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From Varieties of Algebras to Covarieties of Coalgebras

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**Abstract**

Varieties of *F* -algebras with respect to an endofunctor *F* on an arbitrary cocomplete category C are defined as equational classes admitting free algebras. They are shown to correspond precisely to the monadic categories over C. Under suitable assumptions satisfied in particular by any endofunctor on Set and Set*op* the Birkhoff Variety Theorem holds. By dualization, covarieties over complete categories C are introduced, which then correspond to the comonadic categories over C, and allow for a characterization in dual terms of the Birkhoff Variety Theorem. Moreover, the well known conditions of accessibilitly and boundedness for Set-functors *F* , sufficient for the existence of cofree *F* -coalgebras, are shown to be equivalent.

# Introduction

What is a variety?

A classical answer is: an equationally presented class of

finitary algebras (such as groups, lattices, etc). Less classical answer: an equa- tionally presented class of algebras with infinitary operations of possibly un- bounded arities (such as complete semilattices or compact Hausdorff spaces). The first, classical, case corresponds precisely to algebras of a finitary monad

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over Set. The latter one, then, to algebras of an arbitrary monad over Set. This, however, has nothingto do with Set as a base category: we are going to introduce equations for —and equational classes of—*F* -algebras, where *F* is an endofunctor on any cocomplete category C. Those equational classes that have free algebras are called *varieties*. They are proved to precisely corre- spond to monadic categories over C. Although this is a folklore fact, it seems that it has never been really formulated. Our formulation is based on the con- struction of free *F* -algebras as a colimit of “term-objects” (which, in general, diverges: we do *not* assume that free *F* -algebras exist) presented by the first author in [2]. Functors *F* for which free *F* -algebras exist are called *varietors* in [6]. For all varietors on Set (and all varietors on “reasonable” categories preserving regular epimorphisms) the Birkhoff Variety Theorem generalizes to the present context: varieties are precisely the full subcategories of Alg(*F* ) which are closed under products, subalgebras and quotients.

What is a covariety?

It is a simple but important observation that for

every endofunctor on a category C the category Coalg(*F* ) of all *F* -coalgebras is the dual of the category of *Fop*-algebras, where *Fop* is the endofunctor on C*op* acting as *F* . Consequently, by simple dualization we obtain the con- cept of coequation and coequational class of coalgebras. Those coequational classes that have cofree coalgebras (i.e., which are varieties of *Fop*-algebras) are called *covarieties*. And, for complete base categories, those are precisely the comonadic categories. If *F* is a *covarietor*, i.e., if all cofree coalgebras exist, and C = Set or C is “reasonable” and *F* preserves regular monomor- phisms, then covarieties are precisely the full subcategories of Coalg(*F* ) which are closed under coproducts, quotients and subcoalgebras.

Covarieties—for bounded endofunctors on Set only—have been considered by various authors. The equivalence of our approach, when specialized to this particular case, to most of these concepts will be shown below.

Which

Functors are

(Co-)Varietors?

Varietors on Set have been com-

pletely characterized in [6] by the existence of arbitrarily large fixed points. A full characterization of covarietors on Set is not known, but several sufficient conditions are known: M. Barr shows in [9] that every accessible functor is a covarietor, and Y. Kawahara and M. Mori [17] prove that every bounded

functor has a final coalgebra (from which it follows that every bounded func- tor is a covarietor). In the present note we prove that accessibility is, in fact, equivalent to boundedness, and both are equivalent to *F* being *small*, i.e., a small colimit of hom-functors. This seems to be the first time that the impli- cation *accessible* =*⇒ small* has been properly proved (but see e.g. P. Freyd [11], which contains this result implicitly).

# Algebras and coalgebras with respect to a functor

Let *F* : C *→* C be an endofunctor of some category C. Categories Alg(*F* ) and

Coalg(*F* ) are defined as follows.

Objects of Alg(*F* ), called *F-algebras (over* C*)*, are pairs (*C, αC*) where *C* is a C-object and *αC* : *FC → C* is a C-morphism. Morphisms *f* : (*C, αC*) *→* (*D, αD*) of Alg(*F* ), called *F* -*algebra homomorphisms*, are C -morphisms *f* : *C → D* makingthe diagram

*FC Ff*  *F D*

*αC αD*

JJ

commute.

*C f*  *D*

Objects of Coalg(*F* ), called *F-coalgebras (over* C*)*, are pairs (*C, αC*) where *C* is a C-object and *αC* : *C → FC* is a C-morphism. Morphisms *f* : (*C, αC*) *→* (*D, αD*) of Coalg(*F* ), called *F* -*coalgebra homomorphisms*, are C-morphisms *f* : *C → D* makingthe diagram

*C*  *f*  *D*

*αC αD*

J J

commute.

*FC Ff F D*

Composition and identities in Alg(*F* ) and Coalg(*F* ) respectively are those of C.

Alg(*F* ) and Coalg(*F* ) are concrete categories over C in that they are equipped with canonical underlyingfunctors

*F U* : Alg(*F* ) *→* C and *UF* : Coalg(*F* ) *→* C

respectively 5 .

The dual of a functor *F* : C *→* D is the functor *Fop* : C*op →* D*op* acting on objects and morphisms as *F* . With these notations one has

* 1. **Lemma** *For any functor F* : C *→* C *the following hold:*
     1. Coalg(*F* )= (Alg(*F op*))*op*
     2. *UF* = (*F op U* )

*op*

* 1. **Example** Let Ω be a signature in Birkhoff’s sense, i.e., Ω = (Ω*n*)*n∈*N is a countable family of sets Ω*n*. We shall describe the category AlgΩ of Ω-algebras—up to a concrete isomorphism— as (Alg(*F* )*, U* ) for a functor *F* = *F*Ω : Set *→* Set as follows:

*F*Ω assigns to a set *X* the sΣet Σ*n∈*N Ω*n × Xn*. CorresponΣdingly *F*Ω assigns

to a map *f* : *X → Y* the map *n∈*N Ω*n × fn*, i.e., the map *n∈*N Ω*n × Xn →*

Σ

*n∈*N Ω*n ×Y n* mappinga pair (*ω,* (*x*1*,..., xn*)) to the pair (*ω,* (*fx*1*,...,fxn*)).

5 Whenever confusion is unlikely to arise we will omit the subscript *F* .

Functors of the form *F*Ω are called *polynomial functors*.

We collect a number of well known properties of categories of *F* -algebras in the case C = Set as follows:

* 1. **Theorem** *For every endofunctor F of* Set *the following hold:*
     1. Alg(*F* ) *has all limits and these are created by U.*
     2. Alg(*F* ) *has all colimits. Those which are preserved by F are created by U.*
     3. Alg(*F* ) *has regular factorizations of homomorphisms; these are created by U.*
  2. **Remark** The above properties hold in more general situations than just over Set: as an inspection of the essentially standard proofs (see e.g. [3] or [6]) shows, the followingproperties of the category C = Set and the functor *F* respectively are needed only:
* C is complete, cocomplete, (regularly) co-wellpowered and has (regular epi, mono)-factorizations of homomorphisms.
* *F* preserves (regular) epimorphisms.

These conditions can be assumed to be satisfied in particular for every endofunctor on the category Set*op*, too. One only has to recall the following result of V.Trnkova´ (see [6, III.4.5-6]):

*Every endofunctor F on* Set *either preserves monomorphisms, or there is a monomorphism-preserving functor F ' which coincides with F on all non- empty sets and functions, and F '∅ /*= *∅ /*= *F∅.*

It follows that, concerningcategories Alg(*F* ) and Coalg(*F* ) over Set, one al- ways may assume that *F* preserves monomorphisms: Coalg(*F* ) and Coalg(*F'*) are (concretely) isomorphic, while Alg(*F* ) = Alg(*F '*) whenever *F* fails to pre- serve monomorphisms.

By means of Lemma 1.1 one thus gets by dualization the following properties of categories of *F* -coalgebras:

* 1. **Theorem** *Let F be an endofunctor of* Set*. Then the following hold:*
     1. Coalg(*F* ) *has all colimits and these are created by U.*
     2. Coalg(*F* ) *has all limits. Those which are preserved by F are created by U.*
     3. Coalg(*F* ) *has regular factorizations for homorphisms; these are created by U.*

# Free algebras and cofree coalgebras

* 1. **Example** For polynomial endofunctors *F*Ω on Set, the concept of free algebra *X* on a set *X* of generators is well known. We can describe it either recursively as *X* = *∪i<ωX* where

*i*

*X* = *X* + Ω0

0

*i*+1

*X*

= *X* + *F*Ω*∅* terms of depths 0 are variables and nullary operations

= *X* + *{*(*ω, t*0*,..., tn−*1) *| ω ∈* Ω*n, t*0*,..., tn−*1 *∈ X}*

*i*

= *X* + *F*Ω*X*

*i*

terms of depths i+1

Or directly: *X* is the algebra of all finite “properly” labeled trees. “Properly” means that a node with *n >* 0 children is labeled by an *n*-ary operation, and a leaf is labeled by a variable or a nullary operation. We have the universal arrow *ηX* : *X → X*, embedding *X* into *X*.

* 1. **Remark** Free *F* -algebras on *X* for an object *X* (of “variables”) of C can be defined for all functors *F* : C *→* C as pairs consistingof an *F* -algebra

*FX ϕX X*

*−−→*

and a morphism *ηX* : *X → X*

with the universal property that given an *F* -algebra (*C, αC*) and a morphism *f* : *X → C* of C there exists a unique *F* -homomorphism *f* extending *f* , i.e., such that the followingdiagram commutes.

*FX*  *ϕX*  *X* ¸*, X*

*ηX*

*Ff*

J *αC*

*FC*

*f*

*f*

J, *s*

*C*

In other words, *ηX* is a universal arrow of the forgetful functor *U* : Alg(*F* ) *→* C.

* 1. **Lemma** *Let F* : C *→* C *be a functor where* C *has finite coproducts and X a* C*-object. The following are equivalent for a morphism ιX* : *X* + *F IX → IX with components ηX* : *X → IX and αX* : *F IX → IX.*

1. (*IX, ιX*) *is the initial algebra of type X* + *F* (*−*)*.*
2. (*IX, αX*) *is the free F-algebra on X with universal morphism ηX.*
   1. **Corollary** *For a free F-algebra X one has X X* + *FX.*

This is Lambek’s Lemma (saying that initial *F* -algebras are fixed points of *F* ) applied to *FX* = *X* + *F* (*−*).

It is good to have a name for endofunctors *F* such that every object of C

admits a free *F* -algebra, that is, such that *U* has a left adjoint.

* 1. **Definition ([6])** An endofunctor *F* : C *→* C is called a *varietor* provided that a free *F* -algebra exists on every C-object.
  2. **Examples** 1. Polynomial endofunctors on Set are varietors.

2. If, more generally, C has colimits and finite products such that colimits of *ω*-chains commute with finite products, then every polynomial endo- functor *F*Ω on C is a varietor. In fact, it is easy to see that, for all Ω, *F*Ω preserves colimits of *ω*-chains. And then free algebras can be obtained by the following

* 1. **Finitary Free-Algebra Construction** (see [2]): This is an application of the famous construction of an *initial F-algebra* (the free *F* -algebra on 0, an initial object of C) as a colimit of the chain

0 *−→*!

*F* 0 *−F→*!

*F* 20 *F* 2!

*F* 30 *···*

to the functor *FX* = *X* + *F* (*−*) (see Lemma 2.3 above). Let C have countable colimits. Given an object *X* in C we define an *ω*-chain *X* (*i < ω*) as follows:

*−−→*

*i*

0 *−→*!

*X* + *F* 0 *−X−*+*−F→*!

*X*+*F* (*X*+*F* !)

*X* + *F* (*X* + *F* 0) *−−−−−−−→*

(*X* + *F* (*X* + *F* (*X* + *F* 0)) *···*

That is:

* *X* = 0, *X* = *X* + *F* 0= *X* + *FX*

and *x*

=0 *−→*!

*X* + *F* 0 is the unique

0 1

morphism

0 0*,*1



*X*

*i*+1

= *X* + *FX*

*i*+1*,j*+1

= *X* + *Fx*

for all *i ≤ j*

**Claim:** If *F* —and thus *X* + *F* —preserve a colimit *X* = colim*i<ωX*

*i*

and *x*

*i,j*

*i*

of the

above chain, then *X* is a free *F* -algebra on *X*. More detailed: suppose

(*X −x→i X*) is a colimit cocone. If *X* + *F* preserves that colimit we have a

*i*

unique morphism

*ϕX* : *X* + *FX → X* with *ϕX ◦* (*X* + *Fxi*)= *xi*+1

The two components *ηX* : *X → X* and *αX* : *FX → X* of *ϕX* form a free

*F* -algebra on *X*.

**Proof** For every *F* -algebra (*C, αC*) and any morphism (“assignment to vari- ables”) *f* : *X → C* define a cocone of the above chain ( *computation of terms*) recursively as follows:

*f* =! and *f* = [*f, αC ◦ Ff*]

0

Then the (unique) factorization *X*

*i*

*i*+1

*−x→i*

*i*

*f C* = *f*

*X*

*i*

*−→*

gives the (unique)

homomorphism *f* : (*X, αX*) *→* (*C, αC*) with *f* = *f ◦ ηX*. *✸*

* 1. **Examples** 1. For the endofunctor *FY* = 1 + *Y × Y* (i.e., one constant

and one binary operation) on Set, we know that the terms in *X*

*i*

are

just the binary trees of depths *≤ i* labelled in *X* + 1. This corresponds

precisely to the construction above.

2. For the endofunctor *FY* = *Y* N (i.e., one *ω*-ary operation) on Set we again might form the sets *X* of terms, but here the colimit after *ω* steps does not give a free *F* -algebra, of course: we need *ω*1 steps of the following

*i*

* 1. **Free-Algebra Construction** (see [2] or [6, IV.3.2]): Let C be a co- complete category. For every endofunctor *F* on C and every object *X* (“of

variables”) in C define a transfinite chain of objects *X*

*i*

(*i* any ordinal) and

connectingmorphisms

*x* : *X → X* (*i ≤ j*)

*i,j i j*

by the following transfinite induction:

* = 0*, X*

*X*

1

0

= *X* + *F* 0 with *x*

beingthe unique morphism 0

*−→*!

*X* + *F* 0

0*,*1



*X*

*i*+1

= *X* + *FX*

for all ordinals *i*, *x*

= *X* + *Fx*

for all *i ≤ j*

* *X* = colim*i<jX* for all limit ordinals *j* with colimit cocone *x*

*i*

*i*+1*,j*+1

*i,j*

*, i < j*.

*j i i,j*

**Claim:** If the above chain construction *stops after k steps*, i.e, if *k* is an

ordinal such that *x* : *X → X* + *FX* is an isomorphism, then *X* is a free

*k,k*+1 *k k k*

*F* -algebra on *X*. More detailed: Denoting the inverse of *xk,k*+1 by *ϕX* with

components

*αX* : *FX → X* and *ηX* : *X → X*

*k k k*

these form a free *F* -algebra on *X*.

**Proof** Given an *F* -algebra (*C, αC*) and a morphism *f* : *X → C* we define a cocone *f* : *X → C* (*i* an ordinal) by transfinite induction as above (leaving

*i* *i*

out the limit steps; compatibility *fj ◦x*

*i,j*

= *fi* (*i < j*) implies that the *fi* (*i < j*)

determine *fj* for limit ordinals *j*):

*f* =! and *f* = [*f, αC ◦ Ff*]

0 *i*+1 *i*

Now *f* : *X → C* is the unique homomorphism with *f* = *f ◦ ηX*. *✸*

*k k k*

* 1. **Definition ([6])** A functor *F* : C *→* C is called *constructive varietor*

provided that its Free-Algebra Construction 2.9 stops for each C-object *X*.

A functor can be a varietor, though the above chain-construction fails to stop for every *X* (see e.g. [6, IV.3.A]). However we have the following results:

* 1. *An endofunctor F on a cocomplete category which preserves colimits of λ-chains for some infinite cardinal λ is a constructive varietor.*

In fact, if *F* preserves *X* = colim*j<λX*, then the free algebra construction

*λ j*

stops after *λ* steps.

* 1. **Theorem ([6, 4.3], [4])** *Every varietor on each of the categories* Set*,* Set*op,* Vec*k* 6 *and* Vec*op is a constructive varietor.*

*k*

6 This is the category of vector spaces over some field *k*.

Callingan endofunctor *F* on Set *trivial* iff *F* is constant on nonempty sets one can prove (note that trivial endofunctors clearly are varietors):

* 1. **Theorem ([6])** *A non-trivial endofunctor F on* Set *is a (constructive) varietor iff F has arbitrarily large fixed points.*
  2. **Cofree coalgebras** are the correspondingdualization of free algebras. A cofree *F* -coalgebra (with respect to a functor *F* : C *→* C) on a C- object

*X* (“of colours”) is a coalgebra *ψX* : *X → F X* together with a (“colour- ing”) morphism *ρX* : *X → X* having the universal property that given an *F* -coalgebra (*C, αC*) and a morphism *f* : *C → X* of C there exists a unique *F* -coalgebra homomorphism *f* extending *f* , i.e., such that the diagram

*C*  *αC*  *F C*

*f*

,*s*

*f*

J *ψX*

*Ff*

J

*X* ¸*,ρ X*  *F X*

*X*

commutes. In other words, *ρX* is a couniversal arrow of the forgetful functor

*U* : Coalg(*F* ) *→* C.

* 1. **Definition** An endofunctor *F* : C *→* C is called a *covarietor* provided that a cofree *F* -coalgebra exists on every C-object.

This terminology is justified by the following remark based on 1.1 and 2.3.

* 1. **Remark** The followingare equivalent for any *F* : C *→* C:
* *F* is a covarietor.
* *Fop* is a varietor.

In case C has finite products, another equivalent condition is:

* For every object *X* in C the functor *FX* = *X × F* has a terminal (= final) coalgebra.

Dualization of the free-algebra construction above gives the following

* 1. **Cofree Coalgebra Construction:** Let C be a complete category. For every endofunctor *F* on C and every object *X* (“of colours”) in C define a transfinite cochain of objects *Xi* (*i* any ordinal) and connectingmorphisms *xi,j* : *Xi → Xj* (*i ≥ j*) as follows (where 1 denotes a terminal object of C):

* *X*0 = 1*, Xi* = *X × F* 1 with *x*1*,*0 : *X × F* 1 *−→*!

1 the unique morphism

* *Xi*+1 = *X × FXi* for all ordinals *i*, *xi*+1*,j*+1 = *X × Fxi,j* for all *i ≥ j*
* *Xj* = lim*i<jXi* for all limit ordinals *j* with limit cone *xj,i, i < j*.

If this cochain construction *stops after k steps*, i.e, if *k* is an ordinal such

that *xk,k*+1 : *X × FXk*

*→ Xk*

is an isomorphism, then *Xk*

is a cofree *F* -

coalgebra on *X*. More detailed: Denotingthe inverse of *xk,k*+1 by *ϕX* : *Xk →*

*X × FXk* with components

*αX* : *Xk → FXk* and *ρX* : *Xk → X*

these form a cofree *F* -coalgebra on *X*. For an *F* -coalgebra (*C, αC*) and a morphism *f* : *C → X* the extension *f* of *f* is the *k*-th member of the cocone

*fi* : *C → Xi*

which is defined by transfinite induction (leaving out the limit

steps) as follows:

*f* 0 =! and *fi*+1 = *⟨f, Ffi ◦ αC⟩.*

* 1. **Definition** A functor *F* : C *→* C is called *constructive covarietor* pro- vided that its Cofree-Coalgebra Construction 2.17 stops for each C-object *X*.

As the dual of Corollary 2.11 the followingholds:

* 1. **Corollary** *An endofunctor F on a complete category which preserves limits of λ-cochains for some infinite cardinal λ is a constructive covarietor.*
  2. **Examples** 1. Polynomial functors on Set are covarietors (here *λ* = *ω*).

1. *Generalized polynomial functors on* Set, i.e., functors *FY* = Σ *Y Ci*

*i∈I*

for a given family (*Ci*)*i∈I* of (not necessarily finite) sets are covarietors

(again, *λ* = *ω*).

1. Every endofunctor on Set which has arbitrarily large *exponential fixed points* (i.e., there are arbitrarily large sets *X* such that each set *Y* with card*X ≤* card*Y ≤* card exp*X* is a fixed point of *F* ) is a covarietor (see [4]). Compare with Theorem 2.13.

# Varieties and Covarieties

The followingdefinition generalizes concepts from [1]:

* 1. **Definitions** Let *F* be an endofunctor of a cocomplete category C. Using the notation *X* and *f* as in 2.9 we define:

*i* *i*

* + 1. An *equation arrow over X* is a regular epimorphism *e* : *X → E* for some ordinal *i*. An *F* -algebra (*C, αC*) is said to *satisfy e* provided that for every morphism *f* : *X → C* the morphism *f* factors through *e*:

*i*

*i*

*X*  *e*  *E*

*i* ,,,

,,,,

*f* ,,,

*i* ,vJ*z*

*C*

* + 1. For any class *E* of equation arrows, Alg(*F, E* ) denotes the full subcate- gory of Alg(*F* ) spanned by all *F* -algebras satisfying every *e ∈ E*. Such categories are called *equational categories (of F-algebras)* over C.
    2. An equational category Alg(*F, E* ) over C will be called a *variety (of F- algebras)* over C provided that the underlyingfunctor

*UE* = *U|*Alg(*F,E*) : Alg(*F, E* ) *→* C

has a left adjoint.

* 1. **Remarks** 1. Equations in classical (finitary) universal algebra are pairs of terms, i.e, parallel pairs of morphisms

*u, v* : 1 *→ X* = *X*

*ω*

An algebra (*C, αC*) satisfies this equation iff for every morphism *f* : *X →*

*C* the unique homomorphism *f* = *f*

*ω*

extending *f* merges *u* and *v*, i.e,

*f ◦ u* = *f ◦ v.*

This is equivalent to the satisfaction, in the above sense, of the equation arrow *e* : *X → E* which is a coequalizer of *u* and *v*.

In general, every pair of parallel morphisms (with C-objects *A, X* and an ordinal *i*)

*u, v* : *A → X*

*i*

in C defines an equation arrow *e* : *X*

*i*

*→ E*, a coequalizer of *u* and *v*.

An algebra (*C, αC*) “satisfies *u* = *v*” (in the expected sense: for every

morphism *f* : *X → C* we have *f ◦ u* = *f ◦ v*) iff (*C, αC*) satisfies *e* in the

*i* *i*

above sense.

1. If the base category C has kernel pairs, then, conversely, equation ar- rows can be substituted by parallel pairs: given a regular epimorphism *e* : *X → E*, denote by *u, v* : *A → X* a kernel pair of *e*. Then an algebra

*i* *i*

satisfies *u* = *v* iff it satisfies *e*.

Observe here that the index *i* can be upgraded arbitrarily: given a

*i,j*

parallel pair *u, v* : *A → X*

*i*

and an ordinal *j > i*, put *u'* = *x*

*u* and

*'*

*v* = *x*

*i,j*

*v*. Then the equations *u* = *v* and *u'* = *v'* are satisfied by the

same algebras.

1. Let C have kernel pairs and let *F* be a *constructive* varietor. It follows from 2. that all equations we have to consider are of the form

*u, v* : *A → X*

for objects *A, X* in C: in fact, upgrade any given parallel pair to an ordinal *j* such that *X* = *X*. Here, the satisfaction of *u* = *v* by (*C, αC*) means that for every *f* : *X → C* the unique homomorphism *f* : (*X, ϕX*) *→*

*j*

(*C, αC*) merges *u* and *v*. Since all homomorphisms *h* on (*X, ϕX*) have the form *h* = *f* (for *f* = *h ◦ ηX*), we see that (*C, αC*) satisfies *u* = *v* iff *hu* = *hv* for all homorphisms *h* : (*X, ϕX*) *→* (*C, αC*). Thus we can, equivalently, work with equation arrows

*e* : *X → E*

which are regular quotients, in C, of free *F* -algebras.

1. Suppose that *F* is a constructive varietor such that Alg(*F* ) has coequal- izers (e.g., whenever C has kernel pairs and regular factorizations of mor- phisms, is regularly cowellpowered and *F* preserves regular epimorphisms, see Remark 1.4). Then instead of regular epimorphisms *e* : *X → E* in C we can work with regular epimorphisms in Alg(*F* ). For that purpose recall the followingnotions:
   1. An object *I* in a category C is called *injective* w.r.t. a given morphism

*e* : *C → D* provided that each morphism *h* : *C → I* factorizes over *e*:

*h* : *C*¸ *e*  *D*

¸¸¸¸¸

*h* ¸¸¸¸¸

¸zJ

*I*

* 1. Given a class *E* of homomorphisms, we denote by Inj*E* the *injectivity class of E* , i.e., the full subcategory of Alg(*F* ) spanned by all algebras injective w.r.t. each *e ∈ E*.
  2. The injectives w.r.t. all regular monomorphisms are called the *regular injectives*.
  3. The dual notions are *(regular) projective* and *projectivity class* Proj*E* . Now observe that for every equation *u, v* : *A → X* we have a new equation *u, v* : *A → X* which is satisfied by precisely the same algebras (*C, αC*) (because, given a homomorphism *h* : (*X, ϕX*) *→* (*C, αC*), then (*hu*)= *hu* and (*hv*)= *hv*). Thus, if *e*¯: (*X, ϕX*) *→* (*E*¯*, αE*¯ ) denotes

a coequalizer of *u* and *v* in Alg(*F* ), then for every algebra (*C, αC*) we

have

(*C, αC*) satisfies *e ⇐⇒* (*C, αC*) is injective w.r.t. *e*¯*.*

1. Consider the base category C = Set. It then follows from 4. that, for any varietor *F* : Set *→* Set,

every variety of *F* -algebras is specified by injectivity to regular epimor- phisms of Alg(*F* ) with regularly projective domains 7 .

The converse is also true:

every class of *F* -algebras specified by injectivity to regular epimor- phisms of Alg(*F* ) with regularly projective domains is a variety.

7 Every free *F* -algebra over Set is regularly projective by the axiom of choice.

In fact, consider such an epimorphism, *e* : (*D, αD*) *→* (*E, αE*), in Alg(*F* ). Since Since the homomorphism *id* : (*D, ϕD*) *→* (*D, αD*) is a regular epimorphism in Alg(*F* ) and (*D, αD*) is regularly projective we have a homomorphism

*m* : (*D, αD*) *→* (*D, ϕD*) with *id ◦ m* = *id.*

Choose a pair of homomorphisms *u, v* with coequalizer *e* in Alg(*F* ). Then an algebra (*C, αC*) is orthogonal to *e* iff for every homomorphism *h* : (*D, αD*) *→* (*C, αC*) we have *h ◦ u* = *h ◦ v*. This is equivalent to stating that for every homomorphism *k* : (*D, ϕD*) *→* (*C, αC*) we have *k ◦* (*m ◦ u*) = *k ◦* (*m ◦ v*): given *k*, put *h* = *k ◦ m*, and given *h*, put

*k* = *h ◦ id*. Thus, if

*e*¯: (*D, ϕD*) *→* (*E*¯*, αE*¯ ) denotes a coequalizer of

*m ◦ u* and *m ◦ v* in Alg(*F* ), then injectivity to *e*¯ and *e*, respectively, is equivalent. (And the former can be substituted by the equations *u*0 = *v*0 obtained by the kernel pair of *e*¯.)

This concept of equation and its satisfaction has already been consid- ered by H. Herrlich and his co-authors in [16] and [8].

* 1. **Example** The power-set functor *P* on Set is not a varietor. However, we can consider equational categories of *P*-algebras. Complete semilattices are an example. In fact, the join-operation of a complete (upper) semilattice C is an arrow *αC* : *PC → C* satisfying (i) *αC{x}* = *x*, and (ii) *αC Mi* = *αC{αCMi | i ∈ I}* for any collection *Mi* in *PC*. Conversely, every *P*-algebra satisfying (i) and (ii) is a (join operation of a unique) complete semilattice. Now (i) can be expressed by the equation arrow *e* : *X → E* where *X* = *{x}* and *e* just merges *x* and *{x}*, whereas (ii) corresponds to the equation arrows *f* : *X → F* where *X* is an arbitrary set and, for a given collection *Mi* in *PX*,

2

3

*f* merges *Mi* with *{Mi | i ∈ I}*. The homomorphisms are precisely the

functions preservingall joins.

The followinglemma—to be proven by an easy transfinite induction—will be used frequently:

* 1. **Lemma** *Homomorphisms of F-algebras preserve computation of terms, i.e., given a homomorphism h* : (*C, αC*) *→* (*D, αD*) *and an assignment of vari- ables f* : *X → C then, for all ordinals i,* (*h ◦ f* )= *h ◦ f.*

*i* *i*

* 1. **Proposition** Alg(*F, E* ) *is always closed in* Alg(*F* ) *under*
     1. *subalgebras and all limits which exist;*
     2. *homomorphic images carried by split epimorphisms in* C*.*

**Proof** 1. is trivial by an obvious diagonal fill-in argument.

2. Let *r* : (*C, αC*) *→* (*D, αD*) be a homomorphism with coretraction *s* in C, where (*C, αC*) satisfies the equation arrow *e* : *X → E*. Given *f* : *X → D*, one

*i*

has *r ◦* (*s ◦ f* )= (*r ◦ s ◦ f* )= *f*. Thus, since (*s ◦ f* )factorizes through *e* so

*i i i* *i*

does *f*. *✸*

*i*

* 1. **Theorem** *Monadic categories over a cocomplete category* C *are precisely the categories concretely equivalent to varieties over* C*.*

**Proof** I. Sufficiency: By Beck’s Theorem we have to verify that the underlying functor of a variety Alg(*F, E* ) creates split coequalizers. Since Alg(*F, E* ) is closed under quotients splittingin C, it suffices to prove that *U* : Alg(*F* ) *→* C creates absolute coequalizers. This is proved exactly as in the proof of Becks’s Theorem (see [18]).

II. For the converse it suffices to show that, for any monad T = (*T, η, µ*) on a cocomplete category C, the Eilenberg-Moore category CT of T-algebras coincides with the subcategory Alg(*T, E* ) of Alg(*T* ) for a suitable class *E* of equation arrows. For doingso consider, for every C-object *X*, the coproduct

*mi*+1

*X*

*ni*+1

*−−−→ X* + *T Xi* = *Xi*+1 *←−− T Xi .*

A class *E*1 of equation arrows now is defined as follows: for every C-object

*X* let *eX* : *X → EX* be a coequalizer of the pair *m*2*, n*2 *◦η*

2

*◦m*1. A *T* -algebra

1

*X*

(*C, αC*) satisfies *eX* iff, for every morphism *f* : *X → C*, the morphism

*f* = [*f, αC ◦ Tf*]: *X* + *TX → C*

2 1 1

satisfies *f ◦ m*2 = *f ◦ n*2 *◦ η ◦ m*1 or, equivalently, *f* = *αC ◦ Tf ◦ η ◦ m*1.

2 2 *X*1 1 *X*1

Since *η* is natural, this is equivalent to *f* = *αC ◦ ηC ◦ f* which, for *X* = *C* and

*f* = 1*C* yields satisfaction of the T-algebra axiom *αC ◦ ηC* = 1*C*. Conversely, *αC ◦ ηC* = 1*C* yields *f* = *αC ◦ ηC ◦ f* by composition with *f* . Thus, the satisfaction of *E*1 is equivalent *αC ◦ ηC* = 1*C*.

Next we define a class *E*2 of equation arrows as follows: for every C-object

3

*X* let *dX* : *X* + *TX µX, n*3 *◦ Tn*2 *◦ T* 2*m*1.

2

= *X*

*→ DX* be a coequalizer of the pair *n*3 *◦ Tm*2 *◦*

*T* *X,*¸¸

*µX*

¸¸

*T* 2*X*¸¸

¸¸¸¸¸

¸

¸¸*T m*2

¸¸z

*TX*

*,*2

2

*X*

*n*3 *X* + *TX*

*dX* *D*

*T* 2*m*1

¸z *T n*2

*T* 2*X*

1

A *T* -algebra (*C, αC*) satisfies *dX* iff, for every morphism *f* : *X → C*, the morphism

*f* = [*f, αC ◦ T* [*f, αC ◦ Tf*]] : *X* + *TX → C*

3 1 2

satisfies *f ◦ n*3 *◦ Tn*2 *◦ T* 2*m*1 = *f ◦ n*3 *◦ Tm*2 *◦ µX*. This is equivalent to

3 3

*αC ◦ T αC ◦ T* 2*f* = *αC ◦ Tf ◦ µX* or, since *µ* is natural, to *αC ◦ T αC ◦ T* 2*f* =

*αC ◦ µC ◦ T* 2*f* . Choosing *f* = 1*C* this yields satisfaction of the T-algebra axiom *αC ◦T αC* = *αC ◦µC*. Conversely, *αC ◦T αC* = *αC ◦µC* yields *αC ◦T αC ◦ T* 2*f* = *αC ◦ µC ◦ T* 2*f* by composition with *T* 2*f* . Thus, the satisfaction of *E*2 is equivalent *αC ◦ T αC* = *αC ◦ µC*.

Chosing *E* = *E*1 *∪ E*2 one thusgets CT = Alg(*T, E* ). *✸*

* 1. **Theorem** *Let* C *be a cocomplete, regularly co-wellpowered category with regular factorizations and kernel pairs. If F* : C *→* C *is a constructive vari- etor preserving regular epimorphisms, the following are equivalent for any full subcategory* K *of* Alg(*F* )*:*

1. K *is a variety.*
2. K *is closed under subalgebras, products and homomorphic images carried by split epimorphisms.*

**Proof** In view of Proposition 3.5 we only have to show that (ii) implies (i). By Remark 1.4, Alg(*F* ) has regular factorizations. Thus (ii) implies that K is a reflective subcategory whose reflection-arrows are regular epimorphisms in Alg(*F* ), see [3, 16.8]. Let now *E* be the class of all reflection-arrows of free algebras. We claim K = Inj*E* . Trivially each algebra in K is injective w.r.t. all reflection arrows. Conversely, if (*C, αC*) is injective w.r.t. the reflection *r* of

the free algebra (*F, αF* ) over *C*, then the homomorphic extension *id*

*C*

of the

identity of *C* factors as *id*

*C*

= *g ◦ r*. This shows that *g* is—as a C-morphism—

a retraction. Thus, (*C, αC*) is a split-epi carried quotient of the K-reflection of (*F, αF* ), hence belonging to K by hypothesis. Thus, K = Inj*E* . From Remark

3.2.4 above we conclude that K is a variety. *✸*

* 1. **Corollary (Birkhoff Variety Theorem)** *For every varietor F on the category* Set*, varieties of F-algebras are precisely the full subcategories closed under products, subalgebras and homomorphic images.*
  2. **Remark** In the special case C = Set it suffices, in the proof of Theorem

3.7 above, to take as *E* the *set* of reflection arrows of free algebras on sets of cardinality less than *λ* (*λ* a regular cardinal), provided that *F* preserves *λ*-directed colimits (see e.g. proof of [5, 3.9]).

By formally dualizingDefinition 3.1, see Lemma 1.1, we obtain the follow-

ing

* 1. **Definitions** Let *F* be an endofunctor of a complete category C.
     1. An *coequation arrow over X* is a regular monomorphism *m* : *M → Xi* for some ordinal *i*. An *F* -coalgebra (*C, αC*) is said to *satisfy m* provided that for every morphism *f* : *C → X* the morphism *fi* factors through *m*.

* + 1. For any class *M* of coequation arrows Coalg(*F, M*) is the full subcategory of Coalg(*F* ) spanned by all *F* -coalgebras satisfying every *m ∈ M*. Such categories are called *coequational categories (of F-coalgebras)* over C.
    2. A coequational category Coalg(*F, M*) will be called a *covariety (of F- coalgebras)* over C provided that the underlyingfunctor

*UM* = *U|*Coalg(*F,M*) : Coalg(*F, M*) *→* C

has a right adjoint (that is, if Alg(*F op, M*) isa variety over C*op*).

By dualizing the respective results on equational categories and varieties we immediately obtain the followingresults.

* 1. **Corollary** *Comonadic categories over a complete category* C *are pre- cisely the categories concretely equivalent to covarieties over* C*.*
  2. **Corollary (Birkhoff Covariety Theorem)** *For every covarietor F on the category* Set*, covarieties of F-algebras are precisely the full subcategories closed under coproducts, subalgebras and homomorphic images* 8 *.*
  3. **Remarks** 1. Clearly, by duality, an equivalent condition for a full subcategory of Coalg(*F* ) (over Set) to be a covariety is to be a projec- tivity class w.r.t. of some class of regular monomorphisms M with cofree codomains.

2. Moreover, as in the case of varieties over Set (see Remark 3.9), also covari- eties of *F* -coalgebras over Set can be specified by projectivity w.r.t. a *set* of regular monomorphisms—in fact a single one—with cofree codomains, provided that the functor *F* preserves *λ*-directed colimits for some reg- ular cardinal *λ*. This follows easily from the boundedness property (see Theorem 4.1 below) of these functors (see [20,12]). Note, however, that this observation *cannot* be obtained by dualization of Remark 3.9.

While Theorem 3.11 shows that the dual of a covariety over Set is a variety over Set*op* it moreover implies the followingadditional dualization principle:

* 1. **Proposition** *The dual of a covariety over* Set *is equivalent to a variety over* Set*.*

**Proof** By means of the contravariant power-set functor *P'* the category Set*op* is monadic over Set. Let *V* : Coalg(*F, M*) *→* Set be the composite of (*UM*)*op* and *P'*. We need to show that *V* is monadic. Since *V* has a left adjoint and creates limits it suffices to prove that *V* creates coequalizers of congruence relations (= kernel pairs). Hence let *r, s* : (*C, αC*) *→* (*D, αD*) bea pair of Coalg(*F, M*)-morphisms sich that *V r, V s* is a congruence relation and

8 Observe that, in Coalg(*F* ), the homorphic images are given by (plain) epimorphims while the embeddings of subalgebras are the regular monomorphisms.

let *q* : *P'*(*D*) *→ X* be its coequalizer. Since *P'* reflects congruence relations and creates their coequalizers there is a unique Set*op*-morphism *q'* : *D → X'* with *P'*(*q'*)= *q* and this is a coequalizer of the congruence relation *UMr, UMs*. If *X' /*= *∅* this will even be a split coequalizer such that *UM* creates from it a coequalizer of *r, s*. The remaining case *X'* = *∅* is trivial: the unique *F* - coalgebra structure on *∅* obviously does the job. *✸*

* 1. **Remarks** 1. Coequations and their satisfaction have the following simple interpretation in the case of coalgebras over Set: define, for every “coterm” *x ∈ Xi*, the coequation [*x*] as the followingembedding

*Xi \ {x} ‹→ Xi.*

A coalgebra (*C, αC*) satisfies [*x*] iff *x* does not lie in the image of *fi* : *C → Xi* for any colouring *f* : *C → X*. These are all the coequations needed: we can substitute an arbitrary coequation

*m* : *M → Xi*

by the set of coequations *{*[*x*] *| x ∈ Xi \ m*[*M* ]*}*.

1. Various concepts of covariety of *F* -coalgebras—all restricted to the case of a bounded endofunctor on Set (thus, a varietor—see Section 4)—have already been discussed in the literature:
   * subcategories of Alg(*F* ) closed w.r.t. coproducts, subcoalgebras and homomorphic images ([20]).
   * projectivity classes in Alg(*F* ) w.r.t. collections of embeddings of sub- coalgebras of cofree coalgebras ([13]).
   * projectivity classes in Alg(*F* ) w.r.t. collections of embeddings of sub- coalgebras of regularly injective coalgebras ([14] 9 , [7]).

Theorem 3.12 (in connection with Remark 3.2) shows in particular that all of them are equivalent to the concept introduced here if specialized to bounded Set-functors.

1. A more complicated concept of coequation appears in [10]; this probably is not equivalent to the one above.

# Properties of Set-functors

Every endofunctor *F* of Set preservingcolimits of *λ*-chains (or, equivalently, *λ*- filtered colimits) for some regular cardinal *λ* is a varietor by 2.11. Such functors are called *accessible*, see [19]. An accessible functor is also a covarietor, as observed by M. Barr [9]. A different criterion is due to Y. Kawahara and M. Mori (see [17] or also [20] and [15]): recall that *F* is called *bounded* if there

9 Here, regular injectivity is called *extension property*

exists an infinite cardinal *λ* such that for every *F* -coalgebra (*C, αC*) and every element *x* of *C* there is a coalgebra homomorphism *h* : (*D, αD*) *→* (*C, αC*) with *x ∈ h*[*D*] and card*D ≤ λ*. We are going to prove that this, however, is equivalent to accessibility and both are equivalent to *F* being *small*, i.e., a small colimit of hom-functors:

* 1. **Theorem** *For an endofunctor F of* Set *the following conditions are equiv- alent:*

1. *F is small;*
2. *F is accessible;*
3. *F is bounded.*

**Proof** I. Suppose first that the given endofunctor *F* preserves finite intersec- tions (i.e., pullbacks of monomorphisms).

(iii) =*⇒* (i): For the above cardinal *λ* let D be the (essentially small) category of all pairs (*X, x*) where *X* is a set of cardinality *≤ λ* and *x ∈*

*' ' '*

*FX*, with morphisms *f* : (*X, x*) *→* (*X ,x* ) all functions *f* : *X → X* with

*'*

*Ff*(*x* ) = *x*. We prove that *F* is a colimit of the diagram *V* : D *→* [Set*,* Set]

where *V* (*X, x*)= hom(*X, −*) with the colimit cocone *f*(*X,x*) havingcomponents

*Y*

*f*

(*X,x*)

: hom(*X, Y* ) *→ FY, q −→ Fq*(*x*) for all *q* : *X → Y.*

That is, we prove that for every set *Y*

* 1. the maps *f Y*

(*X,x*)

are collectively epimorphic, and

* 1. whenever *f Y*

(*X,x*)

(*q*) = *f Y ' '*

(*q* ) then *q* is connected with *q* by a zig–zag

in the diagram of elements of *V* composed with the evaluation–at–*Y* ,

(*X ,x* )

*' '*

*evalY* : [Set*,* Set] *→* Set.

Proof of (a): Given *y ∈ FY* , for the coalgebra (*Y, const*(*y*)) there exists a homomorphism *h* : (*D, αD*) *→* (*Y, const*(*y*)) with card *D ≤ λ* which fulfills *D /*= *∅* if *Y /*= *∅*. For *Y /*= *∅* choose *d*0 *∈ D*, then (*D, d*) *∈*D with *d* = *αD*(*d*0)

*Y*

*f*

(*D,d*)

(*h*) = *F h*(*αD*(*d*)) = *const*(*y*) *· h*(*d*)= *y*.

The case *Y* = *∅* is trivial since (*Y, y*) *∈*D.

Proof of (b): We have *Fq*(*x*)= *Fq'* (*x'* ) for some *q* : *X → Y* and *q'* : *X' →*

*Y* . Factor *q* as an epimorphism *e* : *X → Z* followed by a monomorphism *m* : *Z → Y* and put *z* = *Fq*(*x*); analogously *e'*, *m'*, and *z'*. By assumtion, *F* preserves the pullback

*u* *P* ,,,*u'*

*j*

*Z* ,,

,

,,

,,v*z*

*Z'*

*m* v*z* *jm'*

*Y*

The equality *Fm*(*z*) = *Fq*(*x*) = *Fq'*(*x'*) = *Fm'*(*z'*) thus guarantees that there exists *p ∈ FP* with *z* = *F u*(*p*) and *z'* = *Fu'*(*p'*). And since card*P ≤* card(*Z × Z'*) *≤* card(*X × X'*) *≤ λ*2 = *λ*, we obtain an object (*P, p*) of D with morphisms

*←−*

*→*

*q*

(*X, x*) *←−*

(*Z, z*) *→u*

(*P, p*) *u'*

(*Z', z'*) *q*

(*X', x'*)

forming the desired zig-zag.

(i) =*⇒* (ii): Every hom–functor is accessible, and a small colimit of accessible functors is accessible, see [11].

(ii) =*⇒* (iii): Let F preserve *λ*–filtered colimits for some regular cardinal *λ*. Since every set is a *λ*–filtered colimit of all subsets of cardinality less than *λ*, we see that

(*∗*) given sets *C* and *T ⊆ FC* with card *T < λ* there exists a subset *m* : *B ‹→ C*

with card*B < λ* and *T ⊆ Fm*[*FB*].

We prove that *F* is bounded: given (*C, αC*) and *x ∈ C* define a *λ*–chain *mi* : *Bi ‹→ C* (*i < λ*) of subsets of cardinality less than *λ* by transfinite induction as follows:

* *B*0 = *{x}*;
* given *Bi*, apply (*∗*) to *T* = *αC*[*Bi*] to get *mi*+1 : *Bi*+1 *→ C* with *mi ⊆*

*mi*+1*, αC*[*Bi*] *⊆ Fmi*+1[*F Bi*+1], and card*Bi*+1 *< λ*;

* given a limit ordinal *i* define *Bi* = *Bj* – due to the regularity of *λ*, if

*j<i*

card*Bj < λ* for all *j < i*, then card*Bi < λ*.

Define *D* = *Bi* and *h* = colim *mi* : *D → A*, then since *F* preserves the

*i<λ*

colimit *D* = colim*Bi*, and since *αC*[*Bi*] is contained in the image of *Fmi*+1 for each *i < λi* we see that *αC*[*D*] is contained in the image of *F h*. Thus, we have *αD* : *D → FD* fΣor which *h* : (*D, αD*) *→* (*C, αC*) is a coalgebra homomorphism.

And card*D ≤ i<λ* card*Bi* = *λ*. Since *x ∈ B*0 *⊆ D*, this proves that *F* is

bounded.

II. For *F* : Set *→* Set arbitrary we use the result of V.Trnkova´ (see [6,

III.4.5-6]) that there exists a functor *F '* preservingfinite intersections and such

that *FX* = *F 'X* for all nonempty sets *X* (and *Fh* = *F 'h* for all nonempty functions h). It is easy to verify that *F* satisfies one of the properties (i)–(iii) iff so does *F '*. *✸*

* 1. **Example** of a covarietor which is not small. Given a class *M* of cardinal numbers, define *PM* : Set *→* Set on objects *X* by *PM X* = *{A ⊆ X*; *A* = *∅* or card*A ∈ M}* and an morphismus *f* : *X → Y* by *PM f* (*A*)= *f* [*A*] if *f/A* is injective, else = *∅*. Then *PM* is small iff *M* is small ( = bounded). But every infinite set *X* with card*X ∈/ M* is, obviously, a fixed point of *PM* . It is easy to find an unbounded class *M* for which *PM* has arbitrarily large exponential fixed points (i.e., *M* is unbounded but has arbitrarily large ”exponential holes”). Then *PM* is a varietor and covarietor but is not small.

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