better understanding of it.

**Definition 2.5 (Strange coloring)** *Given a complete graph G with n vertices and a sequence* ***V*** = (*v*1*,..., vn*) *of V* (*G*)*, ﬁrst consider a coloring ϕj*: *E*(*G*) *→*

*{*red*,* blue*} deﬁned as follows, where X*1 = *{v*1*,.* *, v[n/*2*|} and X*2 = *V* (*G*) *\ X*1*:*

(*i* ) *G*red[*X*1] *is a complete graph;*

(*ii* ) *G*red[*X*2] *contains no edges;*

(*iii* ) *NG*red*,ϕ′* (*vi*)= *{v*1*,.* *, vn—i} for [n/*2*|* +1 *≤ i ≤ n −* 1*;*

(*iv* ) *ϕj*(*v*1*vn*)= red*;*

(*v* ) *ϕj*(*v[n/*2*|*+1*v[n/*2*♩*)= red*;* (*vi* ) *All other edges are blue.*

*The* strange coloring *of G for* ***V*** *is the coloring ϕ obtained from ϕj by changing the color of the following edges, which we call* strange edges*:*

* *v[n/*2*♩v[n/*2*♩—*1 *becomes blue;*
* *v[n/*2*♩—*1*vn—*1 *becomes red;*
* *v*1*v[n/*2*|*+1 *for [n/*2*| even becomes blue;*
* *v*1*v[n/*2*♩*+1 *for [n/*2*| odd becomes blue;*
* *vn—*2*vn—*3*, vn—*4*vn—*5*, . .., vn/*2+4*vn/*2+3 *for n* =0 *(mod 4) become red;*
* *vn—*1*v[n/*2*♩—*1*, vn—*2*v[n/*2*♩, vn—*3*v[n/*2*♩*+1*, vn—*4*vn—*5*, vn—*6*vn—*7*, . . . , v[n/*2*♩*+3*v[n/*2*♩*+2 *for n* =1 *(mod 4) become red;*
* *vn—*2*vn—*3*, vn—*4*vn—*5*, . .., vn/*2+3*vn/*2+2 *for n* =2 *(mod 4) become red;*
* *vn—*2*vn—*3*, vn—*4*vn—*5*, . .., v[n/*2*♩*+4*v[n/*2*♩*+3 *for n* =3 *(mod 4) become red.*

Note that in a strange coloring of a complete graph *G* for a sequence ***V*** = (*v*1*,..., vn*), we have

and

*dG*red*,ϕ* (*vi*)= *n − i* + 1, for 3 *≤ i ≤ n , dG*red*,ϕ* (*v*1)= *n − [n/*2*♩−* 1 *,*

*d* (*v* )= (*n − [n/*2*♩−* 2 if *[n/*2*|* is even

*G*red*,ϕ*

2

*n − [n/*2*♩−* 1 otherwise *.*

Therefore, we know that, for a strange coloring *ϕ* of *G*,

the only vertices with same degree in *G*red*,ϕ* are *v*1 and *v[n/*2*♩*+2 *,* (2)

and

the only vertices with same degree in

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*G*blue*,ϕ*

are *v*2 and *v[n/*2*♩*+3 if *[n/*2*|* is even

*v*2*, v*1*,* and *v[n/*2*♩*+1 otherwise.

From the definition of strange coloring and by ([2](#_bookmark20)) and ([3](#_bookmark20)), we conclude that

*G*red has exactly one conflicting edge *v*1*v[n/*2*♩*+2 while

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*G*blue

has exactly one conflicting edge *v*2*v[n/*2*♩*+3 if *[n/*2*|* is even two conflicting edges *v*2*v[n/*2*♩*+2 and *v*2*v*1 otherwise.

Given a graph *G* and a 2-edge coloring *ϕ*: *E*(*G*) *→ {*red*,* blue*}*, we say an even cycle *C* = (*u*1*,..., uk, uk*+1 = *u*1) is an *alternating cycle in ϕ* if *ϕ*(*uiui*+1) */*= *ϕ*(*ui*+1*ui*+2), for any 1 *≤ i < k*, and *dGγ* (*ui*) */*= *dGγ* (*ui*+1), for any 1 *≤ i ≤ k*

*v*7 (4, 5)*v*1 (4, 5)



*v*7 (5, 5)

*v*2 (4, 5)

*v*6 (5, 4)

*v*8 (4, 6)

*v*2 (4, 6)

*v*1 (5, 5)

*v*6 (6, 4)

*v*8 (3, 6)

*v*9 (2, 7)

*v*5 (6, 3)

*v*4 (7, 2)

*v*9 (3, 7)

*v*10 (2, 8)

*v*5 (7, 3)

*v*4 (8, 2)

*v*10 (1, 8)*v*3 (8, 1)

*v*8 (5, 6*v*)1 (5, 6)



*v*11 (1, 9*v*)3 (9, 1)

*v*8 (6, 6)



*v*9 (4, 7)

*v*2 (4, 7)

*v*10 (3, 8)

*v*11 (2, 9)

*v*7 (6, 5)

*v*6 (7, 4)

*v*5 (8, 3)

*v*4 (9, 2)

*v*2 (6, 6)

*v*9 (5, 7)

*v*10 (4, 8)

*v*11 (3, 9)

*v*12 (2, 10)

*v*1 (6, 6)

*v*7 (7, 5)

*v*6 (8, 4)

*v*5 (9, 3)

*v*4 (10, 2)

*v*12 (1, 1*v*03)(10, 1) *v*13 (1, *v*131)(11, 1)

Figure 2. Strange colorings of *E*(*Kn*) with *n ∈ {*10*,* 11*,* 12*,* 13*}*, where *V* (*Kn*) = *{v*1*,..., vn}* and the sequence is (*v*3*, ..., v[n/*2*♩*+1*, v*1*, v[n/*2*♩*+2*, v[n/*2*♩*+3*, v*2*, v[n/*2*♩*+4*, ..., vn*) if *[n/*2*|* is even, and it is (*v*3*,*

*..., v[n/*2*♩*+1*, v*1*, v[n/*2*♩*+2*, v*2*, v[n/*2*♩*+3*, ..., vn*) otherwise. Strange edges are highlighted, as well as the vertices of the conflicting edges. For better visualization, we omit the other blue edges.

and *γ ∈ {*red*,* blue*}\ ϕ*(*uiui*+1). In other words, an alternating cycle has incident edges with different colors and the endpoints of a red edge (resp. blue edge) have different blue degrees (resp. red degrees). Let *ϕj* be the 2-edge coloring of *E*(*G*) such that *ϕj*(*uv*) *∈ {*red*,* blue*}\ ϕ*(*uv*) if *uv ∈ E*(*C*) and *ϕj*(*uv*)= *ϕ*(*uv*) otherwise. We say *ϕj* is *obtained from ϕ by inverting C*.

Lemma [2.6](#_bookmark22) shows that an inversion on an alternating cycle does not create conflicting edges. It will be used in the proof of Lemma [2.7](#_bookmark23).

**Lemma 2.6** *Let G be a graph, ϕ*: *E*(*G*) *→ {*red*,* blue*} be a* 2*-edge coloring of G. Let C be an alternating cycle in ϕ and let ϕj be obtained from ϕ by inverting C. The set of conflicting edges of Gγ,ϕ′ is a subset of the conflicting edges of Gγ,ϕ, for any γ ∈ {*red*,* blue*}.*

**Proof** First note that, in *ϕ*, every vertex of *V* (*C*) has one incident blue edge and one incident red edge, which remains valid in *ϕj*. No other edge has its color changed. Thus, *dG**γ,ϕ* (*v*)= *dGγ,ϕ′* (*v*) for any *v ∈ V* (*G*) and *γ ∈ {*red*,* blue*}*.

Let *γ ∈ {*red*,* blue*}*, *γ*¯ *∈ {*red*,* blue*}\ {γ}*, and consider any edge *uv ∈ E*(*G*). If

*uv ∈/ E*(*C*), then *uv* is conflicting in *Gγ,ϕ* (resp. *Gγ*¯*,ϕ*) if and only if it is conflicting in *Gγ,ϕ′* (resp. *Gγ*¯*,ϕ′* ). So let *uv ∈ E*(*C*) and let *ϕ*(*uv*) = *γ*. From the definition of alternating cycle, *dGγ*¯*,ϕ* (*u*) */*= *dGγ*¯*,ϕ* (*v*). This means that *uv* is not conflicting in *Gγ*¯*,ϕ*, no matter if it is conflicting in *Gγ,ϕ* or not. *2*

**Lemma 2.7** *Let n ≥* 10 *and let G*(*X, Y* ) *be a split graph with X* = *{v*1*,..., vn}*

*where d*1 *≥ · · ·≥ dn and d[n/*2*♩* = 0*. If d*1 *≥ [n/*2*♩ or d*2 *≥* 1*, then χj*

irr

(*G*) = 2*.*

**Proof** *Sketch.* First note that since *d[n/*2*♩* =0 we have *χj*

irr

(*G*) *≥* 2 due to Fact [1.2](#_bookmark2).

Let *ϕj*: *E*(*G*[*X*]) *→ {*red*,* blue*}* be the strange coloring of *E*(*G*[*X*]) for the se- quence

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*π* = (*v*3*,..., v[n/*2*♩*+1*, v*1*, v[n/*2*♩*+2*, v[n/*2*♩*+3*, v*2*, v[n/*2*♩*+4*,..., vn*) if *[n/*2*|* is even

(*v*3*,..., v[n/*2*♩*+1*, v*1*, v[n/*2*♩*+2*, v*2*, v[n/*2*♩*+3*,..., vn*) otherwise *.*

(3)

We show how to obtain a locally irregular 2-edge coloring of *G* from *ϕj*. We start by extending *ϕj* to a coloring *ϕ* of *E*(*G*) with colors red and blue. Let *X*red =

*{v*1*, v*3*, v*4*,..., v**[n/*2*♩—*1*}*. We give color red to all edges between *X*red and *Y* , and color blue to all edges between *v*2 and *Y* . Note that this is a coloring of *E*(*G*) because *d[n/*2*♩* = 0.

We first show that if there is a conflicting edge in *G*red*,ϕ*, then it is unique and there exists an alternating cycle that removes it. Then we show the same for *G*blue*,ϕ*. *2*

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