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Generic Axiomatized Digital Surface-Structures

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**Abstract**

In digital topology, Euclidean *n*-space R*n* is usually modeled either by the set of points of a discrete grid, or by the set of *n*-cells in a convex cell complex whose union is R*n*. For commonly used grids and complexes in the cases *n* = 2 and 3, certain pairs of adjacency relations (*κ, λ*) on the grid points or *n*-cells (such as (4,8) and (8,4) on Z2) are known to be “good pairs”. For these pairs of relations (*κ, λ*), many results of digital topology concerning a set of grid points or *n*-cells and its complement (such as Rosenfeld’s digital Jordan curve theorem) have versions in which *κ*-adjacency is used to define connectedness on the set and *λ*-adjacency is used to define connectedness on its complement. At present, results of 2D and 3D digital topology are usually proved for one good pair of adjacency relations at a time

— so for each result there are different (but analogous) theorems for different good pairs of adjacency relations. In this paper we take the first steps in developing an alternative approach to digital topology based on very general axiomatic definitions of “well-behaved digital spaces”. This approach gives the possibility of stating and proving results of digital topology as single theorems which apply to all spaces of the appropriate dimensionality that satisfy our axioms. Specifically, this paper introduces the notion of a *generic axiomatized digital surface-structure* (gads) — a general, axiomatically defined, type of discrete structure that models subsets of the Euclidean plane and of other surfaces. Instances of this notion include gads corresponding to all of the good pairs of adjacency relations that have previously been used (by ourselves or others) in digital topology on planar grids and boundary surfaces. We define basic concepts for a gads (such as homotopy of paths and the intersection number of two paths), give a discrete definition of *planar* gads (which are gads that model subsets of the Euclidean plane) and present some fundamental results including a Jordan curve theorem for planar gads.

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# Introduction

In digital topology, Euclidean *n*-space R*n* is usually modeled either by the set of points of a discrete grid, or by the set of *n*-cells in a convex cell complex whose union is R*n*. Connectedness in Euclidean *n*-space is usually modeled by graph-theoretic notions of connectedness derived from adjacency relations defined on the grid points or *n*-cells.

For commonly used grids and complexes in the cases *n* = 2 and 3, certain pairs of adjacency relations (*κ, λ*) on the grid points or *n*-cells are known to be “good pairs”. For these pairs of relations (*κ, λ*), many results of digital topology concerning a set of grid points or *n*-cells and its complement have versions in which *κ*-adjacency is used to define connectedness on the set and *λ*-adjacency is used to define connectedness on its complement.

For example, (4*,* 8) and (8*,* 4) are good pairs of adjacency relations on Z2. Thus Rosenfeld’s digital Jordan curve theorem [10] is valid when one of 4- and 8-adjacency is used to define the sense in which a digital simple closed curve is connected and the other of the two adjacency relations is used to define connected components ofthe digital curve’s complement. The theorem is not valid ifthe same one of4- or 8-adjacency is used for both purposes: (4*,* 4) and (8*,* 8) are not good pairs on Z2.

Some adjacency relations form good pairs with themselves. An example of such a good pair is the pair (6*,* 6) on the grid points of a 2D hexagonal grid. (The grid points are the centers of the hexagons in a tiling of the Euclidean plane by regular hexagons, and two points are 6-adjacent ifthey are the centers of hexagons that share an edge.) Another example is the good pair (*κ*2*, κ*2) on Z2, where *κ*2 is Khalimsky’s adjacency relation [6] on Z2, which is defined as follows: Say that a point of Z2 is *pure* if its coordinates are both even or both odd, and *mixed* otherwise. Then two points of Z2 are *κ*2-adjacent ifthey are 4-adjacent, or ifthey are pure points and are 8-adjacent.

In three dimensions, (6*,* 26), (26*,* 6), (6*,* 18), (18*,* 6) are good pairs of adja- cency relations on Z3. A different example of a good pair on Z3 is (*κ*3*, κ*3), where *κ*3 is the 3D analog of *κ*2: Two points of Z3 are *κ*3-adjacent ifthey are 6- adjacent, or ifthey are 26-adjacent and at least one ofthe two is a pure point, where a *pure* point is a point whose coordinates are all odd or all even. (12*,* 12), (12*,* 18) and (18*,* 12) are good pairs ofadjacency relations 6 on the points ofa 3D face-centered cubic grid (e.g., on *{*(*x, y, z*) *∈* Z3 *| x* + *y* + *z ≡* 0(mod 2)*}*) and (14*,* 14) is a good pair on the points of a 3D body-centered cubic grid (e.g., on *{*(*x, y, z*) *∈* Z3 *| x ≡ y ≡ z*(mod 2)*}*).

At present, results of 2D and 3D digital topology are usually proved for one good pair of adjacency relations at a time, and the details of the proof

6 If *α* is an irreflexive symmetric binary relation on the set *G* of all points of a Cartesian or non-Cartesian grid, then *α* is referred to as the *k-adjacency* relation on *G*, and is denoted by the positive integer *k*, if for all *p ∈ G* the set *{q ∈ G | p α q}* contains just *k* points and they are all strictly closer to *p* (in Euclidean distance) than is any other point of *G \ {p}*.

may be significantly different for different good pairs. In the case of 3D grids, even ifwe consider only the nine good pairs ofadjacency relations mentioned above, a result such as a digital Jordan surface theorem would be expected to have nine different versions with nine separate proofs!

This state of affairs seems to us to be unsatisfactory. We have begun to consider an alternative approach to digital topology, in which “well-behaved” digital spaces are defined axiomatically, using axioms that are general enough to admit digital spaces which correspond to the good pairs of adjacency re- lations mentioned above. This approach allows a result of 2D or 3D digital topology to be proved as a *single* theorem for all well-behaved spaces that sat- isfy appropriate hypotheses. (Our Jordan curve theorem, Theorem 4.7 below, illustrates this.)

In this paper we confine our attention to digital spaces that model subsets ofthe Euclidean plane and other surfaces, and give an axiomatic definition of a very general class ofsuch spaces, which includes spaces corresponding to all of the good pairs of adjacency relations that have been used in the literature on 2D digital topology (both in the plane and on boundary surfaces). A space that satisfies our axiomatic definition is called a gads. As will be seen in Section 2.5, a substantial part of the mathematical framework used in our definition ofa gads has previously been used by the third author [4,5].

As first steps in the development of digital topology for these spaces, we define the intersection number of two paths on a gads, and outline a proof that the number is invariant under homotopic deformation of the two paths. This is mostly a generalization, to arbitrary gads, ofdefinitions and theorems given by the first author and Malgouyres in [1,2,3]. We also give a (discrete) definition of *planar* gads, which model subsets of the Euclidean plane, and present a Jordan curve theorem for such gads. In contrast to some earlier work by the second author (e.g., [7,8,9]), this paper does not use any arguments that are based on polyhedral continuous analogs ofdigital spaces, but uses only discrete arguments.

# GADS and pGADS

* 1. *Basic Concepts and Notations*

For any set *P* we denote by *P{*2*}* the set of all unordered pairs of distinct elements of *P* (equivalently, the set of all subsets of *P* with exactly two el- ements). Let *P* be any set and let *ρ ⊆ P{*2*}*. 7 Two elements *a* and *b* of *P* [respectively, two subsets *A* and *B* of *P* ] are said to be *ρ-adjacent* if *{a, b}∈ ρ* [respectively, if there exist *a ∈ A* and *b ∈ B* with *{a, b} ∈ ρ*]. If *x ∈ P* we denote by *Nρ*(*x*) the set of elements of *P* which are *ρ*-adjacent to *x*; these elements are also called the *ρ-neighbors* of *x*. We call *Nρ*(*x*) the *punctured*

7 *ρ* can be viewed as a binary, symmetric and irreflexive relation on *P* , and (*P, ρ*) as an undirected simple graph.

*ρ-neighborhood* of *x*.

A *ρ-path* from *a ∈ P* to *b ∈ P* is a finite sequence (*x*0*,..., xl*) of one or more elements of *P* such that *x*0 = *a*, *xl* = *b* and, for all *i ∈ {*0*,...,l −* 1*}*,

*{xi, xi*+1*} ∈ ρ*. The nonnegative integer *l* is the *length* of the path. A *ρ*-path of length 0 is called a *one-point path*. For all integers *m, n*, 0 *≤ m ≤ n ≤ l*, the subsequence (*xm,..., xn*) of (*x*0*,..., xl*) is called an *interval* or *segment* of the path. For all *i ∈ {*1*,..., l}* we say that the elements *xi−*1 and *xi* are *consecutive* on the path, and also that *xi−*1 *precedes xi* and *xi follows xi−*1 on the path. Note that consecutive elements ofa *ρ*-path can never be equal.

A *ρ*-path (*x*0*,..., xl*) is said to be *simple* if *xi /*= *xj* for all distinct *i* and *j* in *{*0*,..., l}*. It is said to be *closed* if *x*0 = *xl*, so that *x*0 follows *xl−*1. It is called a *ρ-cycle* ifit is closed and *xi /*= *xj* for all distinct *i* and *j* in *{*1*,..., l}*. One-point paths are the simplest *ρ*-cycles. Two *ρ*-cycles *c*1 = (*x*0*,..., xl*) and *c*2 = (*y*0*,..., yl*) are said to be *equivalent* if there exists an integer *k*, 0 *≤ k ≤ l −* 1, such that *xi* = *y*(*i*+*k*) mod *l* for all *i ∈ {*0*,..., l}*.

If *S ⊆ P* , two elements *a* and *b* of *S* are said to be *ρ-connected in S* ifthere exists a *ρ*-path from *a* to *b* that consists only ofpoints in *S*. *ρ*-connectedness in *S* is an equivalence relation on *S*; its equivalence classes are called the *ρ-components of S*. The set *S* is said to be *ρ-connected* if there is just one *ρ*-component of *S*.

Given two sequences *c*1 = (*x*0*,..., xm*) and *c*2 = (*y*0*,..., yn*) such that *xm* = *y*0, we denote by *c*1*.c*2 the sequence (*x*0*,..., xm, y*1*,..., yn*), which we call the *catenation of c*1 *and c*2. Whenever we use the notation *c*1*.c*2, we are also implicitly saying that the last element of *c*1 is the same as the first element of *c*2. It is clear that if *c*1 and *c*2 are *ρ*-paths oflengths *l*1 and *l*2, then *c*1*.c*2 is a *ρ*-path oflength *l*1 + *l*2.

For any sequence *c* = (*x*0*,..., xm*), the *reverse of c*, denoted by *c−*1, is the sequence (*y*0*,..., ym*) such that *yk* = *xm−k* for all *k ∈ {*0*,..., m}*. It is clear that if *c* is a *ρ*-path oflength *l* then so is *c−*1.

A *simple closed ρ-curve* is a nonempty finite *ρ*-connected set *C* such that each element of *C* has exactly two *ρ*-neighbors in *C*. (Note that a simple closed *ρ*-curve must have at least three elements.) A *ρ*-cycle *c* oflength *|C|* that con- tains every element ofa simple closed *ρ*-curve *C* is called a *ρ-parameterization of C*. Note that if *c* and *c'* are *ρ*-parameterizations ofa simple closed *ρ*-curve *C*, then *c'* is equivalent to *c* or to *c−*1.

If *x* and *y* are *ρ*-adjacent elements of a simple closed *ρ*-curve *C*, then we may say that *x* and *y* are *ρ-consecutive* on *C*. If *x* and *y* are distinct elements of a simple closed *ρ*-curve *C* that are not *ρ*-consecutive on *C*, then each of the two *ρ*-components of *C \{x, y}* is called a *ρ-cut-interval* (of *C*) associated with *x* and *y*.

* 1. *Definition of a* gads

**Definition 2.1 (2D digital complex)** *A* 2D digital complex *is an ordered triple* (*V, π, L*)*, where*

* + - *V is a set whose elements are called* vertices *or* spels*,*
    - *π ⊆ V {*2*}, and the pairs of vertices in π are called* proto-edges*,*
    - *L is a set of simple closed π-curves whose members are called* loops*, and the following four conditions hold:*
      1. *V is π-connected and contains more than one vertex.*
      2. *For any two distinct loops L*1 *and L*2*, L*1 *∩L*2 *is either empty, or consists of a single vertex, or is a proto-edge.*
      3. *No proto-edge is included in more than two loops.*
      4. *Each vertex belongs to only a finite number of proto-edges.*

When specifying a 2D digital complex whose vertex set is the set of points ofa grid in R*n*, a positive integer *k* (such as 4, 8 or 6) may be used to denote the set ofall unordered pairs of *k*-adjacent vertices. We write *L*2*×*2 to denote the set of all unit lattice squares in Z2. The triple (Z2*,* 4*, L*2*×*2) is a simple example ofa 2D digital complex.

**Definition 2.2 (GADS)** *A* generic axiomatized digital surface-structure*, or* gads*, is a pair* G = ((*V, π, L*)*,* (*κ, λ*)) *where* (*V, π, L*) *is a 2D digital complex (whose vertices, proto-edges and loops are also referred to as vertices, proto- edges and loops of* G*) and where κ and λ are subsets of V {*2*} that satisfy Axioms 1, 2 and 3 below. The pairs of vertices in κ and λ are called κ*-edges *and λ*-edges*, respectively.* (*V, π, L*) *is called the* underlying complex *of* G*.*

**Axiom 1** *Every proto-edge is both a κ-edge and a λ-edge: π ⊆ κ ∩ λ.* **Axiom 2** *For all e ∈* (*κ ∪ λ*) *\ π, some loop contains both vertices of e.* **Axiom 3** *If x, y ∈ L ∈ L, but x and y are not π-consecutive on L, then*

* + - * 1. *{x, y} is a λ-edge if and only if L \ {x, y} is not κ-connected.*
        2. *{x, y} is a κ-edge if and only if L \ {x, y} is not λ-connected.*

Regarding Axiom 2, note that if *e ∈* (*κ ∪ λ*) *\ π* (i.e., *e* is a *κ*- or *λ*-edge that is not a proto-edge) then there can only be one loop that contains both vertices of *e*, by condition (ii) in the definition ofa 2D digital complex.

As illustrations of Axiom 3, observe that both ((Z2*,* 4*, L*2*×*2)*,* (4*,* 8)) and ((Z2*,* 4*, L*2*×*2)*,* (8*,* 4)) satisfy Axiom 3, but ((Z2*,* 4*, L*2*×*2)*,* (4*,* 4)) violates the “if” parts of the axiom, while ((Z2*,* 4*, L*2*×*2)*,* (8*,* 8)) violates the “only if” parts of the axiom.

A gads is said to be *finite* if it has finitely many vertices; otherwise it is said to be *infinite*. The set ofall gads can be ordered as follows:

**Definition 2.3 (***⊆* **order, subGADS)** *Let* G = ((*V, π, L*)*,* (*κ, λ*)) *and* G*'* =

((*V ', π', L'*)*,* (*κ', λ'*)) *be* gads *such that*

* + - *V ⊆ V ', π ⊆ π' and L⊆ L'.*
    - *For all L ∈ L, κ ∩ L{*2*}* = *κ' ∩ L{*2*} and λ ∩ L{*2*}* = *λ' ∩ L{*2*}.*

*Then we write* G *⊆* G*' and say that* G *is a* subGADS *of* G*'. We also refer to* G *as* the subGADS of G*'* induced by (*V, π, L*)*. We write* G Ç G*' to mean* G *⊆* G*' and* G */*= G*'. We write* G *<* G*' to mean* G Ç G*' and L /*= *L'.*

The following simple but important property of gads is an immediate consequence ofthe symmetry ofAxioms 1, 2 and 3 with respect to *κ* and *λ*:

**Property 2.4** *If* ((*V, π, L*)*,* (*κ, λ*)) *is a* gads *then* ((*V, π, L*)*,* (*λ, κ*)) *is also a* gads*. So any statement which is true of every* gads ((*V, π, L*)*,* (*κ, λ*)) *remains true when κ is replaced by λ and λ by κ.*

* 1. *Interior Vertices and* pgads

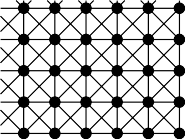
We are particularly interested in those gads that model a surface without boundary. The next definition gives a name for any such gads.

**Definition 2.5 (pGADS)** *A* pgads *is a* gads *in which every proto-edge is included in two loops. (The* p *in* pgads *stands for* pseudomanifold*.)*

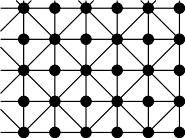
A finite pgads models a closed surface. A pgads that models the Eu- clidean plane must be infinite.

A vertex *v* ofa gads G is called an *interior vertex* of G ifevery proto-edge of G that contains *v* is included in two loops of G. It follows that a gads G is a pgads ifand only ifevery vertex of G is an interior vertex.

Below are pictures ofsome pgads.

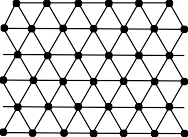
**Example 2.6** Z2 *with the 4- and 8-adjacency relations*

G = ((Z2*,* 4*, L*2*×*2)*,* (4*,* 8))

**Example 2.7** Z2 *with Khalimsky’s adjacency relation*

G = ((Z2*,* 4*, L*2*×*2)*,* (*κ*2*, κ*2)), where *κ*2 consists of all un- ordered pairs of 4-adjacent points and all unordered pairs of8-adjacent pure points.

**Example 2.8** *The hexagonal grid with the 6-adjacency relation*

G = ((*H,* 6*, L*)*,* (6*,* 6))

*H* = *{*(*i* + *j , j√*3 ) *∈* R2 *| i, j ∈* Z *}*

2 2

*L* = *{{p, q, r} ⊂ H |* dst(*p, q*) = dst(*q, r*) = dst(*p, r*)= 1*}*

dst(*x, y*) denotes the Euclidean distance between *x* and *y*.

**Example 2.9** *A torus-like* pgads

*a b c d*



*f*

*h*

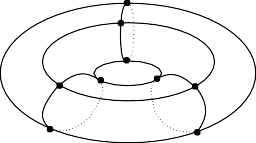
*g*

*i*

*e*

*a*

*b e c*



*d*

*f a*

*b*

*c g*

*h*

*i*

*a* G = ((*V, κ, L*)*,* (*κ, λ*))

*V* = *{a, b, c, d, e, f, g, h, i}*

*d κ* = *{{a, b}, {b, c}, {c, a}, {d, f}, {f, g}, {g, d},*

*e {e, h}, {h, i}, {i, e}, {b, f}, {c, g}, {a, d},*

*{f, h}, {g, i}, {d, e}, {h, b}, {i, c}, {e, a}}*

*a λ* = *{{x, y}| ∃L ∈ L, x, y ∈ L}* (not shown)

*L* = *{{a, b, f, d}, {d, f, h, e}, {e, h, b, a},*

*{b, c, g, f}, {f, g, i, h}, {h, i, c, b},*

*{c, a, d, g}, {g, d, e, i}, {i, e, a, c}}*

* 1. *Strong Connectedness and Singularities*

Let G = ((*V, π, L*)*,* (*κ, λ*)) be a gads. Two loops *L* and *L'* of G are said to be *adjacent* if *L ∩ L'* is a proto-edge of G. A subset *L'* of *L* is said to be *strongly connected* if for any two loops *L* and *L'* in *L'*, there exists a sequence *L*0*,..., Ln* ofloops in *L'* such that *L*0 = *L*, *Ln* = *L'* and, for all *i ∈ {*0*,...,n −* 1*}*, *Li* and *Li*+1 are adjacent. G is said to be *strongly connected* if *L* is strongly connected. (So whether or not G is strongly connected depends only on the underlying complex of G.)

A vertex *x* of G is said to be a *singularity* of G if the set of all loops of G that contain *x* is not strongly connected. Vertices that are not singularities are said to be *nonsingular*. Again, whether or not *x* is a singularity of G depends only on the underlying complex of G.

Even a strongly connected pgads may have a singularity. For example, the pgads obtained from the torus-like pgads ofExample 2.9 above by identifying the vertices *a*, *b* and *c* has a singularity at *a* = *b* = *c* but is strongly connected.

* 1. *Relationship to the Mathematical Framework of [4,5]*

Here we briefly discuss the relationship between our concept of a gads and digital structures previously studied by the third author in [4,5].

If ((*V, π, L*)*,* (*κ, λ*)) is a gads, then, in the terminology of [5], (*V, π*) is a *digital space*, *π* is the *proto-adjacency* of that space, and each of *κ* and *λ* is a *spel-adjacency* ofthe space. The principal new ingredients in our concept ofa gads are the set ofloops *L* and Axioms 2 and 3. In a gads ((*V, π, L*)*,* (*κ, λ*)) with the property that every simple closed *π*-path oflength 4 is a loop ofthe gads, the “if” parts of Axiom 3 make *{κ, λ}* a *normal* pair ofspel-adjacencies.

An important difference between our theory and that of [4,5] is that our theory is restricted to spaces that model subsets of surfaces (though only because ofcondition (iii) in the definition ofa 2D digital complex).

# Homotopic Paths and Simple Connectedness

In this section G = ((*V, π, L*)*,* (*κ, λ*)) is a gads, *ρ* satisfies *π ⊆ ρ ⊆ κ ∪ λ*, and *X* is a *ρ*-connected subset of *V* . (We are mainly interested in the cases where *ρ* = *κ, λ* or *π*.)

Loosely speaking, two *ρ*-paths in *X* with the same initial and the same final vertices are said to be *ρ-homotopic* within *X* in G ifone ofthe paths can be transformed into the other by a sequence of small local deformations within

*X*. The initial and final vertices of the path must remain fixed throughout the deformation process. The next two definitions make this notion precise.

**Definition 3.1 (elementary** G**-deformation)** *Two finite vertex sequences c and c' of* G *with the same initial and the same final vertices are said to be* the same up to an elementary G-deformation *if there exist vertex sequences c*1*, c*2*, γ and γ' such that c* = *c*1*.γ.c*2*, c'* = *c*1*.γ'.c*2*, and* either *there is a proto-edge*

*{x, y} for which one of γ and γ' is* (*x*) *and the other is* (*x, y, x*)*,* or *there is a loop of* G *that contains all of the vertices in γ and γ'.*

**Definition 3.2 (homotopic** *ρ***-paths)** *Two ρ-paths c and c' in X with the same initial and the same final vertices are ρ*-homotopic within *X* in G *if there exists a sequence of ρ-paths c*0*,..., cn in X such that c*0 = *c, cn* = *c' and, for* 0 *≤ i ≤ n −* 1*, ci and ci*+1 *are the same up to an elementary* G*-deformation. Two ρ-paths with the same initial and the same final vertices are said to be ρ*-homotopic in G *if they are ρ-homotopic within V in* G*.*

The next proposition states a useful characterization of *ρ*-homotopy that is based on a more restrictive kind of local deformation than was considered above, which allows only the insertion or removal of either a “*ρ*-back-and- forth” or a cycle that parameterizes a simple closed *ρ*-curve in a loop of G.

**Definition 3.3 (minimal** *ρ***-deformation)** *Two ρ-paths c and c' with the same initial and the same final vertices are said to be* the same up to a minimal *ρ*-deformation in G *if there exist ρ-paths c*1*, c*2 *and γ such that one of c and c' is c*1*.γ.c*2*, the other of c and c' is c*1*.c*2*, and* either *γ* = (*x, y, x*) *for some ρ-edge {x, y}* or *γ is a ρ-parameterization of a simple closed ρ-curve whose vertices are contained in a single loop of* G*.*

This concept of deformation is particularly simple when *ρ* = *π*, because a simple closed *π*-curve whose vertices are contained in a single loop of G must in fact be a loop of G, since a loop of G is a simple closed *π*-curve.

**Proposition 3.4** *Two ρ-paths c and c' in X with the same initial and the same final vertices are ρ-homotopic within X in* G *if and only if there is a sequence of ρ-paths c*0*,..., cn in X such that c*0 = *c, cn* = *c' and, for* 0 *≤ i ≤ n −* 1*, ci and ci*+1 *are the same up to a minimal ρ-deformation in* G*.*

The proof of this proposition is not particularly difficult, and we leave it to the interested reader.

**Definition 3.5 (reducible closed path)** *Let c* = (*x*0*,..., xn*) *be a closed*

*ρ-path in X (so xn* = *x*0*). Then c is said to be ρ*-reducible within *X* in G *if c and the one-point path* (*x*0) *are ρ-homotopic within X in* G*. We say c is ρ*-reducible in G *if c is ρ-reducible within V in* G*.*

**Definition 3.6 (simple connectedness)** *The set X is said to be ρ*-simply connected in G *if every closed ρ-path in X is ρ-reducible within X in* G*. The* gads G *is said to be* simply connected *if V is π-simply connected in* G*.*

Whether or not a gads is simply connected depends only on its underlying complex. If G is simply connected then *V* is *ρ*-simply connected in G for any *ρ* such that *π ⊆ ρ ⊆ κ ∪ λ*. This is because *π ⊆ ρ ⊆ κ ∪ λ* implies that for any *ρ*-path *c* there is a *π*-path *c'* such that *c* and *c'* are *ρ*-homotopic in G, and *π ⊆ ρ* implies that a *π*-reducible *π*-path is also a *ρ*-reducible *ρ*-path.

The final result in this section gives a useful sufficient condition for a gads

to have no singularities:

**Proposition 3.7** *Let* G *be a* gads *that is both simply connected and strongly connected. Then* G *has no singularities.*

**Proof:** Let G = ((*V, π, L*)*,* (*κ, λ*)) and suppose *x* is a singularity of G. Then there exist two nonempty sets of loops of G, *α*1 = *{L*1*,..., Li}* and *α*2 =

*{Li*+1*,..., Ll}*, such that *{L*1*,..., Ll}* is the set of all loops of G that contain

*x*, and such that *L ∩ L'* = *{x}* for all *L* in *α*1 and *L'* in *α*2.

For any *π*-path *c* = (*c*0*, c*1*,..., cn*), let *ν*(*c, x*) be the number of pairs (*ci, ci*+1) for which *ci* belongs to a loop in *α*1 and *ci*+1 = *x*, minus the number of pairs (*ci, ci*+1) for which *ci* = *x* and *ci*+1 belongs to a loop in *α*1. It is easy to verify that if *c'* and *c''* are two *π*-paths which are the same up to a minimal *π*-deformation in G then *ν*(*c', x*) = *ν*(*c'', x*). So, since G is simply connected, *ν*(*c, x*) = 0 for every closed *π*-path *c* (by Proposition 3.4).

Now let *y* be a *π*-neighbor of *x* that belongs to a loop in *α*1, and let *z* be a *π*-neighbor of *x* that belongs to a loop in *α*2. Since G is strongly connected, there must be a *π*-path *c'* from *z* to *y* that does not contain *x*. But the closed *π*-path *c* = (*x, z*)*.c'.*(*y, x*) would satisfy *ν*(*c, x*) = 1, a contradiction. *✷*

# Planar GADS and a Jordan Curve Theorem

In this section we define a class of gads that are discrete models ofsubsets of the Euclidean plane. The definition depends on two concepts which we now present:

**Definition 4.1 (Euler number of a GADS)** *Let* G = ((*V, π, L*)*,* (*κ, λ*)) *be a* finite gads*. Then the integer |V |− |π|* + *|L| is called the* Euler number *of* G*, and is denoted by χ*(G)*.*

Note that the Euler number of a gads depends only on the underlying complex, and that it is not defined for an infinite gads.

**Definition 4.2 (limit of an increasing GADS sequence)** *For all i ∈* N

*let* G*i* = ((*Vi, πi, Li*)*,* (*κ* *i, λi*)) *b* *e a* gad s *and let* G0 *⊆* G1 *⊆* G2 *⊆ .. ..*

*Then*

*i∈*N

G*i denotes* ((

*i∈* N

*Vi,*

*i∈*N

*πi,*

*i∈*N

*Li*)*,* (

*i∈*N

*κi,*

*i∈*N

*λi*))*, which is*

*a* gads *if each element of*

*i∈*N *Vi is contained in only finitely many distinct*

*members of*

*i∈*N *πi.*

We are now in a position to define a planar gads. Whether or not a gads is planar depends only on its underlying complex, as can be deduced quite easily from the following definition.

**Definition 4.3 (planar GADS)** *A* pgads PG *is said to be* planar *if* PG =

*i∈*N

G*i for some infinite sequence of finite* gads G0 *<* G1 *<* G2 *< . . . such*

*that* G*i is strongly connected and χ*(G*i*)=1 *for all i ∈* N*. A* gads G *is said to*

*be* planar *if there exists a planar* pgads PG *such that* G *⊆* PG*.*

It is evident that all planar pgads are infinite and strongly connected. A somewhat less obvious property of planar pgads is that they are all simply connected. This follows quite easily from:

**Proposition 4.4** *Let* G *be a strongly connected* gads *and let* G*' be a finite*

gads *such that* G*' <* G *and χ*(G*'*)= 1*. Then* G*' is simply connected.*

**Sketch of proof:** Let G*'* = ((*V ', π', L'*)*,* (*κ', λ'*)). In the case where *L'* = *∅*,

*|V '|− |π'|* = *χ*(G*'*) = 1 and so (*V ', π'*) is a tree. In this case the result is easily proved by induction on the number ofproto-edges. To prove the result in the case where G*'* has at least one loop, we use induction on the number ofloops. [The induction step is based on the easily established fact that, since G*' <* G and G is strongly connected, there must exist a proto-edge *e* of G*'* that belongs to just one loop of G*'*, *L* say. Readily, G*'* is simply connected ifthe subGADS of G induced by (*V ', π' \ {e}, L' \ {L}*) is simply connected.] *✷*

**Corollary 4.5** *A planar* pgads *is simply connected.*

**Proof:** Let PG = ((*V ∗, π∗, L∗*)*,* (*κ∗, λ∗*)) be a planar pgads, and suppose *c∗* is a *π∗*-path that is not *π∗*-reducible in PG. By the definition ofa planar pgads, there exists a gads G*'* = ((*V ', π', L'*)*,* (*κ', λ'*)) which satisfies the hypotheses of the above proposition when G = PG, such that *c∗* is a *π'*-path of G*'*. Since *c∗* is not *π∗*-reducible in PG, *c∗* is not *π'*-reducible in G*'*, which contradicts the proposition. *✷*

As a consequence ofthis corollary and Proposition 3.7, we deduce:

**Proposition 4.6** *A planar* pgads *has no singularities.*

The next theorem is our main result concerning planar gads. It general- izes Rosenfeld’s digital Jordan curve theorem [10] (for Z2 with (4,8) or (8,4) adjacencies) to every planar gads. We will outline a proofofthis theorem in Section 8.

**Theorem 4.7 (Jordan curve theorem)** *Let* PG = ((*V, π, L*)*,* (*κ, λ*)) *be a planar* gads*. Let C be a simple closed κ-curve that is not included in any loop of* PG*, and which consists entirely of interior points of* PG*. Then V \ C*

*has exactly two λ-components, and, for each vertex x ∈ C, Nλ*(*x*) *intersects both λ-components of V \ C.*

# Local Orientations and Orientability

* 1. *Definitions*

Let *L*1 and *L*2 be adjacent loops of a gads G = ((*V, π, L*)*,* (*κ, λ*)) and let

*{x, y}* = *L*1 *∩ L*2. Then *π*-parameterizations *c*1 of *L*1 and *c*2 of *L*2 are said to be *coherent* if *x* precedes *y* in one of *c*1 and *c*2 but *x* follows *y* in the other of *c*1 and *c*2. A *coherent π-orientation* of a set of loops *L' ⊆ L* is a function Ω with domain *L'* such that:

1. For each loop *L* in *L'*, Ω(*L*) isa *π*-parameterization of *L*.
2. For all pairs of adjacent loops *L* and *L'* in *L'*, the *π*-parameterizations Ω(*L*) and Ω(*L'*) of *L* and *L'* are coherent.

Two coherent *π*-orientations Ω1 and Ω2 of *L'* are said to be *equivalent* if, for every *L* in *L'*, Ω1(*L*) and Ω2(*L*) are equivalent *π*-parameterizations of *L*.

A *coherent orientation of* G is a coherent *π*-orientation of the set *L* of all loops of G. The gads G is said to be *orientable* ifit has a coherent orientation. Evidently, if G*'* and G are gads such that G*' ⊆* G and G is orientable, then G*'* is also orientable. Note that whether or not a gads is orientable depends only on its underlying complex. It is easy to verify that the four pgads shown in the diagrams ofSection 2.3 are all examples oforientable gads.

* 1. *The Cycle* 𝒩Ω*,y*(*x*) *Around a Nonsingular Interior Vertex x of a* gads

Let G = ((*V, π, L*)*,* (*κ, λ*)) be a (not necessarily orientable) gads, and let *x* be a nonsingular interior vertex of G.

A *loop-circuit* of G is a sequence (*L*0*,..., Ll−*1) ofloops of G such that, for all *i ∈ {*0*,...,l −* 1*}*, *Li* is adjacent to *L*(*i*+1) mod *l*. A *loop-circuit of* G *at x* is a loop-circuit of G that is an enumeration of the set of loops of G that contain *x* (with each of those loops occurring just once). Thus if (*L*0*,..., Ll−*1) is a loop-circuit of G at *x* then, for each *i ∈ {*0*,...l −* 1*}*, *Li ∩ L*(*i*+1) mod *l* is a proto-edge of G that contains *x* (by condition (ii) in the definition of a 2D digital complex).

The set of loops of G that contain *x* is strongly connected (since *x* is nonsingular in G), and it is easy to show that each loop in the set is adjacent to exactly two others (since *x* is an interior vertex of G). Therefore a loop- circuit of G at *x* exists.

A *coherent local orientation of* G *at x* is a coherent *π*-orientation of the loops of G that contain *x*. Let Λ = (*L*0*,..., Ll−*1) be a loop-circuit of G at *x*. Then the *coherent local orientation of* G *at x induced by* Λ, denoted by ΩΛ, is defined as follows. For 0 *≤ i ≤ l −* 1 let *ci* be a *π*-parameterization of *Li* that begins and ends at *x*, in which the second vertex is the vertex of *Li ∩*

*x*

*L*(*i−*1) mod *l \{x}*, and the second-last vertex is the vertex of *Li∩L*(*i*+1) mod *l \{x}*. Then ΩΛ is defined by ΩΛ(*Li*)= *ci* for 0 *≤ i ≤ l −* 1.

*x* *x*

Now let Ω*'x* be any coherent local orientation of G at *x*. Then Ω*'x*(*L*0) is

equivalent either to ΩΛ(*L*0) or to (ΩΛ(*L*0))*−*1. It is readily confirmed that Ω*'*

*x x* *x*

must be equivalent to ΩΛ in the former case and to ΩΛ*−*1 in the latter case.

*x* *x*

For any vertex *v* of G, the *punctured loop neighborhood* of *v* in G, denoted by *NL*(*v*), is defined to be the union of all the loops of G which contain *v*, minus the vertex *v* itself.

Let Ω*x* be a coherent local orientation of G at *x*. For each vertex *y* of

*NL*(*x*), we now define a *π*-cycle 𝒩Ω*x,y* (*x*) with the following properties:

1. The vertices of 𝒩Ω*x,y* (*x*) are exactly the vertices of *NL*(*x*).
2. 𝒩Ω*x,y* (*x*) begins and ends at *y*.

Let Λ = (*L*0*,..., Ll−*1) be a loop-circuit of G at *x* such that ΩΛ is equivalent to Ω*x*. For *i ∈ {*0*,...,l −* 1*}* let *pi* be the *π*-path obtained from ΩΛ(*Li*) by removing its first and last vertices (both ofwhich are the vertex *x*). Then we define 𝒩Ω*x,y* (*x*) to be the *π*-cycle that is equivalent to the *π*-cycle *p*0*.p*1 *pl−*1

*x*

*x*

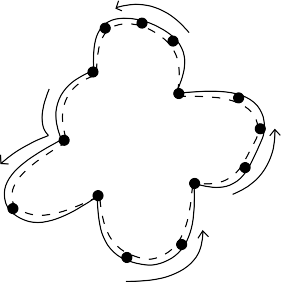
and which begins and ends at *y*. It is readily confirmed that this *π*-cycle is

independent ofour choice ofthe loop-circuit Λ (provided that ΩΛ is equivalent to Ω*x*). If G is orientable, and Ω is a coherent orientation of G, then we write

*x*

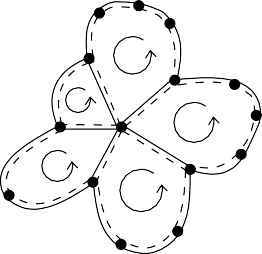
𝒩Ω*,y*(*x*)for 𝒩Ω*x,y* (*x*), where Ω*x* is the coherent local orientation of G at *x* that is given by the restriction ofΩ to the loops of G that contain *x*. The definition of 𝒩Ω*,y*(*x*) is illustrated by Figure 1.

* + *NL*(*x*) *∪ {x}*



(b)

*y*



*x*

(a)

*y*

 *π*

Ω*x*

Fig.1. (a) The set of loops which contain a vertex *x*, and a coherent local orientation Ω*x* of G at *x*. (b) The corresponding *π*-cycle 𝒩Ω*x,y* (*x*).

* 1. *Simply Connected* gads *are Orientable*

In this section we outline a proofofthe following result:

**Proposition 5.1** *Let* G *be a* gads *that is a subGADS of a simply connected*

gads*. Then* G *is orientable.*

**Sketch of proof:** Let G = ((*V, π, L*)*,* (*κ, λ*)) be a subGADS of the simply connected gads G*'* = ((*V ', π', L'*)*,* (*κ', λ'*)). Suppose G is not orientable. It is not hard to show that this implies G has a loop-circuit (*L*0*,..., Ll−*1) whose

set of loops is not *π*-orientable, such that no two *L*’s are equal and, for all

*i, j ∈ {*0*,...,l −* 1*}*, *Li* is not adjacent to *Lj* unless *j* = (*i ±* 1) mod *l*.

The idea now is to construct a *π'*-path in

*Li* that cannot be *π'*-

reducible in G*'*

0*≤i≤l−*1

, and so contradict the simple connectedness of

G*'*. For *i ∈*

*{*0*,...,l −* 1*}*, let *ai, bi ∈ V* be vertices such that *{ai, bi}* is the *π*-edge that is

shared by *Li* and *L*(*i*+1) mod *l*, and such that, for *i ∈ {*1*,...,l −* 1*}*, *ai−*1 and *ai* belong to the same *π*-component of *Li \{bi−*1*, bi}*. (It is possible that *ai−*1 = *ai* or *bi−*1 = *bi*.) Then it is straightforward to verify that, since *{L*0*,..., Ll−*1*}* is not *π*-orientable, *al−*1 and *b*0 must belong to the same *π*-component of *L*0*\{bl−*1*, a*0*}*. For *i ∈ {*1*,..., l−*1*}*, let *ci* be the simple *π*-path in *Li\{bi−*1*, bi}* from *ai−*1 to *ai*. Also, let *cl* be the simple *π*-path in *L*0 *\ {bl−*1*, a*0*}* from *al−*1 to *b*0. Let *γ* be the *π*-path *c*1*.c*2 *cl.*(*b*0*, a*0).

Define the *parity* ofa *π'*-path (*x*0*,.* *, xn*) to be 0 or 1 according to whether

an even or an odd number of terms in its sequence of *π'*-edges (*{xi, xi*+1*} |* 0 *≤ i ≤ n −* 1) lie in the set *{{ai, bi} |* 0 *≤ i ≤ l −* 1*}*. It is readily confirmed that *π'*-paths which are the same up to a minimal *π'*-deformation in G*'* have the same parity. But *γ* has parity 1 whereas a one-point path has parity 0. Hence *γ* is not *π'*-reducible in G*'*, a contradiction. *✷*

Since every planar pgads is simply connected (Corollary 4.5), a special case ofthe above proposition is:

**Corollary 5.2** *Every planar* gads *is orientable.*

# The Structure of Loops in a GADS

Let G = ((*V, π, L*)*,* (*κ, λ*)) be a gads and let *L* be an arbitrary loop of G. In this section we present some properties that *κ ∩ L{*2*}* and *λ ∩ L{*2*}* must have. These properties will be used in the next section.

**Theorem 6.1** *Let C be a simple closed* (*κ ∩ λ*)*-curve in the loop L. Then C*

*has one of the following properties:*

1. *For all distinct x, y ∈ C, {x, y}∈ κ.*
2. *For all distinct x, y ∈ C, {x, y}∈ λ.*

One way to prove this is to use the following lemma, whose proof we leave to the reader. Assertion (ii) ofthis lemma is illustrated by Figure 2.

**Lemma 6.2** *Let C be a simple closed* (*κ ∩ λ*)*-curve in the loop L. Then:*

1. *Assertions (a) and (b) of Axiom 3 hold with C in place of L whenever*

*x, y ∈ C but x and y are not* (*κ ∩ λ*)*-consecutive on C.*

1. *Let ρ = κ or λ and let a, b ∈ C be two vertices which are ρ-adjacent but not* (*κ∩λ*)*-consecutive on C. Let I*1 *and I*2 *be the two* (*κ∩λ*)*-cut-intervals of C associated with a and b. Then if x ∈ I*1 *and y ∈ I*2 *are ρ-adjacent, we have {x, a}∈ ρ, {x, b}∈ ρ, {y, a}∈ ρ and {y, b}∈ ρ.*

*a a*

*y y*



*x*



*x*

*⇒ ρ* = *κ* or *λ*

 *κ ∩ λ*

*b b*

Fig. 2. Illustration of Lemma 6.2(ii).

**Sketch of proof of Theorem 6.1:** Let *x* be a vertex on *C* that belongs to a (*λ \ κ*)- or (*κ \ λ*)-edge of *C{*2*}*. (If no such *x* exists then *|C|* = 3 by Lemma 6.2(i), and the theorem holds.) We first show that if *a* and *b* are vertices of *C* then it is impossible for *{x, a}∈ κ \ λ* and *{x, b}∈ λ \ κ* to both be true. This can be established by induction on the size of the (*κ ∩ λ*)-cut- interval of *C* associated with *a* and *b* that does not contain *x*, using Lemma 6.2. (We begin by verifying that *{x, a} ∈ κ \ λ* and *{x, b} ∈ λ \ κ* cannot both be true if *a* and *b* are (*κ ∩ λ*)-consecutive on *C*; otherwise we could deduce a contradiction of Lemma 6.2.) Next, we deduce from Lemma 6.2 that, if *y* is a (*κ ∩ λ*)-neighbor of *x* on *C* and *{x, a} ∈ κ \ λ* for some *a ∈ C*, then either *y* is also a vertex of a (*κ \ λ*)-edge in *C{*2*}*, or else there is a vertex *a'*, in the (*κ ∩ λ*)-cut-interval of *C* associated with *x* and *a* that contains *y*, such that

*{x, a'}∈ κ\ λ*. It follows from this (by induction on the size of the (*κ∩λ*)-cut- interval of *C* associated with *x* and *a* that contains *y*) that if *{x, a} ∈ κ \ λ* for some *a ∈ C*, then each (*κ ∩ λ*)-neighbor *y* of *x* on *C* is also a vertex of a (*κ \ λ*)-edge in *C{*2*}* — whence (by induction) every vertex of *C* is a vertex of a (*κ \ λ*)-edge in *C{*2*}*. This would imply that no vertex of *C* is a vertex of a (*λ \ κ*)-edge in *C{*2*}*, so that there are no (*λ \ κ*)-edges in *C{*2*}*, whence (by Lemma 6.2(i)) every pair of vertices of *C* are *κ*-adjacent. Symmetrically, if *x* is a vertex of a (*λ \ κ*)-edge in *C{*2*}* then all pairs of vertices of *C* are *λ*-adjacent. *✷*

Any loop of G can be “subdivided” into simple closed (*κ ∩ λ*)-curves, and by Axiom 3 two vertices of the loop cannot be *κ*- or *λ*-adjacent unless one of the simple closed (*κ ∩ λ*)-curves contains both vertices. (Figure 3 shows a loop that can be subdivided into three simple closed (*κ ∩ λ*)-curves.) So the following lemma is a straightforward consequence of Theorem 6.1:

**Lemma 6.3** *Let* (*ρ, ρ*) = (*κ, λ*) *or* (*λ, κ*)*, and let C be any simple closed ρ- curve included in the loop L such that |C| /*= 3*. Then C is a simple closed* (*κ ∩ λ*)*-curve. Moreover, {x, y}∈ ρ*˜ *for all x, y ∈ C.*

˜

The reader can verify that Theorem 6.1 and Lemma 6.3 hold in Figure 3.

The final result of this section implies that for any *ρ* satisfying *π ⊆ ρ ⊆ κ ∪ λ*, a *ρ*-parameterization of a simple closed *ρ*-curve whose vertices are contained in a loop *L* must be a subsequence ofa *π*-parameterization of *L* — loosely speaking, it must proceed in a single direction around *L*, and cannot

* *L ρ ρ*˜

(*κ∩λ*)= *ρ∩ ρ*˜

Fig. 3. Illustration of Theorem 6.1 and Lemma 6.3. This is a possible loop in a gads, if the eight (*κ ∩ λ*)-edges that belong to just one of the three simple closed (*κ ∩ λ*)-curves are proto-edges and the other two (*κ ∩ λ*)-edges are not.

reverse direction at some vertex.

**Lemma 6.4** *Let ρ satisfy π ⊆ ρ ⊆ κ ∪ λ. Let C be a simple closed ρ- curve whose vertices are contained in the loop L. Let x and y be two vertices of C which are ρ-consecutive in C but not π-consecutive in L. Then either C \ {x, y}⊆ I*1 *or C \ {x, y}⊆ I*2 *where I*1 *and I*2 *are the two π-cut-intervals of L associated with the vertices x and y.*

**Proof:** If *|C|* = 3 the result is immediate. If *|C| >* 3 then, by Lemma 6.3, *C* is a simple closed (*κ ∩ λ*)-curve and therefore *{x, y} ∈ κ ∩ λ*, so the result follows from Axiom 3. *✷*

# The Intersection Number

Let G = ((*V, π, L*)*,* (*κ, λ*)) be an orientable gads and let Ω be a coherent orientation of G. In this section we define an *intersection number* ofa (*κ ∪ λ*)- path *p* with a closed (*κ ∪ λ*)-path *c*, which we denote by *I*Ω . The intersection number is defined only ifevery common vertex ofthe two paths is a nonsingular interior vertex of G. Loosely speaking, it is the number of times the path *p* crosses from the right of the closed path *c* to its left, minus the number of times *p* crosses *c* from left to right.

*c,p*

Our intersection number is a generalization to gads of the intersection number between paths of surfels in digital boundaries that was defined and used in [1,2], except that we only define the intersection number when one of the two paths is closed. 8 Our main result about the intersection number (Theorem 7.7) is that in an orientable gads the intersection number of a *λ*- path with a closed *κ*-path is invariant under *λ*-homotopic deformations of the *λ*-path, assuming that all vertices ofthe closed *κ*-path are nonsingular interior vertices of G. As we shall see in the next section, this fact can be used to prove our Jordan curve theorem for planar gads (Theorem 4.7 above).

The definition ofthe intersection number is based on the idea that, for each three-vertex segment (*x*0*, x*1*, x*2) ofa (*κ ∪λ*)-path in which *x*1 is a nonsingular interior vertex of G, we can partition the set *NL*(*x*1) *\{x*0*, x*2*}* into a “left” side and a “right” side with respect to the segment (*x*0*, x*1*, x*2), using the *π*-cycle

8 It is quite easy to extend our definition to two paths that are not closed.

𝒩Ω*,x*0 (*x*1) defined in Section 5.2. The details of this are given in the next definition. Note that since *{x*0*, x*1*}, {x*1*, x*2*} ∈ κ ∪ λ*, Axiom 2 implies that *x*0*, x*2 *∈ NL*(*x*1), so that *x*2 lies on the *π*-cycle 𝒩Ω*,x*0 (*x*1).

**Definition 7.1** *Let* (*x*0*, x*1*, x*2) *be a segment of a* (*κ ∪ λ*)*-path, where x*1 *is a nonsingular interior vertex of* G*, and let* 𝒩Ω*,x*0 (*x*1) = (*v*0*,..., vn*)*, so that v*0 = *vn* = *x*0*. Let h ∈ {*0*,..., n} be the integer such that vh* = *x*2*. Then we define R*Ω(*x*0*, x*1*, x*2)= *{vi |* 0 *< i < h} and L*Ω(*x*0*, x*1*, x*2)= *{vi | h < i < n}.*

Let *c* = (*x*0*,..., xl*) be a closed (*κ ∪ λ*)-path. If *xi* is a nonsingular interior vertex of G, we write Right*c* (*i*) and Left*c* (*i*) for *R*Ω(*x*(*i−*1) mod *l, xi, x*(*i*+1) mod *l*)

Ω

Ω

and *L*Ω(*x*(*i−*1) mod *l, xi, x*(*i*+1) mod *l*), respectively. Now let *{y, z}* be a (*κ ∪ λ*)-

edge. If one of *y* and *z* is not an interior vertex of G or is a singularity of G,

and that vertex is also a vertex of *c*, then *Wc*

(*y,z*)

is undefined. Otherwise, we

define *Wc*

= Σ*l−*1 *Wc*

(*i*), where:

(*y,z*)

*i*=0 (*y,z*)

1. *Wc* (*i*)= *−*0*.*5 if *y* = *xi* and *z ∈* Right*c* (*i*), or if *z* = *xi* and *y ∈* Left*c* (*i*).

(*y,z*) Ω Ω

1. *Wc*

(*i*)= +0*.*5 if *y* = *xi* and *z ∈* Left*c* (*i*), or if *z* = *xi* and *y ∈* Right*c* (*i*).

(*y,z*)

1. *Wc*

(*y,z*)

Ω Ω

(*i*) = 0 otherwise.

**Definition 7.2 (intersection number)** *Let p* = (*y*0*,..., yh*) *be a* (*κ ∪ λ*)*- path, and c a closed* (*κ ∪ λ*)*-path, such that every common vertex of c and p*

*is a nonsingular interior vertex of* G*. Then the* intersection number of *p* with

*c, denoted by I*Ω *, is defined to be* Σ*h−*1 *Wc .*

*c,p*

*i*=0

(*yi,yi*+1)

The next two lemmas state fundamental properties of the intersection num- ber that follow without much difficulty from this definition.

**Lemma 7.3** *Let c be a closed* (*κ ∪ λ*)*-path, and let p', p*1 *and p*2 *be* (*κ ∪ λ*)*- paths such that p'* = *p*1*.p*2*. Suppose further that every common vertex of c and*

*p' is a nonsingular interior vertex of* G*. Then I*Ω *'* = *I*Ω + *I*Ω *.*

*c,p*

*c,p*1

*c,p*2

**Lemma 7.4** *If c*1 *and c*2 *are closed* (*κ ∪ λ*)*-paths and every common vertex*

*of c*1 *and c*2 *is a nonsingular interior vertex of* G*, then I*Ω

*c*1*,c*2

= *−I*

Ω

*c*2*,c*1 *.*

The next Lemma can be proved using Lemma 6.4: It is a consequence of Axiom 3 and the fact that a *λ*-parameterization of a simple closed *λ*-curve that lies in a loop of G must proceed in a single direction around that loop.

**Lemma 7.5** *Let C be a simple closed λ-curve whose vertices all lie in a single loop of* G*. Let c* = (*x*0*,..., xl*) *be a λ-parameterization of C. Then C has one of the following two properties:*

1. *For each i such that xi is a nonsingular interior vertex of* G*,*
   1. *Nκ*(*xi*) *\ C ⊆* Right*c* (*i*)*, and*

Ω

* 1. *Nκ*(*xi*) *∩ C \ {x*(*i−*1) mod *l, x*(*i*+1) mod *l}⊆* Left*c* (*i*)*.*

Ω

1. *For each i such that xi is a nonsingular interior vertex of* G*,*
   1. *Nκ*(*xi*) *\ C ⊆* Left*c* (*i*)*, and*

Ω

* 1. *Nκ*(*xi*) *∩ C \ {x*(*i−*1) mod *l, x*(*i*+1) mod *l}⊆* Right*c* (*i*)*.*

Ω

This lemma can be used to prove the following important result:

**Proposition 7.6** *Let c be a λ-parameterization of a simple closed λ-curve whose vertices all lie in a single loop of* G*, and let c' be a closed κ-path such that every common vertex of c and c' is a nonsingular interior vertex of* G*. Then I*Ω *'* = 0*.*

*c,c*

**Sketch of proof:** Let *c* = (*x*0*,..., xl*), let *C* be the simple closed *λ*-curve parameterized by *c*, and let *c'* = (*y*0*,..., yh*). (Thus *xl* = *x*0 and *yh* = *y*0.) For

all *j* such that *yj* and *yj*+1 both lie on *C*, *Wc*

(*yj,yj*+1)

= 0. (Indeed, *Wc*

*j j*+1

(*y ,y* )

(*i*) =

0 except possibly at the two values of *i* in 0*,...,l−* 1 for which *xi ∈ {yj, yj*+1*}*.

*c*

*W*

(*yj,yj*+1)

(*i*) = 0 for both of those values of *i* if *yj* and *yj*+1 are *λ*-consecutive

on *C*, and by Lemma 7.5 *Wc*

(*y ,y* )

*j j*+1

(*i*) is +0*.*5 for one value of *i* and *−*0*.*5 for

the other if *yj* and *yj*+1 are not *λ*-consecutive on *C*.) Lemma 7.5 also implies

*c*

that *W*

(*yj,yj*+1)

has one nonzero value (*±*0*.*5) for all *j* such that *yj ∈ C* and

*yj*+1 *∈/ C*, and has the opposite nonzero value for all *j* such that *yj ∈/ C* and

*yj*+1 *∈ C*. Since *c'* is a closed *λ*-curve, there are exactly as many values of *j* in 0*,...,h −* 1 for which *yj ∈ C* and *yj*+1 *∈/ C* as there are values of *j* for which

*yj ∈/ C* and *yj*+1 *∈ C*. Hence *I*Ω *'* = Σ*h−*1 *Wc* = 0. *✷*

*c,c*

*j*=0

(*yj,yj*+1)

Using this proposition and Proposition 3.4, we now prove:

**Theorem 7.7** *Let* G = ((*V, π, L*)*,* (*κ, λ*)) *be an orientable* gads*, and let* Ω *be a coherent orientation of* G*. Let c be a closed κ-path all of whose vertices are nonsingular interior vertices of* G*, and let p and q be two λ-paths which are*

= *I .*

*λ-homotopic in* G*. Then I*Ω

*c,p*

Ω

*c,q*

**Corollary 7.8** *Under the hypotheses of Theorem 7.7, I*Ω *'* =0 *for any closed*

*c,c*

*λ-path c' that is λ-reducible in* G*.*

**Proof of Theorem 7.7:** By Proposition 3.4, it is sufficient to prove Theo- rem 7.7 when *p* and *q* are the same up to a minimal *λ*-deformation in G. There are two cases. First suppose *p* = *p*1*.*(*x, y, x*)*.p*2 and *q* = *p*1*.p*2, where *{x, y}∈ λ*.

Then (by Lemma 7.3) *I*Ω

*c,p*

Ω

*c,p*1

= *I*

Ω

*c,*(*x,y*)

+ *I*

Ω

*c,*(*y,x*)

+ *I*

*c,p*

Ω

*c,p*2

+ *I*

. But it is immediate

from Definition 7.2 that *I*Ω

*c,*(*x,y*)

Ω

*c,*(*y,x*)

+*I*

= 0, so *I*Ω

Ω

*c,p*1

= *I*

Ω

*c,p*2

+*I*

Ω

*c,p*1*.p*2

= *I*

Ω

*c,q*

= *I* .

Next, suppose *p* = *p*1*.γ.p*2 and *q* = *p*1*.p*2, where *γ* is a simple closed *λ*-

curve included in a loop of G. Now *I*Ω

*c,p*1*.γ.p*2

*p*2

*c,γ*

Ω

*c,p*1

= *I*

Ω

*c,γ*

+ *I*

+ *I*Ω . But Propo-

sition 7.6 implies that *I*Ω

*γ,c*

= 0 and so, by Lemma 7.4, *I*Ω

= 0. Hence

Ω

*I*

*c,p*

Ω

*c,p*1*.γ.p*2

= *I*

Ω

*c,p*1

= *I*

Ω

*c,p*2

+ *I*

Ω

*c,p*1*.p*2

= *I*

= *I*Ω . *✷*

Note that, by Property 2.4, this theorem, Lemma 7.5 and Proposition 7.6 all remain true when *κ* is replaced by *λ* and *λ* by *κ*.

*c,q*

# A Proof of the Jordan Curve Theorem

As an application of the intersection number, we now outline a proof of the Jordan curve theorem for planar gads (Theorem 4.7 above). Since a planar pgads is orientable (Corollary 5.2), has no singularities (Proposition 4.6), and

is simply connected (Corollary 4.5), this theorem follows from the following more general result:

**Theorem 8.1** *Let* G = ((*V, π, L*)*,* (*κ, λ*)) *be a* gads *that is a subGADS of an orientable* pgads G*'* = ((*V ', π', L'*)*,* (*κ', λ'*)) *which has no singularities. Let c be a κ-parameterization of a simple closed κ-curve C of* G *such that*

1. *C is not included in any loop of* G*.*
2. *Every vertex in C is an interior vertex of* G*.*
3. *c is κ'-reducible in* G*'.*

*Then V \ C has exactly two λ-components, and, for each vertex x ∈ C, Nλ*(*x*)

*intersects both λ-components of V \ C.*

It is perhaps worth mentioning that in this theorem the hypothesis that G*'* is orientable is not really necessary, but is included because we wish to give a proofofthe theorem that uses the intersection number (which is only defined in orientable gads).

Regarding condition (ii), note that an interior vertex *v* of G cannot be a vertex of a (*π' \ π*)-edge, and cannot be a singularity of G, for in both cases *v* would be a singularity of G*'*, contrary to the hypothesis that G*'* has no singularities.

A first step in proving Theorem 8.1 is to prove:

**Lemma 8.2** *Under the hypotheses of Theorem 8.1, let* Ω *be a coherent ori- entation of* G*', and let c* = (*x*0*,..., xl*)*, so that xl* = *x*0*. Then, for all*

*i ∈ {*0*,..., l}, each of the sets* Left*c* (*i*) *\ C and* Right*c* (*i*) *\ C is nonempty*

Ω Ω

*and λ-connected, and contains at least one vertex in Nλ*(*x*)*.*

Note that Left*c* (*i*) and Right*c* (*i*) are sets ofvertices of G, since the vertices

Ω Ω

of *c* are interior vertices of G. This lemma can be proved using Theorem 6.1.

(Subdivide the loops of G into simple closed (*κ ∩ λ*)-curves.) We omit the details here. Using this lemma, it is not hard to prove the following result:

**Proposition 8.3** *Under the hypotheses of Theorem 8.1, V \ C has at least two λ-components.*

**Proof:** Suppose *V \ C* is *λ*-connected. Since *V '* is *π'*-connected (by condition

(i) in the definition ofa 2D digital complex), for each vertex of *V ' \C* there is a shortest *π'*-path from that vertex to a vertex in *C*, and the second-last vertex on such a path must be in *V \ C* because all vertices of *C* are interior vertices of G. So the fact that *V \ C* is *λ*-connected implies *V ' \ C* is *λ'*-connected.

Let Ω be a coherent orientation of G*'*, let *c* = (*x*0*,..., xl*) (so that *xl* = *x*0), and pick *i ∈ {*0*,..., l}*. By Lemma 8.2 there exist vertices *y ∈* Left*c* (*i*) *∩ Nλ*(*xi*) *\ C* and *z ∈* Right*c* (*i*) *∩ Nλ*(*xi*) *\ C*. As *V ' \ C* is *λ'*-connected, there is a *λ'*-path *α* in *V ' \ C* from *y* to *z*. The closed *λ'*-path *α'* = *α.*(*z, xi, y*) satisfies

Ω

Ω

*I*Ω *'* = 1. But *c* is *κ'*-reducible in G*'*, so *I*Ω

= 0 by Theorem 7.7 (with *κ*

*c,α*

*α',c*

replaced by *λ'* and *λ* by *κ'*) and, by Lemma 7.4, *I*Ω *'* = 0, a contradiction. *✷*

*c,α*

The next proposition will be used to prove that the set *V \C* in Theorem 8.1 cannot have more than two *λ*-components. For any set *ρ* of unordered pairs of vertices of a gads, we say that a set *A* of vertices of the gads is a *ρ-arc* if *A* is a singleton set, or if *A* is a finite *ρ*-connected set with the following property: There are two (and only two) elements of *A* that each have just one *ρ*-neighbor in *A*, and all other elements of *A* have exactly two *ρ*-neighbors in

*A*. Note that if *C* is any simple closed *ρ*-curve and *p ∈ C* then *C − {p}* is a *ρ*-arc. Each element of a *ρ*-arc *A* that does not have two *ρ*-neighbors in *A* is called an *extremity* of *A*.

**Proposition 8.4** *Let* G = ((*V, π, L*)*,* (*κ, λ*)) *be a* gads*. Let A be a κ-arc such that every vertex in A is an interior vertex of* G *and no vertex in A is a singularity of* G*. Then V \ A is λ-connected.*

**Sketch of proof:** Our first step is to prove that *NL*(*x*) *\ A* is nonempty and *λ*-connected if *x* is an extremity of *A*. This assertion, like Lemma 8.2, can be proved using Theorem 6.1. Again, we omit the details.

Having established this assertion, we prove the proposition by induction on *|A|*. When *|A|* = 1, the result follows from the assertion, since *V* is *π*- connected. Assume the result holds when *|A|* = *k*, and suppose *|A|* = *k* + 1. Let *x* be an extremity of *A*, and let *A'* = *A\{x}*. Let *v* be any vertex in *V \A*. By the induction hypothesis *v* is *λ*-connected in *V \A'* to *x*, and hence to some vertex of *Nλ*(*x*) *\ A'*. A shortest *λ*-path in *V \ A'* from *v* to *Nλ*(*x*) *\ A'* does not pass through *x*. Hence *v* is *λ*-connected even in *V \ A* to some vertex of *Nλ*(*x*) *\ A' ⊆ NL*(*x*) *\ A*. Since *v* is an arbitrary vertex in *V \ A* and *NL*(*x*) *\ A* is *λ*-connected (by the above assertion), *V \ A* is *λ*-connected. *✷*

**Proof of Theorem 8.1:** From Proposition 8.3 we know that *V \ C* has at least two *λ*-components. Now let *x* be a vertex of *C*, and let *A* be the *κ*-arc *C \ {x}*. Let *v* be any vertex in *V \ C*. By Proposition 8.4, *v* is *λ*-connected in *V \ A* to *x*, and hence to some vertex in *Nλ*(*x*) *\ A*. A shortest *λ*-path in *V \ A* from *v* to *Nλ*(*x*) *\ A* does not pass through *x*, so *v* is *λ*-connected even in *V \ C* to some vertex in *Nλ*(*x*) *\ A*. Since this applies to any vertex *v* in *V \ C*, every *λ*-component of *V \ C* intersects *Nλ*(*x*) *\ A*.

Moreover, since *Nλ*(*x*) *\ A ⊆ NL*(*x*) *\ C*, we can deduce that *V \ C* has no more *λ*-components than *NL*(*x*) *\ C*. But if *c* = (*x*0*,..., xl*), so that *xl* = *x*0, and *i* is the integer in *{*0*,...,l −* 1*}* such that *x* = *xi*, then *NL*(*x*) *\ C* = (Left*c* (*i*) *∪* Right*c* (*i*)) *\ C* does not have more than two *λ*-components, by

Ω Ω

Lemma 8.2. Hence *V \ C* cannot have more than two *λ*-components. *✷*

# Concluding Remarks

A new approach to 2D digital topology (including the digital topology of boundary surfaces), based on an axiomatization of the spaces being studied, has been presented. A space that satisfies our axioms is called a gads. In-

stances of this very general concept include gads corresponding to all of the good pairs ofadjacency relations that have previously been used (by ourselves or others) in digital topology on planar grids and boundary surfaces.

Some results that have been established in the literature for certain specific digital spaces have been generalized to gads (e.g., a homotopy invariance theorem for intersection numbers of digital paths, and a digital Jordan curve theorem). There are many other results of digital topology for which this could be done, such as results about simple points and boundary tracking. The problem ofdeveloping a 3D version ofthis theory seems more challenging.

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