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Effective Dini’s Theorem on Effectively Compact Metric Spaces

# Hiroyasu Kamo[1](#_bookmark0)

*Dept. of Information and Computer Sciences, Faculty of Science, Nara Women’s University,*

*Nara, Japan*

Abstract

We show that if a computable sequence of real-valued functions on an effectively compact metric space converges pointwise monotonically to a computable function, then the sequence converges effectively uniformly to the function. This is an effectivized version of Dini’s Theorem.

*Keywords:* Dini’s Theorem, effectively compact metric space, computable function, effective uniform convergence

# Introduction

If a sequence of real-valued continuous functions on a compact space con- verges pointwise monotonically to a continuous function, then the sequence converges uniformly to the function. It is called *Dini’s theorem* and one of the fundamental theorems in functional analysis and general topology.

From the viewpoint of computability, the question arises: whether we can effectivize Dini’s theorem, in other words, whether there is a theorem which is a Dini’s theorem with some topological concepts replaced by their computa- tional counterparts. In this paper, we show a positive answer to this question in the case of metric spaces, more precisely, the theorem that if a computable sequence of real-valued functions on an effectively compact metric space con-

1 Email: [wd@ics.nara-wu.ac.jp](mailto:wd@ics.nara-wu.ac.jp)

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verges pointwise monotonically to a computable function, then the sequence converges effectively uniformly to the function.

Meanwhile, if a computable sequence of real numbers converges monoton- ically to a computable real number, then the sequence converges effectively to the real number. It is called the *monotonic convergence theorem* [[6](#_bookmark14)]. The main theorem in this paper is not only an effectivization of Dini’s thorem but also an extension of the monotonic convergence theorem to real-valued continuous functions on effectively compact metric spaces. The monotonic convergence theorem is considered a special case of our theorem on *C*({0}), the space of all real-valued continuous functions on a singlton.

# Preliminaries

For a relation *S* ⊂ *X*1 × ··· × *Xm* × *Y*1 × ··· × *Yn* and a tuple (*x*1*,... , xm*) ∈ *X*1×·· ·×*Xm*, the set {(*y*1*,... , yn*) ∈ *Y*1×·· ·×*Yn* | (*x*1*,..., xm, y*1*,... , yn*) ∈ *S*} is denoted by *S*(*x*1*,... , xm*).

We fix some standard tuple functions and corresponding projection func-

tions on N. ⟨-*,... ,* -⟩ denotes the *n*-tuple function. (-)*n*

*k*

denotes the cor-

responding *k*th projection function. We often identify an *n*-tuple sequence

(*xk*1*,...,kn* ) with its serialization (*x*(*k*)*n,...,*(*k*)*n* )*k*∈N.

1 *n*

We use the terminology and the notation on computability of real numbers

and of real functions that Pour-El and Richards have used in [[6](#_bookmark14)].

The following four definitions are introduced in [[9](#_bookmark15)] by Yasugi, Mori, and Tsujii.

Definition 2.1 Let (*M, d*) be a metric space. A set S ⊂ *Mω* is a *computability structure* on (*M, d*) if the following three conditions hold.

* 1. If (*xn*)*,* (*yn*) ∈ S, then (*d*(*xn, yn*' ))*n,n*' forms a computable double sequence of real numbers.
  2. If (*xn*) ∈ S and *σ* : N → N is a recursive function, then (*xσ*(*n*)) ∈ S.
  3. If (*xn,k*) ∈ S, (*x*' ) ∈ *Mω*, and (*xn,k*) converges to (*x*' ) effectively in *n*

*n n*

and *k* as *k* → ∞, then (*x*' ) ∈ S.

*n*

An element of S is called a *computable sequence* in *M*.

Definition 2.2 A metric space with a computability structure (*M, d,* S) is *effectively totally bounded* if there exists a computable sequence (*el*) ∈ S and a recursive function *γ* : N → N such that

*γ*(*i*)

*M* = {*el* | *l* ∈ N} and (∀*i* ∈ N) *M* = *B*(*el,* 1*/*2*i*)*.*

*l*=0

(*M, d,* S) is *effectively compact* if it is effectively totally bounded and *d* is a complete metric.

Definition 2.3 Let (*M, d,* S) be a metric space with a computability struc- ture. A subset *K* ⊂ *M* is an *effectively compact subset* of *M* if (*K, d*|*K,* S∩*Kω*) is an effectively compact metric space.

In other words, *K* is an effectively compact subset of (*M, d,* S) iff it is a compact subset of (*M, d*) and there exists a computable sequence (*el*) ∈ S and a recursive function *γ* : N → N such that

*γ*(*i*)

*K* = {*el* | *l* ∈ N} and (∀*i* ∈ N) *K* ⊂ *B*(*el,* 1*/*2*i*)*.*

*l*=0

Definition 2.4 Let (*M, d,* S) be an effectively compact metric space. A se- quence of functions (*fn*), *fn* : *M* → R, is *computable* if the following two conditions hold.

1. (*Sequential computability*) For any computable sequence (*xk*) in *M*, (*fn*(*xk*))*n,k* forms a computable sequence of real numbers.
2. (*Effective uniform continuity* ) There exists a recursive function *α* :

N2 → N such that for any *n, j* ∈ N and any *x, y* ∈ *M*,

*d*(*x, y*) *<* 1*/*2*α*(*n,j*) =⇒ |*fn*(*x*) − *fn*(*y*)| *<* 1*/*2*j.*

A function *f* : *M* → R is a *computable function* if (*f* )*n*∈N, the sequence all of whose elements are equal to *f* , is a computable sequence of functions.

The recursive function *α* in Definition [2.4](#_bookmark1) ([ii](#_bookmark2)) is called an *effective modulus of continuity* of (*fn*).

# Effective Dini’s Theorem

First, we show two propositions with no assumptions on computability.

Proposition 3.1 *Let* (*M, d*) *be a metric space. Let K* ⊂ *M be a nonempty compact subset with* (*el*) ∈ *Mω and γ* : N → N *such that K* = {*el* | *l* ∈ N} *and* (∀*i*) *K* ⊂ *γ*(*i*) *B*(*el,* 1*/*2*i*)*. Then, for any ﬁnite sequence of open balls* (*B*(*xk, rk*))*k*=0*,...,m, the following holds:*

*l*=0

*K* ⊂ *B*(*x ,r* ) ⇐⇒ (∃*i*)(∀*l* ≤ *γ*(*i*))(∃*k* ≤ *m*) *d*(*x ,e* )+ 1

*m*

*k*

*k*

*k*

*l*

2*i*

*k*

*k*=0

*< r .*

Proof. (⇐=) Suppose *y* ∈ *K*. Then, (∃*l* ≤ *γ*(*i*)) *d*(*y, el*) *<* 1*/*2*i*. It follows that

(∃*i*)(∃*l* ≤ *γ*(*i*))(∃*k* ≤ *m*) *d*(*y, e* ) *<*  1

*l*

2*i*

1

∧ *d*(*x ,e* )+

*k*

*l*

2*i*

*< r* *.*

This implies (∃*k* ≤ *m*) *d*(*y, xk*) *< rk*, i.e., *y* ∈ *m*

*k*

*B*(*xk, rk*). This shows

*K* ⊂ *m*

*k*=0

*k*

*k*

*B*(*x ,r* ).

*k*=0

is:

(=⇒) We will prove the contraposition. The negation of the conclusion

1

(∀*i*)(∃*l* ≤ *γ*(*i*))(∀*k* ≤ *m*) *d*(*xk, el*)+ 2*i* ≥ *rk.*

By choosing an *l* for each *i*, we obtain a function *γ*' : N → N such that

1

(∀*i*)(∀*k* ≤ *m*) *d*(*xk, eγ*'(*i*))+ 2*i* ≥ *rk.*

Since *K* is sequentially compact, there exists a subsequence (*eγ*'(*θ*(*i*))) that converges to a point in *K*. Let *y* be the limit. A simple manipulation of

limits yields (∀*k* ≤ *m*) *d*(*xk, y*) ≥ *rk*, i.e., *y* /∈ *m B*(*xk, rk*). This shows

*K* /⊂ *m*

*k*=0

*k*

*k*

*B*(*x ,r* ).

*k*=0

Proposition 3.2 *Let* (*M, d*) *be a separable metric space with a dense, at most countable subset* {*el* | *l* ∈ N}*. Let a be a real number such that a* ≥ 1*. Let* (*εi*) *be a sequence of positive real numbers converging to* 0*. Then,*

*B*(*x, r*)=

*l,i*∈N*, d*(*x,el*)+*aεi<r*

*B*(*el, εi*)*.*

Proof. Comparison of the distance between the centers with the difference between the radii yields *B*(*x, r*) ⊃ *B*(*el, εi*) if *d*(*x, el*)+ *aεi < r*. This implies *B*(*x, r*) ⊃ *l,i*∈N*, d*(*x,el*)+*aεi<r B*(*el, εi*).

Suppose *y* ∈ *B*(*x, r*). There exists an *εi* such that *d*(*x, y*) *< r* −

(*a* + 1)*εi*. Furthermore, there exists an *el* such that *d*(*y, el*) *< εi*. From these two inequalities as well as *d*(*x, e**l*) ≤ *d*(*x, y*) + *d*(*y, el*), we obtain *d*(*x, e* ) + *aε < r*. Therefore *y* ∈ *B*(*e , ε* ). This shows

*l i*  *l,i*∈N*, d*(*x,el*)+*aεi<r l i*

*B*(*x, r*) ⊂ *l,i*∈N*, d*(*x,el*)+*aεi<r B*(*el, εi*).

Next, we show two lemmata. Lemma [3.3](#_bookmark5) is on a consequence of effective compactness. Lemma [3.4](#_bookmark6) is on another characterization of computable real- valued functions.

Lemma 3.3 *Let* (*M, d,* S) *be a metric space with a computability structure. For any effectively compact subset K of M and any computable double sequence*

(*B*(*xn,k, rn,k*)) *of open balls in M, there exists a recursive partial function*

*α* : N *~* N *such that α*(*n*) *is deﬁned and K* ⊂ *α*(*n*) *B*(*xn,k, rn,k*) *holds if*

∞ *k*=0

*K* ⊂

*k*=0 *B*(*xn,k, rn,k*)*, and α*(*n*) *is undeﬁned otherwise.*

Proof. We choose a computable sequence (*el*) and a recursive function *γ* : N → N such that *K* = {*el* | *l* ∈ N} and (∀*i*) *K* ⊂ *γ*(*i*) *B*(*el,* 1*/*2*i*). By applying Proposition [3.1](#_bookmark3) to each finite sequence (*B*(*xn,k, rn,k*))*k*=0*,...,m* for *n* = 0*,* 1*,* 2*,...* , we obtain

*l*=0

*m*

*K* ⊂ *B*(*x*

*n,k*

*k*=0

*,r* ) ⇐⇒ (∃*i*)(∀*l* ≤ *γ*(*i*))(∃*k* ≤ *m*) *d*(*x*

1

*,e* )+ *< r .*

*n,k*

*l*

2*i*

*n,k*

It is clear that the right-hand side is recursively enumerable in *n* and *m*. Therefore, there exists a primitive recursive predicate *ϕ* on N3 such that

*n,k*

(∃*i*')*ϕ*(*n, m, i*') iff *K* ⊂ *m B*(*xn,k, rn,k*). Hence, we can construct *α* by

*k*=0

*α*(*n*) (min{*m*' | *ϕ*(*n,* (*m*')2*,* (*m*')2)})2.

1 2 1

Lemma [3.3](#_bookmark5) corresponds to “*δ*' ≤ *δ*

|K∗ ” shown by Brattka and

range

Haine−Borel

Presser in [[1](#_bookmark12)]. The proof here is essentially the same as that in [[1](#_bookmark12)] with some

correction for a minor error.

Lemma 3.4 *Let* (*M, d,* S) *be an effectively compact metric space. Let* (*el*) ∈ S *be dense in M. For any sequence* (*fn*) *of real-valued functions on M, the following two conditions are equivalent.*

* 1. (*fn*) *is a computable sequence of functions.*
  2. *There exists a recursively enumerable set S* ⊂ N × Q × Q+ × N × N *such that*

(∀*n* ∈ N)(∀*c* ∈ Q)(∀*r* ∈ Q+) *f* −1((*c* − *r, c* + *r*)) = *B*(*el,* 1*/*2*i*)*.*

*n*

(*l,i*)∈*S*(*n,c,r*)

Proof. [([i](#_bookmark7))⇒([ii](#_bookmark8))] Let *α* : N2 → N be an effective modulus of continuity of (*fn*). Let *S* ⊂ N × Q × Q+ × N × N be the set defined by

(*n, c, r, l, i*) ∈ *S*

⇐⇒ (∃*j* ∈ N)(*i* = *α*(*n, j*) ∧ *c* − *r < fn*(*el*) − 1*/*2*j* ∧ *fn*(*el*)+ 1*/*2*j < c* + *r*)*.*

Since *α* is a recursive function and (*fn*(*el*)) is a computable double sequence of real numbers, it follows immediately from the definition of *S* that *S* is a recursively enumerable set.

Let (*l, i*) ∈ *S*(*n, c, r*). Then we have, for some *j* ∈ N,

*fn*(*B*(*el,* 1*/*2*i*)) = *fn*(*B*(*el,* 1*/*2*α*(*n,j*)))

⊂ (*fn*(*el*) − 1*/*2*j, fn*(*el*)+ 1*/*2*j*)

⊂ (*c* − *r, c* + *r*)*.*

Therefore, *B*(*el,* 1*/*2*i*) ⊂ *f* −1((*c* − *r, c* + *r*)). This shows *f* −1((*c* − *r, c* + *r*)) ⊇

*n n*

*i*

(*l,i*)∈*S*(*n,c,r*) *B*(*el,* 1*/*2 ).

Suppose *x* ∈ *f* −1((*c* − *r, c* + *r*)), i.e., |*fn*(*x*) − *c*| *< r*. Then, for some *j* ∈ N, it holds that 2*/*2*j* ≤ *r* − |*fn*(*x*) − *c*|. For such a *j*, there exists an *el* such that *d*(*x, el*) *<* 1*/*2*α*(*n,j*). Hence |*fn*(*el*) − *fn*(*x*)| *<* 1*/*2*j*. By using *fn*(*el*) − 1*/*2*j < fn*(*x*), we obtain

*n*

*fn*(*el*)+ 1*/*2*j < fn*(*x*)+ 2*/*2*j* ≤ *fn*(*x*)+ *r* − |*fn*(*x*) − *c*|≤ *c* + *r.*

Analogously, by using *fn*(*el*)+ 1*/*2*j > fn*(*x*), we obtain *fn*(*el*) − 1*/*2*j > c* − *r*. The conjunction of the obtained two inequalities implies (*n, c, r, l, α*(*n, j*)) ∈

*S*. Therefore, *x* ∈ *B*(*el,* 1*/*2*i*) for some (*l, i*) ∈ *S*(*n, c, r*). This shows

*f* −1((*c* − *r, c* + *r*)) ⊂

*B*(*el,* 1*/*2*i*).

*n* (*l,i*)∈*S*(*n,c,r*)

[([ii](#_bookmark8))⇒([i](#_bookmark7))] (*Sequential computability*) Suppose (*xk*) ∈ S. From ([ii](#_bookmark8)), we

obtain

|*fn*(*xk*) − *c*| *<* 1*/*2*j* ⇐⇒ (∃*l*)(∃*i*)[(*n, c,* 1*/*2*j, l, i*) ∈ *S* ∧ *d*(*xk, el*) *<* 1*/*2*i*]*.*

Hence {(*n, k, j, c*) ∈ N × N × N × Q | |*fn*(*xk*) − *c*| *<* 1*/*2*j*} is a recursively enumerable set. Meanwhile, (∀*n*)(∀*k*)(∀*j*)(∃*c* ∈ Q) |*fn*(*xk*) − *c*| *<* 1*/*2*j* holds since Q is dense in R. Therefore, there exists a computable triple sequence of rational numbers (*cn,k,j*) such that (∀*n*)(∀*k*)(∀*j*) |*fn*(*xk*) − *cn,k,j*| *<* 1*/*2*j*, i.e., (*fn*(*xk*))*n,k* is a computable double sequence of real numbers.

(*Effective uniform continuity*) Using Proposition [3.2](#_bookmark4), we obtain that

*B*(*el,* 1*/*2*i*)=

*l*'*,* *i*'∈N*,*

*i*'

*B*(*el*' *,* 1*/*2*i* )*.*

*i*

'

*d*(*el,el*' )+2*/*2

*<*1*/*2

It is clear that *d*(*el, el*' )+ 2*/*2*i <* 1*/*2*i* is a recursively enumerable predicate

'

' ' ' '

*i*' *i*

of *l, i, l ,i* . It is also clear that (∀*l*)(∀*i*)(∃*l* )(∃*i* ) *d*(*el, el*' )+ 2*/*2

*<* 1*/*2

holds.

Hence there exist recursive functions *σ, ρ* : N3 → N such that for any *l, i* ∈ N,

' ' *i*' *i*

{(*l, i, l ,i* ) ∈ N × N × N × N | *d*(*el, el*' )+ 2*/*2 *<* 1*/*2 }

= {(*l, i, σ*(*l, i, k*)*, ρ*(*l, i, k*)) | *k* ∈ N}*.*

Therefore, for any *l, i* ∈ N,

∞

*B*(*el,* 1*/*2*i*)= *B*(*eσ*(*l,i,k*)*,* 1*/*2*ρ*(*l,i,k*)) (1)

*k*=0

and

(∀*k*) *d*(*el, eσ*(*l,i,k*))+ 2*/*2*ρ*(*l,i,k*) *<* 1*/*2*i.* (2)

Since *S* is a recursively enumerable set, so is {(*n, j, l, i, c*) ∈ N × N ×

N × N × Q | (*n, c,* 1*/*2*j*+1*, l, i*) ∈ *S*}. On the other hand, since *M* =

*n*

*c*∈Q

*f* −1((*c* − 1*/*2*j*+1*, c* + 1*/*2*j*+1)), it holds that

*M* =

*B*(*el,* 1*/*2*i*)*.*

*c*∈Q (*l,i*)∈*S*(*n,c,*1*/*2*j*+1)

Hence (∀*n*)(∀*j*)(∃*l*)(∃*i*)(∃*c* ∈ Q) (*n, c,* 1*/*2*j*+1*, l, i*) ∈ *S*. Therefore, there exist recursive functions *σ*'*, ρ*' : N3 → N and a computable triple sequence of rational numbers (*cn,j,k*) such that for any *n, j* ∈ N,

{(*n, j, l, i, c*) ∈ N × N × N × N × Q | (*n, c,* 1*/*2*j*+1*, l, i*) ∈ *S*}

= {(*n, j, σ*'(*n, j, k*)*, ρ*'(*n, j, k*)*, cn,j,k*) | *k* ∈ N}*.*

Thus, for any *n, j* ∈ N,

∞

*M* =

*k*=0

*B*(*eσ*'(*n,j,k*)*,* 1*/*2

*ρ*'(*n,j,k*)

) (3)

and

'

*ρ* (*n,j,k*) *j*+1 *j*+1

(∀*k*) *fn*(*B*(*eσ*'(*n,j,k*)*,* 1*/*2 )) ⊂ (*cn,j,k* − 1*/*2 *, cn,j,k* + 1*/*2 )*.* (4)

From ([1](#_bookmark9)) and ([3](#_bookmark10)), we obtain

∞

*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )

1

1

2

*k*=0

Application of Lemma [3.3](#_bookmark5) yields that there exists a recursive function *γ* :

*M* =

*B*(*eσ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )*,* 1*/*2

1

1

2 )*.*

N2 → N such that

*γ*(*n,j*)

*M* =

*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )

*B*(*eσ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )*,* 1*/*2

1

1

2 )*.*

*k*=0

1

1

2

Let *α* : N2 → N be a recursive function defined by

*α*(*n, j*)= max

*ρ*(*σ*'(*n, j,* (*k*)2)*, ρ*'(*n, j,* (*k*)2)*,* (*k*)2)*.*

*k*≤*γ*(*n,j*)

1 1 2

Suppose points *x, y* ∈ *M* satisfy *d*(*x, y*) *<* 1*/*2*α*(*n,j*). Then there exists some

*k* ≤ *γ*(*n, j*) such that

*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )

*d*(*x, eσ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )) *<* 1*/*2

1 1 2 *.*

1 1 2

With such a *k*,

*d*(*x, eσ*'(*n,j,*(*k*)2))

1

≤ *d*(*eσ*'(*n,j,*(*k*)2)*, eσ*(*σ*' (*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 ))+ *d*(*x, eσ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 ))

1 1 1 2 1 1 2

*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )

*< d*(*eσ*'(*n,j,*(*k*)2)*, eσ*(*σ*' (*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 ))+ 1*/*2

1 1 2

1 1 1 2

*<* 1*/*2*ρ*'(*n,j,*(*k*)2) − 1*/*2*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 )*.*

1

1

1

2

Furthermore,

*d*(*y, eσ*'(*n,j,*(*k*)2)) ≤ *d*(*x, eσ*'(*n,j,*(*k*)2))+ *d*(*x, y*)

1 1

*<* 1*/*2*ρ*'(*n,j,*(*k*)2) − 1*/*2*ρ*(*σ*'(*n,j,*(*k*)2)*,ρ*'(*n,j,*(*k*)2)*,*(*k*)2 ) + 1*/*2*α*(*n,j*)

1

≤ 1*/*2*ρ*'(*n,j,*(*k*)2)*.*

1

Therefore, *x, y* ∈ *B*(*eσ*'(*n,j,*(*k*)2)*,* 1*/*2

1

*j*+1

1 1 2

*ρ*'(*n,j,*(*k*)2)). Due to ([4](#_bookmark11)), this implies

1

*j*+1

1

*fn*(*x*)*, fn*(*y*) ∈ (*cn,j,*(*k*)2 − 1*/*2

1

*, cn,j,*(*k*)2 + 1*/*2

). Hence |*f* (*x*) − *f* (*y*)| *<*

1*/*2*j*. This shows that *α* is an effective modulus of continuity of (*fn*).

The condition ([ii](#_bookmark8)) in Lemma [3.4](#_bookmark6) is equivalent to *δ*6-computability defined by Weihrauch in [[7](#_bookmark16)]. Lemma [3.4](#_bookmark6) shows that for a real-valued function on an effectively compact metric space, computability defined by Mori, Tsujii, and Yasugi in [[5](#_bookmark13)] and used in this paper is equivalent to *δ*6-computability.

Now we are ready to show the main theorem.

Theorem 3.5 *Let* (*M, d,* S) *be an effectively compact metric space. Let* (*fn*) *be a computable sequence of real-valued functions on M and f a computable real-valued function on M. If fn converges pointwise monotonically to f as n* → ∞*, then fn converges effectively uniformly to f.*

Proof. Let (*el*) be a computable sequence dense in *M*.

Let *Uj,n* = (*fn* − *f* )−1((−1*/*2*j,* 1*/*2*j*)). Since *fn* converges pointwise to *f* , it

holds that (∀*j*) *M* = ∞ *Uj,n*. Due to Lemma [3.4](#_bookmark6), there exists a recursively

*n*=0

enumerable set *S* ⊂ N4 such that (∀*j*)(∀*n*) *Uj,n* = (*i,l*)∈*S*(*j,n*) *B*(*el,* 1*/*2*i*). From these two equalities, we obtain

(∀*j*) *M* =

(*n,i,l*)∈*S*(*j*)

*B*(*el,* 1*/*2*i*)*.*

This implies (∀*j*) *S*(*j*) /= ∅. Therefore, there exist recursive functions *θ, ρ, σ* : N2 → N such that *S* = {(*j, θ*(*j, k*)*, ρ*(*j, k*)*, σ*(*j, k*)) | *j, k* ∈ N}. Using these functions, we can rewrite the formula above as follows:

∞

(∀*j*) *M* = *B*(*eσ*(*j,k*)*,* 1*/*2*ρ*(*j,k*))*.*

*k*=0

Since *M* itself is a compact subset of *M*, application of Lemma [3.3](#_bookmark5) yields that there exists a recursive total function *β* : N → N such that

*β*(*j*)

(∀*j*) *M* = *B*(*eσ*(*j,k*)*,* 1*/*2*ρ*(*j,k*))*.*

*k*=0

Since *B*(*e ,* 1*/*2*ρ*(*j,k*)) ⊂ *U* , this implies (∀*j*) *M* = *β*(*j*) *U* . Since

*σ*(*j,k*)

*j,θ*(*j,k*)

*k*=0

*j,θ*(*j,k*)

*fn* converges monotonically to *f* , it holds that *Uj,n* ⊂ *Uj,n*' if *n* ≤ *n* . Let *α* : N → N be a recursive function defined by *α*(*j*) = max*k*≤*β*(*j*) *θ*(*j, k*). We have (∀*j*)(∀*n* ≥ *α*(*j*)) *M* = *Uj,n*, which is equivalent to:

'

(∀*j*)(∀*n* ≥ *α*(*j*))(∀*x* ∈ *M*) |*fn*(*x*) − *f* (*x*)| *<* 1*/*2*j.*

Thus *fn* converges effectively uniformly to *f* as *n* → ∞.

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