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Graphs with Girth at Least 8 are b-continuous [1](#_bookmark0)

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**Abstract**

A b-coloring of a graph is a proper coloring such that each color class has at least one vertex which is adjacent to each other color class. The b-spectrum of *G* is the set *Sb*(*G*) of integers *k* such that *G* has a b-coloring with *k* colors and *b*(*G*) = max *Sb*(*G*) is the b-chromatic number of *G*. A graph is b-continous if *Sb*(*G*) = [*χ*(*G*)*, b*(*G*)] *∩* Z. An infinite number of graphs that are not b-continuous is known. It is also known that graphs with girth at least 10 are b-continuous. In this work, we prove that graphs with girth at least 8 are b-continuous, and that the b-spectrum of a graph *G* with girth at least 7 contains the integers between 2*χ*(*G*) and *b*(*G*). This generalizes a previous result by Linhares-Sales and Silva (2017), and tells that graphs with girth at least 7 are, in a way, almost b-continuous.

*Keywords:* b-chromatic number; b-continuity; girth; bipartite graphs.

# Introduction

Let *G* be a simple graph (for basic terminology on graph theory, we refer the reader to [[4](#_bookmark17)]). A function *ψ*: *V* (*G*) *→* N is a *proper k-coloring of G* if *|ψ*(*V* (*G*))*|* = *k* and *ψ*(*u*) */*= *ψ*(*v*) whenever *uv ∈ E*(*G*). Because we only deal with proper colorings in this text, from now on we refer to them as simply a coloring. We call the elements of *ψ*(*V* (*G*)) *colors*. Given a color *i ∈ ψ*(*V* (*G*)), the set *ψ−*1(*i*) is called *color class*

*i*. We say that *u ∈ V* (*G*) is a *b-vertex in ψ*(of color *ψ*(*u*)) if *ψ*(*N* [*u*]) = *ψ*(*V* (*G*)). If for some color *c ∈ ψ*(*V* (*G*)), the color class *c* does not contain b-vertices, we can obtain a (*k −* 1)-coloring by changing the color of each vertex *v ∈ ψ−*1(*c*) to another color in *ψ*(*V* (*G*)) *\ ψ*(*N* [*v*]). We say that this new coloring is obtained from the first

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one by *cleaning color c*. In a coloring such that we cannot apply this procedure, all color classes have at least one b-vertex. Such a coloring is called a *b-coloring of G*. Observe that an optimal coloring cannot have the number of colors decreased by the described algorithm; therefore every optimal coloring is also a b-coloring. In [[8](#_bookmark22)], the authors define the *b-chromatic number of G*, denoted by *b*(*G*), as the largest natural *k* for which *G* has a b-coloring with *k* colors. In the same article, the authors demonstrated that the problem of finding *b*(*G*) is NP-complete in general.

Another interesting aspect about b-colorings concerns its existence for every

possible value between *χ*(*G*) and *b*(*G*). In [[8](#_bookmark22)], the authors observe that the cube has a b-coloring using 2 colors and 4 colors, but has no b-coloring using 3 colors. Inspired by this result, in [[9](#_bookmark23)] it is shown that for any integer *n ≥* 4 the graph obtained from the complete bipartite graph *Kn,n* by deleting the edges from a perfect matching has a b-coloring using 2 and *n* colors, but has no b-coloring using a number of colors between 2 and *n*. This motivates the definition of the *b-spectrum of G*, that is the set *Sb*(*G*) containing every integer *k* such that *G* has a b-coloring with *k* colors. A graph *G* is *b-continous* if *Sb*(*G*) = [*χ*(*G*)*, b*(*G*)] *∩* Z. In [[2](#_bookmark15)], they prove that for each finite subset *S ⊂* N *− {*1*}*, there exists a graph *G* such that

*Sb*(*G*)= *S*, and also that deciding if a graph is b-continuous is NP-complete even if colorings with *χ*(*G*) and *b*(*G*) colors are given.

Now, given a b-coloring with *k* colors, since each b-vertex has at least *k −* 1 neighbors, there exists *k* vertices with degree at least *k −* 1 (this would be a subset of *k* b-vertices of the *k* colors). So if we define *m*(*G*) as the largest positive integer *k* such that there exist at least *k* vertices with degree at least *m*(*G*) *−* 1 in *G*, we have that *b*(*G*) *≤ m*(*G*). This upper bound was introduced in [[8](#_bookmark22)], where the authors show that one can find *m*(*G*) in polynomial time using the degree list of the graph. Also, they prove that if *G* is a tree, then *b*(*G*) *≥ m*(*G*) *−* 1, and that one can decide if *b*(*G*) = *m*(*G*) in polynomial time. Their result was later generalized for graphs with girth at least 7 [[6](#_bookmark20)] (the *girth of G* is the minimum lenght of a cycle in *G*). We also mention that there are many results that say that regular graphs with large girth have high b-chromatic number [[3,](#_bookmark18)[5](#_bookmark19),[13,](#_bookmark27)[3](#_bookmark18)]. Indeed, the following conjecture is still open.

**Conjecture 1.1** *If G is a d-regular graph with girth at least 5 and G is not the Petersen graph, then b*(*G*)= *d* + 1*.*

Because of these results, it makes sense to investigate the b-continuity of graphs with large girth. Indeed, in [[1](#_bookmark16)] the authors prove that regular graphs with girth at least 6 and without cycles of length 7 are b-continuous, and in [[11](#_bookmark25)], they prove that every graph with girth at least 10 are b-continuous. Here, we improve their result to graphs with girth at least 8.

**Theorem 1.2** *If G is a graph with girth at least 8, then G is b-continuous.*

In addition, we prove that graphs with girth at least 7 are, in way, almost b-continuous.

**Theorem 1.3** *If G is graph with girth at least 7, then* [2*χ*(*G*)*, b*(*G*)] *∩* Z *⊆ Sb*(*G*)*.*

Given a graph *G* and a b-coloring of *G* with *k* colors, *k ≥ χ*(*G*) + 1, the proof of Theorems [1.2](#_bookmark1) and [1.3](#_bookmark2) consists in trying to obtain a b-coloring with *k −* 1 colors using simple recoloring procedures; when this is not possible, we get that the graph has a special structure and apply non-constructive arguments to obtain the desired b-coloring. We mention that the coloring problem is NP-complete for graphs with

girth at least *k*, for every fixed *k ≥* 3 [[12](#_bookmark26)]. This is why any proof of a result like

Theorem [1.2](#_bookmark1) is expected to have a non-constructive part. In the next section, we present the basic definitions and results, in Section [3](#_bookmark5) we present our proofs, and in Section [4](#_bookmark12), we make some further comments on the proof and state some open questions.

# Preliminaries

In [[1](#_bookmark16)], a vertex *u ∈ V* (*G*) is called a *k-iris* if there exists *S ⊂ N* (*u*) such that

*|S|≥ k −* 1 and *d*(*v*) *≥ k −* 1 for every *v ∈ S* (observe Figure [1](#_bookmark3)).



*u*

Fig. 1. In the figure, we presente a 4-iris.

This definition is important because of the following important lemma. Observe that the lemma also implies that if *G* and *k* satisfies the conditions, then *b*(*G*) *≥ k*.

**Lemma 2.1 ([**[**1**](#_bookmark16)**])** *Let G be a graph with girth at least 6 and without cycles of lengh*

*7. If G has a k-iris with k ≥ χ*(*G*)*, then G has a b-coloring with k colors.*

As we said before, given a b-coloring of *G* with *k* colors, *k > χ*(*G*) + 1, we try to obtain a b-coloring of *G* with *k −* 1 colors. However, this is not always possible, and when this happens, it is because we have a *k*-iris. Our theorem then follows from the lemma above. We mention that the constraint about not having cycles of length 7 appears only in the above lemma, but not on our proof. We now introduce the further needed definitions.

From now on, let *G* be a simple graph and *ψ* be a b-coloring of *G* with *k > χ*(*G*) + 1 colors. We say that *u realizes color i* if *ψ*(*u*) = *i* and *u* is a b-vertex. We also say that color *i* is *realized by u*. For *x ∈ V* (*G*) and *i ∈ {*1*,..., k}*, let *Nψ,i*(*x*) be the set of vertices of color *i* in the neighborhood of *x*, i.e., *Nψ,i*(*x*) = *N* (*x*) *∩ ψ−*1(*i*); in fact, we omit *ψ* in the superscript since it is always clear from the context. This is also done in the next definitions. For a subset *X ⊆ V* (*G*), let

*Ni*(*X*) = (S*x∈X Ni*(*x*)) *\ X*. Let *B*(*ψ*) denote the set of b-vertices in *ψ* and, for

each *i ∈ {*1*,..., k}*, let *Bi* = *B*(*ψ*) *∩ ψ−*1(*i*) be the set of b-vertices in color class *i*.

Given a set *K* such that *K ⊆ ψ—*1(*i*) for some *i ∈ {*1*,..., k}*, we say that a color *j ∈ {*1*,..., k}\ {i}* is *dependent on K* if *Ni*(*Bj*) *⊆ K*; denote by *U* (*K*) the set of colors depending on *K*. If *K* = *{x}*, we write simply *U* (*x*). Given *x ∈ V* (*G*) *\ B*(*ψ*), if *|U* (*x*)*| ≥* 2 we call *x* a *useful* vertex; otherwise, we say that *x* is *useless*. For *j ∈ {*1*,..., k}*, we say that *x ∈ V* (*G*) is *j-mutable* if *x* is useless and there exists a color *c* such that we can change the color of *x* to *c* without creating any b-vertex of color *j*; we also say that color *c* is *safe for* (*x, j*). If there is no safe color for (*x, j*), we say that *x* is *j*-imutable.

# Proofs

The next lemma is the main ingredient in our proof. Combined with Lemma [2.1](#_bookmark4), it immediatly implies Theorem [1.2](#_bookmark1).

**Lemma 3.1** *Let G* = (*V, E*) *be a graph with girth at least 7. If G has b-coloring with k colors where k ≥ χ*(*G*)+ 1*, then either G has a b-coloring with k −* 1 *colors, or G contains a* (*k −* 1)*-iris.*

**Proof.** Our proof is similar to that made in [[11](#_bookmark25)], but we concentrate in one color that we want to eliminate.

Suppose that *G* does not have a b-coloring with *k −* 1 colors; we prove that

*G* has a (*k −* 1)-iris. For this, let *ψ* be a b-coloring with *k* colors that minimizes

*|B*1*|* and then minimizes *|ψ—*1(1)*|* (i.e., it firstly minimizes the number of b-vertices of color 1, then it minimizes the number of vertices of color 1). First, we prove that every *x ∈ ψ—*1(1) *\ B*1 is useful. Suppose otherwise and let *x* be a useless vertex in color class 1, i.e., *|U* (*x*)*| ≤* 1. If *U* (*x*)= *∅*, then we can recolor *x* without losing any b-vertex, a contradiction since *ψ* minimizes *|ψ—*1(1)*|*. And if *U* (*x*)= *{d}*, then we can obtain a b-coloring with *k −* 1 by recoloring *x* and cleaning *d*, again a contradiction. Therefore, the following holds:

* 1. Every *x ∈ ψ—*1(1) *\ B*1 is useful.

Now, we choose any *u ∈ B*1 and analyse its vicinity in order to obtain the desired (*k −* 1)-iris. For this, the following two claims are essential.

**Claim 3.2** *Let j ∈ {*2*,..., k}. If every x ∈ Nj*(*u*) *\ Bj is 1-mutable, then one of the following holds:*

* 1. *N* (*u*) *∩ Bj /*= *∅; or*
  2. *There exists a color d ∈ {*2*,..., k}\ {j} such that d depends on Nj*(*u*)*, i.e.,*

*Nj*(*Bd*) *⊆ Nj*(*u*)*.*

*Proof of claim:* Suppose that neither (ii) nor (iii) holds, and let *ψj* be obtained from *ψ* by changing the color of each *x ∈ Nj*(*u*) to a color *c* safe for (*x,* 1). Because (*iii*) does not hold, we get that *U* (*Nj*(*u*)) *⊆ {*1*}*. Therefore, at most one color loses all of its b-vertices, namely color 1, and since every *x ∈ Nj*(*u*) is 1-mutable, no b-vertices of color 1 is created. But because *u* is not a b-vertex in *ψj* (it is not adjacent to color *j* anymore) and *ψ* minimizes *|B*1*|*, we get that *ψj* cannot be a b-coloring, which

means that we can obtain a b-coloring with *k −* 1 colors by cleaning color 1.*♦* The following claim tells us that (ii) or (iii) actually always hold.

**Claim 3.3** *(iv) Every x ∈ Nj*(*u*) *\ Bj is 1-mutable, for every j ∈ {*2*,..., k}.*

*Proof of claim:* Suppose, without loss of generality, that *d ∈ {*2*,..., k}* is such that the colors in *{d* + 1*,..., k}* are exactly the colors that contains some 1-imutable vertex. We count the number of colors with b-vertices in the vicinity of *u* to get that in fact *d ≥ k*. So, for each *i ∈ {d* + 1*,..., k}*, let *wi ∈ Ni*(*u*) be a 1-imutable vertex. By definition, this means that, for each *i ∈ {d* + 1*,..., k}*, there exists some neighbor of *wi* that would be turned into a b-vertex of color 1 in case we change the color of *wi*; let *vi* be such a vertex. We then know that *vi ∈ ψ—*1(1) *\ B*1, which by (i) gets us that *|U* (*vi*)*| ≥* 2. By the definition of *U* (*x*) and the fact that every *x ∈ {vd*+1*,..., vk}* is colored with color 1, we get:

* + 1. *U* (*vi*) *∩ U* (*vl*)= *∅*, for every *i, l ∈ {d* + 1*,..., k},i /*= *l.*

Now, we investigate the b-vertices around colors *{*2*,..., d}*. By Claim [3.2](#_bookmark7),

suppose, without loss of generality, that *p ∈ {*2*,..., d}* is such that (ii) holds for colors in *{*2*,..., p}*, while (iii) holds for colors in *{p* + 1*,..., d}*. For each *i ∈ {p* + 1*,..., d}*, let *ci ∈ {*2*,..., k}\ {i}* be a color depending on *Ni*(*u*), which means that *Bc ⊆ N* (*Ni*(*u*)). Observe that, since *G* has no cycles of length 3, we get:

*i*

* + 1. *{*2*,..., p}∩ {cp*+1*,..., cd}* = *∅*

Also, because *G* has no cycles of length 4, we get *ci /*= *cl* for every *i /*= *l*, i.e.:

* + 1. *|{cp*+1*,..., cd}|* = *d − p*

Finally, because *G* has no cycles of length smaller than 6, we get that:

*k*

* + 1. ​

*{*2*,..., p, cp*+1*,..., cd}∩*

*i*=*d*+1

*U* (*vi*)= *∅.*

Now, recall that *ψ*(*vi*) = 1 for every *i ∈ {d* + 1*,..., k}*, and that *ci /*=1 for every

*i ∈ {p* +1*,..., d}*. This means that 1 *∈/ {cp*+1*,..., cd}∪* S*p U* (*vi*). By combining

*i*=*d*+1

Equations ([1](#_bookmark8)) through ([4](#_bookmark9)), we get the following, which implies *d ≥ k* as desired:

*k −* 1 *≥ |{*2*,..., p}∪ {cp*+1*,..., cd}∪ U* (*vd*+1) *∪ ... ∪ U* (*vk*)*|*

= *d −* 1+ Σ*k*

*i*=*d*+1

*|U* (*vi*)*| ♦*

*≥ d −* 1+ 2(*k − d*)*.*

Now, let *N* = (*N* (*u*) *∪ N* (*N* (*u*))) *\ {u}*. Observe that because (ii) or (iii) holds for every color *l ∈ {*2*,..., k}*, we get that *B*(*ψ*) *⊆ N* . Suppose that *N* [*u*] does not contain a (*k −* 1)-iris, otherwise the proof is done. This means that at least one color in *{*2*,..., k}*, say *k*, is such that (ii) does not hold for *k*, which by Claim [3.2](#_bookmark7) implies that (iii) holds, i.e., that there exists a color in *{*2*,...,k −* 1*}*, say 2, such that *N* 2(*Bk*) *⊆ N* 2(*u*) (Observe Figure [2](#_bookmark10)). Now, let *w ∈ N* 1(*Bk*); it exists since the vertices in *Bk* are b-vertices. By (i), there exists at least two colors in *{*2*,..., k}*

that depend on *w*. But because *B*(*ψ*) *⊆ N* , we get a cycle of length at most 6, a contradiction.

*N*



*u*

*w N* 1(*Bk*)

*Bk*

*Nk*(*u*)

*...*

*N* 3(*u*)

*N* 2(*u*)

Fig. 2. Structure around *u* when color *k* does not satisfy Claim [3.2](#_bookmark7).(ii).

*2*

Now, to prove Theorem [1.3](#_bookmark2), we apply Lemma [3.1](#_bookmark6) and the next lemma. A *star* is a tree that has at most one vertex with degree bigger than 1, and the *diameter* of a graph *G* is the maximum number of edges in a shortest path of *G*. Here, as happens in Lemma [2.1](#_bookmark4), we get that the existence of a *k*-iris in *G* implies *b*(*G*) *≥ k*.

**Lemma 3.4** *If a graph G has girth at least 7 and a k-iris where k ≥* 2*χ*(*G*)*, then*

*G has a b-coloring with k colors.*

**Proof.** Let *u ∈ V* (*G*) be a *k*-iris with *k ≥* 2*χ*(*G*). Let *u*2*,..., uk* be neighbors of *u* such that *d*(*ui*) *≥ k−*1 for every *i ∈ {*2*,..., k}*; let *Ni* be a subset of *k−*2 neighbors of *ui* different from *u*. Start by coloring *u* with 1 and, for each *i ∈ {*2*,..., k}*, give color

*i* to *ui* and colors *{*2*,..., k}\{i}* to *Ni*. Denote by *T* the set *{u, u*2*,..., uk}∪*S*k Ni*,

*i*=2

i.e., *T* denotes the set of colored vertices. Observe Figure [3.](#_bookmark11)



|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 *u*  *u*2 2 *u*3 3 *... uk k N*2 *N*3 *Nk*  1 2 *... k* 1 2 *... k ...* 1 2 *... k* | | | | | | | | | | | | | |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| *{w ∈ V* (*G*) *\ T | ∃x ∈ N* (*w*) *s.t. ψ*(*x*) *∈ {*1*,..., χ*(*G*)*}}* | | | | | | | | | | | | | |

*T*

*B*

Fig. 3. Subset of vertices around *u*. The label inside a vertex denotes its color.

Observe that the coloring can be easily done since *G*[*T* ] *−u* is a forest formed by *k −* 1 stars. Also, note that we already have *k* b-vertices of distinct colors, and thus it only remains to extend the partial coloring to the rest of the graph. For this, let *B* be the set of vertices adjacent to some color class in *{*1*,..., χ*(*G*)*}*. We claim that

*|N* (*w*) *∩ T|≤* 1 for every *w ∈ B*; indeed, because *T* induces a tree of diameter 4, if this was not true, then we would get a cycle of length at most 6. By the definition of *B*, we then get that every *w ∈ B* has no neighbors in color classes *χ*(*G*)+1*,..., k*. Thus, since *k ≥* 2*χ*(*G*), we can color *G*[*B*] with colors *χ*(*G*)+ 1*,...,* 2*χ*(*G*). Finally, by the definition of *B*, we know that every *w ∈ V* (*G*) *\* (*B ∪ T* ) has no neighbors of color 1 through *χ*(*G*), which means that we can color *G−T −B* with these colors.*2*

# Conclusion

We have proved that every graph with girth at least 8 is b-continuous, and that graphs with girth at least 7 are in way almost b-continuous. This improves the result presented in [[11](#_bookmark25)], where they prove that graphs with girth at least 10 are b-continuous. There, the authors also pose the following questions:

**Question 1** *What is the minimum g*ˆ *such that G is b-continuous whenever G is a graph with girth at least g*ˆ*?*

**Question 2** *Are bipartite graphs with girth at least 6 b-continuous?*

Recall that the graph obtained from the complete bipartite graph *Kn,n* by re- moving a perfect matching is not b-continuous, for every *n ≥* 4 [[9](#_bookmark23)]. Hence, by our result we get:

5 *≤ g*ˆ *≤* 8*.*

We believe that the same techniques might improve this bound to 7, but not further. In particular, we mention that Lemma [3.1](#_bookmark6) works for graphs with girth 7 and that the bound is 8 because of Lemma [2.1](#_bookmark4). Therefore, if the following question is answered “yes”, then we get *g*ˆ *≤* 7.

**Question 3** *Let G be a graph with girth at least 7 such that G has a k-iris, with*

*k ≥ χ*(*G*)+ 1*. Does G admit a b-coloring with k colors?*

As for the case of bipartite graphs, we think it is worth mentioning a known conjecture about their b-chromatic number. Recall the upper bound *m*(*G*) for the b-chromatic number *b*(*G*), which is the maximum value *k* for which there exist *k* vertices with degree at least *k −* 1. The set of all vertices with degree at least *m*(*G*) *−* 1 is denoted by *D*(*G*), and a graph is said to be *tight* if *|D*(*G*)*|* = *m*(*G*); this means that there is only one candidate set for the b-vertices of a b-coloring of *G* with *m*(*G*) colors. Deciding if *b*(*G*) = *m*(*G*) is NP-complete even for bipartite

tight graphs [[9](#_bookmark23)]. In [[7](#_bookmark21)], the authors define the class *Bm* that contains every bipartite

graph *G* = (*A ∪ B, E*) such that *m*(*G*)= *m*, *D*(*G*)= *A* and *G* has girth at least 6. They conjecture the following:

**Conjecture 4.1** *[*[*7*](#_bookmark21)*] For every m ≥* 3*, and every G ∈ Bm, we have that:*

*b*(*G*) *≥ m*(*G*) *−* 1*.*

We mention that, if *G* is a bipartite graph with girth at least 6 and a b-coloring of *G* with *k* colors is given, *k ≥ χ*(*G*)+1, then, with a little more work, one can get from the proof of Lemma [3.1](#_bookmark6) that *G* contains an induced subgraph *H* that has a structure

similar to the structure of a graph in *Bk*. Trying to use this structure to obtain a b- coloring of *H* with *k −* 1 colors could translate into proving Conjecture [4.1](#_bookmark14). And on the other way around, we believe that a strategy to prove Conjecture [4.1](#_bookmark14) could help coloring these graphs, which would imply that the answer to Question [2](#_bookmark13) is “yes”. This means that answering Question [2](#_bookmark13) seems as hard as proving Conjecture [4.1](#_bookmark14). We also mention that in [[10](#_bookmark24)], it is proved that Conjecture [4.1](#_bookmark14) is a consequence of the famous Erdos-Faber-Lova´sz Conjecture, which remains open since 1972 and which is largely believed to hold. This is strong evidence that Conjecture [4.1](#_bookmark14) holds.

Finally, because of the difficulties in obtaining b-continuity already for bipartite graphs with girth at least 6, maybe a good bet would be also to see if the lower bound for *g*ˆ is tight. So, we propose one additional question:

**Question 4** *Does there exist a graph with girth 5 that is not b-continuous?*

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