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Large Immersions in Graphs with Independence Number 3 and 4

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**Abstract**

The analogue of Hadwiger’s conjecture for the immersion order, a conjecture independently posed by Lescure and Meyniel, and by Abu-Khzam and Langston, states that every graph *G* which does not contain the complete graph *Kt*+1 as an immersion satisfies *χ*(*G*) *≤ t*. If true, this conjecture would imply that every graph with *n* vertices and independence number *α* contains *K[ n |* as an immersion (and if *α* = 2, the two

*α*

statements are known to be equivalent).

The immersion conjecture has been tackled with more success than its graph minors counterpart: not only is a linear upper bound known for the chromatic number of *Kt*+1-immersion-free graphs, but the best bound currently known is very close to optimal. Namely, the currently best bound in this respect is due to Gauthier, Le and Wollan, who recently proved that every graph not containing *Kt*+1 as an immersion satisfies *χ*(*G*) *≤* 3*.*54*t* + 7. Their result implies that any graph with *n* vertices and independence number *α* contains *K*  *n* as an immersion, where *c <* 1*.*98. Moreover, the same authors prove that every graph

*[ −c|*

3*.*54*α*

of independence number 2 contains *K*2*[ n ♩* as an immersion.

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We show that any graph with *n* vertices and independence number 3 contains a clique immersion on at least *[* 2*n ♩−*1 vertices, and any graph with *n* vertices and independence number 4 contains a clique immersion on at least *[* 4*n ♩−* 1 vertices. Thus, comparing to the bound from above, in both cases we roughly double the size of the immersion obtained.

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*Keywords:* Graph immersion, independence number, chromatic number, Hadwiger’s conjecture

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# Introduction

For a graph *G*, let *χ*(*G*) denote its chromatic number. A famous conjecture of Hadwiger [[13](#_bookmark23)] states that every graph *G* which does not contain the complete graph *Kt*+1 as a minor satisfies *χ*(*G*) *≤ t*. The conjecture is only known to be true for 1 *≤ t ≤* 5; the proofs for cases *t* = 4 and *t* = 5 depend on the Four Color Theorem. For general *t*, the currently best upper bound on the chromatic number springs from (independent) results of Kostochka [[16](#_bookmark30)] and Thomason [[20](#_bookmark34)] and is

*χ*(*G*) *∈* O(*t√*log *t*).

Letting *α*(*G*) denote the independence number of a graph *G*, it is well known

and easy to see that *χ*(*G*) *≥ [ n |*. Therefore, the Four Color Theorem implies

*α*(*G*)

that every planar graph on *n* vertices has an independent set of size *[ n|*. In 1968, Erd˝os asked whether this corollary could be proved without using the Four Color Theorem. This question is still open (see [[5](#_bookmark19)]).

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Similarly, Hadwiger’s conjecture, if true, implies that a graph *G* on *n* vertices excluding *Kt*+1 as a minor contains an independent set of size *[ n|*, which would be best possible as evidenced by the disjoint union of copies of *Kt*. Alternatively, we could say Hadwiger’s conjecture implies that every *n*-vertex graph contains a clique minor of size at least *[*  *n |*.

*t*

*α*(*G*)

As mentioned above, it is not known whether every graph *G* excluding the complete graph *Kt*+1 as a minor satisfies *χ*(*G*) *≤* 2*t* (and indeed, any linear upper bound on Hadwiger’s conjecture would be a major breakthrough). However, in 1982, Duchet and Meyniel [[9](#_bookmark24)] proved that any graph *G* on *n* vertices contains a clique minor of size *[*  *n |*. There have been several improvements on this bound [[3,](#_bookmark17)[11,](#_bookmark25)[14,](#_bookmark28)[15,](#_bookmark29)[22](#_bookmark36)], the best bound, due to Balogh and Kostochka [[2](#_bookmark16)], being *[ cα*(*G*) *|*, where *c* is a constant with *c <* 1*.*95.

*n*

2*α*(*G*)*−*1

In this extended abstract, we shall focus on the immersion relation on graphs,

which has received extensive attention in recent years. The immersion relation can be seen as a variant of the topological minor relation. A graph *G* is said to contain another graph *H* as an *immersion* if there exists an injective function *φ* : *V* (*H*) *→ V* (*G*) such that:

1. For every *uv ∈ E*(*H*), there is a path in *G*, denoted *Puv*, with endpoints *φ*(*u*) and *φ*(*v*).
2. The paths in *{Puv | uv ∈ E*(*H*)*}* are pairwise edge disjoint.

We call the vertices in *φ*(*V* (*H*)), the *branch vertices* of the immersion. By *splitting off the path Puv* we mean we delete the edges of the path and add an edge *uv* to *G* (unless it is already present). So, *H* is an immersion of *G*, if it can be obtained from

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*G* through operations of vertex deletion, edge deletion and splitting off paths *Puv*

(in any order).

The minor order and the immersion order are not comparable. The class of planar graphs, while excluding *K*5 as a minor, contains all cliques as immersions. On the other hand, the class of graphs with maximum degree at most 3, while excluding *K*4 as an immersion, contains all cliques as minors.

However, the two relations do share some important similarities. Both of them are well-quasi-orders, as proved by Robertson and Seymour [[18,19](#_bookmark33)]. Additionally, if a graph *G* contains a graph *H* as a topological minor then *G* contains *H* both as a minor and as an immersion. Possibly inspired by such similarities, Lescure and Meyniel [[17](#_bookmark31)] and, independently, Abu-Khzam and Langston [[1](#_bookmark15)] proposed the following conjecture.

**Conjecture 1.1 ([**[**1**](#_bookmark15)**,**[**17**](#_bookmark31)**])** *Every graph G excluding the complete graph Kt*+1 *as an immersion satisﬁes χ*(*G*) *≤ t.*

This conjecture has received much attention recently, and has been tackled with more success than its minor order counterpart. The cases 1 *≤ t ≤* 3 follow from the fact that Hajos’ Conjecture (replace ‘immersion’ by ‘topological minor’ in Conjec- ture [1.1](#_bookmark1)) is true for these cases [[8](#_bookmark22)]. The cases 4 *≤ t ≤* 6 were solved independently by Lescure and Meyniel [[17](#_bookmark31)] and DeVos, Kawarabayashi, Mohar, and Okamura [[7](#_bookmark21)]. For general *t*, DeVos, Dvoˇr´ak, Fox, McDonald, Mohar, and Scheide [[6](#_bookmark20)] were the first to give a linear upper bound for Conjecture [1.1](#_bookmark1). They proved that every graph *G* excluding *Kt*+1 as an immersion satisfies *χ*(*G*) *≤* 200*t*. Dvoˇr´ak and Yepre- myan [[10](#_bookmark26)] improved that upper bound to 11*t* + 6. The best upper bound currently known is due to Gauthier, Le and Wollan [[12](#_bookmark27)] who showed that if *G* excludes *Kt*+1

as an immersion, then *G* satisfies *χ*(*G*) *≤* 3*.*54*t* + 7. This implies the following.

**Theorem 1.2 (Gauthier, Le and Wollan [**[**12**](#_bookmark27)**])** *Let G be a graph on n vertices and with independence number α. Then G contains K*  *n as an immersion,*

*[ −c|*

3*.*54*α*

*where c <* 1*.*98 *is a constant.*

It seems natural to ask whether this can be improved without necessarily improv- ing Gauthier, Le and Wollan’s bound on the chromatic number of *Kt*+1-immersion- free graphs, as Duchet and Meyniel did in the context of graph minors.

The first attempt in this direction (actually earlier than [[12](#_bookmark27)]) has been carried out by Vergara [[21](#_bookmark35)]. She conjectured that every graph on *n* vertices and indepen- dence number 2 contains an immersion of *K[ n |* , and showed that this conjecture is equivalent to Conjecture [1.1](#_bookmark1) for graphs of independence number 2.

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In support of her conjecture, Vergara proved that every graph on *n* vertices and independence number 2 contains a *K[ n |*-immersion. This result was improved by Gauthier, Le and Wollan [[12](#_bookmark27)] as follows.

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**Theorem 1.3 (Gauthier, Le and Wollan [**[**12**](#_bookmark27)**])** *Let G be a graph on n vertices with independence number* 2*. Then G contains K*2*[ n ♩ as an immersion.*

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For general independence number *α* we make the following conjecture, extend-

ing the conjecture by Vergara. Our conjecture is best possible, since, as already mentioned earlier, the disjoint union of complete graphs of order *t* does not contain an immersion of *Kt*+1.

**Conjecture 1.4 [**[**4**](#_bookmark18)**]** *Let G be a graph on n vertices with independence number α.* *Then G contains K[ n | as an immersion.*

*α*

Our main results, which give more evidence in support of Conjecture [1.4](#_bookmark4), are the following.

**Theorem 1.5 [**[**4**](#_bookmark18)**]** *Let G be a graph on n vertices with independence number at most*

1. *Then G contains a clique immersion on at least [* 2*n♩−* 1 *vertices.*

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**Theorem 1.6 [**[**4**](#_bookmark18)**]** *Let G be a graph on n vertices with independence number at most*

1. *Then G contains a clique immersion on at least [* 4*n♩−* 1 *vertices.*

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Both results greatly improve on the size of the immersion obtained from Theo- rem [1.2](#_bookmark2). It is our hope that further development of the techniques we implement here may help bring further evidence of Conjecture [1.4](#_bookmark4) and, consequently, of Con- jecture [1.1](#_bookmark1). In fact, while this extended abstract was being reviewed we managed to prove the following result, which improves on Theorem [1.2](#_bookmark2) for every value of *α*.

**Theorem 1.7 [**[**4**](#_bookmark18)**]** *Let G be a graph on n vertices with independence number at most*

*α. Then G contains K*  *n as an immersion.*

*[ |*

3*α−*2

The remainder of this extended abstract is dedicated to the proof of Theo- rem [1.5](#_bookmark5). Theorem [1.6](#_bookmark6) relies on similar ideas, but is more complicated. We refer the interested reader to [[4](#_bookmark18)] for the proof of Theorem [1.6](#_bookmark6), and the proof of Theorem [1.7](#_bookmark7).

# Graphs with independence number 3

In this section we prove Theorem [1.5](#_bookmark5). Before getting to the actual proof, we shall give a quick outline of the basic ideas.

* 1. *An overview of the proof*

We will apply induction on the number of vertices of the graph *G*. In the inductive step, we separate three vertices, *a*1, *a*2 and *a*3, and use the induction hypothesis to find an immersion *H* of a complete graph in *G − {a*1*, a*2*, a*3*}*.

This immersion *H* might be one short of having the size we need. In that case we will try to connect one of *a*1, *a*2, *a*3 to the branch vertices of *H*, making the chosen vertex *ai* a new branch vertex of the larger immersion. The paths used for connecting *ai* to the other branch vertices will belong entirely to the bipartite graph between *{a*1*, a*2*, a*3*}* and *V* (*G*) *− {a*1*, a*2*, a*3*}*, and therefore, they will not interfere with any of the paths from the original immersion *H*.

Which of the three vertices we choose as the new branch vertex *ai* will depend on the sizes of their respective neighborhoods and their intersections in the different

parts of *G−{a*1*, a*2*, a*3*}* (the part containing the branch vertices and the remainder). A more precise description of this choice is given by the first of the two main conditions of Lemma [2.1](#_bookmark8) below. Lemma [2.1](#_bookmark8) will be responsible for finding the connecting paths for the new branch vertex.

It might happen, however, that none of the three vertices is suitable, because there are simply not enough paths to make the necessary connections (that is, if the second of the two main conditions of Lemma [2.1](#_bookmark8) is not met). In this case, we will be able to show that we are in one of the following two scenarios. Either, for one the vertices *ai*, its non-neighborhood in *G − {a*1*, a*2*, a*3*}* (which spans a graph of independence number at most 2) is large enough to apply Theorem [1.3](#_bookmark3) and find the desired immersion there. Or, there are two vertices, *ai* and *aj*, such that their joint non-neighborhood (which spans a complete graph) is of the size we need for our immersion. In either case, this finishes the proof.

* 1. *The proof of Theorem* [*1.5*](#_bookmark5)

Let us now make this idea more precise. We need the following lemma for making the connections to the new branch vertex. (Later on, in the application of the lemma, the set *M* will be the set of ‘old’ branch vertices.)

For a graph *G* and *v ∈ V* (*G*), let *N*¯ (*v*) := *V* (*G*) *\* (*N* (*v*) *∪ v*). A set of vertices

*A dominates* a set of vertices *B*, if every vertex in *B* has a neighbor in *A*.

**Lemma 2.1** *Let G be a bipartite graph with bipartition* (*M ∪Q, {a*1*, a*2*, a*3*}*)*, where*

*M and Q are disjoint, and where the set {a*1*, a*2*, a*3*} dominates M. Suppose that*

*k* =3 *maximizes*

*N* (*ai*) *∩ N* (*aj*) *∩ Q \ N* (*ak*)

*over all choices of distinct i, j, k ∈ {*1*,* 2*,* 3*}. Further, assume that*

*|N*¯ (*a*3) *∩ M|≤* *N* (*a*1) *∪ N* (*a*2) *∩ N* (*a*3) *∩ Q* *.*

*Then, G contains an immersion of a star whose centre is a*3 *and whose leaves are in M.*

**Proof.** Given *G* as in the statement of the lemma, first observe that every vertex from *N*¯ (*a*3) *∩* (*M ∪ Q*) has a neighbor in *{a*1*, a*2*}*. We partition *N*¯ (*a*3) *∩ M* into two

sets *N∗* and *N∗* such that for *i* = 1*,* 2, every vertex in *N∗* is adjacent to *ai*. (This

1 2 *i*

partition need not be unique.)

For each distinct *i, j, k* *∈ {*1*,* 2*,* 3*}*, let

*Yij* := (*N* (*ai*) *∩ N* (*aj*) *∩ Q*) *\ N* (*ak*)*.*

By the hypotheses of the lemma, we know that

*|Y*12*|≥* max*{|Y*13*|, |Y*23*|},* (1)

and

*|N*¯ (*a*3) *∩ M|≤ |Y*13*|* + *|Y*23*|* + *X* *,* (2)

donde *X* := *Q ∩* *j*=1*,*2*,*3 *N* (*aj*).

First, suppose it is possible to partition *X* into two sets, *X*1 and *X*2, such that

*|Yi*3 *∪ Xi| ≥ |Ni |* for both *i ∈ {*1*,* 2*}*. In this case, we can pair, for *i* = 1*,* 2, each vertex *x ∈ N∗* with a vertex *yx ∈ Yi*3 *∪ Xi* in such a way that *yx /*= *yx′* if *x /*= *xj*. This means that for every *x ∈ N∗ ∪ N∗* there is a path of the form *xaiyxa*3, and

*∗*

*i*

1

2

all these paths are edge disjoint. Splitting off all these paths, and also using the edges between *a*3 and *M* , we obtain a star having *a*3 as its center and *M* as its set of leaves, which is as desired.

We will therefore assume from now on that it is not possible to partition *X* into sets *X*1 and *X*2 such that *|Yi*3 *∪ Xi| ≥ |N∗|* for both *i ∈ {*1*,* 2*}*. This implies that, modulo swapping indices 1 and 2, we have

*i*

*|Y*23 *∪ X| < |N*2 *|.*

*∗*

Now, in a similar way as before, we pair each vertex *x ∈ N∗* with a vertex

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*yx ∈ Y*13, and we pair each vertex *x ∈ N∗* with a vertex *yx ∈ Y*13 *∪ Y*23 *∪ X*, in a way that all the vertices *yx* are distinct. This is possible by ([2](#_bookmark10)). For all vertices *x ∈ N∗*, and for each vertex *x ∈ N∗* which has not been paired to a vertex from

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*Y*13, we split off the paths of the form *xaiyxa*3.

For vertices *x ∈ N∗* which have been paired to a vertex from *Y*13, we do the following. Pair each such vertex with a second vertex *vx ∈ Y*12, such that all vertices *vx* are disjoint. This is possible because of ([1](#_bookmark9)). Now, split off the path *xa*2*vxa*1*yxa*3. After doing this for all remaining vertices *x ∈ N∗*, we obtain the desired star immersion. *2*

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We are now ready to prove Theorem [1.5](#_bookmark5).

**Proof.** [Proof of Theorem [1.5](#_bookmark5)] We use induction on *n* to prove the statement of Theorem [1.5](#_bookmark5) for all graphs. The statement is trivially true for all *n <* 13. Thus, we can assume the given graph *G* has order *n ≥* 14.

If *G* has independence number at most 2, we can use Theorem [1.3.](#_bookmark3) Hence, we can assume the independence number of *G* is equal to 3. Therefore, we can remove an independent set *I* = *{a*1*, a*2*, a*3*}* from *G*. By the induction hypothesis, we know that *G − I* contains a immersion of *H* := *K[* 2(*n−*3) *♩*, with branch vertices *M* .

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Set *Q* := *V* (*G*) *\* (*I ∪ M* ). Then clearly,

*|Q| ≥ n −* 3 *−* 2(*n −* 3) *>* 7 *n −* 3*.* (3)

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Our aim is to either add a new branch vertex to the immersion of *H*, or to find a different immersion of the correct size.

Set *Ni* := *N* (*ai*) and *N*¯*i* := *N*¯ (*ai*) for *i* = 1*,* 2*,* 3. Since *G* has independence number 3, we know that *G*[*N*¯*i*] has independence number at most 2, for every

*i* = 1*,* 2*,* 3. Hence, we may assume that

*|N*¯*i|*

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*< n* + 1*,* for every *i* = 1*,* 2*,* 3*,*

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(4)

as otherwise (by Theorem [1.3](#_bookmark3)) we would have that some *N*¯*i* contains an immersion of the complete graph on 2*[ n* + 1*♩≥ [* 2*n♩* vertices.

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We may assume, without loss of generality, that *a*1*, a*2 are such that *{i, j}* =

*{*1*,* 2*}* maximizes *|*(*Ni ∩ Nj ∩ Q*) *\ Nk|* over all choices of distinct *i, j, k* in *{*1*,* 2*,* 3*}*. Since *I* is an independent set of maximum size, *I* dominates *G − I*. If we could,

by means of splitting off paths which are edge disjoint from *G−I*, make *a*3 adjacent to *N*¯3 *∩M* , then we would obtain the desired immersion of *K[* 2*n ♩*. So, by Lemma [2.1](#_bookmark8),

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we can assume that

where

*|N*¯3 *∩ M|≥ |Z|* + 1*,* (5)

*Z* := (*N*1 *∪ N*2) *∩ N*3 *∩ Q.*

Since *Q* is dominated by *{a*1*, a*2*, a*3*}* we have

*Q* = *Z ∪* (*N*¯1 *∩ N*¯2 *∩ Q*) *∪* (*N*¯3 *∩ Q*)*.* (6)

Now, since *G* has independence number 3, the set *N*¯1 *∩ N*¯2 *∪ {a*3*}* induces a clique. We can assume this clique to have size at most *[* 2*n♩−* 1, as otherwise we

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have found the desired immersion.

Therefore, and by ([6](#_bookmark14)), we can deduce that

*|Q| ≤ |Z|* + 2*n* *−* 2+ *|N*¯

9

3

*∩Q| ≤* 2*n* + *|N*¯ *|−* 3 *≤* 2*n*

5 7

*—* 3 *< n−* 3*,*

+ *n*

9

9

where the second inequality comes from ([5](#_bookmark13)), and the third inequality comes from ([4](#_bookmark12)). This contradicts ([3](#_bookmark11)). *2*

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