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Linial’s Conjecture for Arc-spine Digraphs

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Abstract

A *path partition P* of a digraph *D* is a collection of directed paths such that every vertexΣbelongs to precisely

one path. Given a positive integer *k*, the *k*-norm of a path partition *P* of *D* is defined as *P ∈P* min*{|Pi|, k}*.

A path partition of a minimum *k*-norm is called *k*-optimal and its *k*-norm is denoted by *πk*(*D*). A *stable*

*set* of a digraph *D* is a subset of pairwise non-adjacent vertices of *V* (*D*). Given a positive integer *k*, we denote by *αk*(*D*) the largest set of vertices of *D* that can be decomposed into *k* disjoint stable sets of *D*. In 1981, Linial conjectured that *πk*(*D*) *≤ αk*(*D*) for every digraph. We say that a digraph *D* is arc-spine if *V* (*D*) can be partitioned into two sets *X* and *Y* where *X* is traceable and *Y* contains at most one arc in

*A*(*D*). In this paper we show the validity of Linial’s Conjecture for arc-spine digraphs.

*Keywords:* digraph, path partition, partial *k*-coloring, Linial’s conjecture.

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# Introduction

For a digraph *D*, let *V* (*D*) denote its set of vertices and let *A*(*D*) denote its set of arcs. Given an arc *a* = (*u, v*) *∈ A*(*D*), we say that *u* and *v* are *adjacent*. The set of *neighbors* of a vertex *u* in *D*, denoted by *N* (*u*), is the set of all vertices that are adjacent to *u* and distinct from *u*. In this paper, we consider only digraphs without loops and parallel arcs. A *path P* is a sequence of distinct vertices (*v*1*, v*2*,..., vl*) such that for every *i* = 1*,* 2*,...,l −* 1, (*vi, vi*+1) *∈ A*(*D*). We define the order of a path *P* , denoted by *|P |*, as the number of its vertices. A *Hamilton path* is a path containing every vertex in *V* (*D*). We say that a digraph *D* is *traceable* if it contains a Hamilton path. A *cycle C* is a sequence of vertices (*v*0*, v*1*,..., vl*) such that (*vi, vi*+1) *∈ A*(*D*) for every *i* = 0*,* 1*,* 2*,...,l −* 1 and all vertices are distinct except precisely *v*0 and *vl* which coincide. We say digraph *D* is *acyclic* if it contains no cycles. A digraph *D* is *transitive* if whenever (*u, v*) *∈ A*(*D*) and (*v, w*) *∈ A*(*D*), then (*u, w*) *∈ A*(*D*) as well.

Given a path *P* = (*v*1*, v*2*,..., vl*), we denote by *ter(P)* the terminal vertex *vl* of

*P* . The subpath (*v*1*, v*2*,..., vi*) of *P* is denoted by *Pvi*, the subpath (*vi, vi*+1*,..., vl*) of *P* is denoted by *viP* and the subpath (*vi, vi*+1*,..., vj*) of *P* is denoted by *viPvj*. We denote by *λ*(*D*) the order of a longest path in *D*. Given another path *Q* = (*w*1*, w*2*,..., wf* ), where *vl* = *w*1, we denote the concatenation of *P* and *Q* by *P ◦Q* = (*v*1*, v*2*,..., vl* = *w*1*, w*2*,..., wf* ).

A *path partition P* of a digraph *D* is a collection of directed paths such that every vertex belongs to precisely one path and we denote by *|P|* the number of paths in the partition. We say that a path partition *P* in *D* is *optimal* if there is no path partition *Pj* such that *|Pj| < |P|*. We denote by *π*(*D*) the cardinality of an optimal path partition. Given a positive integer *k*, the *k-norm* of a path partition *P* of *D*

is defined as Σ

*P∈r*

min*{|Pi|, k}*. A path partition of *D* with minimum *k*-norm is

called *k-optimal* and its *k*-norm is denoted by *πk*(*D*). Note that *π*(*D*)= *π*1(*D*).

A *stable set S* in *D* is a subset of vertices of *V* (*D*) such that every two vertices of *S* are nonadjacent. A stable set with maximum cardinality is called a *maximum stable set* and its cardinality is denoted by *α*(*D*). Let *k* be a positive integer and *D* be a digraph. A *k-partial coloring Ck* of *D* is a set of *k* disjoint stable sets. Each such stable set is called a *color class*. Note that some vertices may not belong to

any of the *k* color classes. The *weight* of a *k*-partial coloring is defined as Σ

*C∈C*

*k |C|*

and it is denoted by *||Ck||*. We say that *Ck* is an *optimal k-partial coloring* of *D*

if it is a partial coloring of maximum weight and we denote its weight by denoted

*αk*(*D*). Note that *α*(*D*)= *α*1(*D*).

Dilworth [[1](#_bookmark26)] was the first to associate a path partition with stable set in digraphs.

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In 1950, he showed that *π*(*D*)= *α*(*D*) when the digraph *D* is transitive and acyclic. A decade later, in 1960, Gallai and Milgram [[2](#_bookmark27)] generalized Dilworth’s Theorem to arbitrary digraphs by relaxing the equality to the inequality *π*(*D*) *≤ α*(*D*). Note that equality does not always hold; for instance if *D* is a cycle of order 5, then *π*(*D*) = 1 but *α*(*D*) = 2. Much later, in 1976, Greene and Kleitman [[3](#_bookmark28)] found a different way to generalize Dilworth’s theorem by establishing a relation between *πk*(*D*) and *αk*(*D*), i.e., changing the notion of minimality of a path partition and allowing the use of up to *k* disjoint stable sets to cover the maximum number of vertices possible. They showed that for any transitive acyclic digraph *D* and any positive integer *k*, we have *πk*(*D*)= *αk*(*D*).

As much as Gallai-Milgram’s Theorem extends Dilworth’s Theorem by relaxing the equality when dealing with arbitrary digraphs, we may think that the next step is to relax the equality proved by Greene and Kleitman to the inequality *πk*(*D*) *≤ αk*(*D*) in order to generalize their theorem for arbitrary digraphs. However, it is an open problem whether such generalization holds. Such question was raised by Linial [[6](#_bookmark30)] in 1981 and is known as Linial’s Conjecture. Some particular cases of the conjecture were already proved. We highlight the cases *k* = 1 (Gallai-Milgram’s Theorem itself), *k* =2 [[8](#_bookmark32)], acyclic digraphs [[6](#_bookmark30)], bipartite digraphs [[7](#_bookmark31)], digraphs with *λ*(*D*) *≤ k* [[7](#_bookmark31)] and traceable digraphs [[7](#_bookmark31)]. For more details on the state of the art of Linial’s Conjecture we refer the reader to [[10](#_bookmark34), Table 6.1].

There is one recent partial result on Linial’s Conjecture that is particularly relevant to this work. In 2017, Sambinelli, Nunes da Silva and Lee proved Linial’s Conjecture for a class of digraphs called spine digraphs [[9](#_bookmark33)]. A digraph *D* is a *spine digraph* if there exists a partition *{X, Y }* of *V* (*D*) such that *D*[*X*] is traceable and *Y* is a stable set in *D*. Spine digraphs are a superclass of split digraphs. Long before, in 1994, Hartman, Saleh and Hershkowitz [[4](#_bookmark29)] gave a proof of a different (although related) conjecture of Linial which we refer to as Linial’s Dual Conjecture (see [[10](#_bookmark34)] for its statement). The proof of Sambinelli, Nunes da Silva and Lee [[9](#_bookmark33)] has some similarity in structure to that of Hartman, Saleh and Hershkowitz; however some particular technique had to be developed to address Linial’s Conjecture. The recent discovery of this new technique motived this work. Here we present an extension of the use of that technique on a superclass of spine digraphs. The purpose of the work is to start the investigation of more superclasses where the new technique may be applied to solve Linial’s Conjecture. We started with the class of arc-spine digraphs, defined in the next section.

# Arc-spine Digraphs

We say that a digraph *D* is an *arc-spine digraph* if there exists a partition *{X, Y }* in *V* (*D*) where *D*[*X*] is traceable and *D*[*Y* ] contains at most one arc. One such partition *{X, Y }* of an arc-spine digraph is *maximal* if *X* is maximal. We use *D*[*X, Y* ] to denote that *D* has one such partition *{X, Y }* and that it is maximal. We denote by *a* the unique arc of *D*[*Y* ] that may exist and by *u* and *v* the tail and head of *a*, respectively. Let *P* = (*x*1*, x*2*,..., xl*) be a Hamilton path of *D*[*X*].

The proof of Linial’s Conjecture for spine digraphs presented by Sambinelli, Nunes da Silva and Lee [[9](#_bookmark33)] involves defining a canonical path partition and a canon- ical partial *k*-coloring that have *k*-norm and weight differing by exactly one. Then, a subclass of spine digraphs whose optimal partial *k*-coloring has weight higher than that of the canonical one is identified. Those are called *k-loose spine digraphs*. For the remaining digraphs, called *k-tight spine digraphs*, it is shown that the canoni- cal path partition is not *k*-optimal. The proof of Linial’s Conjecture for arc-spine digraphs presented in this paper follows the same structure while adapting the def- initions and arguments for the superclass of arc-spine digraphs. We assume that positive integer *k* is at least 2 throughout the paper; this is fundamental in many steps of our proof. However, since Gallai-Milgram’s Theorem is a proof of Linial’s Conjecture for the case *k* = 1, Linial’s Conjecture does hold for every positive integer *k* for arc-spine digraphs.

Let *D*[*X, Y* ] be an arc-spine digraph. We define a *canonical (path) partition* of *D*[*X, Y* ] with respect to some Hamilton path *P* of *D*[*X*] as *{P,* (*u, v*)*}∪ {*(*y*): *y ∈ Y − {u, v}}*. Clearly, this path partition has *k*-norm equal to min*{|X|, k}* + *|Y |*. Hence, *πk*(*D*) *≤* min*{|X|, k}* + *|Y |*. We say that *P* is *zigzag-free* in *D* if none of the following types of arcs exist in *D*: (*i*) (*y, x*1) or (*ii*) (*xl, y*), where *y ∈ Y* ; (*iii*) (*v, x*2) or (*iv*) (*xl—*1*, u*); (*v*) (*xi—*1*, u*) and (*v, xi*+1) simultaneously or (*vi*) (*xi, u*) and (*v, xi*+1) simultaneously or (*vii*) (*xi, y*) and (*y, xi*+1) simultaneously, where 1 *< i < l* and *y ∈ Y* . The motivation for defining this concept is because if *P* is not zigzag- free, then there is *Xj ⊃ X* such that *D*[*Xj*] is traceable and *Y j* = *V* (*D*) *−Xj* induces at most one arc.

**Proposition 2.1** *Let D*[*X, Y* ] *be an arc-spine digraph and let P* = (*x*1*, x*2*,..., xl*)

*be a Hamilton path of D*[*X*]*. Then, P is zigzag-free.*

**Proof** Follows directly from the definition of *D*[*X, Y* ] as having *X* maximal. When any of the arcs prohibited in the definition of zigzag-free path exist, Hamilton path *P* can be extended. *2*

All of the following properties hold when *P* is zigzag-free:

**Lemma 2.2** *Let D*[*X, Y* ] *be an arc-spine digraph and let P* = (*x*1*, x*2*,.* *, xl*) *be a*

*Hamilton zigzag-free path of D*[*X*]*. For every subpath xqPxr* = (*xq, xq*+1*,.* *, xr*) *of*

*P, if there exists a vertex y ∈ Y such that y is adjacent to all vertices of xqPxr, then for every q ≤ i ≤ r:*

* 1. (*xi, y*) *∈ A*(*D*) *if* (*xq, y*) *∈ A*(*D*) *and,*
  2. (*y, xi*) *∈ A*(*D*) *if* (*y, xr*) *∈ A*(*D*)*.*

**Proof** The proof is by induction on *i*. Consider first the case ([*i*](#_bookmark2)); the base case *i* = *q* is thus verified. Now, suppose that *i > q*. By the induction hypothesis, we have that (*xt, y*) *∈ A*(*D*) for *q ≤ t ≤ i −* 1. If (*y, xi*) *∈ A*(*D*), then *P* is not zigzag-free in *D*; whence (*xi, y*) *∈ A*(*D*). In particular, when *i* = *r*, we have shown that (*xi, y*) *∈ A*(*D*) for *q ≤ i ≤ r*. A symmetric reasoning can be used to prove case

([*ii*](#_bookmark3)). *2*

**Corollary 2.3** *Let D*[*X, Y* ] *be an arc-spine digraph and let P* = (*x*1*, x*2*,..., xl*) *be a zigzag-free Hamilton path of D*[*X*]*. Then there is no vertex y ∈ Y adjacent to all vertices of P.*

**Proof** Assume to the contrary that there is some vertex *y ∈ Y* adjacent to every vertex of *P* . Since *P* is zigzag-free, (*x*1*, y*) *∈ A*(*D*). Then, by Lemma [2.2](#_bookmark1), (*xl, y*) *∈ A*(*D*); a contradiction to the fact that *P* is zigzag-free. *2*

Linial’s Conjecture is valid for spine digraphs, thus we will assume that *D*[*Y* ] contains an arc (*u, v*). Thus, by Corollary [2.3](#_bookmark4), there is some vertex *xv ∈ X* that is not adjacent to vertex *v*. Now let *S* be any subset of *X − xv* containing min*{|X|−* 1*,k −* 2*}* vertices. We may thus define a *canonical k-partial coloring* with respect to *S* as *{Y − v, {v, xv}} ∪ {{x}* : *x ∈ S}}*. Clearly, this *k*-partial coloring has weight

*|Y |−* 1+2+ min*{|X|−* 1*,k −* 2*}*. But min*{|X|−* 1*,k −* 2*}* = min*{|X|,k −* 1*}−* 1; whence *αk*(*D*) *≥ |Y |* + min*{|X|,k −* 1*}* for every arc-spine digraph *D*.

An arc-spine digraph is *k-loose* if either *|X| < k* or there is a subset *S ⊆ X* with *|S|* = *k* such that no vertex *y ∈ Y* is adjacent to every vertex of *S* and there are at least two distinct vertices *xu* and *xv* in *S* such that *{u, xu}* and *{v, xv}* are independent sets. In constrast, an arc-spine digraph is *k-tight* if it is not *k*-loose, i.e., *|X|≥ k* and for every subset *S ⊆ X* with *|S|* = *k* either:

* + 1. there exists *y ∈ Y* : *S ⊆ N* (*y*) or
    2. there exists *x ∈ S* such that *N* (*u*) *∩ S* = *N* (*v*) *∩ S* = *S − {x}*.

In Lemma [2.4](#_bookmark7) we show that there is an analogue of [[9](#_bookmark33), Lemma 1] for *k*-loose arc- spine digraphs. Note, however, that the concept of *k*-loose for arc-spine digraphs presented here is different from the concept of *k*-loose for spine digraphs presented in [[9](#_bookmark33)]. The different definition of *k*-loose was needed as a means to guarantee that there would be perfect analogues of Lemmas 1 and 3 from [[9](#_bookmark33)] for arc-spine digraphs. The analogue of [[9](#_bookmark33), Lemma 3] is Lemma [2.8](#_bookmark16).

**Lemma 2.4** *If D*[*X, Y* ] *is a k-loose arc-spine digraph, then:*

* + - 1. *αk*(*D*) *≥ |Y |* + min*{|X|, k} and*
      2. *πk*(*D*) *≤ αk*(*D*)*.*

**Proof** Recall that *πk*(*D*) *≤ |Y |* + min*{|X|, k}* since this is the *k*-norm of the canonical partition (even when *D* is spine there is a path partition with such *k*- norm). Also, recall that the canonical *k*-partial coloring *|Y |* + min*{|X|,k −* 1*}* (even when *D* is spine there is a *k*-partial coloring with such weight). If *|X| < k*, then min*{|X|,k −* 1*}* = min*{|X|, k}* in this case and both ([i](#_bookmark8)) and ([ii](#_bookmark9)) hold. Thus, we may assume that *|X|≥ k*. So, there exists *S ⊆ X* with *|S|* = *k* such that no vertex *y ∈ Y* is adjacent to every vertex in *S* and there are at least two distinct vertices *xu* and *xv* in *S* such that *{u, xu}* and *{v, xv}* are independent sets. Suppose that *S* = *{x*1*, x*2*,..., xk}* and let *Ck* = *{C*1*, C*2*,.* *, Ck}* be a *k*-partial coloring in which

0

*Cp* = *{xp}*, *p* = 1*,* 2*,..., k*. By the choice of *S*, *{u, xu}* is an independent set for some *xu ∈ S* and *{v, xv}* is an independent set for some *xv ∈ S* such that *xu /*= *xv*. We thus add *u* to the color class of *xu*, *v* to the color class of *xv* and every other

vertex *y ∈ Y − {u, v}* to some color class *Cp* such that *{y, xp}* is an independent set (which exists by the choice of *S*). The *k*-partial coloring *Ck* thus obtained has weight *|Y |* + *k* = *|Y |* + min*{|X|, k}*. Therefore, *αk*(*D*) *≥ ||Ck||* = *|Y |* + min*{|X|, k}*. Hence, we establish that ([i](#_bookmark8)) and ([ii](#_bookmark9)) hold. This finishes the proof. *2*

**Lemma 2.5** *Given a k-tight arc-spine digraph D*[*X, Y* ]*, and a zigzag-free path P* = (*x*1*, x*2*,..., xl*) *of D*[*X*]*, there is an arc* (*xj, y*) *∈ A*(*D*) *such that y ∈ Y for some k −* 1 *≤ j ≤ l.*

**Proof** Consider *S* = *{x*1*, x*2*,..., xk}*, the set of the *k* first vertices of *P* . First assume that condition ([*a*](#_bookmark5)) in the definition of *k*-tight digraphs is valid, that is, there exists *y ∈ Y* such that *S ⊆ N* (*y*). Since *P* is zigzag-free, we have (*x*1*, y*) *∈ A*(*D*). So by Lemma [2.2](#_bookmark1)(*i*), the result follows with *j* = *k*. We may thus assume that exclusively condition ([*b*](#_bookmark6)) is valid. Let *xt* be the only vertex of *S* not adjacent to both *u* and *v* (note that 1 *≤ t ≤ k*). If *t /*= 1, we conclude that (*x*1*, u*) *∈ A*(*D*) and (*x*1*, v*) *∈ A*(*D*) as *P* is zigzag-free. By Lemma [2.2](#_bookmark1) applied to subpath *x*1*Pxt—*1 and *u*, we conclude that (*xt—*1*, u*) *∈ A*(*D*). If *t* = *k*, the result follows (Figure [1](#_bookmark11)); we may thus assume that *t < k*. Thus *xt*+1 is in *S* and *v* is adjacent to *xt*+1. Arc (*v, xt*+1) */∈ A*(*D*), otherwise there would be a zigzag in *P* . The latter assertion is true even if *t* = 1 (Figure [2](#_bookmark12)); in fact, the only difference when *t* = 1 is that we do not know the orientation of the arcs joining vertices of *S* to *u*. Since (*xt*+1*, v*) *∈ A*(*D*) whenever 1 *≤ t < k*, by Lemma [2.2](#_bookmark1) applied to subpath *xt*+1*Pxk* and *v*, (*xk, v*) *∈ A*(*D*) and the result follows (Figure [3](#_bookmark14)). *2*

*x*1

*x*2

...

*xk−*1

*xk*

*u*

*v*

Figure 1. *t* = *k*. The arc (*xk−*1*, v*) *∈ A*(*D*).

*x*1

*x*2

...

*xk−*1

*xk*

*u*

*v*

Figure 2. *t* = 1. The arc (*xk, v*) *∈ A*(*D*).

**Lemma 2.6** *Given a k-tight arc-spine digraph D*[*X, Y* ]*, and a zigzag-free path P* = (*x*1*, x*2*,..., xl*) *of D*[*X*]*, there is an arc* (*y, xi*) *∈ A*(*D*) *such that y ∈ Y for some* 1 *≤ i ≤ l.*

**Proof** Consider *S* = *{xl—k—*1*, xl—k,..., xl}*, the set of the *k* last vertices of *P* . First assume that condition ([*a*](#_bookmark5)) is valid. By Lemma [2.2](#_bookmark1), the result follows immediately.

*x*1

...

*xt−*1

*xt*

*xt*+1

...

*xk*

*u*

*v*

Figure 3. The arc (*xk, v*) *∈ A*(*D*).

We may thus assume that exclusively condition ([*b*](#_bookmark6)) is valid. So let *x* be the unique vertex of *S* not adjacent to *u* and *v*. Consider first the case in which vertex *x /*= *xl*. Then, as *u* is adjacent to every vertex of *S − {x}*, it is adjacent to *xl*. Since *P* is zigzag-free, (*u, xl*) *∈ A*(*D*) in this case. Now, when *x* = *xl*, vertex *u* is adjacent to every vertex of *S − {x}*, it is adjacent to *xl—*1. Since *P* is zigzag-free, (*u, xl—*1) *∈ A*(*D*) and the result follows in both cases. *2*

**Lemma 2.7** *Given a* 2*-tight arc-spine digraph D*[*X, Y* ]*, and a zigzag-free path P* = (*x*1*, x*2*,..., xl*) *of D*[*X*]*, for each vertex xp of P, there must be some vertex yp ∈ Y adjacent to xp, for every p* = 1*,* 2*,.* *, l.*

**Proof** Assume, to the contrary, that there is some *p* such that no vertex of *Y* is adjacent to *xp*. First we show that both *u* and *v* are adjacent to every vertex in *X − {xp}*, let *Sq* = *{xp, xq}* be a set of two vertices of *X* for some *q /*= *p*. Since no vertex of *Y* is adjacent to *xp* and *D* is 2-tight, we conclude ([*b*](#_bookmark6)) holds for *Sq*. Thus, both *u* and *v* are adjacent to *xq*. Such argument holds for every *q*,1 *≤ q ≤ l*, *p /*= *q*. We therefore conclude that both *u* and *v* are adjacent to every vertex in *X − xp*.

Consider first the case in which *p* = 1. As the digraph *D* is 2-tight, *|X|* = *l ≥* 2. Since *P* is zigzag-free, (*u, xl*) *∈ A*(*D*) and (*v, xl*) *∈ A*(*D*). By Lemma [2.2](#_bookmark1), we deduce (*v, x*2) *∈ A*(*D*). But then *P* has a zigzag, a contradiction. We may thus assume *p /*= 1. By a symmetric reasoning, we can deduce *p /*= *l*.

Since *P* is zigzag-free, we conclude that (*x*1*, u*) *∈ A*(*D*), (*x*1*, v*) *∈ A*(*D*), (*u, xl*) *∈ A*(*D*) and (*v, xl*) *∈ A*(*D*). By Lemma [2.2](#_bookmark1), we conclude that (*xq, u*) *∈ A*(*D*), (*xq, v*) *∈ A*(*D*) for 1 *≤ q < p* and (*u, xq*) *∈ A*(*D*) and (*v, xq*) *∈ A*(*D*) for *p < q ≤ l*. But then (*xp—*1*, u, v, xp*+1) is a zigzag in *P* ; again a contradiction. *2*

The most ingenious part of the proof presented by Sambinelli, Nunes da Silva and Lee is the proof of [[9](#_bookmark33), Lemma 3] . Next, we show the proof of its analogue for arc-spine digraphs, Lemma [2.8.](#_bookmark16) Again, note that the concept of *k*-tight for spine digraphs is different from the presented definition of *k*-tight arc-spine digraphs. Fur- thermore, we point out that the proof of the base case of Lemma [2.8](#_bookmark16) is considerably more intricate than the base case of [[9](#_bookmark33), Lemma 3].

**Lemma 2.8** *Let D*[*X, Y* ] *be a k-tight arc-spine digraph and let P* = (*x*1*, x*2*,.* *, xl*)

*be a zigzag-free path of D. Then, there exist paths P*1 *and P*2 *such that:*

1. *V* (*P*1) *∩ V* (*P*2)= ∅*;*
2. *|P*1*|* + *|P*2*|≥ |X|* + *k* + 1*;*
3. *ter*(*P*1) *∪ ter*(*P*2)= *{xl, y}, for some y ∈ Y ;*
4. *X ⊆ V* (*P*1) *∪ V* (*P*2)*.*

**Proof** We begin by stating the following useful claim.

**Claim 2.9** *Let t < l. If there exist yt and yt*+1 *such that* (*xt, yt*) *∈ A*(*D*) *and*

(*yt*+1*, xt*+1) *∈ A*(*D*) *then we may assume that*

* + *yt /*= *yt*+1*;*
  + *yt /*= *u;*
  + *yt*+1 */*= *v;*

*otherwise either P is not zigzag-free or paths P*1 *and P*2 *trivially exist.*

**Proof** We know that *yt /*= *yt*+1 because *P* is zigzag-free. Then, the first condition follows trivially. Assume that *yt* = *u*. Then *yt*+1 */*= *v*, since *P* is zigzag-free. Now, consider the paths *P*1 = *Pxt ◦* (*xt, yt* = *u, v*) and *P*2 = (*yt*+1*, xt*+1) *◦ xt*+1*P* . Paths *P*1 and *P*2 meet conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)). Finally, assume that *yt*+1 = *v*. Then *yt /*= *u*, since *P* is zigzag-free. Consider the two paths *P*1 = *Pxt ◦* (*xt, yt*) and *P*2 = (*u, v* = *yt*+1*, xt*+1) *◦ xt*+1*P* . Paths *P*1 and *P*2 meet conditions ([*i*](#_bookmark17)) through

([*iv*](#_bookmark20)). *2*

The proof of the Lemma is by induction on *k*. The base case is *k* = 2. Given a 2-tight arc-spine digraph *D*, by Lemma [2.5](#_bookmark10) there is a vertex *y ∈ Y* such that (*xj, y*) *∈ A*(*D*) for some *xj ∈ V* (*P* ). Among all arcs (*xj, yj*) *∈ A*(*D*) with *yj ∈ Y* choose an arc *aj* such that *j* is maximum. As *P* is zigzag-free in *D*, we have that *j < l* and so the vertex *xj*+1 exists. Note that by Lemma [2.7](#_bookmark15) and the choice of *aj*, we can claim the existence of some vertex *yj*+1 *∈ Y* such that (*yj*+1*, xj*+1) *∈ A*(*D*). By Claim [2.9](#_bookmark21), we conclude that *yj /*= *yj*+1, *yj /*= *u* and *yj*+1 */*= *v*. By Lemma [2.6](#_bookmark13) there is a vertex *y ∈ Y* such that (*y, x*) *∈ A*(*D*) for some *x ∈ V* (*P* ). Let (*yi, xi*) *∈ A*(*D*) such that *i* is minimum. We shall now show that *i ≤ j*. Let *S* = *{xj, xj*+1*}*. Since *yj*+1 */*= *v*, we know that (*v, xj*+1) *∈/ A*(*D*). If ([*a*](#_bookmark5)) holds for *S*, then there is a vertex *y ∈ Y* adjacent to both vertices in *S*. Thus, (*y, xj*+1) *∈ A*(*D*) by the choice of the arc *aj* and since *P* is zigzag-free, (*y, xj*) *∈ A*(*D*). If ([*b*](#_bookmark6)) holds for the subset *S*, then by the choice of the arc *aj*, (*xj*+1*, v*) *∈/ A*(*D*) and since *v /*= *yj*+1, we deduce that (*v, xj*+1) *∈/ A*(*D*). Therefore, *u* and *v* are adjacent to *xj* where (*u, xj*) *∈ A*(*D*) as *yj /*= *u*. Since *i* is minimum, the analysis of these two cases allow us to conclude *i ≤ j*. Moreover, *i >* 1, otherwise *P* is not zigzag-free. Then, vertex *xi—*1 exists and so does arc (*xi—*1*, yi—*1) by the minimality of *i* and Lemma [2.7.](#_bookmark15) Again, by Claim [2.9](#_bookmark21), we conclude that *yi—*1 */*= *yi*, *yi—*1 */*= *u* and *yi /*= *v*.

To conclude the base case, let *Sj* = *{xi—*1*, xj*+1*}*. Since *yj*+1 */*= *v* and *yi—*1 */*= *u*, neither *xj*+1 nor *xi—*1 can be simultaneously adjacent to *u* and *v*; therefore, ([*b*](#_bookmark6)) cannot hold for *Sj*. Then, ([*a*](#_bookmark5)) holds for *Sj* and there is a vertex *yj* such that it is adjacent to both vertices of *Sj*. By the choice of *j*, we have that (*yj, xj*+1) *∈ A*(*D*). Therefore, *yj /*= *yj* because *P* is zigzag-free. By a symmetric reasoning we deduce that (*xi—*1*, yj*) *∈ A*(*D*) and *yj /*= *yi*. By Claim [2.9](#_bookmark21) we have that *yj /*= *u* and *yj /*= *v*.

Vertices *yi* and *yj* may or may not be the same. Consider first the case in which

*yi /*= *yj*. Then, paths *P*1 = *P xi—*1 *◦* (*xi—*1*, yj, xj*+1) *◦xj*+1*P* and *P*2 = (*yi, xi*) *◦xiPxj ◦*

*y′*

*x*1

*xi−*1

*xi*

...

*xj*

*xj*+1

*xÆ P*1

*yi*

*yj P*2

Figure 4. *yi /*= *yj* . The paths *P*1 = *P xi−*1 *○* (*xi−*1*, y′, xj*+1) *○ xj*+1*P* and *P*2 = (*yi, xi*) *○ xiP xj ○* (*xj , yj* ).



*y′*

*x*1

*xi−*1

*xi*

*xt−*1

*xt*

*xt*+1

*xj*

*xj*+1

*xÆ*

*P*1

*yj*

*yt*

*P*2

Figure 5. *yi* = *yj* and (*xt, yt*) *∈ A*(*D*). The paths *P*1 = *P xt−*1 *○* (*xt−*1*, y′, xj*+1) *○ xj*+1*P* and

*P*2 = (*yt, xt*) *○ xtP xj ○* (*xj , yj* ) satisfy the conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)).

(*xj, yj*) satisfy conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)) (Figure [4](#_bookmark22)).

We may thus assume that *yi* = *yj*. We will now show that some vertex *xt*, *i ≤ t ≤ j*, is not adjacent to *yj*. To do so, assume the contrary. Then, by Lemma [2.2](#_bookmark1), since (*yj, xj*+1) *∈ A*(*D*), we must have arc (*yj, xi—*1) *∈ A*(*D*) as well; a contradiction to the choice of *i*. Thus, choose *i ≤ t ≤ j* such that *t* is minimum and *xt* is not adjacent to *yj*. By Lemma [2.7](#_bookmark15), there is some vertex *yt ∈ Y* adjacent to *xt*. As *yt* is adjacent to *xt*, clearly *yt /*= *yj*. We shall now show that *yt* is distinct from *yi* = *yj*. Assume to the contrary that *yt* = *yi* = *yj* is the only vertex in *Y* adjacent to *xt*. Recall that *yi /*= *u* and *yj /*= *v*. Let *St* = *{xt, xj*+1*}*. Since *yi /*= *u* and *yj /*= *v*, ([*a*](#_bookmark5)) must hold for *St*. Therefore, (*xj, yj* = *yt*) *∈ A*(*D*) and (*yt, xj*+1) *∈ A*(*D*); so *P* has a zigzag, a contradiction. We thus deduce that *yt* is distinct from *yi* = *yj*.

If (*xt, yt*) *∈ A*(*D*), then the paths *P*1 = *Pxi—*1 *◦* (*xi—*1*, yj, xj*+1) *◦ xj*+1*P* and

*P*2 = *xt*+1*Pxj ◦* (*xj, yj* = *yi, xi*) *◦ xiPxt ◦* (*xt, yt*) satisfy the conditions ([*i*](#_bookmark17)) through

([*iv*](#_bookmark20)) (Figure [5](#_bookmark23)).

If (*yt, xt*) *∈ A*(*D*) note that, by the choice of *t* we know that *yj* is adjacent to every vertex in (*xi—*1*,..., xt—*1). Moreover, since (*xi—*1*, yj*) *∈ A*(*D*), by Lemma [2.2](#_bookmark1) (*xt—*1*, yj*) *∈ A*(*D*). Then, the paths *P*1 = *Pxt—*1 *◦* (*xt—*1*, yj, xj*+1) *◦ xj*+1*P* and *P*2 = (*yt, xt*) *◦ xtPxj ◦* (*xj, yj*) satisfy the conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)) (Figure [6](#_bookmark24)).

Finally, the proof of the base case *k* = 2 is complete. Assume thus that *k >* 2.

By Lemma [2.5](#_bookmark10) there is some vertex *xj ∈ V* (*P* ) such that there is some arc (*xj, yj*) *∈ A*(*D*) for *yj ∈ Y* and *j ≥ k −* 1. Among all such arcs choose an arc *aj* with *j* maximum. Since *P* is zigzag-free, we know that *j < l* and thus there is a vertex *xj*+1 in *P* . Let *Xj* = *V* (*P xj*) and let

*y′*

*x*1

*xi−*1

*xi*

*xt−*1

*xt*

*xt*+1

*xj*

*xj*+1

*xÆ*

*P*1

*yj yt*

*P*2

Figure 6. *yi* = *yj* and (*yt, xt*) *∈ A*(*D*). The paths *P*1 = *P xt−*1 *○* (*xt−*1*, y′, xj*+1) *○ xj*+1*P* and

*P*2 = (*yt, xt*) *○ xtP xj ○* (*xj , yj* ) satisfy the conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)).

*Y j* = *{yj* : *yj ∈ Y* and *yj* is adjacent to *xj*+1*}*, if at least one of *{u, v}* is adjacent to *xj*+1;

*Y j* = *{yj* : *yj ∈ Y* and *yj* is adjacent to *xj*+1*}∪ {u}*, otherwise.

Note that *yj /∈ Y j* as *yj /*= *u* by Claim [2.9](#_bookmark21) and *P* is zigzag-free. Morevover, if some vertex of *Y j* is not adjacent to *xj*+1, that vertex is *u* and *v /∈ Y j* in this case. Let *Dj* = *D*[*Xj,Y j*] and let *Pj* = *Pxj*. If *Y j* is a stable set, then *Dj* is a spine digraph and since at least one vertex in *{u, v}* is in *Y j*, it can be shown that *Dj* is (*k −* 1)-tight. According to [[9](#_bookmark33), Lemma 3], *Dj* has paths *Pj* and *Pj* that meet the

1 2

conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)) above. Consider now the case in which *Y j* contains *u*

and *v*.

We shall show that *Pj* is zigzag-free in *Dj*. Assume the contrary. Then, since *P* is zigzag-free in *D*, this implies that there either is an arc (*xj—*1*, u*) *∈ A*(*Dj*) or (*xj, y*) *∈ A*(*Dj*) for some vertex *y ∈ Y j*. However, if (*xj—*1*, u*) *∈ A*(*Dj*), since *v ∈ Y j* then (*v, xj*+1) *∈ A*(*D*) and *P* would not be zigzag-free in *D*, a contradiction. Similarly, if (*xj, y*) *∈ A*(*Dj*) for some vertex *y ∈ Y j*, since *y /*= *u* by Claim [2.9](#_bookmark21), we deduce (*y, xj*+1) *∈ A*(*D*). But then *P* would not be zigzag-free in *D*, again a contradiction.

We shall now show that *Dj* is a (*k −* 1)-tight arc-spine digraph. Let *Sj ⊆ Xj* be an arbitrary set of *k −* 1 vertices; by Lemma [2.5](#_bookmark10) we know that *j ≥ k −* 1. Let *S* = *Sj ∪ {xj*+1*}* be a set of *k* vertices of *D*. Since *D* is *k*-tight, either ([*a*](#_bookmark5)) or ([*b*](#_bookmark6)) holds for *S*. If ([*a*](#_bookmark5)) holds for *S* in *D*, then it is easy to see that ([*a*](#_bookmark5)) also holds for *Sj* in *Dj*. So suppose that ([*b*](#_bookmark6)) holds for *S* in *D*. We do know that *{u, v}∈ Y j*, therefore (*v, xj*+1) *∈ A*(*Dj*). So, the only vertex in *S* not adjacent to *u* and *v* is some vertex of *Sj* and ([*b*](#_bookmark6)) holds for *Sj* in *Dj* (Figure [7](#_bookmark25)).

We have thus shown that *Dj* is a (*k−*1)-tight arc-spine digraph (an assertion that holds even if *Y j* is stable) and *Pj* is zigzag-free. By the induction hypothesis applied to *Dj* and *Pj*, there exist paths *Pj* and *Pj* in *Dj* which satisfy conditions ([*i*](#_bookmark17)) through

1 2

([*iv*](#_bookmark20)). Assume, without loss of generality, that *ter*(*Pj*) = *xj* and *ter*(*Pj*) = *yj*,

1 2

for some *yj ∈ Y j*. If (*yj, xj*+1) *∈ A* and we may take *P*1 = *Pj ◦* (*xj, yj*) and

1

*P*2 = *Pj ◦* (*yj, xj*+1) *◦ xj*+1*P* as the paths of *D*. Condition ([*i*](#_bookmark17)) holds in this case because *Pj* and *Pj* are disjoint by induction hypothesis and neither vertex *yj* nor

2

1 2

*X′*

*x*1

*x*2

...

*xj*

*xj*+1

...

*yj*

*u*

*v* ...

*Y ′*

Figure 7. ([*b*](#_bookmark6)) holds for *S′* in *D′*.

any vertex of *xj*+1*P* are vertices of *Dj*. When (*yj, xj*+1) */∈ A*, it is certain that *yj* = *u*

and *v /∈ Y j*. So we may take *P*1 = *Pj ◦ xjP* and *P*2 = *Pj ◦* (*u, v*) as the paths of

1 2

*D*. Condition ([*i*](#_bookmark17)) also holds in this case. At last, we claim that *P*1 and *P*2 meet all

conditions ([*i*](#_bookmark17)) through ([*iv*](#_bookmark20)) in either case. Conditions ([*iii*](#_bookmark18)) and ([*iv*](#_bookmark20)) obviously hold. Condition ([*ii*](#_bookmark19)) holds because *|P j|* + *|P j|* = *j* + *k* by induction hypothesis. Therefore

1 2

*|P*1*|* + *|P*2*|* = *|P j|* + *|P j|* + *|X|− j* +1 = *|X|* + *k* +1

1 2

and the proof is complete. *2*

**Theorem 2.10** *Let D*[*X, Y* ] *be a arc-spine digraph. Then, πk*(*D*) *≤ αk*(*D*)*.*

**Proof** We may assume that *D* is *k*-tight, otherwise the result follows by Lemma [2.4](#_bookmark7). We know that *αk*(*D*) *≥ |Y |* + min*{|X|,k −* 1*}*. Since *D* is *k*-tight, we have by definition that *|X|≥ k* and, therefore, that *αk*(*D*) *≥ |Y |* + *k −* 1. Now, suppose that *λ*(*D*) *> |X|*. Since *λ*(*D*) *> |X|*, there exists a path *P* in *D* such that *|P |* = *|X|* + 1. Let *P* = *P ∪ {*(*v*) : *v ∈/ V* (*P* )*}*. Clearly, *P* is a path partition of *D* and *|P|k* = min*{|P|, k}*+*|Y |−*1= *|Y |*+*k−*1. Therefore, *πk*(*D*) *≤ |P|k* = *|Y |*+*k−*1 *≤ αk*(*D*) and the result follows. Hence, we may assume that *λ*(*D*)= *|X|*. Let *P* = (*x*1*, x*2*,.* *, xl*)

be a Hamilton path in *D*[*X*]. Since *λ*(*D*)= *|X|*, we have that *P* is a longest path in *D*; as such, it must be zigzag-free. By Lemma [2.8](#_bookmark16), there exist disjoint paths *P*1 and *P*2 in *D* such that *|P*1*|* + *|P*2*|* = *|X|* + *k* + 1. Note that *|Pi| > k*, for *i* = 1*,* 2, otherwise *P*3*—i* would be larger than *|X|*. Let *P* = *{P*1*, P*2*}∪ {*(*y*) : *y ∈/ V* (*P*1)*∪V* (*P*2)*}*. It is easy to see that *P* is a path partition in *D*. The *k*-norm of *P* is

*|P|k* = min*{|P*1*|, k}*+min*{|P*2*|, k}*+*|Y |−k−*1= *|Y |*+*k−*1. So, *πk*(*D*) *≤ |Y |*+*k−*1 and the result follows. *2*

# 3 Conclusion

Sambinelli, Nunes da Silva and Lee adapted the technique of the proof of Linial’s Dual Conjecture for split digraphs to prove Linial’s Conjecture for spine digraphs. In this paper we were able to apply the same technique to a superclass of spine digraphs. The most important statement proved is Lemma [2.8,](#_bookmark16) whose assertion and structure of the inductive proof has similar elements to that of [[9](#_bookmark33), Lemma 3]. Even though spine and arc-spine digraphs are very similar in structure, the proof of the base case for arc-spine digraphs happens to be a lot more complex than the base case for spine digraphs. It is hard to understand at this moment what does that represent. Intuition suggests that it might be possible to adapt the structure of

the proof presented here to deal with superclasses of spine digraphs more complex in structure than arc-spine digraphs.

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