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From Iterative Algebras to Iterative Theories (Extended Abstract)†

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Abstract

Iterative theories introduced by Calvin Elgot formalize potentially infinite computations as solu- tions of recursive equations. One of the main results of Elgot and his coauthors is a description of a free iterative theory as the theory of all rational trees. Their algebraic proof of this fact is extremely complicated. In our paper we show that by starting with “iterative algebras”, i. e., alge- bras admitting a unique solution of all systems of flat recursive equations, a free iterative theory is obtained as the theory of free iterative algebras. The (coalgebraic) proof we present is dramatically simpler than the original algebraic one. And our result is, nevertheless, much more general: we describe a free iterative theory on any finitary endofunctor of every locally presentable category ffi. This allows us, e. g., to consider iterative algebras over any equationally specified class ffi of finitary algebras.

*Keywords:* free iterative theory, rational monad, coalgebra

† The full version of this paper containing all proofs can be found at the URL <http://www.iti.cs.tu-bs.de/~milius>

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# Introduction

Iterative theories have been introduced by Calvin C. Elgot [[9](#_bookmark36)] as a model of computation formalized as a sequence of instantaneous descriptions of an abstract machine. He and his co-authors then proved that for every signature Σ a free iterative theory on Σ exists [[7](#_bookmark34)] and that it consists of all rational Σ-trees [[10](#_bookmark37)]. Recall that a Σ-tree (i. e., a tree, possibly infinite, labelled by operation symbols in Σ so that every node with *n* children is labelled by an *n*- ary symbol) is *rational* if it has up to isomorphism only finitely many subtrees, see [[13](#_bookmark38)].

In the present paper we introduce *iterative algebras* rather than iterative theories, and we show that the theory formed by all free iterative algebras is Elgot’s free iterative theory. In the classical case of Σ-algebras, iterativity has been introduced by Evelyn Nelson [[16](#_bookmark42)] as follows: given a Σ-algebra *A*, let us consider an arbitrary system of recursive equations

(1.1) *xi* ≈ *ti, i* = 1*,...,n,*

where *X* = {*x*1*, x*2*,... , xn*} is a finite set of variables and *t*1, *t*2, ..., *tn* are terms over *X* + *A*, none of which is a single variable *xi*. The algebra *A* is called *iterative* provided that for every such system of equations there exists a unique *solution*. That is, there exists a unique *n*-tuple *x*1†, *x*2†, ..., *xn*† of elements of *A* such that each of the formal equations in ([1.1](#_bookmark1)) becomes an equality after the substitution *x*1†*/x*1, *x*2†*/x*2, ..., *xn*†*/xn*:

*xi*† = *ti*(*x*1†*/x*1*, x*2†*/x*2*,..., xn*†*/xn*) *, i* = 1*,... , n.*

Example: let Σ consist of a single binary operation symbol, ∗, then the algebra

*A* of all (finite and infinite) binary trees is iterative. For example, the system

(1.2)

*x*1 ≈ *x*2 ∗ *t*

*x*2 ≈ (*x*1 ∗ *s*) ∗ *t*

where *s* and *t* are trees in *A* has the unique solution *x*1† = ·· ·∗*s*)∗*t*)∗*t*)∗*s*)∗*t*)∗*t*, and analogously for *x*2†.

Every system ([1.1](#_bookmark1)) above can be modified to a *flat system*, i.e., one where each right-hand side is either a *flat term*

*ti* = *σ*(*x*1*,... , xk*) *,* for *σ* ∈ Σ*k*, *x*1*,... , xk* ∈ *X*, or an element of *A*

*ti* ∈ *A.*

For example, the above system ([1.2](#_bookmark2)) has the following modification to a flat

system:

*x*1 ≈ *x*2 ∗ *x*3 *x*3 ≈ *t x*5 ≈ *s x*2 ≈ *x*4 ∗ *x*3 *x*4 ≈ *x*1 ∗ *x*5

Therefore, an algebra is iterative iff every flat equation system has a unique

solution.

Now Σ-algebras are a special case of algebras for an endofunctor *H* : ffi −→ ffi (which are pairs consisting of an object *A* of ffi and a morphism *α* : *HA* −→ *A*): here ffi is the category of sets and *H* = *H*Σ is the *polynomial functor* given on objects by *H*Σ*X* = Σ0 + Σ1 × *X* + Σ2 × *X*2 + ·· ·. For an algebra (*A, α*) observe that a flat equation system has its right-hand sides in *H*Σ*X* + *A*, thus, it can be represented by a morphism

*e* : *X* −→ *H*Σ*X* + *A, e*(*xi*)= *ti.*

A *solution* of *e* is then a morphism

*e*† : *X* −→ *A, e*†(*xi*)= *xi*† *,*

with the property that the following diagram

*X e*† *A*

,*,*

(1.3)

*e*

J

[*α,A*]

commutes.

*H X* + *A*

*H*Σ

Σ

*e*†+*A*

*H*Σ*A* + *A*

Definition 1.1 A Σ-algebra *A* is called *iterative* provided that for every flat equation morphism *e* : *X* −→ *H*Σ*X* + *A*, where *X* is a finite set, there exists a unique solution *e*† : *X* −→ *A*.

“Classical” algebras are seldom iterative. But there are enough interesting iterative algebras. For example, the algebra

*T*Σ

of all (finite and infinite) Σ-trees is iterative. And so is its subalgebra

*R*Σ

of all rational Σ-trees. In fact, the full subcategory Alg*it* Σ of Alg Σ formed by all iterative Σ-algebras is rich enough: a limit or a filtered colimit of iterative algebras is always iterative, thus Alg*it* Σ is reflective in Alg Σ (see Reflection Theorem in [[6](#_bookmark33)]). From this it follows that every set generates a free iterative algebra, i. e., the forgetful functor Alg*it* Σ −→ Set is a right-adjoint. This defines a monad *R*Σ on Set. We prove that

1. *R*Σ is a free iterative monad on *H*Σ,

and

1. *R*Σ assigns to every set *X* the algebra *R*Σ*X* of all rational Σ-trees on *X*.

In this way a new proof of the result of Elgot et al. describing a free iterative monad (or theory) is achieved.

In our proof we work with an arbitrary endofunctor *H* of the category of sets which is *ﬁnitary*, i. e., preserves filtered colimits. As in Definition [1.1](#_bookmark4), an algebra *α* : *HA* −→ *A* is called *iterative* if for every flat equation morphism *e* : *X* −→ *HX* + *A*, where *X* is a finite set, there exists a unique solution,

i. e., a unique morphism *e*† : *X* −→ *A* with *e*† = [*α, A*] · (*He*† + *A*) · *e*, compare with ([1.3](#_bookmark3)). The main technical result is coalgebraic: in order to describe a free iterative algebra on a set *Y* , we form the diagram Eq*Y* of all coalgebras *e* : *X* −→ *HX* + *Y* of the endofunctor *H*(−)+ *Y* on finite sets *X*. We prove that a colimit of that diagram

*RY* = colim Eq*Y*

carries naturally the structure of an algebra, and that *RY* is a free iterative algebra on *Y* . From that we derive that the monad *R*(−) is a free iterative monad on *H*. In our proof the fact that *H* is a finitary endofunctor of Set plays no roˆle: the same result holds for finitary endofunctors of all locally finitely presentable categories. Thus, if we start with e. g. an equational class ffi of finitary algebras then, again, for every finitary endofunctor *H* the free iterative algebras *RY* are constructed as colimits of coalgebras of *H*(−)+ *Y* on finitely presentable objects of ffi, and they form a free iterative theory on *H*.

Related Work. In the classical setting, i. e., for polynomial endofunctors of Set, iterative algebras were introduced by Evelyn Nelson [[16](#_bookmark42)] to obtain a short proof of Elgot’s free iterative theories. Our paper can be seen as a categorical generalization of that paper with distinctive coalgebraic “flavour”. Also Jerzy Tiuryn introduced a concept of iterative algebra in [[17](#_bookmark43)] with the same aim as ours: to relate iterative theories of Elgot to properties of algebras. But the approach of [[17](#_bookmark43)] is fundamentally different from ours; e.g., the trivial, one-element, algebra is not iterative in the sense of Tiuryn, thus, his iterative algebras are not closed under limits.

The description of the rational monad as a colimit is also presented in [[12](#_bookmark39)]. The present paper is a dramatic improvement of our previous descrip- tion of the rational monad in [[3](#_bookmark30)], [[4](#_bookmark31)] where we assumed that the endofunctor preserves monomorphisms and the underlying category satisfies three rather technical conditions, and the proof was much more involved. The current ap- proach includes all equationally defined algebraic categories as base categories

(whereas in [[4](#_bookmark31)] we still needed strong side conditions which only hold in very few algebraic categories). All proofs have been omitted, the reader can find them in the full version of our paper [[5](#_bookmark32)].

# Iterative Algebras

Notation 2.1 Throughout the paper all categories are assumed to have finite coproducts. We denote by inl and inr the coproduct injections of *A* + *B*.

In order to define the concept of a flat equation morphism as in the intro- duction (a morphism *e* : *X* −→ *HX* + *A* in Set where *X* is finite) in a general category, we need the appropriate generalization of finiteness. Recall that a functor is called *ﬁnitary* provided that it preserves filtered colimits. A set is finite if and only if its hom-functor is finitary. This has inspired Gabriel and Ulmer [[11](#_bookmark40)] to the following

Definition 2.2 An object of *A* a category ffi is *ﬁnitely presentable* if its hom- functor ffi(*A,* −): ffi −→ Set is finitary.

A category ffi is called *locally ﬁnitely presentable* provided that it has col- imits and a (small) set of finitely presentable objects whose closure under filtered colimits is all of ffi.

Examples 2.3

* 1. Set, the category of posets, and every variety of finitary algebras are locally finitely presentable.
  2. Let *H* be a finitary endofunctor of a locally finitely presentable category ffi. Then the category Alg *H* of *H*-algebras and homomorphisms is also locally finitely presentable, see [[6](#_bookmark33)].

Definition 2.4 Given an endofunctor *H* : ffi −→ ffi, by a *ﬁnitary flat equation morphism* (later just: *equation morphism*) in an object *A* we mean a morphism *e* : *X* −→ *HX* + *A* of ffi, where *X* is a finitely presentable object of ffi.

Suppose that *A* is an underlying object of an *H*-algebra *α* : *HA* −→ *A*. Then by a *solution* of *e* in the algebra *A* is meant a morphism *e*† : *X* −→ *A* in ffi such that the square

*X e*† *A*

,*,*

(2.1)

*e*

*HX* J

[*α,A*]

commutes.

+ *A He*†+*A*

*HA* + *A*

An *H*-algebra is called *iterative* provided that every finitary flat equation morphism has a unique solution.

Example 2.5

1. Groups, lattices etc. considered as Σ-algebras are seldom iterative. For example, if a group is iterative, then its unique element is the unit element 1, since the recursive equations *x* ≈ *x* · *y, y* ≈ 1 have a unique solution. If a lattice is iterative, then it has a unique element: consider *x* ≈ *x* ∨ *x*.
2. The algebra of addition on the set

 = { 1*,* 2*,* 3*,...* }∪ { ∞ }

is iterative (and “almost classical”). (Observe that 0 is not included. This is forced by the uniqueness of solutions of *x* = *x* + *x*.)

1. The algebras *T*Σ and *R*Σ (see Introduction) are iterative.

Remark 2.6 We denote by

Alg*it H*

the category of all iterative algebras and all homomorphisms.

Proposition 2.7 *Iterative algebras are closed under limits and ﬁltered colim- its in* Alg *H.*

The proof is a rather simple calculation based on the fact that Alg *H* has limits and filtered colimits formed on the level of the base category. Using the Reflection Theorem of [[6](#_bookmark33)] we derive:

Corollary 2.8 *The category* Alg*it H is a reflective subcategory of* Alg *H.*

Corollary 2.9 *Every object of* ffi *generates a free iterative H-algebra.*

In other words, the natural forgetful functor *U* : Alg*it H* −→ ffi has a left adjoint.

Definition 2.10 The finitary monad on ffi formed by free iterative *H*-algebras is called the *rational monad* of *H* and is denoted by R = (*R, η, µ*).

Thus, R is the monad of the above adjunction

*R*

¸*,*

Alg*it H*  ⊥ ffi

*U*

More detailed, for every object *Z* of ffi we denote by *RZ* a free iterative *H*- algebra on *Z* with the universal arrow

*ηZ* : *Z* −→ *RZ ,*

and the algebra structure

*ρZ* : *HRZ* −→ *RZ .*

Then *µZ* : *RRZ* −→ *RZ* is the unique homomorphism of *H*-algebras with

*µZ* · *ηRZ* = *id* .

Proposition 2.11 *An initial iterative algebra of the endofunctor H*(−)+ *Z*

*is precisely a free iterative H-algebra on Z.*

Example 2.12 *The rational monad of H*Σ : Set −→ Set*.*

Recall from the introduction the algebra *T*Σ of all (finite and infinite) Σ-trees. This algebra is iterative — this is folklore. For every set *Z* the algebra *T*Σ*Z* of all Σ-trees over *Z* (i. e., trees with nodes having *n >* 0 children labelled by *n*-ary operation symbols and leaves labelled by constant symbols

or variables from *Z*) is also iterative, since *T*Σ*Z* = *T*Σ' , where Σ'

0

= Σ0 + *Z*,

and Σ' = Σ*i*, for *i >* 0.

*i*

As proved in [[16](#_bookmark42)] the subalgebra *R*Σ*Z* of all rational Σ-trees, i.e., Σ-trees over *Z* which have only finitely many subtrees (up to isomorphism), is a free iterative Σ-algebra on *Z*.

Corollary 2.13 *The rational monad* RΣ *of the polynomial endofunctor H*Σ *of* Set *is given by the formation of the* Σ*-algebras R*Σ(*Z*) *of all rational* Σ*-trees over Z.*

Example 2.14 The rational monad of Pfin : Set −→ Set, the finite power-set functor has been described in [[2](#_bookmark28)]: it assigns to a set *X* the algebra *A*(*X*)*/*∼, where *A*(*X*) is the algebra of all rational extensional finitely-branching trees (where “extensional” means that every pair of distinct siblings define non- isomorphic subtrees). And ∼ is the largest bisimulation of *A*(*X*) defined as follows: *t* ∼ *s* iff the cuttings at level *n* have the same extensional quotients, for all natural numbers *n*.

# A Coalgebraic Construction

The aim of this section is to describe an initial iterative *H*-algebra as a colimit of a simple diagram Eq in the given base category ffi. We assume throughout this section that

1. ffi is a locally finitely presentable category, see Definition [2.2](#_bookmark7), and
2. *H* is a finitary endofunctor of ffi.

We choose a set ffi*fp* of representatives of finitely presentable objects of ffi w.r.t. isomorphism.

The initial iterative algebra is proved to be a colimit of the diagram

Eq : EQ −→ ffi

whose objects are all *H*-coalgebras carried by finitely presentable objects of

ffi:

*e* : *X* −→ *HX* with *X* in ffi*fp*,

with the usual coalgebra homomorphisms as morphisms, and with Eq the obvious forgetful functor *e* '−→ *X*.

A colimit

*R*0 = colim Eq

of this diagram (with colimit morphisms *e* : *X* −→ *R*0 for all *e* : *X* −→ *HX*

in EQ) yields a canonical morphism

*i* : *R*0 −→ *HR*0

Namely, *i* is the unique morphism such that every *e* becomes a coalgebra homomorphism, i.e., the squares

*X*  *e*  *H X*

(3.1)

*e He*

J J

*R*0 *i H R*0

commute. (In fact, the forgetful functor Coalg *H* −→ ffi creates colimits.)

Theorem 3.1 *R*0 *is the initial iterative H-algebra. More precisely, the mor- phism i is an isomorphism and i*−1 : *HR*0 −→ *R*0 *is an initial iterative H- algebra.*

Sketch of Proof. (a) It is easy to see that the diagram Eq is filtered, and the morphisms *He* · *e* form a cocone, thus, *i* is well-defined. We now construct a morphism *j* : *HR*0 −→ *R*0 and prove that it is inverse to *i*. We use the fact that in a locally finitely presentable category the given object *HR*0 is a colimit of the diagram of all arrows *p* : *P* −→ *HR*0 where *P* is in ffi*fp*. More precisely, let ffi*fp/HR*0 denote the comma-category (of all these arrows *p*), then the forgetful functor *DHR*0 : ffi*fp/HR*0 −→ ffi has, in ffi, the colimit cocone formed by all *p* : *P* −→ *HR*0. Thus, in order to define *j* we need to define morphisms *jp* : *P* −→ *R*0 forming a cocone of the diagram *DHR*0 . We know that *HR*0 is a filtered colimit of *H* · Eq and that ffi(*P,* −) preserves this colimit, since *P* is in ffi*fp*. Therefore, *p* factors through one of the colimit morphisms

*P ¸ p*  *H R*

(3.2)

*¸¸*

*¸¸¸¸*

*¸¸¸*

,0*,*

*Hg*

*p*' *¸¸¸*z*˛*

*HW*

for some *g* : *W* −→ *HW* in EQ. We form a new object

*ep*' ≡ *P* + *W*

[*p*'*,g*]

*HW*

*H*inr *H* (*P* + *W* )

of EQ and define *j* to be the unique morphism such that the following square

*P*  inl *P* + *W*

(3.3)

*p ep*'

JJ

*HR*0 *j*  *R* 0

commutes for every *p* in ffi*fp/HR*0. To prove that this is well-defined we need to show that

* 1. *ep*'inl is independent of the choice of factorization ([3.2](#_bookmark12)), and
  2. the morphisms *ep*'· inl form a cocone of ffi*fp/HR*0. And then we verify *j* = *i*−1.
     1. To prove that (*R*0*, i*−1) is an iterative algebra we just show existence of solutions, leaving out uniqueness. For every equation morphism

*e* : *X* −→ *HX* + *R*0 = colim(*HX* + Eq)

there exists, since *X* is finitely presentable, a factorization through the colimit morphism *HX* + *f* (for some *f* : *V* −→ *HV* in EQ):

*X ¸¸¸*

*e*  *H X* +,*,R*0

(3.4)

*¸¸¸¸*

*e*0 *¸¸¸*

*HX*+*f*

*¸*z*˛*

*¸¸¸*

*HX* + *V*

Recall from [2.1](#_bookmark5) that can : *HX* + *HV* −→ *H*(*X* + *V* ) denotes the canonical morphism. Define a new object, *e*, of EQ as follows:

(3.5)

*e* ≡ *X* + *V* [*e*0*,*inr ] *H X* + *V HX*+*f* *H X* + *HV* can *H* (*X* + *V* )

Observe that (3.6)

*f* = *e* · inr

because inr : *V* −→ *X* + *V* is a coalgebra morphism (in EQ) from *f* to *e*. We define a solution of *e* by

0

(3.7)

*e*† ≡ *X*  inl *X* + *V e* *R .*

In fact, in the following diagram

†

*X*  *R*

*e*

r~ *,,,,* ,*¸*0 *,*

*e i*−1*,,,*

*,,*

0

*,,,*

J *HX*+ *f* [*He*† *,Hf* ] *H R ,*

(3.8) *e*

*HX* + *V*

*HX* + *H¸V*

*¸¸¸*

*¸¸ ¸¸*

*¸¸*

,0*,*

[*He*† *,HR*0]

[*i*−1*,R*0]

*HX*+*f*

(i)

*HX*+*Hf*

*¸¸*z*˛*

*H*˛*X* + *HR*0

,¸ *H X* J

*HX,*+*,i,,,,,,,,,*

*,,,,,,,,,,*

+ *R*0

*He*†+*R*0

*H R*0 + *R*0

all inner parts commute: see ([3.4](#_bookmark13)) for the left-hand part, ([3.1](#_bookmark10)) for part (i), whereas the lower part commutes trivially (analyze the two components sep- arately) and so does the middle triangle. It remains to verify the upper part: here we use ([3.1](#_bookmark10)) and ([3.5](#_bookmark14)) to conclude that the following diagram

*X*  inl *X* + *V e*  *R*

[*e*0*,V,,*] *,,,,,*

*e*0

*e*

J *,,,,,* J

,0 *˛*

*s,*

*HX* + *V*

*H*(*X* + *V* )

can*,,,,,* *¸*

*i*−1

*HX*+*f*

*,,,,,*

(ii)

*He*

J *,,*

J

*HX* + *HV*

†  *H R*0

[*He ,Hf* ]

commutes. In fact, the left-hand component of (ii) commutes by definition of *e*† and the right-hand one does by ([3.6](#_bookmark15)). Thus, ([3.8](#_bookmark16)) commutes, proving that *e*† is a solution of *e*.

* + 1. Initiality. Let *α* : *HA* −→ *A* be an iterative *H*-algebra. We prove first that there is at most one *H*-algebra homomorphism from *R*0. Let

*HR*0

*i*−1 *R*

*Hh h*

0

J J

*HA α A*

be a homomorphism. For every object *e* : *X* −→ *HX* of EQ the following diagram

*X e*  *R*  *h*  *A*

*ccc* ,*,,*

0

*cccc*

*ccccα*

*e* *i*

(3.9)

JJ

*cccc*

[*α,A*]

*HX*

inl

*He*

*H R*0 *Hh* *H A*

*HX* J

+ *A H*(*he* )+*A*

*HA* + *A*

commutes, see ([3.1](#_bookmark10)), which proves that *he* is a solution of inl *e* in *A*.

This determines *h* uniquely, since the *e*’s form a colimit cocone of *R*0 = colim Eq.

Conversely, let us define a morphism *h* : *R*0 −→ *A* by the above rule

*he* = (inl *e*)† for all *e* : *X* −→ *HX* in EQ

where (−)† is the unique solution in *A*. This is well-defined since the mor- phisms (inl *e*)† form a cocone of the diagram Eq: in fact, let

*X*  *e*  *H X*

*p Hp*

JJ

*Y f HY*

be a morphism of EQ. We prove that (inl *f* )†*p* is a solution of inl *e* by consid- ering the corresponding diagram:

*X*  *p*  *Y e f*

(inl *f* )†

*A*,*,*

J *Hp* J

[*α,A*]

*HX*

inl

*HY*

inl

*HX* J J

This proves

+ *A Hp*+*A*

*HY* + *A* † *HA* + *A*

*H*(inl *f* ) +*A*

(inl *e*)† = (inl *f* )†*p.*

The morphism *h* above is a homomorphism of algebras because the dia- gram ([3.9](#_bookmark17)) commutes: the outward square commutes by definition of *h*, the upper left-hand square by ([3.1](#_bookmark10)), and the lower part is obvious. This shows that the upper right-hand part commutes when precomposed with *e*, *e* in EQ. Since the *e*’s form a colimit cocone, it follows that *h* is a homomorphism.

Corollary 3.2 *A free iterative H-algebra RZ is a colimit,*

*RZ* = colim Eq*Z*

*of the diagram*

Eq*Z* : EQ*Z* −→ ffi

*where* EQ*Z consists of all equation morphisms e* : *X* −→ *HX* + *Z, X* ∈ ffi*fp (and the connecting morphisms are the coalgebra homomorphisms w.r.t. H*(−)+ *Z) and* Eq*Z sends e to X.*

In fact, this is a consequence of Proposition [2.11](#_bookmark9) and Theorem [3.1](#_bookmark11).

Remark 3.3 We denote, again, the colimit morphisms of Eq*Z* by

*e* : *X* −→ *RZ*

for all *e* : *X* −→ *HX* + *Z* in EQ*Z*. The appropriate isomorphism is denoted by

*iZ* : *RZ* −→ *HRZ* + *Z*

It is characterized by the fact that the two coproduct injections of *HRZ* + *Z*

are (in the notation of Definition [2.10](#_bookmark8))

inl = *iZρZ* and inr = *iZηZ*

In other words, *iZ* = [*ρZ, ηZ*]−1.

# An Alternative Definition of Iterativity

In the Introduction we considered non-flat systems ([1.1](#_bookmark1)) of recursive equations for Σ-algebras. And we argued that, due to the possibility of flattening such a system, we will just have to consider the flat equation morphism *e* : *X* −→ *H*Σ*X* + *A*. We are going to make that statement precise by showing that in iterative algebras (in general, not only in Set) much more general systems of recursive equations are uniquely solvable. This implies that, for polynomial endofunctors of Set, our definition of iterative algebras coincides with that presented by Evelyn Nelson [[16](#_bookmark42)]. And as we explain in the next section, it also implies that the rational monad is iterative in the sense of Calvin Elgot [[9](#_bookmark36)]. Let us first remark that the condition stated in the Introduction for ([1.1](#_bookmark1)), that no right-hand side be a single variable, is substantial: the equation *x* ≈ *x* has a unique solution only in the trivial terminal algebras. Systems satisfying

the above condition are called *guarded*.

We first consider guarded systems where the right-hand sides live in the free *H*-algebra (i. e., finite trees in case *H* = *H*Σ). Such systems are called *ﬁnitary*.

Since *H* is finitary we have for every object *X* in ffi a free algebra *ϕ*0 :

*X*

*HFX* −→ *FX* on *X* with universal arrow *η*0

*X*

monad F = (*F, η*0*, µ*0) where the component *µ*0

*X*

: *X* −→ *FX*. This defines a

is the unique homomorphism

0 : *FFX* −→ *FX* with *µ*0 0 = *id* . It is easy to see that *FX* is an initial

*µ*

*X*

· *η*

*X*

*FX*

algebra of *H*(−)+ *X*; thus, by Lambek’s Lemma [[14](#_bookmark41)] the morphism

*jX* = [*ϕ*0 *, η*0 ]: *HFX* + *X* −→ *FX*

*X X*

is an isomorphism. For every *H*-algebra *α* : *HA* −→ *A* we have the unique homomorphism

*α*^ : *FA* −→ *A* with *α*^ · *ηA* = *id*

(which, in case of *H*Σ, is the computation of (finite) terms over *A* in the Σ- algebra *A*). This allows us to define solutions of finitary equations morphisms in *A* as follows:

Definition 4.1

* 1. By a *ﬁnitary equation morphism* in an object *Y* (of parameters) is meant a morphism

*e* : *X* −→ *F* (*X* + *Y* )*, X* finitely presentable*.*

* 1. Given an *H*-algebra *α* : *HA* −→ *A* and a morphism *f* : *Y* −→ *A* (inter- preting the parameters in *A*), we say that the finitary equation morphism *e* has a *solution e*† : *X* −→ *A*, *induced by f* provided that the square

*f*

*e*†

*X*  *f*  *A*

,*,*

(4.1)

commutes.

*e*

J

*F* (*X* + *Y* )

*F* [*e*† *,f* ]

*f*

*α*b

*F A*

* 1. We call *e guarded* provided that it factors through the summand

*HF* (*X* + *Y* )+ *Y* of *F* (*X* + *Y* )= *HF* (*X* + *Y* )+ *X* + *Y* (see *jX*+*Y* above):

*e*  *F* (*X* + *Y* )

*X*

,*,*

[*ϕ*0*,η*0·inr ]

z*˛*

*HF* (*X* + *Y* )+ *Y*

Remark 4.2

1. The square ([4.1](#_bookmark22)) in Definition [4.1](#_bookmark21) means, for polynomial functors, that

the assignment *e*†

*f*

of variables *x* ∈ *X* to elements of *A* has the following

property: form the “substitution” mapping [*e*† *,f* ]: *X* + *Y* −→ *A* (which interprets the variables as *e*† does, and the parameters as *f* does). Extend it to the unique homomorphism

*f*

*f*

*α*^ · *F* [*e*† *,f* ]: *F* (*X* + *Y* ) −→ *A*

*f*

of the free algebra. Then the (formal) equations *x* ≈ *e*(*x*) become actual identities in *A* after the substitution *x* '−→ *e*† (*x*) for all *x* ∈ *X*.

*f*

1. The next result states that in an iterative algebra *A* every finitary guarded

equation morphism *e* : *X* −→ *F* (*X* + *Y* ) defines unique function *e*† :

(−)

ffi(*Y, A*) −→ ffi(*X, A*) such that ([4.1](#_bookmark22)) commutes for every *f* ∈ ffi(*Y, A*).

Theorem 4.3 *An H-algebra A is iterative if and only if every ﬁnitary guarded equation morphism has, for any interpretation of the parameters in A, a unique solution.*

Remark 4.4 The proof of Theorem [4.3](#_bookmark23) follows from the next result, gener- alizing “finitary” to “rational”. That is, let *α* : *HA* −→ *A* be an iterative algebra. We denote (analogously to *α*^ above) by

*α*˜ : *RA* −→ *A*

the unique homomorphism of *H*-algebras with *α*˜·*ηA* = *id* . We define a *rational equation morphism* on an object *Y* as a morphism

*e* : *X* −→ *R*(*X* + *Y* )*, X* finitely presentable*.*

Given a morphism *f* : *Y* −→ *Z*, the *solution* of *e induced by f* is a morphism

*e*† : *X* −→ *A* such that the square

*f*

*X*

*e*

J

*R*(*X* + *Y* )

*e*†

*f*  *A*

,*,*

*α*e

† *R A*

*R*[*ef ,f* ]

commutes. Finally, *e* is called *guarded* if it factors through the summand

*HR*(*X* + *Y* )+ *Y* of *R*(*X* + *Y* )= *HR*(*X* + *Y* )+ *X* + *Y* (see Remark [3.3](#_bookmark19)).

Theorem 4.5 *In an iterative algebra, for every guarded rational equation morphism e and every interpretation f of its parameters there exists a unique solution e*† *.*

*f*

Sketch of Proof. Let *α* : *HA* −→ *A* be an iterative algebra. Given a guarded rational equation morphism

*e*  *R* (*X* + *Y* )

*X*

,*,*

*e*0

[*ρX*+*Y ,ηX*+*Y* ·inr ]

z*˛*

*HR*(*X* + *Y* )+ *Y*

and a morphism *f* : *Y* −→ *A*, we will prove that *e* has a solution induced by *f* ; we leave out the proof of the uniqueness.

Recall from Corollary [3.2](#_bookmark18) that *R*(*X* + *Y* ) = colim Eq*X*+*Y* with colimit cocone *g* : *W* −→ *R*(*X* + *Y* ) for all *g* : *W* −→ *HW* + *X* + *Y* in EQ*X*+*Y* . Since this colimit is filtered and *H* is finitary, this implies that

*HR*(*X* + *Y* )+ *Y* = colim *H*Eq*X*+*Y* + *Y*

with the colimit cocone formed by all *Hg*+*Y* . Since *X* is a finitely presentable object, the morphism

*e*0 : *X* −→ colim *H*Eq*X*+*Y* + *Y*

factors through the colimit cocone:

*X ¸¸¸*

*e*0 *H R*(*X* + *Y* )+ *Y*

*¸¸¸¸¸*

*¸¸¸*

*w ¸¸¸¸*

,*,*

*Hg* +*Y*

*¸¸*z*˛*

*HW* + *Y*

for some object *g* : *W* −→ *HW* + *X* + *Y* of EQ*X*+*Y* and some morphism *w*.

We define a finitary flat equation morphism as follows:

(4.2)

⟨*e*⟩≡*W* +*X* [*g,*inm ] *HW* +*X*+*Y* [inl *,w,*inr ] *HW* +*Y H*inl +*f* *H*( *W* +*X*)+*A*

where inm : *X* −→ *HW* + *X* + *Y* is the middle coproduct injection. We obtain a unique solution ⟨*e*⟩† : *W* + *X* −→ *A* and prove that the following morphism

(4.3)

*e*† ≡ *X*

inr *W* + *X*

⟨*e*⟩†

is a solution of *e* induced by *f* .

*A*

Indeed, consider the following diagram:

†

*e*

*X,¸¸¸¸¸¸*

*f*

*,,,* ˛ *A*,*,,*

*,¸,¸¸*

*¸¸¸¸¸¸¸¸¸¸*inr

⟨*e*⟩†*,,,,,,,,*

*ccc*

*, ¸*

*, ¸¸*

*,,, ¸¸¸w*

*,*

*¸¸¸¸¸¸¸¸¸¸¸¸*

*,,,,,,,,,,*

*cc*

[*α,A*] *c*

*c*

*,, ¸¸¸*

*¸*z

*W* +*X*

*ccc*

*,,, ¸¸¸*

⟨*e*⟩

*ccc*

*e*0 *,,*

*¸¸*z

J † *cc*

*e ,,*

*,*

*H*inl +*f*

*H*⟨*e*⟩ +*A*

(4.4)

*HW* +*Y*

*,,**,*

*H*(*W* +*X*)+*A HA*+,*A, α*e

(i) *Hα*e+*A*

*, Hg* +*Y*

*,*

zJ*z*

*¸*

*f*

*¸¸*[*¸ρ,η*]

*HR*(*X*+*Y* )+*Y*

[*ρ,η*·inr *,*] *,,,,*

J *,* *,,,,,*

*HR*[*e*† *,f* ]+*f*

*HRA*+*A¸¸*

*¸¸¸¸*z

*R*(*X*+*Y* ) *s*

† *RA*

*R*[*ef ,f* ]

All of its parts, except, perhaps, for the square (i), commute. The right-hand component of (i) is obvious. To prove the commutativity of the left-hand component of (i), we remove *H* and show that the equation

(4.5)

⟨*e*⟩† · inl = *α*˜ · *R*[*e*† *,f* ] · *g*

holds. To this end observe first that *α*˜ · *R*[*e*† *,f* ] : *R*(*X* + *Z*) −→ *A* is an *H*-algebra homomorphism between iterative algebras extending [*e*† *,f* ]. Pre- composing this homomorphism with the colimit injection *g* : *W* −→ *R*(*X* +*Z*) yields the unique solution of an equation morphism *g* in the iterative algebra

*f*

*f*

*f*

*A*. Thus, to establish ([4.5](#_bookmark25)) it suffices to show that ⟨*e*⟩† · inl is a solution of *g*. In fact, the outward square of the following diagram

r~ *W*

inl *W* + *X*

⟨*e*⟩†

*A*,*,*

*g*

J

*HW* + *X* + *Y*

*¸*

*¸¸¸*[*¸*in*¸*l *,w,*inr ]

*g ¸¸¸¸¸*z*˛*

⟨*e*⟩

[*α,A*]

*HW* +[*e*† *,f* ]

*f*

*HW* + *Y*

*,,,,,,*

,¸ J *,,,,H,W* +*f*

*¸¸¸¸¸H¸¸*inl +*f*

*¸¸¸¸¸*z*˛*J

*H W* +

*s,*  *H* (*W* + *X*)+ *A*

*H A* + *A*

*A H*inl +*A*

*H*⟨*e*⟩† +*A*

commutes.

Corollary 4.6 *Every rational guarded equation morphism e* : *X* −→ *R*(*X* +

*Y* ) *has a unique solution in the algebra RY , i. e., there exists a unique e*‡ :

*X* −→ *RY such that the square*

*X e*‡ *R Y*

,*,*

*e*

J

*R*(*X* + *Y* )

*R*[*e*‡ *,η*]

*µ*

*R RY*

*commutes.*

In fact, apply Theorem [4.5](#_bookmark24) to the iterative algebra *RY* and the morphism

*ηY* : *Y* −→ *RY* .

# Free Iterative Monads

Assumptions 5.1 Throughout this section *H* denotes a finitary endofunctor of a locally finitely presentable category ffi. We suppose, just for convenience, that coproduct injections in ffi are monomorphisms — this assumption can be avoided, see the full version [[5](#_bookmark32)], where we work with arbitrary finitary endofunctors *H* (and with idealized monads, generalizing the ideal monads below) in the last section.

We are going to prove that the rational monad R, introduced in Section [2](#_bookmark6), is iterative in the sense of C. Elgot, and that it can be characterized as a free iterative monad on *H*.

5.2. Iterative Monads. This is a concept that C. Elgot has introduced in [[9](#_bookmark36)] for the base category ffi = Set. He used the language of algebraic theories rather than monads, but we have proved in [[1](#_bookmark29)] that the following concepts are equivalent to those of Elgot.

Definition 5.3 By an *ideal monad* is understood a sixtuple

S = (*S, η, µ, S*'*, σ, µ*')

consisting of a monad (*S, η, µ*), a subfunctor *σ* : *S*' *‹*−→ *S*, and a natural transformation *µ*' : *S*'*S* −→ *S*' such that

1. *S* = *S*' + *Id* with coproduct injections *σ* and *η*

and

1. *µ*' is a restriction of *µ*, i. e., the square

*S*'*S*

*σS*

*µ*' '

*σ*

*S*

J J

*SS µ S*

commutes.

Examples 5.4

* 1. The rational monad R = (*R, η, µ*) on an endofunctor *H* is ideal. Here we consider the subfunctor

*ρ* : *HR ‹*−→ *R*

expressing the *H*-algebra structure *ρZ* : *HRZ* −→ *RZ* of each *RZ*, see Definition [2.10](#_bookmark8). The restriction of *µ* here is

*µ*' = *Hµ* : *HRR* −→ *HR .*

* 1. The free-algebra monad F of Section [4](#_bookmark20) is ideal (again consider *ϕ*0 :

*HF* −→ *F* ).

* 1. Classical algebraic theories (groups, lattices, etc.) are usually not ideal.

Definition 5.5 Let S = (*S, η, µ, S*'*, σ, µ*') be an ideal monad on ffi.

1. By a *ﬁnitary equation morphism* is meant a morphism

*e* : *X* −→ *S*(*X* + *Y* )

in ffi where *X* is a finitely presentable object (“of variables”) and *Y* is any object (“of parameters”).

1. By a *solution* of *e* is meant a morphism

*e*† : *X* −→ *SY*

for which the square

*X e*† *SY*

,*,*

commutes.

*e*

J

*S*(*X* + *Y* )

*S*[*e*†*,ηY* ]

*µY*

*SS Y*

1. The equation morphism *e* is called *guarded* if it factors through the sum- mand *S*'(*X* + *Y* )+ *Y* of *S*(*X* + *Y* )= *S*'(*X* + *Y* )+ *X* + *Y* :

*e*  *S*( *X* + *Y* )

*X*

,*,*

*S*'(*X*

[*σX*+*Y ,ηX*+*Y* inr ]

z*˛*

+ *Y* )+ *Y*

1. The ideal monad S is called *iterative* provided that every guarded finitary equation morphism has a unique solution.

Example 5.6 The rational monad of every finitary endofunctor is iterative, see Corollary [4.6](#_bookmark26).

Definition 5.7 An *ideal monad morphism* from an ideal monad (*S, η, µ, S*'*, σ, µ*') to another one (*T, ηT , µT ,T* '*, τ, µ*'*T* ) is a monad mor- phism *λ* : (*S, η, µ*) −→ (*T, ηT , µT* ) which has a domain-codomain restriction to the ideals. That is, there is a natural transformation *λ*' : *S*' −→ *T* ' with *λ* · *σ* = *τ* · *λ*'.

Given a functor *H*, a natural transformation *λ* : *H* −→ *S* is called *ideal*

provided that it factors through *σ* : *S*' *‹*−→ *S*.

Example 5.8 For the rational monad R, the natural transformation

*κ* ≡ *H*  *Hη*  *H R*  *ρ*  *R*

is ideal.

Theorem 5.9 (Rational Monad as a Free Iterative Monad.) *For every iterative monad* S *and every ideal natural transformation λ* : *H* −→ *S there exists a unique ideal monad morphism λ* : R −→ S *with λ* = *λ* · *κ.*

Remark. Let us form the category Fin(ffi*,* ffi) of all finitary endofunctors and natural transformations. For the category

FIM(ffi)

of all finitary iterative monads (i.e., iterative monads (*S, η, µ, S*'*, σ, µ*') with *S*

and *S*' finitary) and ideal monad morphisms we have a forgetful functor

*U* : FIM(ffi) −→ Fin(ffi*,* ffi)*,* S '−→ *S*'

The theorem states that *U* has a left adjoint, viz, the functor *H* '−→ R.

Sketch of Proof. (1) For every object *Z* consider *SZ* as an *H*-algebra

*HSZ*  *λSZ*  *SS Z*  *µZ*  *SZ .*

It is iterative. In fact, every equation morphism *e* : *X* −→ *HX* + *SZ*, *X* in

ffi*fp*, yields the following guarded equation morphism w.r.t. S:

*e* ≡ *X*  *e*  *H X* + *SZ*  *λX* +*SZ*  *SX* + *SZ*  can *S*( *X* + *Z*) *,*

and it is not difficult to prove that a morphism *e*† : *X* −→ *SZ* is a solution of *e* in the *H*-algebra *SZ* if and only if it is a solution of *e* w.r.t. the iterative monad S.

1. Denote by *λZ* : *RZ* −→ *SZ* the unique homomorphism of *H*-algebras with

*λZ* · *ηZ* = *ηS.* Then *λ* : *R* −→ *S* is a monad morphism with *λ* = *λ* · *κ*. And *λ*

*Z*

is ideal:

*HRZ*  *ρZ*  *R Z*

*HλZ*

J

*HSZ¸¸¸¸*

*¸¸*

'

*λ*

*SZ*

J

'

*S SZ σ*

*λSZ*

*¸¸¸*z *λZ*

*SS Z¸*

*¸¸*

*SZ*

' *¸µZ*

*µ*

*¸¸*

*Z* J *¸¸¸*zJ

*S*'*Z σ*

*Z*

*SZ*

We see that *µ*'*S* · *λ*'*S* · *Hλ* : *HR* −→ *S*' is the desired restriction of *λ*.

1. *Uniqueness of λ.* Suppose that *λ* : R −→ S is an ideal monad morphism with *λ*·*κ* = *λ*. For any object *Z*, *λZ* is an *H*-algebra homomorphism extending *ηS*, thus the freeness of *RZ* as an iterative *H*-algebra establishes the desired uniqueness.

Remark 5.10 For polynomial endofunctors on Set, the freeness of R special- izes to *second order substitution*, see [[8](#_bookmark35)], i. e., substitution of rational trees for operation symbols.

*Z*

For example, consider a signature Σ with a binary operation symbol *b*, and a unary one *u*, and another signature Γ with two binary operation symbols + and ∗ and a constant symbol 1. The assignment

∗*,*

*sss ,,,* +*,*

(5.1)

*b*(*x, y*) '−→ 1*s*

+*, u*(*x*) '−→

*sss*

*,,,,*

*ssss*

*xs*

*,,,*

*y*

*xss x*

of operation symbols in Σ to rational trees over Γ gives rise to a natural transformation *λ* : *H*Σ −→ *R*Γ. The induced ideal monad morphism *λ* : RΣ −→ RΓ replaces, for any set of variables *X*, the operation symbols in trees of *R*Σ*X* according to *λ*. Example:

∗*,*

*sss*

*b,,*

*,*

*,*

*ss ,,*

*s*

*s*

*,*

1 +*,*

*λ*({ *h, k* }): *us*

*k* '−→

*ssss*

+*,,*

*,,,*

*k*

*h ssss ,,*

*h h*

The requirement that *λ* be an ideal transformation means that no operation

symbol of Σ is replaced by a single variable, i. e., that *λ* is a so-called *non- erasing* substitution.

# Conclusions and Future Work

Our paper shows that finitary endofunctors *H* generate free iterative mon- ads without any restriction on *H*. Our proof, simpler and clearer than any presented before, is based on the concept of an iterative algebra. The main technical result is a description of an initial iterative algebra as a colimit of all *H*-coalgebras carried by finitely presentable objects. From this result we derived that the algebraic theory formed by all free iterative *H*-algebras is iterative in the sense of Calvin Elgot. In fact, that theory can be charac- terized as a free iterative theory on *H*. For polynomial endofunctors of the category of sets this approach has already been taken by Evelyn Nelson [[16](#_bookmark42)], but our proof is independent of hers. It substantially clarifies and simplifies the original proof (which occupies most of the papers [[9](#_bookmark36),[7](#_bookmark34),[10](#_bookmark37)]) as well as the coalgebraic proof we have found previously [[3](#_bookmark30),[4](#_bookmark31)]. The freeness of the ratio- nal monad can be used to formulate clearly the “second-order substitution” described for rational Σ-trees by Bruno Courcelle [[8](#_bookmark35)], see Remark [5.10](#_bookmark27).

Our result can be applied to arbitrary base categories which are locally finitely presentable. For example, to the category of all finitary endofunctors of Set. In the future we intend to use this in an attempt to describe the monad of algebraic trees, see Courcelle [[8](#_bookmark35)], categorically.

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