

Electronic Notes in Theoretical Computer Science 221 (2008) 141–152

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Integral of Two-dimensional Fine-computable Functions

Takakazu Mori, [1](#_bookmark0)*,*[4](#_bookmark0) Mariko Yasugi [2](#_bookmark0)*,*[5](#_bookmark0) and Yoshiki Tsujii [3](#_bookmark0)*,*[6](#_bookmark0)

*Kyoto Sangyo University Kyoto, Japan*

**Abstract**

We discuss effective integrability and effectivization of Fubini’s Theorem for a Fine-computable func-

tion *F* (*x, y*) on the upper-right open unit square. The core objective is Fine-computability of *f* (*x*) =

R

[0*,*1) *F* (*x, y*)*dy* as a function on [0*,* 1).

*Keywords:* Two-dimensional Fine-computable function, effective integrability, Fubini’s Theorem, integral operator

# Introduction

Notions of Fine-continuity and of Fine-computabilities on [0*,* 1) are defined with respect to the Fine topology (cf. Section [2](#_bookmark3), [[2](#_bookmark24),[3](#_bookmark25),[6](#_bookmark28)]). We have defined effective integrability for Fine-computable functions on [0*,* 1) ([[6](#_bookmark28),[8](#_bookmark30)]). In this article, we in- vestigate the notions of Fine-computabilities and effective integrability of functions on the upper-right open unit square [0*,* 1)2.

In classical analysis, the integral operator with a kernel *F* (*x, y*), which maps a function *g*(*x*) on *X* to (*Tg*)(*x*) = *X g*(*y*)*F* (*x, y*)*dy*, is a central subject. Measura- bility and integrability of *Tg* are fundamental properties to be proved and Fubini’s Theorem is a fundamental tool to deal with investigations of such an operator.

∫

1 Email: [morita@cc.kyoto-su.ac.jp](mailto:morita@cc.kyoto-su.ac.jp)

2 Email: [yasugi@cc.kyoto-su.ac.jp](mailto:yasugi@cc.kyoto-su.ac.jp)

3 Email: [tsujiiy@cc.kyoto-su.ac.jp](mailto:tsujiiy@cc.kyoto-su.ac.jp)

4 Faculty of Science. This work has been supported in part by Research Grant from KSU (2008, \*\*\*).

5 Graduate School of Ecocomy. The work is supported in part by JSPS Grant-in-Aid No. 20540143.

6 Faculty of Science. This work has been supported in part by Research Grant from KSU (2008, \*\*\*).

1571-0661 © 2008 Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2008.12.013

**Theorem 1.1** (Fubini’s Theorem) *Let F* (*x, y*) ≥ 0 *be a measurable and integrable function on the upper-right open unit square* [0*,* 1)2*. Then the following hold.*

* 1. *For almost all x, F* (*x,* ) *is measurable and integrable.*

*·*

* 1. ∫[0*,*1) *F* (*x, y*)*dy and* ∫[0*,*1) *F* (*x, y*)*dx are measurable.*
  2. ∫∫[0*,*1)2 *F* (*x, y*)*dxdy* = ∫[0*,*1) ∫[0*,*1) *F* (*x, y*)*dy* *dx* = ∫[0*,*1) ∫[0*,*1) *F* (*x, y*)*dx* *dy.*

In this article, we discuss an effectivization of Fubini’s Theorem for uniformly Fine-computable functions on [0*,* 1)2 (Definition [3.3](#_bookmark14)) and make an introductory con- sideration on Fine-computable functions (Definition [4.1](#_bookmark18)). We also make some obser- vations on the transformation *T* . In effectivization, *Fine-computability* and *effective integrability* correspond to classical *measurability* and *integrability* respectively.

From the standpoint of computable analysis, it is expected that

*f* (*x*) = ∫[0*,*1) *F* (*x, y*)*dy* is defined everywhere on [0*,* 1) and *f* (*x*) is Fine-computable

for a Fine-computable function *F* (*x, y*) on [0*,* 1)2. To secure the former, we assume that *F* (*x, y*) is integrable with respect to *y* for all *x ∈* [0*,* 1).

Since Fine-computable functions are continuous at all dyadically irrational points with respect to the Euclidean topology, they are measurable, and Fubini’s Theorem holds classically for integrable Fine-computable functions. Therefore, effectivization of Fubini’s Theorem boils down to the proof of Fine-computability of *f* (*x*). Hence the proof of this property is the main objective of this paper.

Roughly speaking, continuity of *Tg* is deduced from that of *F* (*x, y*). Hence, by modifying the proof of Fine-computability of *f* (*x*), we can easily prove Fine- computability of *Tg* under some suitable conditions on integrability.

Our main assertions are that Fine-computability of *f* (*x*) holds for a “uniformly Fine-computable” function *F* (*x, y*) and for a “bounded Fine-computable” function *F* (*x, y*), and that we need some additional conditions on general Fine-computable functions.

We make introductory speculations with some examples concerning Fine-com- putability of *f* (*x*).

**Example 1.2** (Suggested by Yagishita) Let us define *F* (*x, y*) = 1 *e−*( *x* )2 . Then

1*−y*

1*−y*

*F* (*x, y*) is positive and continuous on R*×*[0*,* 1). It is easy to prove that the restriction

of *F* (*x, y*) to [0*,* 1) *×* [0*,* 1) is Fine-computable.

It holds that ∫ 1 (

0

0

*F*

1

1*−y*

*e*

1*−y*

*dx*

1*−y e−x dx <*

*π*

) = ∫ 1

*−*( *x* )2

= ∫ 1 2 *√*.

Hence ∫ 1 *dy* ∫ 1 *F* (*x, y*)*dx < ∞*.

*−*1

0

*x, y dx*

0

On the other hand, *F* (0*, y*) = 1 is not integrable, that is, *f* (0) = ∫

1*−y*

[0*,*1)

*F* (0*, y*)*dy*

is not defined.

Example [1.2](#_bookmark1) shows that Fine-computability and integrability of *F* (*x, y*) do not assure that *f* (*x*) is a total function.

**Example 1.3** Let *α*(*k*) be a recursive injection whose range is not recursive. Then

*ϕ*(*y*) = 2*k*2*−α*(*k*) if 1 *−* 2*−*(*k−*1) ≤ *y <* 1 *−* 2*−k,k* = 1*,* 2*,...*

is Fine-computable and integrable but not effectively integrable (Brattka, [[1](#_bookmark23)]).

Define *F* (*x, y*) = *ϕ*(*y*)(1 *− x*)*ϕ*(*y*)*−*1 and *f* (*x*) = ∫

[0*,*1)

*F* (*x, y*)*dy*.

Then, *F* (*x, y*) is Fine-computable and not bounded. It holds that ∫[0*,*1) *F* (*x, y*)*dx* =

1, ∫∫[0*,*1)2 *F* (*x, y*)*dxdy* = 1 and *f* (*x*) is total.

On the other hand, *f* (0) = ∫

[0*,*1)

*F* (0*, y*)*dy* = Σ*∞*

2*−α*(*k*) is not a computable

number, and hence sequential computability for *f* (*x*) does not hold.

*k*=1

Example [1.3](#_bookmark2) shows that Fine-computability of *F* (*x, y*) and computability of

∫∫[0*,*1)2 *F* (*x, y*)*dxdy* do not imply Fine-computability of *f* (*x*) even if it is total.

In Section [2](#_bookmark3), we review Fine-computability and effective integrability for a func- tion on [0*,* 1).

In Section [3](#_bookmark12), we define the two-dimensional Fine-space and notions of Fine- computability and prove that *f* (*x*) is uniformly Fine-computable if *F* (*x, y*) is uni- formly Fine-computable (Theorem [3.6](#_bookmark17)).

In Section [4](#_bookmark19), we prove that *f* (*x*) is Fine-computable for a bounded Fine-computable *F* (*x, y*) (Theorem [4.6](#_bookmark22)). We give also a sufficient condition for Fine-computability of *f* (*x*), where *F* (*x, y*) is Fine-computable but not necessarily bounded. This is an effectivization of a well known classical result.

Consult [[2](#_bookmark24)] as to Fine-continuous functions.

# Preliminaries

We summarize Fine-computability properties on [0*,* 1) and effective integrability of such functions. (See [[6](#_bookmark28),[7](#_bookmark29),[8](#_bookmark30)].) We assume basic knowledge of computability on the Euclidean space (cf. [[9](#_bookmark31)]). We use the notations N = *{*0*,* 1*,.. .}* and N+ = *{*1*,* 2*,.. .}*. A left-closed right-open interval with dyadic end points is called a *dyadic interval*.

We call *I*(*n, k*) = [*k*2*−n,* (*k* + 1)2*−n*) a *fundamental dyadic interval* (of level *n*)

and *J*(*x, n*), the unique fundamental dyadic interval *I*(*n, k*) which contains *x*, the

*fundamental dyadic neighborhood* of *x* (of level *n*).

**Lemma 2.1** ([[6](#_bookmark28)]) (1) *The following three properties are equivalent for any x, y*

*∈*

[0*,* 1) *and any nonnegative integer n.*

(i) *y ∈ J*(*x, n*)*.* (ii) *x ∈ J*(*y, n*)*.* (iii) *J*(*x, n*) = *J*(*y, n*)*.*

(2) *If xm is Fine-computable, then we can decide effectively whether xm I*(*n, k*) *or not for all m.*

*{ } ∈*

*J*(*x, n*) satisfies the axioms of the effective uniformity (cf. [[10](#_bookmark32)]). We call the topology generated by *I*(*n, k*) the *Fine topology* and put prefix *Fine-* to such notions. We put no prefix to notions which are defined by means of Euclidean topology.

*{ }*

*{ }*

A double sequence of dyadic rationals *rn,m* is said to be *recursive* if there exist recursive functions *α*(*n, m*)*, β*(*n, m*) such that *rn,m* = *β*(*n, m*)2*−α*(*n,m*).

*{ }*

**Definition 2.2** (1) (Effective Fine-convergence of reals) A double sequence

*{xn,m}* is said to *Fine-converge effectively* to *{xn}* if there exists a recursive function

*α*(*n, k*) which satisfies that *m* ≥ *α*(*n, k*) implies *xn,m ∈ J*(*xn, k*).

(2) (Fine-computable sequence of reals) A sequence of real numbers *xm* in [0*,* 1) is said to be *Fine-computable* if there exists a recursive double sequence of dyadic rationals *{rn,m}* which Fine-converges effective ly to *{xm}*.

*{ }*

If *xn,m* = *xm* and *xn* = *x*, we obtain the definition of effective Fine-convergence of *{xm}* to *x*.

**Remark 2.3** (1) The original definition of a Fine-computable sequence of real numbers is that *rn,m* be a recursive sequence of rational numbers (cf. [[11](#_bookmark33)]). The present definition is equivalent to the original one.

*{ }*

1. The set of computable numbers and that of Fine-computable numbers co- incide.
2. A Fine-computable sequence is (Euclidean) computable.
3. *ei* will denote an effective enumeration of all nonnegative dyadic rationals in [0*,* 1). It is an effective separating set of the Fine-space [0*,* 1) (cf. [[5](#_bookmark27)]).

*{ }*

**Definition 2.4** (Uniformly Fine-computable sequence of functions, [[3](#_bookmark25),[6](#_bookmark28)]) A se- quence of functions *fn* is said to be *uniformly Fine-computable* if (i) and (ii) below hold.

*{ }*

* 1. (Sequential Fine-computability) The double sequence *{fn*(*xm*)*}* is com- putable for any Fine-computable sequence *{xm}*.
  2. (Effectively uniform Fine-continuity) There exists a recursive function *α*(*n, k*) such that, for all *n, k* and all *x, y ∈* [0*,* 1), *y ∈ J*(*x, α*(*n, k*)) implies *|fn*(*x*) *− fn*(*y*)*| <* 2*−k.*

**Definition 2.5** (Effectively uniform convergence of functions, [[3](#_bookmark25),[6](#_bookmark28)]). A double sequence of functions *{gm,n}* is said to *converge effectively uniformly* to a sequence of functions *{fm}* if there exists a recursive function *α*(*m, k*) such that, for all *m, n* and *k*, *n* ≥ *α*(*m, k*) implies *|gm,n*(*x*) *− fm*(*x*)*| <* 2*−k* for all *x ∈* [0*,* 1).

**Definition 2.6** (Fine-computable sequence of functions, [[6](#_bookmark28)]) A sequence of func- tions *{fn}* is said to be *Fine-computable* if it satisfies the following.

1. *{fn}* is sequentially Fine-computable.
2. (Effective Fine-Continuity) There exists a recursive function *α*(*n, k, i*) such that

(ii-a) *x ∈ J*(*ei, α*(*n, k, i*)) implies *|fn*(*x*) *− fn*(*ei*)*| <* 2*−k*,

(ii-b) *∞ J*(*ei, α*(*n, k, i*)) = [0*,* 1) for each *n, k*.

*i*=1

**Definition 2.7** (Effective Fine-convergence of functions, [[6](#_bookmark28)]) We say that a double sequence of functions *{gm,n} Fine-converges effectively* to a sequence of functions

*{fm}* if there exist recursive functions *α*(*m, k, i*) and *β*(*m, k, i*), which satisfy

* 1. *x ∈ J*(*ei, α*(*m, k, i*)) and *n* ≥ *β*(*m, k, i*) imply *|gm,n*(*x*) *− fm*(*x*)*| <* 2*−k*,
  2. *∞ J*(*ei, α*(*m, k, i*)) = [0*,* 1) for each *m* and *k*.

*i*=1

**Definition 2.8** (Computable sequence of dyadic step functions, [[3](#_bookmark25),[6](#_bookmark28)]) A sequence of functions *{ϕn}* is called a *computable sequence of dyadic step functions* if there exist a recursive function *α*(*n*) and a computable sequence of reals *{cn,j}* (0 ≤ *j <*

2*α*(*n*), *n* = 1*,* 2*,.. .*) such that

*ϕn*

(*x*) = Σ2*α*(*n*)*−*1 *c*

*n,j*

*χI*(*α*(*n*)*,j*)

(*x*)*,*

where *χA* denotes the indicator (characteristic) function of *A*.

*j*=0

**Proposition 2.9** ([[6](#_bookmark28)]) *Let f be a Fine-computable function. The computable sequence of dyadic step functions {ϕn}, which is deﬁned by*

(1)

*ϕ* (*x*) = Σ2*n−*1 *f* (*j*2*−n*)*χ*

*I*(*n,j*)

(*x*)*,*

*Fine-converges effectively to f.*

*j*=0

*n*

*Moreover, if f is uniformly Fine-computable, then ϕn converges effectively uniformly to f.*

*{ }*

We will briefly review effective integrability of functions on [0*,* 1). See [[6](#_bookmark28),[7](#_bookmark29),[8](#_bookmark30)] for details.

**Definition 2.10** (Effective integrability of a sequence of functions, [[7](#_bookmark29),[8](#_bookmark30)])

A sequence of Fine-computable functions *{fn}* is called *effectively integrable* if each

*{*∫

*fn* is integrable and of real numbers.

*n*

*n*

[0*,*1)

*f* +(*x*)*dx}* and *{*∫

[0*,*1)

*f−*(*x*)*dx}* are computable sequences

A Fine-computable function is said to be *effectively integrable* if the sequence

*f, f, . . .* is effectively integrable.

Integral on a finite union of fundamental dyadic intervals *E* is defined to be

∫[0*,*1) *f* (*x*)*χE*(*x*)*dx*.

It is easy to prove that a computable sequence of dyadic step functions is effec- tively integrable.

**Theorem 2.11** (Effective bounded convergence theorem, [[7](#_bookmark29),[8](#_bookmark30)]) *Let gn be a uni-*

*{ }*

*formly bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to f. Then, f is Fine-computable and {*∫[0*,*1) *gn*(*x*)*dx} converges effectively to* ∫[0*,*1) *f* (*x*)*dx. As a consequence, f is effectively integrable.*

**Theorem 2.12** ([[7](#_bookmark29),[8](#_bookmark30)]) *A bounded Fine-computable function is effectively inte- grable.*

**Theorem 2.13** ([[7](#_bookmark29),[8](#_bookmark30)]) *Let {fn} be Fine-computable and effectively bounded, that is, there exists a computable sequence of reals {Mn} such that |fn*(*x*)*|* ≤ *Mn for all*

*x. Then {fn} is effectively integrable.*

**Theorem 2.14** (Effective dominated convergence theorem, [[7](#_bookmark29),[8](#_bookmark30)]) *Let gm,n be an effectively integrable Fine-computable sequence which Fine-converges effectively to . Suppose that there exists an effectively integrable Fine-computable func-*

*{ }*

*{fm}*

*tion h such that |gm,n*(*x*)*|* ≤ *h*(*x*)*. Then, {*∫[0*,*1) *gm,n*(*x*)*dx} converges effectively to*

∫[0*,*1) *fm*(*x*)*dx.*

**Proposition 2.15** ([[7](#_bookmark29),[8](#_bookmark30)]) *Let f be an effectively integrable Fine-computable func-*

*tion and let In be a computable sequence of dyadic intervals such that* *∞ In* =

*n*=1

[0*,* 1)*. Put En* = *n Ii. Then,* ∫ *f* (*x*)*dx converges effectively to* ∫ *f* (*x*)*dx, or*

*i*=1

*n*

*En* [0*,*1)

*equivalently,* ∫*E C f* (*x*)*dx converges effectively to zero.*

# Uniformly Fine-computable functions on [0*,* 1)2

The main objective of this section is to prove uniform Fine-computability of *f* (*x*) =

∫[0*,*1) *F* (*x, y*)*dy* for a uniformly Fine-computable function *F* (*x, y*) on [0*,* 1)2.

On the upper-right open unit square [0*,* 1)2, we denote [*k*2*−n,* (*k* + 1)2*−n*) [*l*2*−m,* (*l* + 1)2*−m*) with *I*2(*n, m*; *k, l*) and call it a *fundamental dyadic rectangle*. We also denote *J*(*x, n*) *J*(*y, m*) by *J*2(*x, y*; *n, m*) and call it a *fundamental dyadic neigh- borhood* of (*x, y*). We call the topology generated by the set *J*2(*ei, ej*; *n, m*) *i,j,n,m* the *Fine-topology* on [0*,* 1)2 and the space [0*,* 1)2 with this topology the *two-dimensional Fine-space*. Notions of computability on [0*,* 1)2 are defined with respect to the Fine- topology.

*×*

*{ }*

*×*

Note that *{J*2(*x, y*; *n, n*)*}* satisfies the axioms of the effective uniformity (cf. [[10](#_bookmark32)]), and the topology generated by the set *{J*2(*ei, ej*; *n, n*)*}* is equivalent.

**Definition 3.1** (1) A double sequence *{*(*xp,q, yp,q*)*}* from [0*,* 1)2 is said to *Fine- converges effectively* to *{*(*xp, yp*)*}* if there exists a recursive function *α*(*p, n, m*) such that *q* ≥ *α*(*p, n, m*) implies (*xp,q, yp,q*) *∈ J*2(*xp, yp*; *n, m*).

(2) A sequence *{*(*xp, yp*)*}* is said to be *Fine-computable* if there exist recursive sequences of dyadic rationals *{sp,q}* and *{tp,q}* such that *{sp,q}* and *{tp,q}* Fine- converge effectively to *{xp}* and *{yp}* respectively.

**Lemma 3.2** (cf. Lemma [2.1](#_bookmark4)) (1) *The following three properties are equivalent for any* (*x, y*)*,* (*z, w*) *∈* [0*,* 1)2 *and any positive integers n, m.*

(i) (*z, w*) *∈ J*2(*x, y*; *n, m*)*.* (ii) (*x, y*) *∈ J*2(*z, w*; *n, m*)*.*

1. *J*(*x, y*; *n, m*) = *J*(*z, w*; *n, m*)*.*
2. *If {*(*xp, yp*)*} is Fine-computable, then we can decide effectively whether*

(*xp, yp*) *∈ I*2(*n, m*; *k, l*) *or not.*

In the following, we use the notation *F* (*x,* ) to designate the function *F* (*x, y*) regarded as a function of *y* (for each fixed *x*).

*·*

**Definition 3.3** (Uniform Fine-computability) A function *F* (*x, y*) on [0*,* 1)2 is said to be *uniformly Fine-computable* if it satisfies the following two conditions.

* 1. (Sequential computability) *{F* (*xn, ym*)*}* is a computable double sequence of reals for every Fine-computable sequence *{*(*xn, ym*)*}*.
  2. (Effective uniform Fine-continuity) There exist recursive functions *α*1(*k*) and *α*2(*k*) such that (*x, y*) *∈ J*2(*z, w*; *α*1(*k*)*, α*2(*k*)) implies *|F* (*x, y*) *−F* (*z, w*)*| <* 2*−k*.

**Proposition 3.4** *Let F* (*x, y*) *be uniformly Fine-computable as a function of* (*x, y*)*. Then the following hold.*

* + 1. *If xn is a Fine-computable sequence, then fn*(*y*) = *F* (*xn, y*) *is a uniformly Fine-computable sequence of functions on* [0*,* 1) *(Deﬁnition* [*2.4*](#_bookmark5)*).*

*{ } { } { }*

* + 1. *If a Fine-computable sequence {xm,n} Fine-converges effectively to {xm}, then {F* (*xm,n, ·*)*} converges effectively uniformly to {F* (*xm, ·*)*} (Deﬁnition* [*2.5*](#_bookmark6)*).*

*Proof* Let *α*1(*k*) and *α*2(*k*) be as in Definition [3.3](#_bookmark14).

1. Let *ym* be a Fine-computable sequence of reals. Then *fn*(*ym*) = *F* (*xn, ym*) is a computable sequence of reals due to the sequential computability of *F* (*x, y*). Then, *fn*(*y*) *fn*(*z*) = *F* (*xn, y*) *F* (*xn, z*) *<* 2*−k* if *y J*(*z, α*2(*k*)), and hence follows effective uniform Fine-continuity.

*| − | | − | ∈*

*{ }*

*{ } { }*

1. From the effective Fine-convergence of *{xm,n}* to *{xm}*, there exists a re- cursive function *β*(*m, l*) such that *n ≥ β*(*m, l*) implies *xm,n ∈ J*(*xm, l*).

If we take *δ*(*m, k*) = *β*(*m, α*1(*k*)), then *|F* (*xm,n, y*) *− F* (*xm, y*)*| <* 2*−k* for *n ≥*

*δ*(*m, k*) and all *y ∈* [0*,* 1).

It is pointed out in [[4](#_bookmark26)] that a uniformly Fine-computable function *g*(*y*) on [0*,* 1) is bounded and has a computable supremum. The latter property holds for a uniformly Fine-computable sequence of functions. These properties are easily deduced from Theorem 2 in [[3](#_bookmark25)]. We denote the supremum of *|g|* by *||g||*.

Similarly, we can prove that a uniformly Fine-computable function *F* (*x, y*) takes a computable supremum.

Regarding uniform Fine-computability of *F* (*x, y*), we obtain the following theo- rem.

**Theorem 3.5** *For a function F* (*x, y*)*, the following* (i) *and* (ii) *are equivalent.*

* 1. *F* (*x, y*) *is uniformly Fine-computable.*
  2. (ii-a) *{F* (*xn, ·*)*} is a uniformly Fine-computable sequence of functions on*

[0*,* 1) *for any Fine-computable sequence {xn}.*

(ii-b) *There exists a recursive function α*(*k*) *such that, y ∈ J*(*x, α*(*k*)) *implies*

*||F* (*x, ·*) *− F* (*y, ·*)*|| <* 2*−k for all k.*

*Proof* (i)*⇒*(ii): (ii-a) follows immediately from Proposition [3.4](#_bookmark15) (1).

To prove (ii-b), let us take *α*1(*k*) and *α*2(*k*) in Definition [3.3](#_bookmark14). If *x ∈ J*(*y, α*1(*k* + 1)), then (*x, z*) *∈ J*2(*y, z*; *α*1(*k* + 1)*, α*2(*k* + 1)) for all *z ∈* [0*,* 1). So, *|F* (*x, z*) *− F* (*y, z*)*| <* 2*−*(*k*+1) and *||F* (*x, ·*) *− F* (*y, ·*)*|| <* 2*−k*.

(ii)*⇒*(i): Let *α*(*k*) be the recursive function in (ii-b). Then, *z ∈ J*(*x, α*(*k*)) implies *||F* (*x, ·*) *− F* (*z, ·*)*|| <* 2*−k*. Put *rk,j* = *j*2*−α*(*k*) for *j* = 0*,* 1*,...,* 2*α*(*k*) *−* 1. By (ii-a), the sequence *{F* (*rk,j, ·*)*}* is a uniform Fine-computable sequence of functions on [0*,* 1). So, there exists a recursive function *β*(*k, j*) such that *y ∈ J*(*w, β*(*k, j*)) implies *|F* (*rk,j, y*) *− F* (*rk,j, w*)*| <* 2*−k*.

Define *γ*(*k*) = max *α*(*k* + 2)*, β*(*k* + 2*,* 0)*, β*(*k* + 2*,* 1)*,..., β*(*k* + 2*,* 2*α*(*k*+2) 1) and suppose that (*x, y*) *J*2(*z, w*; *γ*(*k*)*, γ*(*k*)). Since *z J*(*x, α*(*k* + 2)), there exists a *j*, such that [*j*2*−α*(*k*+2)*,* (*j* + 1)2*−α*(*k*+2)) contains both *x* and *z*. Therefore, we obtain

*∈ ∈*

*{ − }*

*|F* (*x, y*) *− F* (*z, w*)*|*

≤ *|F* (*x, y*) *− F* (*rk*+2*,j, y*)*|* + *|F* (*rk*+2*,j, y*) *− F* (*rk*+2*,j, w*)*|* + *|F* (*rk*+2*,j, w*) *− F* (*z, w*)*|*

*<* 3 *·* 2*−*(*k*+2) *<* 2*−k.*

This shows effective uniform Fine-continuity of *F* (*x, y*).

Let *xn* and *ym* be Fine-computable sequences. Then *F* (*xn,* ) is a uni- formly Fine-computable sequence of functions. This implies that *F* (*xn, ym*) is a computable sequence of reals.

*{ }*

*{ } { } { · }*

It is easy to check that a uniformly Fine-computable function on [0*,* 1)2 is Lebesgue integrable and that its integral is a computable number, similarly to the case of uniformly Fine-computable functions on [0*,* 1) ([[7](#_bookmark29)]).

**Theorem 3.6** (Effective Fubini’s Theorem for uniformly Fine-computable func- tions)

*Let F* (*x, y*) *be a uniformly Fine-computable function. Then the following hold.*

1. *If xn is Fine-computable, then F* (*xn,* ) *and F* ( *, xn*) *are uniformly Fine-computable sequences of functions on* [0*,* 1)*.*

*{ } { · } { · }*

1. ∫[0*,*1) *F* (*x, y*)*dy and* ∫[0*,*1) *F* (*x, y*)*dx are uniformly Fine-computable functions.*
2. ∫∫[0*,*1)2 *F* (*x, y*)*dxdy is a computable number and*

∫∫[0*,*1)2 *F* (*x, y*)*dxdy* = ∫[0*,*1) *dx* ∫[0*,*1) *F* (*x, y*)*dy* = ∫[0*,*1) *dy* ∫[0*,*1) *F* (*x, y*)*dx.*

*Proof.* (1) is Proposition [3.4](#_bookmark15) (1). The equation in (3) is a consequence of classical Fubini’s Theorem.

(2) To prove sequential computability, let *xn* be a Fine-computable sequence. Then ( ) is a uniformly bounded uniformly Fine-computable sequence of

*{F xn, · }*

*{ }*

functions. Hence, *{*∫[0*,*1) *F* (*xn, y*)*dy}* is a computable sequence of reals by Theorem

[2.13](#_bookmark10).

Effective uniform Fine-continuity follows from the inequality

*|* ∫[0*,*1) *F* (*x, y*)*dy −* ∫[0*,*1) *F* (*z, y*)*dy|* ≤ *||F* (*x, ·*) *− F* (*z, ·*)*||*

and Theorem [3.5](#_bookmark16) (ii-b).

We can easily extend (2) above as follows.

**Theorem 3.7** *Let F* (*x, y*) *be a uniformly Fine-computable function on* [0*,* 1)2 *and let g be an effectively integrable Fine-computable function on* [0*,* 1)*. Then* (*Tg*)(*x*) =

∫[0*,*1) *g*(*y*)*F* (*x, y*)*dy is uniformly Fine-computable.*

*Especially, the operator T maps any uniformly Fine-computable function to a uniformly Fine-computable function.*

*Proof* First, we note that *M* = sup(*x,y*)*∈*[0*,*1)2 *F* (*x, y*) is computable if *F* (*x, y*) is uniformly Fine-computable on [0*,* 1)2.

*| |*

Let *xm* be Fine-computable. Then *g*(*y*)*F* (*xm, y*) is a Fine-computable se- quence of functions of *y* by Theorem [3.6](#_bookmark17) (1). If we take the approximating com- putable sequence of dyadic step functions *ϕm,n*(*y*) which Fine-converges effectively to *g*(*y*)*F* (*xm, y*) , obtained by Proposition [2.9](#_bookmark9), then, it is obviously an effectively integrable Fine-computable sequence and satisfies ( ) ( ) . Hence,

*{ } { }*

*|ϕm,n y |* ≤ *M|g y |*

*{ }*

*{ }*

*{*∫[0*,*1) *ϕm,n*(*y*)*dy}* converges effectively to *{*∫[0*,*1) *g*(*y*)*F* (*xm, y*)*dy}* by Theorem [2.14](#_bookmark11).

Therefore, *{*∫[0*,*1) *g*(*y*)*F* (*xm, y*)*dy}* is a computable sequence.

Effective uniform continuity follows from the following inequality;

*|* ∫[0*,*1) *g*(*y*)*F* (*x, y*)*dy −* ∫[0*,*1) *g*(*y*)*F* (*z, y*)*dy|* ≤ *||F* (*x, ·*) *− F* (*z, ·*)*||* ∫[0*,*1) *|g*(*z*)*|dz.*

# Fine-computable functions on [0*,* 1)2

In the following, we treat Fine-computability of *f* (*x*) = ∫[0*,*1) *F* (*x, y*)*dy* for a Fine- computable function *F* (*x, y*). First we define Fine-computability of functions on [0*,* 1)2, which is weaker than uniform Fine-computability (Definition [3.3](#_bookmark14)), as follows.

**Definition 4.1** (Fine-computable functions on [0*,* 1)2) Let *F* (*x, y*) be a function on [0*,* 1)2. *F* is said to be *Fine-computable* if it satisfies the following (i) and (ii).

* 1. *F* is sequentially computable.
  2. (Effective Fine-continuity) There exist recursive functions *α*1(*k, i, j*) and

*α*2(*k, i, j*) which satisfy

(ii-a) (*x, y*) *∈ J*2(*ei, ej*; *α*1(*k, i, j*)*, α*2(*k, i, j*)) implies *|F* (*x, y*) *− F* (*ei, ej*)*| <* 2*−k*,

(ii-b) *∞ J*2(*ei, ej*; *α*1(*k, i, j*)*, α*2(*k, i, j*)) = [0*,* 1)2 for each *k*.

*i,j*=1

We state the Proposition 3.1 in [[6](#_bookmark28)] for the case *{ri}* = *{ei}*.

**Proposition 4.2** *A function g on* [0*,* 1) *is effectively Fine-continuous if and only if there exist a recursive sequence of dyadic rationals rk,q and a recursive function δ*(*k, q*) *which satisfy the following.*

*{ }*

* + 1. *x ∈ J*(*rk,q, δ*(*k, q*)) *implies |g*(*x*) *− g*(*rk,q*)*| <* 2*−k.*
    2. *∞ J*(*rk,q, δ*(*k, q*)) = [0*,* 1) *for each k.*

*q*=1

* + 1. *The intervals in J*(*rk,q, δ*(*k, q*)) *are mutually disjoint with respect to q for* *each k.*

*{ }*

In the proof of Proposition 3.1 in [[6](#_bookmark28)], the crucial properties are those of Lemma [2.1](#_bookmark4), whose two-dimensional version is Lemma [3.2](#_bookmark13), and the fact that the complement of a finite (disjoint) union of fundamental dyadic intervals can be represented as a finite disjoint union of fundamental dyadic intervals. A similar fact also holds for fundamental dyadic rectangles. So, we can prove the following proposition.

**Proposition 4.3** *Effective Fine-continuity of a function F on* [0*,* 1)2 *is equiva- lent to the following: There exist a recursive sequence of pairs of dyadic rationals* (*sk,p, tk,p*) *and recursive functions β*1(*k, p*)*, β*2(*k, p*) *which satisfy the following three conditions.*

*{ }*

1. (*x, y*) *∈ J*2(*sk,p, tk,p*; *β*1(*k, p*)*, β*2(*k, p*)) *implies |F* (*x, y*) *−F* (*sk,p, tk,p*)*| <* 2*−k.*
2. *∞ J*2(*sk,p, tk,p*; *β*1(*k, p*)*, β*2(*k, p*)) = [0*,* 1)2 *for each k.*

*p*=1

1. *The fundamental dyadic neighborhoods in J*2(*sk,p, tk,p*; *β*1(*k, p*)*, β*2(*k, p*))

*{ }*

*are mutually disjoint with respect to p for each k.*

**Remark 4.4** The conditions (b) and (c) in Proposition [4.3](#_bookmark20) signify that the upper- right open square [0*,* 1)2 is partitioned into (infinitely many) disjoint rectangles

*{J*2(*sk,p, tk,p*; *β*1(*k, p*)*, β*2(*k, p*))*}* for each *k*. Hence, the following hold:

1. For any *k* and any (*x, y*) *∈* [0*,* 1)2, there is the unique number *p*(*k, x, y*) such that (*x, y*) is contained in *J*2(*sk,p*(*k,x,y*)*, tk,p*(*k,x,y*); *β*1(*k, p*(*k, x, y*))*, β*2(*k, p*(*k, x, y*))).

Moreover, (*z, w*) *J*2(*sk,p*(*k,x,y*)*, tk,p*(*k,x,y*); *β*1(*k, p*(*k, x, y*))*, β*2(*k, p*(*k, x, y*))) implies

*∈*

*p*(*k, x, y*) = *p*(*k, z, w*).

1. If (*xn, yn*) is Fine-computable, then *i*(*k, n*) = *p*(*k, xn, yn*) is a recursive function.

*{ }*

**Proposition 4.5** *Let F* (*x, y*) *be Fine-computable. Then the following hold.*

1. *If xm is a Fine-computable sequence of reals, then F* (*xm,* ) *is a Fine- computable sequence of functions.*

*{ } { · }*

1. *If {xm,n} is a Fine-computable sequence of reals and Fine-converges effec- tively to {xm}, then {F* (*xm,n, ·*)*} Fine-converges effectively to {F* (*xm, ·*)*}.*

*Proof.* Let us take *{*(*sk,p, tk,p*)*}* and *β*1(*k, p*), *β*2(*k, p*) in Proposition [4.3](#_bookmark20).

*Proof of* (1): We prove (i) and (ii) in Definition [2.6](#_bookmark7) for *{F* (*xm, ·*)*}*.

(i): Sequential computability of *F* (*xm,* ) is an easy consequence of sequential computability of *F* (*x, y*).

*{ · }*

(ii-a): For each *m*, *k* and *j*, we can find effectively and uniquely such *p* = *p*(*m, k, j*) that (*xm, ej*) is contained in *J*2(*sk*+1*,p, tk*+1*,p*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)) by Remark [4.4](#_bookmark21). Define *α*(*m, k, j*) = *β*2(*k* + 1*, p*(*m, k* + 1*, j*)) and suppose that *y ∈ J*(*ej, α*(*m, k, j*)).

Then (*xm, y*) is also contained in *J*2(*sk*+1*,p, tk*+1*,p*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)). So, we obtain

*|F* (*xm, y*) *− F* (*xm, ej*)*|*

≤ *|F* (*xm, y*) *− F* (*sk*+1*,p, tk*+1*,p*)*|* + *|F* (*sk*+1*,p, tk*+1*,p*) *− F* (*xm, ej*)*| <* 2*−k.*

(ii-b): Let us take *p* = *p*(*k, xm, y*) for arbitrary *y* [0*,* 1), as in Remark

*∈*

* 1. Then, *J*2(*sk*+1*,p, tk*+1*,p*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)) contains (*xm, ej*) for some dyadic rational *ej*. By Remark [4.4](#_bookmark21) (a), we obtain *p*(*k, xm, y*) = *p*(*k, xm, ej*) and *J*2(*sk*+1*,p, tk*+1*,p*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)) = *J*2(*xm, ej*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)).

Hence, *∞ J*(*ej, α*(*m, k, j*)) = [0*,* 1) holds.

*j*=1

*Proof of* (2): We note first that *xm* is a Fine-computable sequence. Let *γ*(*m, l*) be an effective modulus of effective Fine-convergence. That is, it satisfies that *n* ≥ *γ*(*m, l*) implies *xm,n ∈ J*(*xm, l*).

*{ }*

For any *m* and *ej*, we can find effectively and uniquely such *p* = *p*(*m, j*) that *J*2(*sk*+1*,p, tk*+1*,p*; *β*1(*k* + 1*, p*)*, β*2(*k* + 1*, p*)) contains (*xm, ej*). We note that *J*(*sk*+1*,p, β*1(*k* + 1*, p*)) = *J*(*xm, β*1(*k* + 1*, p*)) by Lemma [2.1](#_bookmark4).

If *n* ≥ *γ*(*m, β*1(*k* + 1*, p*)) and *y ∈ J*(*tk*+1*,p, β*2(*k* + 1*, p*)) = *J*(*ej, β*2(*k* + 1*, p*)), then

*|F* (*xm,n, y*) *− F* (*xm, y*)*|*

≤ *|F* (*xm,n, y*) *−F* (*sk*+1*,p, tk*+1*,p*)*|* + *|F* (*sk*+1*,p, tk*+1*,p*) *−F* (*xm, y*)*| <* 2 *·* 2*−*(*k*+1) = 2*−k.*

By Proposition [4.3](#_bookmark20) (b), *J*(*sk*+1*,p,β*1(*k*+1*,p*))*$x J*(*tk*+1*,p, β*2(*k* + 1*, p*)) = [0*,* 1) for all *k*, and hence, *j J*(*ej, β*2(*k* + 1*, p*)) = [0*,* 1). This proves the effective Fine- convergence of *F* (*xm,n,* ) to *F* (*xm,* ) with respect to *α*(*k, j*) = *β*2(*k*+1*, p*(*m, j*)) and *δ*(*k, i*) = *γ*(*m, β*1(*k* + 1*, p*(*m, j*))) (cf. Definition [2.7](#_bookmark8)).

*{ · } { · }*

In the rest of this section, we investigate Fine-computability of the function

*f* (*x*) = ∫[0*,*1) *F* (*x, y*)*dy* for a bounded Fine-computable function *F* (*x, y*).

**Theorem 4.6** *If F* (*x, y*) *is bounded and Fine-computable on* [0*,* 1)2*, then f* (*x*) =

∫[0*,*1) *F* (*x, y*)*dy is Fine-computable on* [0*,* 1)*.*

*Outline of the proof of Effective Fine-continuity*: Let us take (*sk,p, tk,p*) and *β*1(*k, p*), *β*2(*k, p*) in Proposition [4.3](#_bookmark20). Then, we construct a function *N* (*k, x*), on N+ [0*,* 1), functions *h*(*k, x, l*), *α*1(*k, x, l*), *α*2(*k, x, l*) on N+ [0*,* 1) 1*,* 2*,...,N* (*k, x*) and sequences *uk,x,l*, *vk,x,l* for each *k*, *x* and 1 ≤ *l* ≤ *N* (*k, x*), which satisfy the following:

*× × { }*

*×*

*{ }*

* + 1. Dyadic intervals *{J*(*vk,x,l, α*2(*k, x, l*))*}*1≤*l*≤*N*(*k,x*) are mutually disjoint.
    2. Σ*N* (*k,x*) 2*−α*2(*k,x,l*) *>* 1 *−* 2*−k*.

*l*=1

* + 1. *y ∈ J*(*vk,x,l, α*2(*k, x, l*)) and *z ∈ J*(*uk,x,l, α*1(*k, x, l*)) imply

*|F* (*x, y*) *− F* (*z, y*)*| <* 2*−k* due to Proposition [4.3](#_bookmark20) (a) for 1 ≤ *l* ≤ *N* (*k, x*).

* + 1. *uk,x,l* ≤ *x < uk,x,l* + 2*−α*1(*k,x,l*) for 1 ≤ *l* ≤ *N* (*k, x*).

Define *ξ*(*k, x*) = max1≤*l*≤*N* (*k,x*) *uk,x,l* and *η*(*k, x*) = min1≤*l*≤*N* (*k,x*) *uk,x,l*+2*−α*1(*k,x,l*).

Then, [*ξ*(*k, x*)*, η*(*k, x*)) is a dyadic interval and contains *x*. So, we can define

*γ*(*k, x*) as min*{l| J*(*x, l*) *⊂* [*ξ*(*k, x*)*, η*(*k, x*))*}*.

Properties of *γ*(*k, x*) and *N* (*k, x*):

* + - 1. If *z J*(*x, γ*(*k, x*)), then *N* (*k, z*) = *N* (*k, x*). Moreover, *uk,x,l* = *uk,z,l*,

*∈*

*vk,x,l* = *vk,z,l* and *αi*(*k, x, l*) = *αi*(*k, z, l*) for 1 ≤ *l* ≤ *N* (*k, x*).

* + - 1. If *y ∈* *N* (*k,x*) *J*(*vk,x,l, α*2(*k, x, l*)) and *z ∈ J*(*x, γ*(*k, x*)), then

*l*=1

*|F* (*x, y*) *− F* (*z, y*)*| <* 2*−k*.

* + - 1. *|* *N* (*k,x*) *J*(*vk,x,l, α*2(*k, x, l*))*|* = Σ*N* (*k,x*) 2*−α*2(*k,x,l*) *>* 1 *−* 2*−k*.

*n*=1

*n*=1

Moreover, *N* (*k, ei*), *αi*(*k, ei, l*) (*i* = 1*,* 2) and *γ*(*k, ei*) can be regarded as recursive functions. While, *uk,ei,l* and *vk,ei,l* can be regarded as computable sequences of dyadic rationals.

From boundedness of *F* (*x, y*), there exists an integer *K* such that *F* (*x, y*) *<* 2*K* for all (*x, y*). Now, if we define *δ*(*k, i*) = *γ*(*k* + *K* + 2*, ei*), then *δ* is a recur- sive function. Suppose that *x J*(*e , δ*(*k, i*)) = *J*(*e , γ*(*k* + *K* + 2*, e* )), and put

*| |*

*∈ i i* *i*

*Ek,i* = *N* (*k,ei*) *J*(*vk,e ,l, α*2(*k, ei, l*)). Then, *Ek,i* = *N* (*k,x*) *J*(*vk,x,l, α*2(*k, x, l*)), and

*l*=1 *i*

we obtain

*l*=1

*|f* (*x*) *− f* (*ei*)*|* ≤ ∫*Ek,i |F* (*x, y*) *− F* (*ei, y*)*|dy* + ∫(*Ek,i*)*C |F* (*x, y*)*|dy* + ∫(*Ek,i*)*C |F* (*ei, y*)*|dy*

*<* 2*−*(*k*+*K*+2) +2 *·* 2*K* 2*−*(*k*+*K*+2) *<* 2*−k.*

For all *x ∈* [0*,* 1), *J*(*x, δ*(*k, x*)) contains a dyadic rational, say, *ei*. By property (i),

*J*(*x, δ*(*k, i*)) = *J*(*ei, δ*(*k, i*)). So *x ∈ J*(*ei, δ*(*k, i*)) and we obtain *∞ J*(*ei, δ*(*k, i*)) =

*i*=1

[0*,* 1). This proves effective Fine-continuity of *f* (*x*).

Example [1.3](#_bookmark2) in Introduction shows that the conclusion of Theorem [4.6](#_bookmark22) does not hold for a Fine-computable function in general. Therefore, for general Fine- computable functions, we need an additional condition on integrability.

We give a sufficient condition that assures the Fine-computability of *f* (*x*) for a

Fine-computable function *F* (*x, y*).

**Theorem 4.7** *If F* (*x, y*) *is Fine-computable and there exists an effectively inte- grable Fine-computable function* ( ) *which satisﬁes* ( ) ( ) *for all , then*

*g y |F x, y |* ≤ *g y* *x*

*f* (*x*) = ∫[0*,*1) *F* (*x, y*)*dy is Fine-computable.*

# References

1. Brattka, V. Some Notes on Fine Computability. *Journal of Universal Computer Science*, 8:382-395, 2002.
2. Fine, N. J. On the Walsh Functions. *Trans. Amer. Math. Soc.*, 65:373-414, 1949.
3. Mori, T. On the computability of Walsh functions. *Theoretical Computer Science*, 284:419-436, 2002.
4. Mori, T. Computabilities of Fine continuous functions. *Acta Humanistica et Scientifica Universitatis Sangio Kyotiensis, Natural Science Series I,* 31:163-220, 2002. (in Japanese)
5. Mori, T., Y. Tsujii and M. Yasugi. Computability Structures on Metric Spaces. *Combinatorics, Complexity and Logic* (*Proceedings of DMTCS’96*), ed. by Bridges *et al.,* 351-362. Springer, 1996.
6. Mori, T., Y. Tsujii and M. Yasugi. Fine computable functions and effective Fine convergence. (accepted by *Mathematics Applied in Sciencd and Technology*)
7. Mori, T., Y. Tsujii and M. Yasugi. Integral of Fine Computable functions and Walsh Fourier series.

*ENTCS* 202:279-293, 2008.

1. Mori, T., Y. Tsujii and M. Yasugi. Effective Fine Convergence of Walsh Fourier series. (accepted by

*Mathematical Logic Quarterly*)

1. Pour-El, M.B. and J. I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1989.
2. Tsujii, Y., M. Yasugi and T. Mori. Some Properties of the Effective Uniform Topological Space. *Computability and Complexity in Analysis,* (*4th International Workshop, CCA2000. Swansea*), Wed. by Blanck, J. *et al.,* 336-356. Springer, 2001.
3. Yasugi, M., Y. Tsujii and T. Mori. Sequential computability of a function - Effective Fine space and limitting recursion -, *Journal of Universal Computer Science*, 11-12, 2179-2191, 2005.