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[](http://crossmark.crossref.org/dialog/?doi=10.1016/j.eij.2022.06.002&domain=pdf)Novel linear programming models based on distance measure of IFSs and modified TOPSIS method for portfolio selection

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Due to the unavailability of time series data for newly listed stocks or products, it is a challenge for inves- tors to make rational portfolio selection under uncertain circumstances. To solve the problem, a new approach is put forward in this paper. Firstly, the problem is considered as a multi-criteria decision mak- ing (MCDM) problem based on the assumption that the assessments are given in intuitionistic fuzzy set (IFS) form, which can better describe the uncertainty. Then, the TOPSIS method, which is widely used in MCDM problem, has been modified in two aspects. On the one hand, the new definitions of Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed to represent the best returns and the greatest cost/risk in portfolio selection. They are more meaningful than the def- initions of Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS), which could only express the best and the worst case of portfolio but cannot achieve a balance between returns and risk. On the other hand, the weighted closeness coefficient is refined to offer the more appropriate results that are consis- tent with investors’ demands or preferences. In addition, based on distance measure of IFSs, several novel linear programming models with different constraints are proposed to allocate the investment ratios according to investors’ demands. The models can make up for the disadvantage of TOPSIS method which only considers the ranking of investments but neglects the investment ratios. Finally, compared with the conventional TOPSIS method and the IF ELECTRE Method in a numerical example, our new approach is demonstrated to be more effective and more flexible. Particularly, it can provide a more appropriate strategy in accordance with investors’ preferences and demands.

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1. Introduction

Modern portfolio selection theory and the mean–variance prob- abilistic model, which were proposed by Markowitz [[1](#_bookmark57)], estab- lished the relationship between portfolio selection and mathematics. This theory revealed that investors aim to gain more returns while trying to avoid risks as far as possible, and then the model provided a way for investors to balance the returns and risks in accordance with their risk preferences. On this basis, portfolio

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selection has developed rapidly. With considerable researches held in this field, more and more achievements have sprung up. These achievements extended the portfolio selection theory and trans- formed the model into kinds of new forms with different con- straints according to investors’ demands. For instance, Davis and Norman [[2](#_bookmark58)] studied an investment model which contains linear transaction costs; Zhou and Li [[3](#_bookmark46)] paid attention to a continuous- time mean–variance portfolio selection model; Ledoit and Wolf

[[4](#_bookmark46)] put forward the improved estimation of the co-variance matrix of stock returns; Liu [[5](#_bookmark46)] did a research on portfolio selection in stochastic environments; Tofighian et al. [[6](#_bookmark46)] dealt with multi- period project portfolio selection problem and presented a mathe- matical model considering risks, stochastic incomes, and possibil- ity of investing extra budget in each time period.

However, it should be noticed that the conventional model for portfolio selection is becoming more and more unavailable and useless due to the complexity of portfolio circumstance which con- tains both objective and subjective uncertainties. Zhou and Xu [[7](#_bookmark46)] pointed out that time series data may be unavailable for newly

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listed stocks or products. Besides, Yazdi [[8](#_bookmark46)] mentioned that the actual circumstance is not definitive or predictable because of the existing uncertainties. More generally, it is difficult to obtain the precise probability distributions of returns as well as enough data just by relying on the past statistics due to the ever- changing economic environment [[9](#_bookmark46)]. Thus, it has become a top pri- ority to look for a more appropriate way to deal with portfolio selection problem under the uncertain circumstances.

Fuzzy set theory, which is proposed by Zadeh [[10](#_bookmark46)] in 1965 to describe uncertainties in mathematical language, plays an impor- tant role in the modern society that is full of all kinds of uncer- tainty. As a powerful tool to deal with uncertainty, it has received a great concern and has been studied by many researchers for some relative practical applications. Intuitionistic fuzzy set (IFS), which presented by Atanassov [[11](#_bookmark46)], is doubtless one of the greatest extensions of fuzzy set theory. Since it can describe uncer- tainties from positive and negative perspectives by using the mem- bership degree and non-membership degree, respectively, it is more effective than fuzzy set. In intuitionistic fuzzy set theory, dis- tance measure, the representation of the deviation between IFSs, is widely used in practice, such as pattern recognition [[12,13](#_bookmark46)], deci- sion making [[14,15,16](#_bookmark46)] and medical diagnosis [[17](#_bookmark47)]. It was first defined by Szmidt and Kacprzyk [[18](#_bookmark48)] in 2000. And then, in order to pursue higher accuracy and more practical significance, various distance measures have been proposed. In this paper, we will adopt H-max distance measure [[15](#_bookmark46)] to quantify the differences between investments.

The advantage of intuitionistic fuzzy set in dealing with uncer- tainties implies that it can effectively solve the portfolio selection problem in uncertain environment. Yue et al. [[19](#_bookmark49)] proposed a fuzzy linear higher order moment portfolio selection model; Guo et al.

[[9](#_bookmark46)] studied the fuzzy multi-period portfolio selection problem with V-Shaped transaction cost; and Chen et al. [[20](#_bookmark50)] put forward a mean–variance-skewness fuzzy portfolio selection model based on intuitionistic fuzzy optimization. To help investors make a selection among several newly listed investments and then allo- cate the investment ratios for each selected investment, in this paper, the portfolio selection under uncertain circumstances is regarded as a multi-criteria decision making (MCDM) problem. This kind of problem requires the decision maker to rank the pro- vided alternatives according to the assessments with respect to several criteria [[21](#_bookmark51)]. In order to describe the uncertain circum- stances better, all these assessments will be given in IFS form.

For the sake of ranking the investments conveniently, the method called technique for order preference by similarity to ideal solution (TOPSIS) is introduced. TOPSIS is presented by Hwang and Yoon [[22](#_bookmark52)] in 1981, which points out the best alternative is one that has not only the shortest distance from the Positive Ideal Solution (PIS) but also the farthest distance from the Negative Ideal Solution (NIS) [[23](#_bookmark53)]. In recent years, TOPSIS has been extended to fuzzy envi- ronment and widely applied to solve MCDM problems [[8,16,21,23](#_bookmark46)]. However, there is still not substantially improvement on its model. As pointed out by Dwivedi et al. [[16](#_bookmark54)], one of the limitations of the conventional TOPSIS method is that it does not take care of the weights of PIS and NIS distances. Although there have been some modifications on closeness coefficient, which plays a significant role in TOPSIS method, it is a pity that these modifications are not perfect enough. Besides, for portfolio selection, the conven- tional TOPSIS method cannot give investors a hand to allocate investment ratios for each selected investment, which cannot effectively spread the risk and is fatal in portfolio selection.

To pursue more adaptability and flexibility, the conventional TOPSIS method is modified and refined in this paper. Firstly, the new definitions of Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed to represent the best returns and the greatest cost/risk in portfolio selection.

Compared with PIS and NIS, APIS and ANIS can achieve a balance between returns and risk, which is important for a successful investment. Then, considering that investment is a kind of decision making activity which largely depends on the decision maker’s preference, the weighted closeness coefficient is refined to offer the more appropriate results that are consistent with the investor’s demand or preference. Finally, the modified TOPSIS method is combined with linear programming model to allocate the invest- ment ratios according to the investor’s demand, which could spread risk to some extent.

The rest of this paper is organized as follows. In [Section 2](#_bookmark1), some basic conceptions of IFS and TOPSIS method are introduced for preparation. The conventional TOPSIS method is modified in [Sec-](#_bookmark2) [tion 3](#_bookmark2). In [Section 4](#_bookmark19), three kinds of linear portfolio optimization pro- gramming models based on distance measure are given. [Section 5](#_bookmark27) presents a numerical application to check the effectiveness and practicality of the models and then a sensitivity analysis is held. Finally, the conclusion is drawn in [Section 6](#_bookmark45).

1. Preliminaries

This section will introduce some basic concepts of IFS and TOP- SIS method.

* 1. *The intuitionistic fuzzy set theory*

Definition 1. . [[10](#_bookmark46)] Let *X* be a universe of discourse. Then set *A* is called a fuzzy set in *X* if it has a form as:

*A* = *x*; *lA*(*x*) |*x* ∈ *X* (1)

}

where the function *lA* (*x*) represents the membership degree of the element *x* to the set *A* with the range of [0; 1].

Definition 2. . [[11](#_bookmark46)] Let *X* be a universe of discourse. Then set *A* is called an intuitionistic fuzzy set (IFS) in *X* if it has a form as:

}

*A* = *x*; *lA*(*x*); *mA*(*x*) |*x* ∈ *X* (2)

where the function *lA* (*x*) represents the membership degree and *mA* (*x*) represents the non-membership degree of the element *x* to the set *A* with the range of [0; 1]. Meanwhile, they should satisfy 0 6 *lA* (*x*) + *mA* (*x*) 6 1, 6*x* ∈ *X*.

Let *pA* (*x*) = 1 — *lA* (*x*) + *mA* (*x*) , *pA* (*x*) is called the hesitancy degree of *x* to the IFS *A* and determined by both *lA* (*x*) and *mA* (*x*). We can easily get 0 6 *pA* (*x*) 6 1, 6*x* ∈ *X*. Then, IFS *A* can be

described as the following form:

}

*A* = *x*; *lA*(*x*); *mA*(*x*); *pA*(*x*) |*x* ∈ *X* (3)

Hesitancy degree plays an important role in many applications of IFS, such as measuring the distance of two IFSs [[13](#_bookmark46)] and reflect- ing the uncertainty of IFS [[24](#_bookmark55)]. It is also the major difference between IFS and fuzzy set. Considering that the membership

degree *lA* (*x*) represents the acceptance degree and the non-

membership degree *mA* (*x*) represents the rejection degree of a deci-

sion maker, hesitancy degree could be regarded as the neutrality degree of the decision maker, which may affect the decision mak- ing results to some extent.

For convenience, *IFS*(*X*) is used to denote the collection of all IFSs in *X* in this paper.

Entropy acts as an important role in the intuitionistic fuzzy the- ory for the reason that it represents the degree of uncertainty to IFS. Thus, it is also applied in many fields. In this paper, we will use entropy to assign weights to each criterion in TOPSIS method [[25](#_bookmark56)]. The greater the entropy, the more the uncertainty and the less

important the IFS is. Considering the simplicity of the formula and

8>>< min n*l* o; max *mij*} ; if *Cj* ∈ *C*+;

the effective utilization of *lA* (*x*), *mA* (*x*) and *pA* (*x*), the intuitionistic fuzzy entropy proposed by Wei et al. [[24](#_bookmark55)] is adopted in this paper:

*j*

*j*

*j*

*D*— =

*l*—;

*m*—

16*i*6*m ij*

## = > n o

>:

max

16*i*6*m*

*lij*

; min

16*i*6*m*

*mij*

; if *Cj* ∈ *C*—.

16*i*6*m*

}

(7)

set, *A* is an IFS in*X*, where*A* = *x*; *lA*(*x*); *mA*(*x*); *pA*(*x*) |*x* ∈ *X* , and *E*(*A*) is a mapping from *IFS*(*X*) to [0; 1], which is expressed as follows:

Definition 3. . [[24](#_bookmark55)] Suppose *X* = {*x*1; *x*2; ··· ; *xn*} is a finite fixed

}

*n*

where *D*+ and *D*— represent the assessments of *A*+ and *A*— under the criterion *Cj* in intuitionistic fuzzy set form, respectively, 1 6 *j* 6 *n*.

After assigning the weights to each criterion and adopting a

*j*

*j*

suitable distance measure in accordance with actual demand, we

*E*(*A*) = 1 X cos *lA*(*xi* ) — *mA* (*xi* ) *p* (4)

2(1 + *pA* (*xi*))

can get the weighted distances *d*+ = P*n w d*+ and *d*— = P*n w d*—

*i*=1

*n*

between the alternative *Ai* and PIS, *Ai* and NIS, respectively. Here,

*wj* denotes the weight assigned to criterion*Cj*, and *d*+, *d*— denote

*i*

*j*=1

*j*

*ij*

*i*

*j*=1

*j*

*ij*

*E*(*A*) is called an intuitionistic fuzzy entropy over *X*.

We will use the H-max distance measures [[15](#_bookmark46)] to describe the deviations of IFSs. This distance measure adopts a proper inclusion relation of IFS and has many great properties. It is given as follows:

Definition 4. . [[15](#_bookmark46)] Suppose *X* = {*x*1; *x*2; ··· ; *xn*} is a finite fixed

*ij ij*

the distances between *Ai* and PIS, *Ai* and NIS with respect to the cri- terion *Cj*, respectively, *n wj* = 1, 1 6 *i* 6 *n*, 1 6 *j* 6 *m*. Then, the closeness coefficient *CCi* of alternative *Ai* is expressed in the follow- ing way:

*j*=1

P

set, *A* and *B* are two IFSs in *X*, where*A* = *x*; *lA* (*x*); *mA*(*x*) |*x* ∈ *X*},

—

*CCi* = *i*

*d*

(8)

and *B* = *x*; *l* (*x*); *m* (*x*) |*x* ∈ *X*}. The H-max distance *d*(*A*; *B*)

*B*

*B*

*d*+ + *d*—

*i*

*i*

between *A* and *B* is expressed as follows:

!

Since the greater the *d*— and the smaller the *d*+, the better the

*i* *i*

### *d A B* 1

*n*

( ; ) = X

3*n i*=1

*lA*(*xi*) — *lB* (*xi*) + |*mA*(*xi*) — *mB*(*xi*)|+

max *lA* (*xi*); *mB*(*xi*)} — max *lB* (*xi*); *mA*(*xi*)}

alternative *Ai*, we have the conclusion that the greater the close- ness coefficient, the better the alternative.

1. The modified TOPSIS method

(5)

* 1. *The conventional TOPSIS method*
     1. *The steps of the conventional TOPSIS method*

The steps of the conventional TOPSIS method are briefly shown as follows [[22](#_bookmark52)]:

Step 1: Acquire the assessments of the alternatives with respect to each criterion.

Step 2: Define the PIS and the NIS for comparison.

Step 3: Assign weights to each criterion.

Step 4: Calculate the weighted distance between every alterna- tive and PIS, NIS.

Step 5: Calculate the closeness coefficient of every alternative.

Step 6: Rank the alternatives by the closeness coefficient.

* + 1. *PIS, AIS and closeness coefficient*

In the intuitionistic fuzzy environment, assume that there is a set *A* = {*A*1; *A*2; ··· ; *Am*} consisting of *m* alternatives which wait for ranking and a set *C* = {*C*1; *C*2; ··· ; *Cn*} consisting of *n* criteria

used for assessment. In order to deal with the values of assessment more conveniently, we denote the assessments as an IFS matrix *D* = *Dij m*×*n* , where *Dij* = *lij* ; *mij* represents the assessment of the alternative *Ai* in intuitionistic fuzzy set form. This form uses

the membership degree *lij* to describe the decision maker’s satis-

faction while using the non-membership degree *mij* to describe the decision maker’s dissatisfaction with respect to the criterion *Cj*, 1 6 *i* 6 *m*, 1 6 *j* 6 *n*. In some cases, the criteria set *C* is consid-

ered to comprise two incompatible sets *C*+ and *C*—, where *C*+ con- sists only of benefit/returns criteria and *C*— consists only of cost/

risk criteria, respectively. Then, the PIS *A*+ = *D*+; *D*+; ··· ; *D*+

1

2

*n*

Multi-criteria decision making (MCDM) problem requires deci- sion makers to rank the provided alternatives according to the assessments under several given criteria [[21](#_bookmark51)]. Thus, it is rational to regard the portfolio selection problem, which is based on the assessments under several criteria rather than the past statistical data, as a kind of MCDM problem. As one of the common methods applied in the ranking of alternatives, TOPSIS works on a simple but logical principle that the best alternative should have the shortest distance from the Positive Ideal Solution (PIS) and the far- thest distance from the Negative Ideal Solution (NIS) [[23](#_bookmark53)].

However, the conventional TOPSIS method can only provide the decision maker with a reasonable ranking of the alternatives, but cannot offer an appropriate result according to the decision maker’s preference, thus it is inflexible in the portfolio selection problem with different preferences. In order to make the method more flexible and make up for the drawbacks of the closeness coef- ficient, two improvements are made on the TOPSIS method from the following aspects:

* + - 1. Two new definitions called Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed;
      2. The new definition called weighted closeness coefficient is presented.
  1. *The Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS)*

Let us consider the following question: is the word ‘‘dislike” equivalent to the word ‘‘hate”? Obviously, the answer is no, at

least, the former is less emotional than the latter. Now, there is

and the NIS *A*— = *D*—; *D*—; ··· ; *D*— are usually defined as follows:

1

2

*n*

8> max n*l* o min *m* } if *C C*+

another similar question: can we always transform the risk criteria

into the returns type? This answer is the same as that of the former question. Thus, it is inappropriate to put the returns criteria and

*D*+ = *l*+; *m*+

16*i*6*m*

= ><

16*i*6*m*

*ij* ; *ij* ;

16*i*6*m*

*j* ∈ ;

(6)

risk criteria into together to define the ideal solution when we

*j j j*

>:

*ij*

;

16*i*6*m*

*ij*

;

*j* ∈

.

* min n*l* o

max *m* }

if *C C*—

use TOPSIS method.

From this point of view, the following new definition is presented:

Definition 5. Let *A* = {*A*1; *A*2; ··· ; *Am*} be the set of alternatives that wait for ranking and *C* = {*C*1; *C*2; ··· ; *Cn*} be the set that consists of *n* criteria used for assessment. Consider the set *C*

comprises two incompatible sets *C*+ and *C*—, where

*C*+ = *C*+ ; *C*+ ; ·· · ; *C*+ consists only of returns criteria and

*n*1

*n*2

*np*

Remark 1. *If the returns criteria set is empty*, *it implies that the investor only considers risks but ignores returns*, *i*.*e*. *the investor is completely risk averse*. *Under this condition*, *ANIS is equivalent to NIS but APIS is non*-*existent*; *correspondingly*, *if the cost*/*risk criteria set is empty*, *it also implies that the investor is completely profit*-*seek-*

*C*— = *C*— ; *C*— ; ·· · ; *C*— consists only of cost/risk criteria, respec-

*n*1

*n*2

*nq*

tively, *np* + *nq* = *n*. If *Dij* = *lij* ; *mij* represents the assessment of

the alternative *Ai* with respect to the criterion *Cj* in intuitionistic fuzzy set form, where membership degree *lij* describes the decision maker’s satisfaction and non-membership degree *mij* describes the decision maker’s dissatisfaction, 1 6 *i* 6 *m*, 1 6 *j* 6 *n*, the Absolute Positive Ideal Solution (APIS)

*A*'+ = *D*' +; *D*' +; ··· ; *D*' + and the Absolute Negative Ideal Solu-

*n*1

*n*2

*np*

*existent*.

It is obvious that the definition of APIS and ANIS can be extended to other conditions, whether the environment is uncer- tain or not. A simple numerical example is given to illustrate the flexibility of APIS and ANIS by comparing with PIS and NIS in a cer- tain environment.

*ing*. *Under this condition*, *APIS is equivalent to PIS but ANIS is non*-

Example 1. *Let A*1 *and A*2 *be two alternatives which wait for ranking*, *and C* ={*C*1; *C*2; *C*3; *C*4} *be the set of criteria*, *where C*1 *and C*2 *are*

*returns type and the rest belongs to risk type*. *The assessments of*

tion (ANIS) *A*'— = *D*'—; *D*'—; ··· ; *D*'— are defined as follows:

*n*1

*n*2

*nq*

*D*'+ = *l*'+; *m*'+ = max *l* ; min *m* (9)

*alternatives under each criterion are listed in* [Table 1](#_bookmark10):

Without losing generality, suppose each criterion is allocated

*nj nj nj*

16*i*6*m*

*inj*

16*i*6*m*

*inj*

the same weight. Then, according to [Section 2.2.2](#_bookmark6) and [Table 1](#_bookmark10),

we can find out the PIS, NIS are (0.9, 0.8, 0.3, 0.6) and (0.7, 0.7,

*D*'— = *l*'—; *m*'— = max *l* ; min *min* (10)

*nk*

*nk*

*nk*

16*i*6*m*

*ink*

16*i*6*m*

*k*

0.5, 0.7), respectively. Similarly, according to Definition 5, we get

the APIS and ANIS (0.9, 0.8) and (0.5, 0.7) respectively. On these

where *D*' + represents the assessment of *A*'+ under the criterion *C*+

*nj nj*

and *D*'— represents the assessment of *A*'— under the criterion *C*— in

bases, the differences between the alternatives and PIS, NIS, APIS, ANIS can be obtained and is showed in [Table 2](#_bookmark11).

*nk nk*

intuitionistic fuzzy set form, 1 6 *nj* 6 *np*, 1 6 *nk* 6 *nq*,*np* + *nq* = *n*.

Then, the weighted distances *d*'+ and *d*'— between the alterna-

Remark 2. *It should be noticed that according to Formula* [(11)](#_bookmark7) *and Formula* [(12)](#_bookmark8), *the weight of each criterion is* 0.5 *while we calculating*

*i* *i*

tive *Ai* and APIS, *Ai* and ANIS are defined as follows:

*d*'+ = X *wn d*+ ; (11)

*p*

*the differences between the alternatives and APIS*, *ANIS*.

From [Table 2](#_bookmark11), we can find that the closeness coefficients of the

*j*

*j*=1

*q*

*j inj*

alternatives are the same whether we use PIS, NIS or APIS, ANIS (Which is one of the drawbacks of the closeness coefficient and will

*d*'— = X *wn d*— ; (12)

*j*

*k*=1

*k*

*ink*

be addressed in [Section 3.2](#_bookmark9)). Hence, it is difficult for a neutral deci-

sion maker to make a ranking of these two alternatives.

where *wn* , *wn* denote the weights assigned to criteria *C*+ and*C*— ,

But the superiority of the APIS, ANIS lies in: if the decision

*j k nj*

*k*

*n*

maker is profit-seeking, he/she would prefer *A*

because it is closer

respectively, and *d*+

*in*

*j*

denotes the distances between *Ai* and APIS

1

to APIS than other. Conversely, if the decision maker is risk averse,

with respect to the criterion*C*+ , *d*—

denotes the distances between

*A* will be a better choice. Nevertheless, if the decision maker

*nj ink* 2

In fact, by the definitions of PIS and NIS, it is easy to draw the

*Ai* and ANIS with respect to the criterion *C*— , respectively, adopts PIS and NIS to make decision, he/she will be in a dilemma

*nk*

P*p wn* = 1,P*q wn* = 1, 1 6 *nj* 6 *np*, 1 6 *nk* 6 *nq*, *np* + *nq* = *n*.

*j*=1

*j*

*k*=1

*k*

of selection, whether he/she is profit-seeking or not.

From the definition, we know that APIS represents an imaginary alternative which has the best attributes corresponding to each returns criterion but ignoring all the cost/risk criteria, while ANIS represents an imaginary alternative which has the worst attributes corresponding to each cost/risk criterion but ignoring all the returns criteria. Thus, it is practicable to consider that APIS is better than PIS and ANIS is worse than NIS.

In this paper, we would use APIS and ANIS to replace PIS and NIS for the following reasons:

1. APIS is better than PIS and ANIS is worse than NIS, which is mentioned above, can make the comparison more distinct as to the results of different preferences, which is proved in the sensitivity analysis of the numerical application in [Section 5](#_bookmark27).
2. In the conventional TOPSIS method, PIS and NIS represent the best and the worst alternatives, respectively; in our modified TOPSIS method, APIS and ANIS represent the great- est returns and the highest cost/risk, respectively. In terms of the demand of portfolio selection, the significance of APIS and ANIS is more evident and realistic than PIS and NIS.
3. Compared with PIS and NIS, it can be found that the APIS and ANIS are more flexible than them, because the constraints on APIS and ANIS are less than those on PIS and AIS (they must take both returns and risk criteria into account), which will be illustrated in Example 1.

conclusion that in most cases, the less the distance between the alternative and PIS, the larger the distance between it and NIS. This conclusion points out that TOPSIS method may be unavailable in some cases (such as Example 1).

* 1. *The drawbacks of conventional closeness coefficient*

Dwivedi et al. [[16](#_bookmark54)] pointed out that the closeness coefficient in the conventional TOPSIS method does not take care of weights of

Table 1

The assessments of alternatives under each criterion.

assessments *C*1 *C*2 *C*3 *C*4 *A*1 0.9 0.7 0.5 0.6

*A*2 0.7 0.8 0.3 0.7

Table 2

The differences between the alternatives and PIS, NIS, APIS, ANIS.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Assessments | PIS | NIS | APIS | ANIS |
| *A*1 | 0.075 | 0.075 | 0.05 | 0.05 |
| *A*2 | 0.075 | 0.075 | 0.10 | 0.10 |

PIS and NIS distances. This limitation is also revealed in the analy- sis of [Table 2](#_bookmark11). Example 2 points it out from another perspective.

Example 2. *Let A*1 *and A*2 *be two investments which wait for selection and assume that the investor is highly profit*-*seeking rather than risk averse*. *The weighted distances d*+ *and d*— *between invest- ment Ai and PIS*, *Ai and NIS are shown as follows*:

*i* *i*

## *d*+ = 0.2, *d*— = 0.3.

* 1. *The new definition of weighted closeness coefficient*

As mentioned above, there are some drawbacks in the conven- tional closeness coefficient and its new form proposed by Dwivedi [[16](#_bookmark54)]. To make up for these drawbacks, the new definition called weighted closeness coefficient is put forward.

* + 1. *The weighted closeness coefficient*

1 1

## *d*+ = 0.3, *d*— = 0.5.

2 2

According to Formula [(8)](#_bookmark3), we have *CC*

= 0.6 and *CC*

= 0.625,

Definition 6. *Suppose A* = {*A*1, *A*2, ··· , *Am*} *is the set of alternatives which wait for ranking*, *d*'+, *d*'— *are the weighted distances between*

1 2 *i* *i*

i.e. *CC*1 < *CC*2. Thus, we may choose *A*2 to invest. Nevertheless, the investor is likely to select *A*1 because of his/her preference

*alternative Ai and APIS*, *Ai and ANIS*, *respectively*,*i* = 1, 2, ··· , *m*. *For*

*any indexi*, *if there are always d*'+ 6 1 *and d*'— 6 1, *then*, *the weighted*

*i* *i*

*i*

for greater returns, which may be a contradiction to our result.

*closeness coefficient Cw is defined as follows*:

The examples stress that taking different weights of PIS and NIS distances into account is necessary, which is also mentioned by

*Cw* =

*i* + *i*

'— 1—*k*

*i*

1+*d*

2

4*k* ×  1 + *d*'— 1—*k*

Dwivedi et al. [[16](#_bookmark54)]. In addition, Dwivedi proved that the general linear weighting method is invalid due to the essence of the close-

+

*d*

ness coefficient — it is always determined by the value of *i* ,

—

*d*

*i*

*i*  1+*d*'+ 1+*k*  1+*d*'— 1—*k*

*i* *i*

2 2

1 + *d*'+ 1+*k* + 4*k* × 1 + *d*'— 1—*k*

## (14)

= *i*

whether it is linear weighted or not. To solve the problem, Dwivedi put forward a new form of closeness coefficient *CC*' that can be written as follows [[16](#_bookmark54)]:

*i*

where *k* is the parameter to adjust the weights of APIS and ANIS distances,—0.5 6 *k* 6 0.5,1 6 *i* 6 *m*.

To transform Formula [(14)](#_bookmark13) into a simpler form, denote

^+ 1+*d*'+ ^—

1+*d*'— *w*

*d*— *w*—

*CC*' = *i* , (13)

*di* = 2 *i* ,*di* = 2 *i* , then, *Ci* can be written as:

*i*  *d*+ *w*+ + *d*— *w*—

*i*

*i*

*d*^— 1—*k*

where *w*+ and *w*— are the exponential weights of PIS and NIS dis- tances, respectively,0 6 *w*+ 6 1,0 6 *w*— 6 1.

It is doubtless that the new closeness coefficient is more flexible

*i*  *d*^+ 1+*k*  ^— 1—*k*

*i*

*i*

*Cw*

=

*i*

(15)

## + *d*

and useful, yet it still has some shortcomings if we apply it into the modified TOPSIS method. On the one hand, it neglects the relation-

Remark 3. *If there is an index i that satisfies d*'+ P 1 *or d*'— P 1, *we can normalize all the distances at first by making*

ship between *w*+ and *w*—, thus it will result in the loss of precise- —

*i* *i*

*d*'+

— '—

ness to some extent. On the other hand, it allows the weights to

*d*+ = *i*

*and d*— = *di* . *Then*, *denote*

*i* max {max {*d*'+,*d*'— }} *i* max {max {*d*'+,*d*'— }}

be zero, and Dwivedi explained that under this condition, the

16*i*6*m i* *i*

—+ ——

+

—

^ 1 ^

16*i*6*m i* *i*

results will only depend on one distance. However, this is the rea-

*di* =

+*di* , *di* = 1+*di* ,1 6 *i* 6 *m*, *Formula* [(15)](#_bookmark14) *is still effective*.

son for it lacks the resistance to extreme data, so that it cannot be applied to all conditions. The following instance is presented to

2 2

1+*d*'+

*i*

*i*

1+*d*'—

'+ '—

illustrate these shortcomings distinctly.

Remark 4. *Using d*^+ =

2 *i and d*^— =

2 *i instead of di and di re-*

Example 3. *Let A*1 *and A*2 *be two investments which wait for selection and assume that the investor is highly profit*-*seeking*. *The weighted distances d*'+ *and d*'— *between investment Ai and APIS*, *Ai and ANIS are shown as follows*:

*i* *i*

## *d*'+ = 0, *d*'— = 0.3,

*spectively can increase the resistance to extreme data*. *And then*, *let*

*the molecular and denominator* (*in left expression*) *multiply* 21+*k at the same time*, *we can obtain the right expression*.

The constraint —0.5 6 *k* 6 0.5 will be explained in Remark 5 of

[Section 3.3.2](#_bookmark15).

* + 1. *Some properties of the weighted closeness coefficient*

1 1

Now, let us analyze the properties and the role of *k* in the

*d*'+ = 0, *d*'— = 0.5. weighted closeness coefficient.

2 2

It is apparent that *A*2 is better than *A*1. Now, according to For- mula [(13)](#_bookmark12), let *w*+ = 1 and *w*— = 1, i.e. let the new closeness coeffi-

cient regress to the conventional one, we have *CC*' = *CC*' = 1,

Theorem 1. *The weighted closeness coefficient Cw is an increasing function of k.*

1 2

which is not in line with our intuition. To make the result consis-

tent with the investor’s preference, we adjust the weight of NIS dis-

' ' ' + '—

— Proof. *Since* 0 6 *d* + 6 1, 0 6 *d* — 6 1, *d*^+ = 1+*d and d*^— = 1+*d* , *we*

tance and try to set *w* = 0.5. However, we still have 2 2

*CC*' = *CC*' = 1, which implies that the new closeness coefficient *have* 1 6 *d*^+ 6 1 *and* 1 6 *d*^— 6 1, *i*.*e*. 0 < *d*^+ × *d*^— 6 1. *Then*,

1 2 2 2

is not precise enough and lacks the resistance to extreme data. Even if we let *w*— = 0, the result will not change at all. The only

way to avoid the inconsistent result is to make *w*+ = 0, then the

*d*^+ × *d*^— *k will decrease if k increases*. Since

— 1—*k*

closeness coefficient will only rely on the ANIS distance. Nonethe- less, under this condition, the weight of ANIS distance will lose its significance no matter how we change its value.

In summary, both the conventional closeness coefficient and its weighting form have some problems. Hence, it is imperative to refine them to fix these problems.

*Cw* =

*d*^

*d*^+ 1+*k*

+ ×

*d* 1

+ *d*^— 1—*k*

1

= =

^+ 1+*k*

1 + (*d*^— )1—*k*

*d*^+ *d*^—

1

*d*^+

,

× *d*^— *k*

then, *Cw* will increase if *d*^+ × *d*^— *k* decreases, i.e. *Cw* will increase if (i) If *k* = 0, since *m*1+*k* *d*^— 2*k* = *m* = *m*1+*k* *d*^— 2*k* , then *Cw* = *Cw*;

1

2

1

2

*k* increases.

1. If *k* < 0, since *m*1+*k* *d*^— 2*k* > *m*1+*k* *d*^— 2*k* , then *Cw* < *Cw*;

Theorem 2. *The weighted closeness coefficient Cw is an increasing function of d*'— *and a decreasing function of d*' + *.*

1. If *k* > 0, since *m*

1 2 1 2

1+*k* *d*^— 2*k* < *m*1+*k* *d*^— 2*k* , then *Cw* > *Cw*.

1 2 1 2

1+*d*' +

1+*d*'— +

Let us look at Formula [(15)](#_bookmark14) again. First, suppose *k* = 0. Since

—0.5 < *k* 6 0.5, 0 < *d*^+ 6 1 and0 < *d*^— 6 1, if we increase the value

' +

Proof. *Since d*^+ = 2 *and d*^— = 2 , *then*, *d*^ *will increase if d*

*in-*

of *k*, *d*^— 1—*k* will increase while *d*^+ 1+*k*

will decrease, which is

*creases and d*^— *will increase if d*'— *increases*. Since

^— 1—*k*

similar to increasing the weight of *d*^—

while decreasing the weight

### *d*

*C*

*w*

=

*i*  *d*^+ 1+*k*

+ *d*^— 1—*k*

1

= ,

*d*^+ 1+*k*

1 + (*d*^— )1—*k*

of *d*^+, i.e. increasing the weighted ANIS distance while decreasing the weighted APIS distance. Thus, it represents that the investor prefers more returns. Combined with Theorem 3, we have

and —0.5 6 *k* 6 0.5, i.e.1 + *k* > 0,1 — *k* > 0, and *d*^+ > 0, *d*^— > 0, then, *Cw* will increase if *d*^+ decreases or *d*^— increases, i.e. *Cw* will increase if *d*' + decreases or *d*'— increases.

Theorem 2 discloses that the weighted closeness coefficient is consistent with the definitions of APIS and ANIS distances and the principle that ‘‘the greater the closeness coefficient, the better the alternative”.

Let us pay attention to the following example.

Example 4. *Suppose A*1 *and A*2 *are two investments which wait for selection and assume that the investor is highly profit*-*seeking*. *The weighted distances d*'+ *and d*'— *between investment Ai and APIS*, *Ai and ANIS are shown as follows*:

*i* *i*

1. If the investor is profit-seeking, there should be *k* > 0; If the investor wants to avoid risk, there should be*k* < 0; If the investor wants to keep neutral between returns and risk,

there should be*k* = 0.

1. The bigger or the smaller the *k*, the more obvious the inves- tor’s preference.

Based on the above conclusions, *k* is called the preference factor of the weighted closeness coefficient.

Remark 5. *For the range of k*, *we stipulate it is* —0.5 6 *k* 6 0.5. *The reason lies in*: *if an alternative has the same APIS and ANIS distances*,

*i*.*e*. *we have d*^+ = *d*^— , *then*, *when k* = 0.5, *we get the weighted close-*

*d*'+

'—

*w* (*d*^— )0.5

1 1

## 1 = 0.1, *d*1 = 0.1,

*ness coefficient C*

*d*^+ 1.5

=

=

+(*d*^— )0.5

^+ *d*^+

*d*^—

1+*d*

0.5 =

1+*d*^

+, *which is totally*

2 2

*d*'+ = 0.3, *d*'— = 0.3.

It is apparent that *A*1 is better than *A*2 according to the investor’s preference. But by using the conventional closeness coefficient, we have *CC*1 = *CC*2 = 0.5. Then, by using the weighted closeness coef- ficient and making *k* = 0, we can get

*dependent on the primitive APIS distance* (*d*^+ *is obtained from APIS distance for improving the resistance to extreme data*), *and implies the decision maker is completely profit*-*seeking*. *It is in line with our*

*judgments mentioned above*. *Similarly*, *when k* = —0.5, *we have*

*d*^— 1.5

^— ^—

*Cw* = ( ) = *d* =

+

*d* —, *which shows that the decision*

*d*^1

+

= 0.55, *d*^—

## = 0.55,

*d*^+ 0.5

+(*d*^— )1.5

0.5

*d*^

*d*^—

+*d*^—

1

1+*d*^

*d*^+ = 0.65, *d*^— = 0.65,

*maker is completely risk averse*.

2 2

i.e. *Cw* = *Cw* = 0.5. It is easy to find that the result is caused by

* + 1. *Validity check of the weighted closeness coefficient*

1 2 Let us apply the weighted closeness coefficient to Example 2

*d*^+ *d*^+ and still not meet our demand. Thus, considering the condi- and Example 3 to verify its effectiveness. According to the inves-

^1 = ^2

— —

*d*

*d*

1 2 *w w* tor’s preference, we set *k* > 0 in both two instances, and then the

tion that *k* = 0.5, we have *C*1 = 0.6452 > 0.6061= *C*2 , which is con-

sistent with the investor’s preference.

In fact, we have the following conclusion:

Theorem 3. Suppose *A*1 and *A*2 are two investments which wait for selection and *d*'+, *d*'— are the weighted distances between

results are shown in [Table 3](#_bookmark16) and [Fig. 1](#_bookmark17), [Table 4](#_bookmark17) and [Fig. 2](#_bookmark18), respectively.

From [Table 3](#_bookmark16) and [Fig. 1](#_bookmark17), [Table 4](#_bookmark17) and [Fig. 2](#_bookmark18), we can find that:

* + - 1. All the values of weighted closeness coefficient are increas-

investment *A*

*i* *i*

*A* and ANIS, respectively, *i*

1 2. Let

ing while the value of *k* is increasing, which conforms to

+ 1+*d*'+

*i* and APIS, *i*

1+*d*'—

+ + — —

=

*d*^+

—

,

*d*^+

—

Theorem 4.

*w w*

*d*^*i* = 2 *i* , *d*^— =

*i*

the alternative *A*

2 *i* , *i* = 1, 2. If *d*^1 < *d*^2 , *d*^1 < *d*^2 , and ^1 = ^2 , i.e.

1 2

*d d*

* + - 1. The value of *C*1 is overtaking the value of *C*2 while *k* is increasing in [Table 3](#_bookmark16) and [Fig. 1](#_bookmark17), even when *k* P 0.4, we have

1 can bring the investor more returns while it also

*w w Cw* > *Cw*. The reason for this phenomenon is that *d*'+ < *d*'+,

brings more risks, we have: If *k* = 0, then *C*1 = *C*2 ; If *k* < 0, then 1 2 1 2

*Cw* < *Cw*; If *k* > 0, then *Cw* > *Cw*.

1 2 1 2

Proof. *Let m*

*d*^+ *d*^+, ^+

^— ^+ ^—

Table 3

= ^1 = ^2

*i*.*e*. *d*1 = *m* × *d*1 *and d*2 = *m* × *d*2 ,*m* > 0.

The values of *Cw* and *Cw* with different values of preference factor *k* in Example 2.

— —

*d*

*d*

1 2 1 2

Since 0 < *d*^— < *d*^— , *m* > 0, then,

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *k* 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| *w* | 0.5667 | 0.5879 | 0.6122 | 0.6343 |
| *Cw* 0.5535 | 0.5712 | 0.5887 | 0.6060 | 0.6230 |

*Cw* =

1 2

*d*^— 1—*k*

*i*

*d*

=

1

=

1

=

1

*C*1 0.5434

2

*i*

*i*  *d*^+ 1+*k*

*i*

1 *i*

+

+ *d*^— 1—*k*

^+ 1+*k*

1 + *i*

(*d*^— )1—*k*

*i*

(*md*^— )1+*k*

(*d*^— )1—*k*

*i*

1 + *m*1+*k*

*i*

*d*^— 2*k*

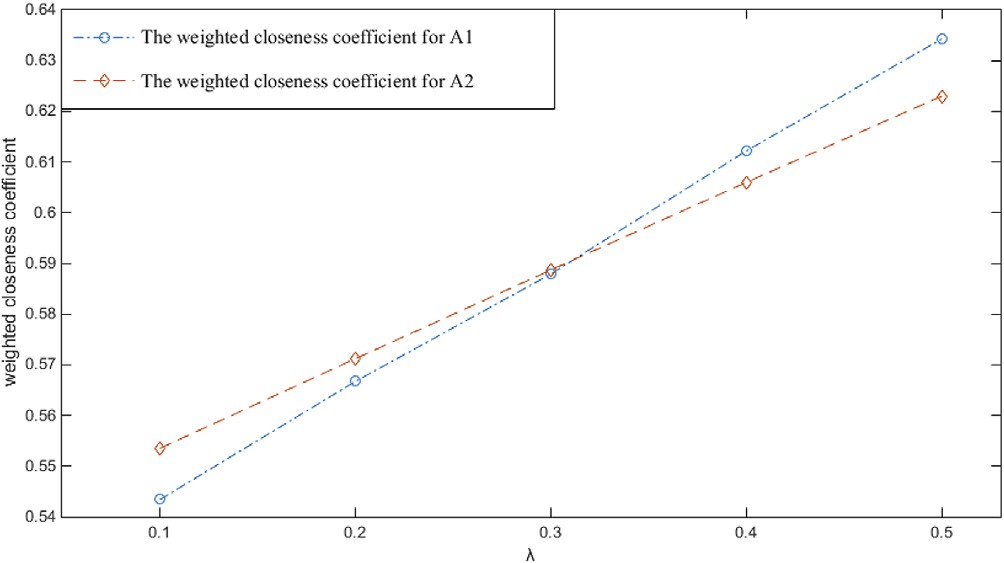


Fig. 1. The values of *Cw* and *Cw* with different values of preference factor *k* in Example 2.

1 2

Table 4

The values of *Cw* and *Cw* with different values of preference factor *k* in Example 3.

several of them rather than only one to invest. How to choose the investments and allocate the investment ratios can help the investor obtain great returns while avoiding the risk to some extent?

In fact, conventional MCDM problem only takes the ranking of the alternatives into account and the investor will choose the best alternative for investment, which is imperfect for the portfolio selection problem mentioned above. Portfolio selection problem not only focus on a satisfactory returns but also focus on an accept- able risk. However, choosing one alternative for investment cannot spread risk well. Next, we will solve the problem by combining the modified TOPSIS method and the linear portfolio optimization pro- gramming models based on the distance measure of IFSs. To meet investors’ different demands, three models with different con- straints are given in this section.

* 1. *The general model for portfolio selection*

Firstly, the steps to solve the aforementioned problem are listed as follows:

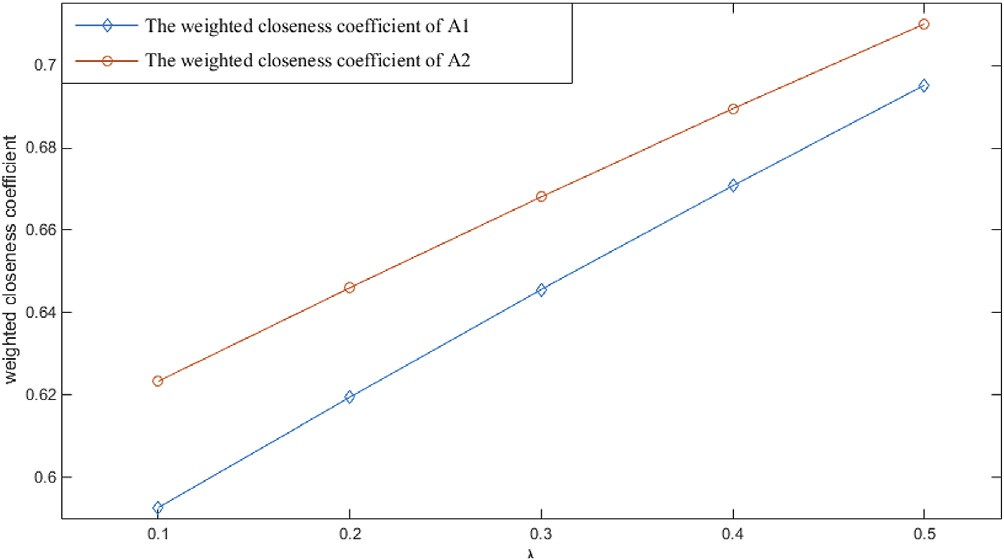
1 2

1

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *k* | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | Step 1: Gain the assessment matrix for investments under the |
| *Cw* | 0.5926 | 0.6194 | 0.6456 | 0.6708 | 0.6952 | given criteria. |
| *Cw* | 0.6233 | 0.6460 | 0.6681 | 0.6895 | 0.7101 | Step 2: Define the APIS and ANIS, and then denote them as the |
| *Cw* — *Cw* | 0.0307 | 0.0266 | 0.0226 | 0.0187 | 0.0149 | greatest returns and the ceiling of the highest risk, respectively. |
|  | | | | | | Step 3: Assign the weights to each criterion by using the |
| entropy weight method [[25](#_bookmark56)]. |
| Step 4: Use the H-max distance measure to calculate the |
| weighted distances between each investment and APIS, ANIS. |
| Step 5: Calculate the weighted closeness coefficient of each |
| investment. |
| Step 6: Rank the investments by their weighted closeness coef- |
| ficients and pick out a certain quantity of them from priority to |
| inferiority. |
| Step 7: Allocate the investment ratios for each selected invest- |
| ment by the linear portfolio optimization programming model |
| which will be mentioned in the following analysis. |
| Step 1 to Step 6 have been analyzed in the above sections, thus |

2

2 1

this section will focus on Step 7. For convenience, let

*A*' = *A*' , *A*' , ··· , *A*' } be the set of the selected investments and

1

2

*k*

*d*'+, *d*'— be the weighted distances between investment *A*' and APIS,

Fig. 2. The values of *Cw* and *Cw* with different values of preference factor *k* in *i i* *i*

1 2

*i*

Example 3.

i.e. in Example 2, the alternative *A*1 can bring the investor more returns. Then, if we increase the value of *k*, which is equivalent to increasing the investor’s preference for returns, the advantage of *A*1 will become more and more obvious. It is consistent with our intuition.

* + - 1. In Example 3, there is *Cw* < *Cw* when *k* > 0, which is in line

*A*' and ANIS, respectively, *i* = 1, 2, ··· , *k*. In consideration of the

investment intention, we should make the returns of portfolio as great as possible while making the risk as small as possible. Since the APIS and ANIS are regarded as the greatest returns and the ceil- ing of the highest risk, respectively, it is taken for granted to pursue the maximization of total weighted ANIS distance and the mini- mization of total weighted APIS distance. In addition, each selected investment should be invested so that all the investment ratios should be positive. Hence, a linear portfolio optimization program-

1 2 *w w* ming model based on the weighted distances is proposed as

with our judgment. Besides, the value of *C*2 — *C*1 is decreas-

ing while the value of *k* is increasing due to *d*'+ = *d*'+. It can

follows:

1 2 *k*

min *D* (*X*) = P *xid*

be expressed that with the increase of *k*, the investor’s pref- erence for returns is increasing and the weight of risk is gradually decreasing. Thereby, in the perspective of the

+ '+

*i*

*i*=1

*k*

max *D*—(*X*) = P *xid*'—

investor, the difference between two alternatives is reduced, which is realistic and reasonable.

## s.t.8><

*k*

P

### *x*

*i*=1

*i*

*i*=1

*i* = 1,

## (16)

1. Several novel linear programming models based on distance measure for portfolio selection

*li* 6 *xi* 6 *ui*.

where *d*'+, *d*'— are the weighted distances between the selected

*i* *i*

Consider such a problem: If *A* = {*A* , *A* , ..., *A* } is a set of invest- investment *A*' and APIS, *A*' and ANIS, respectively, and *xi* is the

>:

1 2 *m i* *i*

ments which wait for selection and the investor wants to choose investment ratio allocated to the selected investment *A*', *li*, *ui* repre-

*i*

sent the minimum investment ratio and the maximal investment

min *D*(*X*) = P *x* *d*'+ — *d*'—

ratio for *A*', respectively, 0 6 *ui* 6 *li* 6 1, *i* = 1, 2, ··· , *k*.

*i*

8>

Model [(16)](#_bookmark20) is quite a simple linear programming model. Obviously, it can be transformed into the following simpler

*i*

*i*=1

*k*

P *x d*'+ 6

*k*

*i* *i*

*D*+(*h* ),

model:

*i i* D 1

*i*=1

# P

*k*

*xid*'— P *D*—(*h*2),

*k*

'+

'—

*i*

D

P >

## min *D*(*X*) =

8

*i*=1 *k*

*xi di* — *di*

## (17)

s.t.>><

*i*=1

P *xi* = 1,

*k*

## s.t.><

P *xi* = 1,

*i*=1

* *l* 6 *x* 6 *u* ,

*i*=1

>

: *l* 6 *x* 6 *u* . >

*i*

*i*

*i*

*i i* *i*

> *D*+(*h* ) = max *d*'+} — *h*

## × max *d*'+} — min *d*'+} ,

If the investor is satisfied with the model, we just need to solve

D 1 16*i*6*n* *i*

>

1 16*i*6*n* *i*

16*i*6*n i*

the model. Otherwise, he/she may put forward some other require-

## >>: *D*—(*h*2) = min *d*'—} + *h*2 × max *d*'—} — min *d*'—} .

ments, such as desiring more returns in spite of greater risks, or asking for a return which is higher than a certain limit while keep- ing the risk does not exceed a set ceiling. Under this condition, we

D 16*i*6*n i*

16*i*6*n* *i*

16*i*6*n i*

## (20)

can add some constraints to Model [(17)](#_bookmark21) and get the more suitable

where *d*'+, *d*'— are the weighted distances between the selected

models. investment *A*' and APIS, *A*' and ANIS, respectively, and *xi* is the

*i*

*i*

*i* *i*

* 1. *The weighted model for portfolio selection*

investment ratio allocated to the selected investment *A*',

*D*+(*h*1), *D*—(*h*2) represent the requirements for returns and risk

*i*

D D

If the investor desires more returns in spite of greater risks, or prefers to avoid the risk as far as possible even if he/she would get fewer returns, it is feasible to modulate the weights of the total weighted APIS distance and the total weighted ANIS distance. Then, Model [(17)](#_bookmark21) can be refined as the following form:

min *D*(*X*) = P *xi* *ad*'+ — *bd*'—

*k*

and satisfy that 0 6 *h*1 6 1, 0 6 *h*2 6 1, *li*, *ui* represent the minimum

investment and the maximal investment for *A*', respectively, 0 6 *ui* 6 *li* 6 1, *i* = 1, 2, ··· , *k*.

*i*

It is evident that when *h*1 = *h*2 = 0, Model [(20)](#_bookmark26) regresses to Model [(17)](#_bookmark21).

Note that comparing with Model [(19)](#_bookmark25), we cannot ensure that there is a feasible solution for Model [(20)](#_bookmark26). To avoid the problem as far as possible, it is judicious to let *h*1 = 0 or *h*2 = 0 to reduce the restriction. Then, the model will be transformed into one of

## s.t.8><>

P

*i* *i*

*i*=1

*k*

*xi* = 1,

*i*=1

## (18)

the following two types:

1. The risks baseline type:

min *D*+(*X*) = P *xid*'+

*i*

*k*

: *li* 6 *xi* 6 *ui*, *a* P 0, *b* P 0.

*i*=1

8 *k*

* P *xid*'— P *D*—(*h*),

where *d*'+, *d*'— are the weighted distances between the selected

*i* D

*i*=1

*k*

>

>

P

*i* *i*

' ' <

investment *Ai* and APIS, *Ai* and ANIS, respectively, *a* and *b* denote

the weights of APIS, ANIS distances, *x* is the investment ratio allo-

*xi* = 1,

*s*.*t*.

*i*=1

## (21)

* *l* 6 *x* 6 *u* ,

*i*

cated to the selected investment *A*', *li*, *ui* represent the minimum

*i*

investment ratio and the maximal investment for ratio *A*', respec-

D

*i*

*D* (*h*) = min

16*i*6*n*

*di*

+ *h* × max

16*i*6*n*

*di*

— min

16*i*6*n*

*di*

.

tively, 0 6 *ui* 6 *li* 6 1,*i* = 1, 2, ·· · , *k*.

>>: —

*i*

*i*

'—}

*i*

'—}

'—}

It is evident that when *a* = *b* = 1, Model [(18)](#_bookmark22) regresses to Model

[(17)](#_bookmark21).

If the investor is profit-seeking, he/she can set *a* > *b*, or else, he/

she can set *a* 6 *b*. In order to make the model more standardized, it

1. The returns baseline type:

max *D*—(*X*) = P *xid*'—

*i*=1

*i*

8>

*k*

is better to stipulate a rational bound on *a* and *b*. Thus, some con-

straints can be imposed on the model and then transformed into

*k*

'+ *D*+(*h*),

P *x d* 6

*i*

the form as follows:

min *D*(*X*) = P *xi* *ad*'+ — *bd*'—

*k*

*s*.*t*.>>

>

<

*i*=1

*i*

*i*P=1

*k*

D

*xi* = 1,

## (22)

*i* *i*

*i*=1

## 8

*li* 6 *xi* 6 *ui*,

>

D

16*i*6*n*

*i*

16*i*6*n*

*i*

16*i*6*n*

*i*

*a* + *b* = 2,

>

*k*

*s*.*t*.<

P *x*

*i* =

1,

## (19)

>: *D*+(*h*) = max *d*'+} — *h* × max *d*'+} — min *d*'+} .

>>:

*i*=1

*li* 6 *xi* 6 *ui*, *a* P 0, *b* P 0.

Although we still cannot ensure that both Model [(21)](#_bookmark23) and Model [(22)](#_bookmark24) have at least one feasible solution, we increase the probability. Moreover, the models have the advantage that they

* 1. *The baseline model for portfolio selection*

If the investor asks for a return higher than a certain limit or he/ she wants to avoid the risk which may exceed a set ceiling, we can add the corresponding constraints on the model and then get the following model:

can ensure the risks will not exceed a certain ceiling or the returns will be higher than a certain baseline, which is exactly the short- coming of Model [(19)](#_bookmark25).

If Model [(21)](#_bookmark23) or Model [(22)](#_bookmark24) is adopted, we can set the value of *h* within the range [0, 1] to obtain a result that is consistent with the investor’s preference. It is apparent that in Model [(21)](#_bookmark23), the bigger the value of *h*, the more the investor wants to avoid risk; whereas

in Model [(22)](#_bookmark24), the bigger the value of *h*, the greater the investor’s preference for returns. Therefore, we have

* + 1. If the investor is highly risk averse, we can make *h* > 0.7 in Model [(21)](#_bookmark23) or *h* < 0.3 in Model [(22)](#_bookmark24);
    2. If the investor is slightly risk averse, we can make

0.7 P *h* > 0.5 in Model [(21)](#_bookmark23) or 0.3 6 *h* < 0.5 in Model [(22)](#_bookmark24);

* + 1. If the investor is neutral, we can use Model [(19)](#_bookmark25) or make

*h* = 0.5 in Model [(21)](#_bookmark23) or Model [(22)](#_bookmark24);

* + 1. If the investor is slightly profit-seeking, we can make

0.3 6 *h* < 0.5 in Model [(21)](#_bookmark23) or 0.7 P *h* > 0.5 in Model [(22)](#_bookmark24);

* + 1. If the investor is highly profit-seeking, we can make *h* < 0.3 in Model [(21)](#_bookmark23) or *h* > 0.7 in Model [(22)](#_bookmark24).

In next section, we will use a numerical example to show how the results vary with different values of *h* in Model [(21)](#_bookmark23) or Model [(22)](#_bookmark24).

From [Table 5](#_bookmark30), we can identify the PIS *A*+, the NIS *A*—, the APIS

*A*'+, the ANIS *A*'—, respectively. They are presented as follows:

The PIS: *A*+ = ((0.7, 0.1), (0.6, 0.1), (0.9, 0), (0.8, 0),

(0.1, 0.8), (0, 0.8), (0.2, 0.8)),

The NIS: *A*— = ((0.1, 0.6), (0.1, 0.5), (0.1, 0.7), (0, 0.5),

(0.5, 0.1), (0.5, 0.2), (0.7, 0.1)),

The APIS: *A*'+ = ((0.7, 0.1), (0.6, 0.1), (0.9, 0), (0.8, 0)),

The ANIS: *A*'— = ((0.5, 0.1), (0.5, 0.2), (0.7, 0.1)).

The next step is to assign weights to each criterion by using the entropy weight method [[25](#_bookmark56)]. Firstly, we calculate the entropy *E Cj* of each criterion according to Formula [(4)](#_bookmark4). It should be noticed that if we use the conventional TOPSIS method, the weight *wj* of crite- rion *Cj* should be assigned according to the following formula [[25](#_bookmark56)]:

*j*

*j*

7

*t*=1

1 — *E* *Cj*

1. Numerical application

*w* = P 1 — *E* *Cj*

## (23)

To verify the validity and applicability of the linear program- ming models proposed in [Section 4](#_bookmark19), an example of portfolio selec- tion in the intuitionistic fuzzy environment is given in this section.

However, if we use the modified TOPSIS method, the weight *w*' of criterion *Cj* should be assigned according to the following formula:

Example 5. *Assuming that ten companies which hold different*

*newest projects are looking for investment*, *and an investor is*

8> P 1—*E*(*Cj* )

## , if 1 6 *j* 6 4,

*interested in the projects so that he*/*she intends to choose three of* *them to invest*. *As these projects are new*, *there is no historical data*

*j*  1—*E*(*Cj* )

7

>: P (1—*E*(*C* ))

*w*' =

<

4

*t*=1

(1—*E*(*Cj* ))

*t*=5 *j*

(24)

## , if 5 6 *j* 6 7.

*available for reference*. *In order to make a sensible investment*, *the investor assesses the capabilities of each company according to the criteria listed as follows*:

* The company is in a large scale (*C*1).
* The company has a promising prospect (*C*2).
* The company has a great influence in the market (*C*3).
* The project is important for the company (*C*4).
* The company had suffered a huge economic blow (*C*5).
* The company has many failed projects (*C*6).
* The company had a terrible reputation (*C*7).

are used to denote

where 1 6 *j* 6 7. The results are shown in [Table 6](#_bookmark31).

After assigning the weights, we use the H-max distance mea- sure shown in Formula [(5)](#_bookmark5) to calculate the weighted distances between investment *Ai* and PIS *A*+, *Ai* and NIS *A*—, *Ai* and APIS *A*'+, *Ai* and ANIS *A*'—. Then, the results are shown in [Table 7](#_bookmark31).

From [Table 7](#_bookmark31), we can find that the data of APIS distances is little different from that of PIS distances while the data of ANIS distances is little different from that of NIS distances.

Let us rank the investments by their weighted closeness coeffi- cients with various values of *k* according to Formula [(15)](#_bookmark14). To be convenient, we denote the weighted closeness coefficient of PIS

and NIS distances as *Cw* *A*+, *A*— , the weighted closeness coeffi-

the companies and

1 2 10

cient of APIS and ANIS distances as *Cw**A*'+, *A*'— . Then, the results

*C*1, *C*2, ··· , *C*7 are used to denote the above

For the sake of convenience, *A* , *A* , ··· , *A*

criteria. We can find that *C*1, *C*2, *C*3, *C*4 are returns criteria and the rest is risk criteria. The investor can express his/her acceptance and rejection degree of each company under different criteria, so that it is easy to get the assessment matrix in intuitionistic fuzzy form.

* 1. *The ranking of the investments*

According to the steps proposed in [Section 4](#_bookmark19), an intuitionistic fuzzy assessment matrix *D* = *Dij* 10×7 =*lij* , *mij* 10 7 is

obtained from the investor and shown in [Table 5](#_bookmark30).

×

with various values of *k* are shown in [Table 8](#_bookmark31).

From [Table 8](#_bookmark31), we can find that:

1. When *k* = 0 or *k* = 0.5, there is little difference in the ranking of *Cw A*+, *A*— and the ranking of*Cw A*'+, *A*'— , which shows that APIS and ANIS are as effective as PIS and NIS.

1. When *k* = —0.5, the ranking of *Cw A*+, *A*— is hugely differ- ent from the ranking of *Cw A*'+, *A*'— . According to the assess- ment matrix shown in [Table 5](#_bookmark30), it is not difficult to find that

the assessments of *A*4 are comparatively better than those of

Table 5

The intuitionistic fuzzy assessment matrix provided by the investor.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *C*1 | *C*2 | *C*3 | *C*4 | *C*5 | *C*6 | *C*7 |
| *A*1 | (0.3, 0.4) | (0.4, 0.4) | (0.6, 0.3) | (0.5, 0.5) | (0.1, 0.7) | (0.4, 0.2) | (0.2, 0.5) |
| *A*2 | (0.2, 0.6) | (0.3, 0.4) | (0.2, 0.3) | (0.7, 0.1) | (0.1, 0.2) | (0.1, 0.3) | (0.2, 0.1) |
| *A*3 | (0.2, 0.1) | (0.3, 0.3) | (0.1, 0.4) | (0.2, 0.1) | (0.3, 0.1) | (0.2, 0.4) | (0.3, 0.3) |
| *A*4 | (0.5, 0.2) | (0.4, 0.2) | (0.5, 0.1) | (0.5, 0.3) | (0.3, 0.2) | (0.2, 0.6) | (0.2, 0.7) |
| *A*5 | (0.4, 0.3) | (0.4, 0.4) | (0.6, 0.2) | (0.6, 0.1) | (0.4, 0.5) | (0.3, 0.3) | (0.2, 0.8) |
| *A*6 | (0.7, 0.1) | (0.5, 0.2) | (0.9, 0.0) | (0.5, 0.2) | (0.5, 0.3) | (0.5, 0.5) | (0.3, 0.6) |
| *A*7 | (0.1, 0.6) | (0.3, 0.5) | (0.1, 0.7) | (0.8, 0.0) | (0.4, 0.4) | (0.2, 0.3) | (0.2, 0.4) |
| *A*8 | (0.3, 0.3) | (0.6, 0.3) | (0.4, 0.4) | (0.5, 0.5) | (0.1, 0.8) | (0.1, 0.7) | (0.7, 0.2) |
| *A*9 | (0.5, 0.3) | (0.4, 0.1) | (0.4, 0.2) | (0.6, 0.2) | (0.2, 0.5) | (0.0, 0.8) | (0.5, 0.4) |
| *A*10 | (0.1, 0.1) | (0.1, 0.3) | (0.1, 0.4) | (0.0, 0.5) | (0.2, 0.1) | (0.4, 0.2) | (0.3, 0.1) |

The entropy and weights of each criterion.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Criterion | *C*1 | *C*2 | *C*3 | *C*4 | *C*5 | *C*6 | *C*7 |
|  | 0.9278 | 0.9710 | 0.8537 | 0.8659 | 0.9105 | 0.8973 | 0.8885 |
| *wj* | 0.1053 | 0.0423 | 0.2135 | 0.1957 | 0.1306 | 0.1498 | 0.1627 |
| *w*' | 0.1892 | 0.0760 | 0.3834 | 0.3515 | 0.2947 | 0.3382 | 0.3672 |

*E* *Cj*

*j*

Table 7

The weighted H-max distances between each investment and PIS, NIS, APIS, ANIS.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Distances | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*+ | 0.3055 | 0.3836 | 0.4528 | 0.2491 | 0.2316 | 0.1994 | 0.3887 | 0.3165 | 0.2414 | 0.5595 |
| *A*— | 0.3088 | 0.3175 | 0.2666 | 0.3909 | 0.3920 | 0.4034 | 0.2671 | 0.2926 | 0.3944 | 0.1629 |
| *A*'+ | 0.3310 | 0.3698 | 0.4912 | 0.2646 | 0.2290 | 0.1013 | 0.4264 | 0.3759 | 0.2716 | 0.6077 |
| *A*'— | 0.3510 | 0.3010 | 0.2519 | 0.3689 | 0.3356 | 0.2126 | 0.2871 | 0.3469 | 0.3755 | 0.1794 |

Table 8

The weighted closeness coefficients of each investment with various values of *k*.

*k k* = 0 *k* = 0.5 *k* = —0.5

*Cw Cw A*+, *A*— *Cw A*'+, *A*'— *Cw A*+, *A*— *Cw A*'+, *A*'— *Cw A*+, *A*— *Cw A*'+, *A*'—

*A*1 0.5027 0.5147 0.7670 0.7568 0.2369 0.2655

*A*2 0.4529 0.4487 0.7034 0.7093 0.2241 0.2136

*A*3 0.3706 0.3390 0.6289 0.5931 0.1698 0.1528

*A*4 0.6108 0.5823 0.8341 0.8169 0.3287 0.3034

*A*5 0.6286 0.5944 0.8489 0.8409 0.3377 0.2889

*A*6 0.6692 0.6773 0.8770 0.9346 0.3646 0.2355

*A*7 0.4073 0.4024 0.6808 0.6580 0.1813 0.1907

*A*8 0.4804 0.4799 0.7523 0.7188 0.2196 0.2500

*A*9 0.6203 0.5803 0.8411 0.8124 0.3352 0.3063

*A*10 0.2255 0.2279 0.4909 0.4720 0.0808 0.0888

Ranking *A*6 > *A*5 > *A*9 > *A*4 > *A*1 > *A*8 > *A*2 > *A*7 > *A*3 > *A*10

*A*6 > *A*5 > *A*4 > *A*9 > *A*1 > *A*8 > *A*2 > *A*7 > *A*3 > *A*10

*A*6 > *A*5 > *A*9 > *A*4 > *A*1 > *A*8 > *A*2 > *A*7 > *A*3 > *A*10

*A*6 > *A*5 > *A*4 > *A*9 > *A*1 > *A*8 > *A*2 > *A*7 > *A*3 > *A*10

*A*6 > *A*5 > *A*9 > *A*4 > *A*1 > *A*2 > *A*8 > *A*7 > *A*3 > *A*10

*A*9 > *A*4 > *A*5 > *A*1 > *A*8 > *A*6 > *A*2 > *A*7 > *A*3 > *A*10

*A*5 with respect to the risk criteria, while the assessments of *A*5 are comparatively better than those of *A*6 with respect to the risk criteria. Thus, it is more rational to think *A*4 is better than *A*5 and *A*5 is better than *A*6 when the investor is extre- mely risk averse. Moreover, it can be found that *A*8 is a little better than *A*6 under the same condition. In consequence, the ranking of *Cw* *A*'+, *A*'— should be regarded as more

Remark 6. In [Table 9](#_bookmark32), the value in 1st (horizontal) row (*A*1) and 3rd (vertical) column (*A*3) is ‘‘3, 5, 7”. It means that compared with *A*3, the strong concordance set of *A*1 is {*C*3, *C*5, *C*7}. Similarly, the values in [Tables 10–14](#_bookmark32) can also be described in this way. Note that

‘‘—” denotes an empty set.

Remark 7. *Since the criteria C*5, *C*6, *C*7 *are risk criteria*, *when we*

rational than the ranking of *Cw A*+, *A*— , i.e. APIS and ANIS are better than PIS and NIS in this case.

*compare the investments*, *we will transform the corresponding assess- ments into returns type*: *if the assessment is denoted by an intuition- istic fuzzy number* (*l*, *m*), *then its value will be converted to* (*v*, *l*)

* 1. *Comparison with IF ELECTRE method* ( —

ELECTRE approach, proposed by Benayoun et al. [[26](#_bookmark59)], is also a MCDM method widely used [[27](#_bookmark60)]. This method is initially applied in some MCDM problems with crisp assessments. And then, to solve uncertainty problems, Wu and Chen [[28](#_bookmark61)] developed the IF ELECTRE method.

In this part, we will adopt IF ELECTRE to rank the alternatives *A*1, *A*2, ··· , *A*10 and then make a comparison with our modified TOPSIS method. According to the steps given by Wu and Chen [[28](#_bookmark61)], the ranking process is shown as follows:

Step 1: Determine the decision matrix. This step is finished in [Section 5.1](#_bookmark29), please see [Table 5](#_bookmark30).

Step 2: Determine the concordance and discordance sets. After the pairwise comparison, we obtain the results shown in [Tables](#_bookmark32) [9–14](#_bookmark32).

*refer to the definition of complementary set A in* [[28](#_bookmark61)]).

Step 3: Calculate the concordance matrix. Here, the investor gives the relative weights for the concordance sets as [*W*(*CS*), *W*(*CM*), *W*(*CW* )] = 1, 2 , 1 (refer to the weights in [[28](#_bookmark61)]), and then we have the concordance matrix shown in [Table 15](#_bookmark33).

3

3

For instance, in 3rd (horizontal) row (*A*3) and 5th (vertical) col- umn (*A*5), the value is

X

X

*Conc*(3, 5) = *W*(*CS*) × *wi* + *W*(*CM*) × *wi*

*i*∈*CS*(3, 5) *i*∈*CM* (3, 5)

X

+ *W*(*CW* ) × *wi*

*i*∈*CW* (3, 5)

Table 9

The strong concordance set *CS* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | — | 3, 5, 7 | 5 | — | — | 3 | 3 | 5 | 3, 5, 7 |
| *A*2 | — | — | — | — | — | 4 | — | — | — | 4 |
| *A*3 | — | — | — | — | — | — | — | — | — | — |
| *A*4 | 6 | 3 | 3, 7 | — | 6 | — | — | 1 | — | 2, 3, 4, 6, 7 |
| *A*5 | — | 3 | 3, 7 | — | — | 5, 7 | — | 7 | 7 | 3, 4, 7 |
| *A*6 | 1 | 3 | 2, 3 | 1, 3 | 1, 3 | — | 1, 3 | 1, 3 | 3 | 2, 3, 4 |
| *A*7 | — | — | 4 | — | 4 | 4 | — | — | — | 4, 7 |
| *A*8 | 2, 6 | 2 | 5, 6 | 5 | 2, 6 | 5 | 2, 5, 6 | — | 5 | 5, 6 |
| *A*9 | 1, 6 | 3, 6 | 3, 5, 6 | 5 | 6 | — | 1, 6 | — | — | 2, 3, 4, 6 |
| *A*10 | — | — | — | — | — | — | — | — | — | — |

Table 10

The moderate concordance set *CM* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 1 | — | — | 5 | 5 | 1, 2, 5 | 7 | 7 | — |
| *A*2 | 4, 6 | — | 3, 5 | 4, 5 | 6 | — | 2, 3, 6 | 4 | 4 | 3, 5, 6, 7 |
| *A*3 | 6 | 1, 2 | — | — | 6 | — | 1, 2, 3 | 7 | — | 4, 6 |
| *A*4 | 1, 2, 4 | 1, 2 | 2 | — | 1, 2 | 6, 7 | 1, 2, 3 | 3, 4, 7 | 1, 3, 7 | — |
| *A*5 | 1, 3, 4, 6 | 1 | — | 4 | — | 4 | 1, 2, 3 | 3, 4 | 4 | 6 |
| *A*6 | 2, 3, 4 | 1, 2 | — | 4 | 2 | — | 2 | 4, 7 | 1, 7 | — |
| *A*7 | 4, 6 | 4 | 7 | 4 | 6 | 5 | — | 4, 7 | 4, 7 | 6 |
| *A*8 | 1 | 1 | — | 6 | 5 | 6 | 1, 3 | — | — | — |
| *A*9 | 2, 4 | 1, 2 | 2 | 2, 4, 6 | 2, 5 | 5, 6 | 2, 3, 5 | 3, 4, 6, 7 | — | — |
| *A*10 | — | — | 5 | — | — | — | 1, 3 | — | — | — |

Table 11

The weak concordance set *CW* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 2, 3, 5, 7 | 1, 2, 4 | 2, 3, | 2, 3 | 4 | 7 | 1, 4 | 2, 3 | 1, 2, 4, 6 |
| *A*2 | — | — | 1, 2, 4 | — | 4 | — | 1 | — | — | 1, 2 |
| *A*3 | — | 6, 7 | — | — | — | — | 6 | — | — | 1, 2, 3, 5, 7 |
| *A*4 | 7 | 5, 6, 7 | 1, 4, 5, 6 | — | — | 4 | 6, 7 | — | 2 | 1, 5 |
| *A*5 | 2, 7 | 2, 5, 6, 7 | 1, 2, 4, 5 | 2, 3, 5, 7 | — | — | 5, 6, 7 | 1 | 2, 3, 5 | 1, 2, 5 |
| *A*6 | 6, 7 | 5, 6, 7 | 1, 4, 5, 6, 7 | 2, 5 | 6 | — | 6, 7 | — | 2 | 1, 5, 6, 7 |
| *A*7 | — | 2, 5, 6, 7 | 2, 3, 5 | 5 | — | — | — | — | — | 1, 2, 3, 5 |
| *A*8 | 4, 5 | 3, 5, 6, 7 | 1, 2, 3, 4 | 2, 4 | — | 2, 4 | — | — | 2, 3 | 1, 2, 3, 4, 7 |
| *A*9 | — | 5, 7 | 1, 4, 7 | 1 | 1, 4 | 4 | 7 | 1 | — | 1, 5, 7 |
| *A*10 | 6 | 7 | 3 | — | — | — | — | — | — | — |

Table 12

The strong discordance set *DS* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | — | 6 | 1, 6, 7 | 1, 6, 7 | 1, 3 | — | 2, 5, 6 | 1, 6 | — |
| *A*2 | 2, 3, 5, 7 | — | — | 3, 7 | 2, 3, 7 | 1, 2, 3 | 4, 7 | 2, 5, 6 | 1, 3, 6 | — |
| *A*3 | 3, 5, 7 | 3, 4 | — | 2, 3, 5, 6, 7 | 3, 4, 7 | 1, 2, 3, 7 | 4, 7 | 2, 3, 5, 6 | 3, 5, 6 | — |
| *A*4 | 5 | 4 | — | — | 7 | 1, 2, 3 | 4 | 5, 6 | 4, 5, 6 | — |
| *A*5 | — | 4 | 6 | 1, 6 | — | 1, 3 | 4 | 2, 5, 6 | 1, 6 | — |
| *A*6 | 5 | 4 | — | 7 | 4, 5, 7 | — | 4, 5 | 5 | 4 | — |
| *A*7 | 1, 2, 3, 5, 7 | 1 | 6 | 1, 6, 7 | 1, 2, 3, 5, 7 | 1, 3 | — | 2, 3, 5, 6 | 1, 6 | — |
| *A*8 | 3 | — | — | 1, 7 | 1, 3, 7 | 1, 3, 7 | — | — | 1, 6, 7 | — |
| *A*9 | 5 | 4 | — | 3, 7 | 3, 7 | 1, 3, 7 | 4 | 5 | — | — |
| *A*10 | 3, 4, 5, 7 | 3, 4, 5 | 1, 2, 6, 7 | 2, 3, 4, 6, 7 | 3, 4, 6, 7 | 1, 2, 3, 4, 7 | 4, 7 | 2, 3, 4, 5, 6 | 2, 3, 4, 5, 6 | — |

## 2 1 2

= 1 × 0 + × *w*6 + × 0 = × 0.1498 = 0.0999,

3

3

3

and, in 6th (horizontal) row (*A*6) and 9th (vertical) column (*A*9), the value is

*Conc*(6, 9) = *W*(*CS*) × X *wi* + *W*(*CM*) × X *wi*

*i*∈*CS*(6, 9)

*i*∈*CM* (6, 9)

+ *W*(*CW* ) × *wi*

X

*i*∈*CW* (6, 9)

2 1 Remark 8. *Since IF ELECTRE does not take the returns criteria and*

## = 1 × *w*3 + 3 × (*w*1 + *w*7) + 3 × *w*2

*risk criteria apart*, *thus*, *we use the formula* [(23)](#_bookmark28) *to calculate the*

2 1 *weights of criteria*. *In addition*, *note that the formula* [(23)](#_bookmark28) *is irrelevant*

## = 1 × 0.2135 + 3 × 0.2680 + 3 × 0.0423 = 0.4063.

Where *wi* is the weight of the criterion *Ci* and is mentioned in [Table 6](#_bookmark31).

*to our modified TOPSIS method*, *thus it is also the difference between our method and IF ELECTRE*.

Table 13

The moderate discordance set *DM* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 4, 6 | — | — | 4 | 2 | 4, 6 | — | 4 | — |
| *A*2 | 1 | — | — | 1, 2 | 1 | — | — | 1 | 2 | — |
| *A*3 | — | 5 | — | — | — | — | — | — | 2 | — |
| *A*4 | — | — | — | — | 4 | — | — | — | — | — |
| *A*5 | 5 | — | — | — | — | 2 | — | — | — | — |
| *A*6 | — | — | — | 6 | — | — | — | 6 | 5, 6 | — |
| *A*7 | — | 3 | 1 | 2, 3 | — | 2 | — | 1 | 2, 3, 5 | — |
| *A*8 | 7 | 4 | 7 | 3 | 4 | — | 4, 7 | — | 4 | — |
| *A*9 | 7 | — | — | — | — | — | — | — | — | — |
| *A*10 | — | 6 | 4 | — | — | — | 6 | — | — | — |

Table 14

The weak discordance set *DW* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | — | — | — | — | 6, 7 | — | — | — | — |
| *A*2 | — | — | 6, 7 | 6 | 5 | 5, 6, 7 | 5 | 3, 7 | 5, 7 | — |
| *A*3 | 1, 2, 4 | — | — | 1, 4 | 1, 2, 5 | 4, 5, 6 | 5 | 1, 4 | 1, 4, 7 | — |
| *A*4 | 3 | — | — | — | 3, 5 | 5 | 5 | 2 | — | — |
| *A*5 | — | — | — | — | — | 6 | — | — | — | — |
| *A*6 | — | — | — | — | — | — | — | 2 | — | — |
| *A*7 | — | — | — | — | — | 6, 7 | — | — | — | — |
| *A*8 | — | — | — | — | — | — | — | — | — | — |
| *A*9 | 3 | — | — | — | — | 2 | — | 2 | — | — |
| *A*10 | 1, 2 | 1, 2 | — | 1, 5 | 1, 2, 5 | 5, 6 | 2, 5 | 1, 7 | 1, 7 | — |

Table 15

The concordance matrix*Conc* .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0.2532 | 0.6212 | 0.2811 | 0.1723 | 0.1523 | 0.4532 | 0.4223 | 0.3243 | 0.6712 |
| *A*2 | 0.2303 | — | 0.3438 | 0.2175 | 0.1651 | 0.1957 | 0.3055 | 0.1305 | 0.1305 | 0.6826 |
| *A*3 | 0.0999 | 0.2026 | — | 0.0000 | 0.0999 | 0.0000 | 0.2907 | 0.1085 | 0.0000 | 0.4485 |
| *A*4 | 0.4329 | 0.4596 | 0.5982 | — | 0.2482 | 0.2736 | 0.3449 | 0.4866 | 0.3351 | 0.8426 |
| *A*5 | 0.5112 | 0.4455 | 0.5342 | 0.3135 | — | 0.4238 | 0.3884 | 0.4706 | 0.4220 | 0.7645 |
| *A*6 | 0.5105 | 0.4596 | 0.5038 | 0.5069 | 0.3969 | — | 0.4512 | 0.5577 | 0.4063 | 0.6343 |
| *A*7 | 0.2303 | 0.2923 | 0.4330 | 0.1740 | 0.2956 | 0.2828 | — | 0.2389 | 0.2389 | 0.6222 |
| *A*8 | 0.3711 | 0.3314 | 0.4660 | 0.3098 | 0.2792 | 0.3098 | 0.5352 | — | 0.2159 | 0.5202 |
| *A*9 | 0.4138 | 0.5595 | 0.6767 | 0.4242 | 0.3654 | 0.2522 | 0.5669 | 0.5162 | — | 0.7342 |
| *A*10 | 0.0499 | 0.0542 | 0.1582 | 0.0000 | 0.0000 | 0.0000 | 0.2125 | 0.0000 | 0.0000 | — |

Table 16

The discordance matrix *Disc*.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0.5333 | 0.4286 | 0.8333 | 0.8571 | 0.9167 | 0.5641 | 1.0000 | 1.0000 | 0.0000 |
| *A*2 | 1.0000 | — | 0.1000 | 1.0000 | 1.0000 | 1.0000 | 0.5556 | 1.0000 | 1.0000 | 0.0000 |
| *A*3 | 1.0000 | 1.0000 | — | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 |
| *A*4 | 1.0000 | 0.5455 | 0.0000 | — | 0.3810 | 1.0000 | 0.5625 | 0.9333 | 0.7778 | 0.0000 |
| *A*5 | 0.6667 | 0.1538 | 0.2727 | 1.0000 | — | 1.0000 | 0.3333 | 0.5882 | 1.0000 | 0.0000 |
| *A*6 | 1.0000 | 0.2941 | 0.0000 | 0.3704 | 0.6250 | — | 0.3478 | 1.0000 | 0.6111 | 0.0000 |
| *A*7 | 1.0000 | 0.6667 | 0.5641 | 0.7500 | 1.0000 | 1.0000 | — | 0.9167 | 0.9231 | 0.0000 |
| *A*8 | 0.6667 | 0.4444 | 0.3750 | 1.0000 | 1.0000 | 1.0000 | 0.6667 | — | 0.8571 | 0.0000 |
| *A*9 | 0.3125 | 0.2727 | 0.0000 | 1.0000 | 0.8462 | 1.0000 | 0.4615 | 1.0000 | — | 0.0000 |
| *A*10 | 1.0000 | 1.0000 | 0.6667 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | — |

Remark 9. *The value* ‘‘—” *means that Ai does not need to be com- pared with itself*. *In the next steps*, *this value will have the same meaning*.

Step 4: Calculate the discordance matrix. Here, the investor gives the relative weights for the discordance sets as [*W*(*DS*), *W*(*DM*), *W*(*DW* )] = 1, 2 , 1 (refer to the weights in [[28](#_bookmark61)]), and then we have the discordance matrix shown in [Table 16](#_bookmark34).

3

3

For instance, in 7th (horizontal) row (*A*7) and 8th (vertical) col- umn (*A*8), the value is

Step 6: Determine the discordance dominance matrix. Since *D* = *i*–*j Disc*(*i*, *j*) = 0.6849, the concordance dominance matrix is obtained and shown in [Table 18](#_bookmark36).

Step 7: Determine the aggregate dominance matrix. The result

10×(10—1)

—

P

is shown in [Table 19](#_bookmark36).

Step 8: Eliminate the less favorable alternatives. According to [Table 19](#_bookmark36), the over-ranking relationship is illustrated in [Fig. 3](#_bookmark37).

From [Fig. 3](#_bookmark37) and [Table 19](#_bookmark36), we can easily find that *A*6 is the only best investment because no investment dominates it. And corre- spondingly, *A*10 is the only worst investment because it is domi- nated by all other investments. By eliminate the best and the worst investments recursively, the ranking of these investments

max *W*(*DS*) × *i* max ) *d*(*D*7*i*, *D*8*i*), *W*(*DM*) × *i*∈max8) *d*(*D*7*i*, *D*8*i*), *W*(*DW* ) × *i*∈ max8) *d*(*D*7*i*, *D*8*i*)

*Disc*(7, 8) =

∈*DS*(7, 8

max

*DM* (7,

*d*(*D*7*i*, *D*8*i*)

*DW* (7,

*i*=1, 2, ... , 7

## max 1 × max (*d*(*D*72, *D*82), *d*(*D*73, *D*83), *d*(*D*75, *D*85), *d*(*D*76, *D*86)), 2 × *d*(*D*71, *D*81)

=

3

4000

( .

,

.

,

.

,

.

,

.

,

.

, 0.

= 0.9167,

max 0 2667

0 2667

0 3000

0 3667

0 3667

0 3000

)

## max 1 × max (0.2667, 0.3000, 0.3667, 0.3000), 2 × 0.2667

= max

3

(0.2667, 0.2667, 0.3000, 0.3667, 0.3667, 0.3000, 0.4000)

=

0.3667

0.4000

Similarly, in 9th (horizontal) row (*A*9) and 1st (vertical) column (*A*1), the value is.

*Disc* 9 1

( ,

) = max

max 1 × *d*(*D*95, *D*15), 2 × *d*(*D*97, *D*17), 1 × *d*(*D*93, *D*13)

=

= 0.3125,

(0.1667, 0.1000, 0.1667, 0.1667, 0.1667, 0.5333, 0.1667)

max 1 × 0.1667, 2 × 0.1667, 1 × 0.1667

3

3

= max

3

3

(0.1667, 0.1000, 0.1667, 0.1667, 0.1667, 0.5333, 0.1667)

0.1667

0.5333

where *Dij* is the assessment of *Ai* under the criterion *Cj* (Please see [Table 5](#_bookmark30)), and *d*(*D*7*i*, *D*8*i*) is the distance between *D*7*i* and*D*8*i*, which is defined by Definition 4.

Step 5: Determine the concordance dominance matrix. Since

*C* = P*i*–*j Conc*(*i*, *j*) = 0.3428, the concordance dominance matrix is

10×(10—1)

—

is *A*6 > *A*4, *A*5, *A*9 > *A*2, *A*8 > *A*1 > *A*7 > *A*3 > *A*10, which is mostly similar to the ranking of *Cw A*'+, *A*'— in [Table 8](#_bookmark31) (when *k* = 0, i.e. the investor is risk neutral).

However, according to [Fig. 3](#_bookmark37) and [Table 19](#_bookmark36), we cannot judge which of *A*4, *A*5, *A*9 is the best. It is the same for *A*2 and*A*8. To solve this problem, Wu and Chen [[28](#_bookmark61)] utilized TOPSIS index to rank

them. The next steps are:

obtained and shown in [Table 17](#_bookmark35).

Table 17

The concordance dominance matrix *Cond*.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| *A*2 | 0 | — | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| *A*3 | 0 | 0 | — | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| *A*4 | 1 | 1 | 1 | — | 0 | 0 | 1 | 1 | 0 | 1 |
| *A*5 | 1 | 1 | 1 | 0 | — | 1 | 1 | 1 | 1 | 1 |
| *A*6 | 1 | 1 | 1 | 1 | 1 | — | 1 | 1 | 1 | 1 |
| *A*7 | 0 | 0 | 1 | 0 | 0 | 0 | — | 0 | 0 | 1 |
| *A*8 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | — | 0 | 1 |
| *A*9 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | — | 1 |
| *A*10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | — |

Table 18

The discordance dominance matrix *Disd*.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| *A*2 | 0 | — | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| *A*3 | 0 | 0 | — | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| *A*4 | 0 | 1 | 1 | — | 1 | 0 | 1 | 0 | 0 | 1 |
| *A*5 | 1 | 1 | 1 | 0 | — | 0 | 1 | 1 | 0 | 1 |
| *A*6 | 0 | 1 | 1 | 1 | 1 | — | 1 | 0 | 1 | 1 |
| *A*7 | 0 | 1 | 1 | 0 | 0 | 0 | — | 0 | 0 | 1 |
| *A*8 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | — | 0 | 1 |
| *A*9 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | — | 1 |
| *A*10 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | — |

Table 19

The aggregate dominance matrix *AGG*.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| *A*2 | 0 | — | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| *A*3 | 0 | 0 | — | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| *A*4 | 0 | 1 | 1 | — | 0 | 0 | 1 | 0 | 0 | 1 |
| *A*5 | 1 | 1 | 1 | 0 | — | 0 | 1 | 1 | 0 | 1 |
| *A*6 | 0 | 1 | 1 | 1 | 1 | — | 1 | 0 | 1 | 1 |
| *A*7 | 0 | 0 | 1 | 0 | 0 | 0 | — | 0 | 0 | 1 |
| *A*8 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | — | 0 | 1 |
| *A*9 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | — | 1 |
| *A*10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | — |





Fig. 3. The over-ranking relationships of the investments.

Step 5b: Determine the concordance dominance matrix *Conc*'. It use the positive-ideal solution of TOPSIS, if *C*\* is the biggest value in the concordance matrix, then calculate

Step 7b: Determine the aggregate dominance matrix *AGG*', where

*Conc*'(*i*, *j*) = *C*\* — *Conc*(*i*, *j*) (25)

*Agg*'

*Donc*'(*i*, *j*)

## (*i*, *j*) = (27)

*Conc*'(*i*, *j*) + *Donc*'(*i*, *j*)

Step 6b: Determine the discordance dominance matrix*Donc*'. Let *D*\* is the biggest value in the discordance matrix, then calculate

*Donc*'(*i*, *j*) = *D*\* — *Donc*(*i*, *j*) (26)

Step 8b: Rank the investments and determine the best invest- ment based on the following formula:

*m*

( *i*) = 1 X ( , ) ( )

*P A Agg*' *i j* 28

*m* — 1 *j*–*i*

The bigger the value of *P*(*Ai*) is, the better the investment is. The results are shown in [Tables 20–22](#_bookmark38), where *C*\* = 0.8426,

*D*\* = 1.

From [Table 22](#_bookmark39), the ranking of these investments is

*A*6 > *A*9 > *A*5 > *A*4 > *A*1 > *A*8 > *A*7 > *A*2 > *A*3> *A*10. And then, let’s

compare the process and results between IF ELECTRE and our pro- posed method. To be convenient, we denote the ranking obtained

from [Fig. 3](#_bookmark37) and [Table 19](#_bookmark36) as *R*', the ranking obtained from [Table 22](#_bookmark39)

1. Similarly, we can find that *A*1 is better than *A*2, which is pointed out by *R*'', *R* and the conventional TOPSIS method. However, *R*' reverses the ranking of *A*1 and*A*2. From this per- spective, we can conclude that *R*' is also not effective enough.
2. Whether we use *R*,*R*', *R*'' or the conventional TOPSIS method, we can always get the same result: *A*6 is the best investment

and *A* , *A* have the terrible performances. Thus, our

3 10

as *R*'', and the ranking of *Cw A*'+, *A*'— in [Table 8](#_bookmark31) (when *k* = 0) as *R*,

* 1. *R*' : *A*6 > *A*4, *A*5, *A*9 > *A*2, *A*8 > *A*1 > *A*7 > *A*3 > *A*10, *R*'' : *A*6 > *A*9 > *A*5 > *A*4 > *A*1> *A*8 > *A*7 > *A*2 > *A*3 > *A*10, and *R* : *A*6 > *A*5 > *A*4 > *A*9 > *A*1 > *A*8 > *A*2 > *A*7 > *A*3 > *A*10.
     1. Obviously, *R*' is inconsistent with*R*'', such as the ranking of *A*1, *A*8. It means that IF ELECTRE may provide investors with unreliable results that contradict the facts.
     2. From the assessments in [Table 5](#_bookmark30) and the weights of criteria in [Table 6](#_bookmark31), we can find that *A*2 is better than *A*7, which is pointed out by *R*', *R* and the conventional TOPSIS method (Please see the ranking of *Cw A*+, *A*— in [Table 8](#_bookmark31), when *k* = 0). However, *R*'' reverses the ranking of *A*2 and *A*7. From

this perspective, we can conclude that *R*'' is not effective enough.

method is in line with the conventional TOPSIS method and IF ELECTRE to some extent.

1. According to the steps of IF ELECTRE, we can find that it is more complicate than our modified method. It needs to cal- culate three concordance sets and three discordance sets for the construction of the concordance matrix and discordance matrix by pairwise comparison. In contrast, our method just needs to compare each investment with APIS and ANIS, which is more time-saving.
2. In IF ELECTRE, investors need to give the weights for concor- dance sets and discordance sets, respectively. However, Wu and Chen [[28](#_bookmark61)] did not provide a method to determine the weights of concordance sets and discordance sets. In addi- tion, the practical significance of the weights of strong con- cordance/discordance set, moderate concordance/ discordance set and weak concordance/ discordance set is

Table 20

The concordance dominance matrix *Conc*'.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0.5894 | 0.2214 | 0.5615 | 0.6703 | 0.6903 | 0.3894 | 0.4203 | 0.5183 | 0.1714 |
| *A*2 | 0.6123 | — | 0.4988 | 0.6251 | 0.6775 | 0.6469 | 0.5371 | 0.7121 | 0.7121 | 0.1600 |
| *A*3 | 0.7427 | 0.6400 | — | 0.8426 | 0.7427 | 0.8426 | 0.5519 | 0.7341 | 0.8426 | 0.3941 |
| *A*4 | 0.4097 | 0.3830 | 0.2444 | — | 0.5944 | 0.5690 | 0.4977 | 0.3560 | 0.5075 | 0.0000 |
| *A*5 | 0.3314 | 0.3971 | 0.3084 | 0.5291 | — | 0.4188 | 0.4542 | 0.3720 | 0.4206 | 0.0781 |
| *A*6 | 0.3321 | 0.3830 | 0.3388 | 0.3357 | 0.4457 | — | 0.3914 | 0.2849 | 0.4363 | 0.2083 |
| *A*7 | 0.6123 | 0.5503 | 0.4096 | 0.6686 | 0.5470 | 0.5598 | — | 0.6037 | 0.6037 | 0.2204 |
| *A*8 | 0.4715 | 0.5112 | 0.3766 | 0.5328 | 0.5634 | 0.5328 | 0.3074 | — | 0.6267 | 0.3224 |
| *A*9 | 0.4288 | 0.2831 | 0.1659 | 0.4184 | 0.4772 | 0.5904 | 0.2757 | 0.3264 | — | 0.1084 |
| *A*10 | 0.7927 | 0.7884 | 0.6844 | 0.8426 | 0.8426 | 0.8426 | 0.6301 | 0.8426 | 0.8426 | — |

Table 21

The discordance dominance matrix *Donc*'.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 |
| *A*1 | — | 0.4667 | 0.5714 | 0.1667 | 0.1429 | 0.0833 | 0.4359 | 0.0000 | 0.0000 | 1.0000 |
| *A*2 | 0.0000 | — | 0.9000 | 0.0000 | 0.0000 | 0.0000 | 0.4444 | 0.0000 | 0.0000 | 1.0000 |
| *A*3 | 0.0000 | 0.0000 | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 1.0000 |
| *A*4 | 0.0000 | 0.4545 | 1.0000 | — | 0.6190 | 0.0000 | 0.4375 | 0.0667 | 0.2222 | 1.0000 |
| *A*5 | 0.3333 | 0.8462 | 0.7273 | 0.0000 | — | 0.0000 | 0.6667 | 0.4118 | 0.0000 | 1.0000 |
| *A*6 | 0.0000 | 0.7059 | 1.0000 | 0.6296 | 0.3750 | — | 0.6522 | 0.0000 | 0.3889 | 1.0000 |
| *A*7 | 0.0000 | 0.3333 | 0.4359 | 0.2500 | 0.0000 | 0.0000 | — | 0.0833 | 0.0769 | 1.0000 |
| *A*8 | 0.3333 | 0.5556 | 0.6250 | 0.0000 | 0.0000 | 0.0000 | 0.3333 | — | 0.1429 | 1.0000 |
| *A*9 | 0.6875 | 0.7273 | 1.0000 | 0.0000 | 0.1538 | 0.0000 | 0.5385 | 0.0000 | — | 1.0000 |
| *A*10 | 0.0000 | 0.0000 | 0.3333 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | — |

Table 22

The aggregate dominance matrix *Agg*' and the value of *P*(*Ai*).

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Assessments | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 | *A*6 | *A*7 | *A*8 | *A*9 | *A*10 | *P*(*Ai*) | Ranking |
| *A*1 | — | 0.4419 | 0.7207 | 0.2289 | 0.1757 | 0.1077 | 0.5282 | 0.0000 | 0.0000 | 0.8537 | 3.0568 | 5 |
| *A*2 | 0.0000 | — | 0.6434 | 0.0000 | 0.0000 | 0.0000 | 0.4528 | 0.0000 | 0.0000 | 0.8621 | 1.9583 | 8 |
| *A*3 | 0.0000 | 0.0000 | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.7173 | 0.7173 | 9 |
| *A*4 | 0.0000 | 0.5427 | 0.8036 | — | 0.5101 | 0.0000 | 0.4678 | 0.1578 | 0.3045 | 1.0000 | 3.7865 | 4 |
| *A*5 | 0.5014 | 0.6806 | 0.7022 | 0.0000 | — | 0.0000 | 0.5948 | 0.5254 | 0.0000 | 0.9276 | 3.9320 | 3 |
| *A*6 | 0.0000 | 0.6483 | 0.7469 | 0.6522 | 0.4569 | — | 0.6250 | 0.0000 | 0.4713 | 0.8276 | 4.4282 | 1 |
| *A*7 | 0.0000 | 0.3772 | 0.5156 | 0.2722 | 0.0000 | 0.0000 | — | 0.1213 | 0.1130 | 0.8194 | 2.2186 | 7 |
| *A*8 | 0.4141 | 0.5208 | 0.6240 | 0.0000 | 0.0000 | 0.0000 | 0.5202 | — | 0.1857 | 0.7562 | 3.0210 | 6 |
| *A*9 | 0.6159 | 0.7198 | 0.8577 | 0.0000 | 0.2437 | 0.0000 | 0.6614 | 0.0000 | — | 0.9022 | 4.0007 | 2 |
| *A*10 | 0.0000 | 0.0000 | 0.3275 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | — | 0.3275 | 10 |

not clear enough. Most important of all, IF ELECTRE cannot provide investors with more flexible results according to

their risk appetite, which is just an advantage of our method.

Case 1. *By using Model* [(19)](#_bookmark25) *and settinga* = 1.5, *we have*

## min *D*(*X*) = 0.2125*x*4 + 0.1757*x*5 + 0.0456*x*6

In summary, our modified TOPSIS method is more effective and

*s*.*t*.

*x*4 + *x*5 + *x*6 = 1,

*xi* P 0.1, *i* = 4, 5, 6.

## (32)

more flexible than IF ELECTRE.

* 1. *The ratios of each selected investment*

Now, stipulate each investment ratio should be higher than 0.1 and then choose the investments according to the ranking of *Cw A*'+, *A*'— shown in [Table 8](#_bookmark31), and we can allocate the investment ratios in line with the following conditions:

Then, we can get the optimal investment ratios *x*4 = *x*5 = 0.1, *x*6 = 0.8, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.1304 and*D*— = 0.2405, respec- tively. This result is the same as that of Case 1 in condition (i).

Case 2. *By using Model* [(21)](#_bookmark23) *and settingh* = 0.2, *we have*

## min *D*+(*X*) = 0.2646*x*4 + 0.2290*x*5 + 0.1013*x*6

1. If the investor is risk neutral (*k* = 0), *A* , *A* , *A*

4

5

6

will be the

## 8>< 0.3689*x*4 + 0.3356*x*5 + 0.2126*x*6 P 0.2439,

(33)

selected investments for the investor. Let *xi* be the invest-

ment ratio allocated to the investment*Ai*,*i* = 4, 5, 6. :

*s*.*t*.

*x*4 + *x*5 + *x*6 = 1,

>

*xi* P 0.1, *i* = 4, 5, 6.

Case 1. *By using Model* [(19)](#_bookmark25) *and settinga* = 1, *we have*

## min *D*(*X*) = —0.1043*x*4 — 0.1066*x*5 — 0.1113*x*6

Then, we can get the optimal investment ratios *x*4 = 0.1000, *x*5 = 0.1274, *x*6 = 0.7726, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.1339 and *D*— = 0.2439, respectively.

*s*.*t*.

*x*4 + *x*5 + *x*6 = 1,

*xi* P 0.1, *i* = 4, 5, 6.

## (29)

Case 3. *By using Model* [(22)](#_bookmark24) *and settingh* = 0.8, *we have*

Then, we can get the optimal investment ratios *x*4 = *x*5 = 0.1,

*x*6 = 0.8, while the total weighted APIS distance and the total

weighted ANIS distance are *D*+ = 0.1304 and*D*— = 0.2405,

## max *D*—(*X*) = 0.3689*x*4 + 0.3356*x*5 + 0.2126*x*6

8>< 0.2646*x*4 + 0.2290*x*5 + 0.1013*x*6 6 0.1340,

*s*

## (34)

respectively.

.*t*.

>:

*x*4 + *x*5 + *x*6 = 1,

*xi* P 0.1, *i* = 4, 5, 6.

Case 2. *By using Model* [(21)](#_bookmark23) *and setting h* = 0.5, *we have*

## min *D*+(*X*) = 0.2646*x*4 + 0.2290*x*5 + 0.1013*x*6

Then, we can get the optimal investment ratios *x*4 = 0.1000, *x*5 = 0.1282, *x*6 = 0.7718, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.1340 and

*D*— = 0.2440, respectively.

## 8>< 0.3689*x*4 + 0.3356*x*5 + 0.2126*x*6 P 0.2908,

*s*.*t*.

*x*4 + *x*5 + *x*6 = 1,

(30)

(iii) If the investor is highly risk averse (

*k* =— .

0 5), *A*

4,

*A A* will

5,

9

## >: *x* P 0.1, *i* = 4, 5, 6.

be the selected investments for the investor. Let *xi* be the

*i*

Then, we can get the optimal investment ratios *x*4 = 0.1000, *x*5 = 0.5087, *x*6 = 0.3913, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.1826 and*D*— = 0.2908, respectively.

investment ratio allocated to the investment *Ai*, *i* = 4, 5, 9.

Case 1. *By using Model* [(19)](#_bookmark25) *and settinga* = 0.5, *we have*

## min *D*(*X*) = —0.4211*x*4 — 0.3889*x*5 — 0.4275*x*9

Case 3. *By using Model* [(22)](#_bookmark24) *and settingh* = 0.5, *we have*

## max *D*—(*X*) = 0.3689*x*4 + 0.3356*x*5 + 0.2126*x*6

*s*.*t*.

*x*4 + *x*5 + *x*9 = 1,

*xi* P 0.1, *i* = 4, 5, 9.

## (35)

8> 0.2646*x*4 + 0.2290*x*5 + 0.1013*x*6 6 0.1830,

Then, we can get the optimal investment ratios *x*4 = *x*5 = 0.1,

## < (31)

*x*9 = 0.8, while the total weighted APIS distance and the total

*s*.*t*.

>:

*x*4 + *x*5 + *x*6 = 1,

*xi* P 0.1, *i* = 4, 5, 6.

weighted ANIS distance are *D*+ = 0.2666 and*D*— = 0.3708, respectively.

Then, we can get the optimal investment ratios *x*4 = 0.1000,

*x*5 = 0.5119, *x*6 = 0.3881, while the total weighted APIS distance

Case 2. *By using Model* [(21)](#_bookmark23) *and settingh* = 0.8, *we have*

and the total weighted ANIS distance are *D*+ = 0.1830 and

min *D*+ *X* 0 2646*x*

0 2290*x*

0 2716*x*

*D*— 0 2912, respectively.

( ) = .

## 4 + .

5 + . 9

= . 8> 0.3689*x*

+ 0.3356*x*5

+ 0.3755*x*9

# P 0.3675,

1. If the investor is highly profit-seeking (*k* = 0.5), *A*4, *A*5, *A*6

>:

*s*.*t*.<

4

*x*4 + *x*5

+ *x*9

## = 1,

(36)

will be the selected investments for the investor. Let *xi* be the investment ratio allocated to the investment *Ai*,*i* = 4, 5, 6.

*xi* P 0.1, *i* = 4, 5, 9.

Then, we can get the optimal investment ratios *x*4 = 0.1000, *x*5 = 0.1840, *x*9 = 0.7160, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.2631 and *D*— = 0.3675, respectively.

Case 3. *By using Model* [(22)](#_bookmark24) *and settingh* = 0.2, *we have*

## max *D*—(*X*) = 0.3689*x*4 + 0.3356*x*5 + 0.3755*x*9

8>< 0.2646*x*4 + 0.2290*x*5 + 0.2716*x*9 6 0.2631,

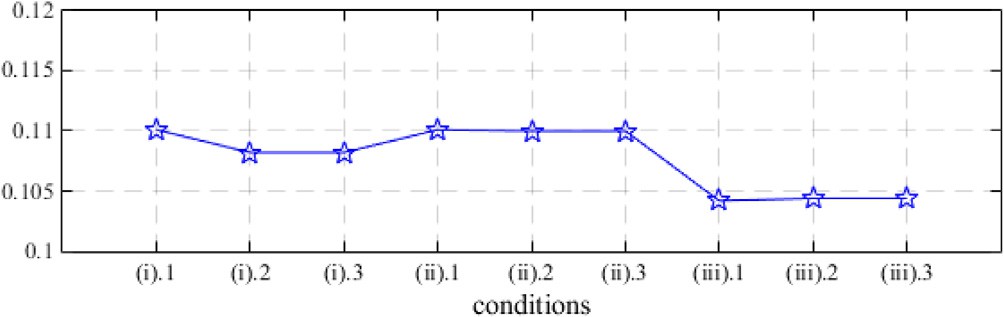


Fig. 4. The values of *D*— — *D*+ under different conditions.

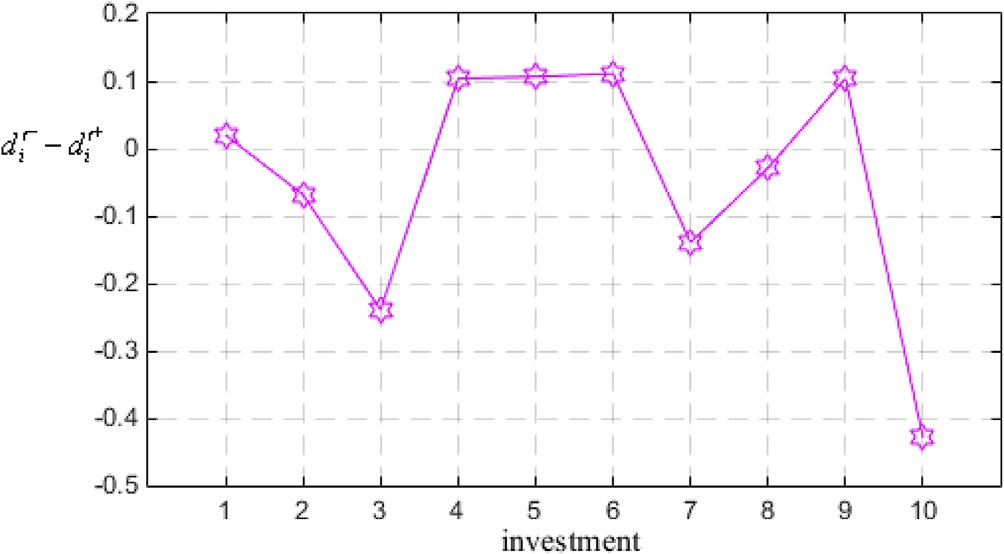
*s*.*t*.

>:

*x*4 + *x*5 + *x*9 = 1,

*xi* P 0.1, *i* = 4, 5, 9.

## (37)

Then, we can get the optimal investment ratios *x*4 = 0.1000, *x*5 = 0.1831, *x*9 = 0.7169, while the total weighted APIS distance and the total weighted ANIS distance are *D*+ = 0.2631 and *D*— = 0.3675, respectively.

Remark 10. *In fact*, *although it seems that there is little difference in the results of Case* 2 *and Case* 3 *in condition* (*iii*), *their essences are lar- gely different*. *After calculating in more detail*, *we can find that the accurate total weighted APIS distance is higher than* 0.2631 *in Case* 2, *but it is lower than* 0.2631 *in Case* 3, *which is due to their different constraints*.

Now, let us analyze the above results:

Fig. 5. The value of *d*'— — *d*'+ of each investment.

*i* *i*

1. From [Table 23](#_bookmark42), it can be found that when the investor is neu-

tral, the results of three cases are different, especially the dif-

ference between Case 1 and Case 2 or Case 3. As a matter of to mention whether their value of *d*'— — *d*'+ is more than

*i* *i*

fact, when *a* = 1, Model [(19)](#_bookmark25) regresses to Model [(17)](#_bookmark21), which

0.1. Here, *d*'— represents the ANIS distance while *d*'+ repre-

*i* *i*

is completely neutral for the reason that the investor does not take any preference of returns or risks into account. Whereas, Model [(21)](#_bookmark23) can ensure that the risk will not exceed a certain ceiling and Model [(22)](#_bookmark24) can ensure that the returns will not fall below a certain baseline. Therefore, to the inves- tor, these two models still indicate his/her preferences to some extent.

1. Comparing the data of each condition in [Table 23](#_bookmark42), we can find that for Model [(21)](#_bookmark23), the bigger the value of *h*, the greater the total weighted ANIS distance and the more the investor wants to avoid risk; for Model [(22)](#_bookmark24), the bigger the value of *h*, the greater the investor’s preference for returns. These results are consistent with the conclusions mentioned at the end of [Section 4](#_bookmark19).
2. From [Table 23](#_bookmark42), we can also find the greater the total weighted APIS distance, the greater the total weighted ANIS distance, which means that the greater the returns, the greater the risks. It is rational and realistic in portfolio selection.
3. From [Fig. 4](#_bookmark40), we can find that for all the conditions, the value of *D*— — *D*+ is always more than 0.1. According to the actual meanings of APIS distance and ANIS distance, it is evident that the greater the value of *D*— — *D*+, the better the portfolio selection. However, [Fig. 5](#_bookmark41) shows that not all the investments

can satisfy that the value of *d*'— — *d*'+ is more than zero, not

sents the APIS distance of investment *Ai*, *i* = 1, 2, ·· · , 10. In summary, we can ensure that the results given by the pro- posed models are effective to a large extent.

What kind of models should be used to obtain the optimal investment ratios depends on the investor’s preference. The above models are just presented to offer a more flexible selection.

*5.4. Sensitivity analysis*

To illustrate the superiority of the modified TOPSIS method bet- ter, a sensitivity analysis is carried out to study the influence of *k* in the ranking of weighted closeness coefficients. In [Section 3](#_bookmark2), it had been pointed out that the parameter *k* indicates the decision maker’s preference as to returns and cost/risk, which is the reason that it is called the preference factor of the weighted closeness coefficient. Thus, we will also do an analysis with reference to the relationship between its value and decision makers’ preferences.

To be convenient, we denote the weighted closeness coefficient of APIS distance and ANIS distance as*CA*, the weighted closeness coefficient of PIS distance and NIS distance as*Cc*. By using the data shown in [Table 7](#_bookmark31), the weighted closeness coefficients of various values of *k* are calculated and the results are presented in [Fig. 6](#_bookmark43)

*i i* and [Fig. 7](#_bookmark44).

Table 23

The total weighted APIS distances and ANIS distances in example 5.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Condition | (i) |  |  |  | (ii) |  |  |  | (iii) |  | | |
|  | Case 1 | Case 2 | Case 3 |  | Case 1 | Case 2 | Case 3 |  | Case 1 | Case 2 | Case 3 |  |
| APIS distance | 0.1304 | 0.1826 | 0.1830 |  | 0.1304 | 0.1339 | 0.1340 |  | 0.2666 | 0.2631 | 0.2631 |  |
| ANIS distance | 0.2405 | 0.2908 | 0.2912 |  | 0.2405 | 0.2439 | 0.2440 |  | 0.3708 | 0.3675 | 0.3675 |  |

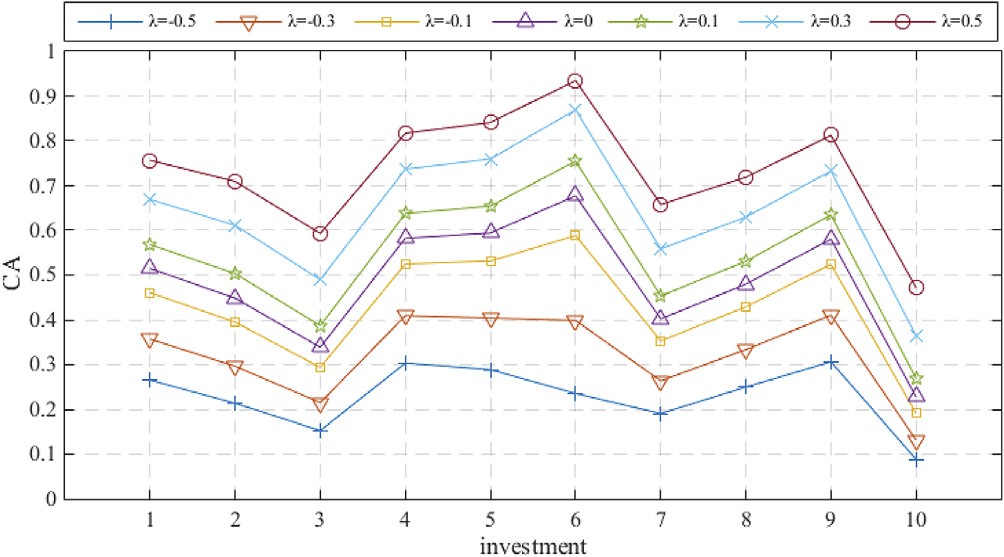


Fig. 6. *CA* with different values of*k*.

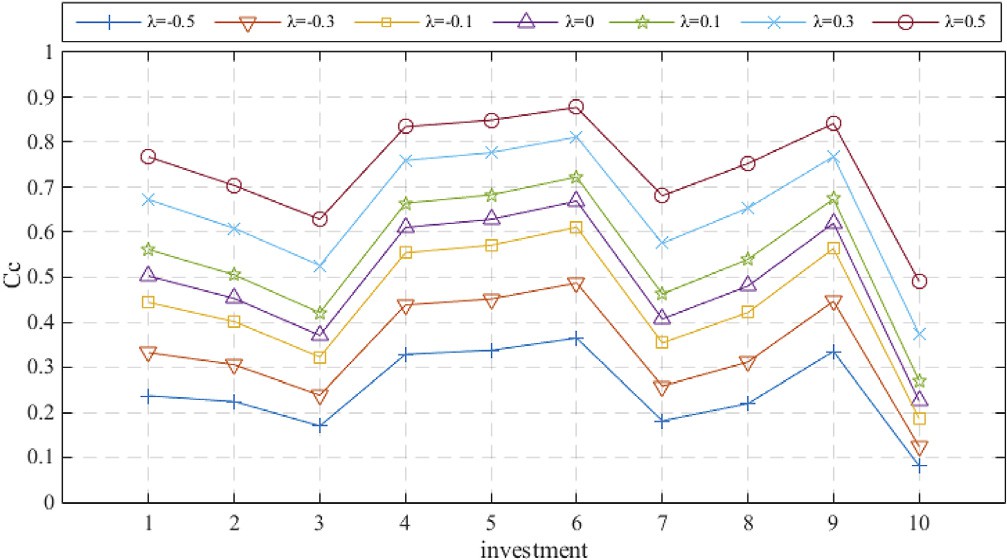


Fig. 7. *Cc* with different values of*k*.

From [Fig. 6](#_bookmark43) and [Fig. 7](#_bookmark44), we can find that:

1. All the values of weighted closeness coefficients increase with the increase of the value of*k*, which is in line with Theorem 1.
2. The shapes of lines in [Fig. 7](#_bookmark44) are similar, whereas in [Fig. 6](#_bookmark43), they are obviously different. It implies that APIS and ANIS can fit in the requirement of the investor’s different prefer- ences, while PIS and NIS cannot provide enough flexibility to meet the demand because they can only offer the same results. In addition, according to the analysis of [Table 8](#_bookmark31) in [Section 5](#_bookmark27), it is pointed out that the rankings of *Cw A*'+, *A*'— are more rational than those of *Cw A*+, *A*— . In summary, APIS and ANIS are better than PIS and AIS.

1. In [Fig. 6](#_bookmark43), the influence of *k* is evident in the change of the rankings, especially in the rankings of *A*5 and*A*6. With the analysis of [Table 5](#_bookmark30), we can find that the assessments of *A*5 and *A*6 with respect to returns criteria are much better than other investments while their assessments with respect to risk criteria are much worse. It means that their differences between advantage and disadvantage are great. Compara- tively, the differences of assessments of *A*3 and *A*10 between returns criteria and risk criteria are smaller so that it can be found that their rankings change little in [Fig. 6](#_bookmark43). Hence, we can point out that the ranking of the investment may be more sensitive to the change of the value of *k* if the differ- ence in its assessments of returns criteria and risk criteria is greater.

*5.5. The benefits and goals of our approach*

To make the benefits and goals of our approach more clear, a solid discussion is provided as follows:

1. The modified TOPSIS can help investors to properly estimate the returns and risks of the stocks when the historical data is insufficient or unavailable. That is important, because the estimation result will directly affect the effectiveness and rationality of the portfolio strategy offered by the model. In fact, without the modified TOPSIS method or other effec- tive evaluation method to quantify the returns and risks as input data, investors cannot construct the corresponding portfolio model, not to mention to make rational portfolio selection.
2. The weighted closeness coefficient can provide a more appropriate evaluation result for investors. It should be noted that different investors have different risk preferences. If the approach can only provide the same result for different investors, it is not enough flexible and cannot be widely used. However, if the approach provides risk-averse inves- tors with results that the best stock has the highest risk, or it provides profit-seeking investors with results that the best stock has the least return, doubtlessly, that will be more ter- rible. Thus, an approach that can effectively provide inves- tors with suitable evaluation results, which are in line with their preferences, is necessary.
3. The new kind of intuitionistic fuzzy portfolio selection model is put forward to make up for the disadvantage of TOPSIS method. As we known, TOPSIS or most of other MCDM methods can help people to find out the best alterna- tive or rank the alternatives according to multiple criteria. Although these methods can quantitatively evaluate the alternatives, they cannot tell people how to combine these alternatives and distribute their resource to make profit as much as possible while avoiding the potential risk. Compar- atively, portfolio selection can finish this task well. In addi- tion, referring to the classical M—V model proposed by Markowitz, we construct a new kind of bi-objective model, which utilizes the APIS distance and ANIS distance as the expected return and risks of stocks, rather than the single objective model based on the weighted closeness coefficient. The advantage is that we can better balance the return and risk of portfolio (if we use the weighted closeness coefficient, all the resource will be occupied by the best stock, which will lead to a poor ability to disperse risk).
4. Conclusion

In this paper, the portfolio selection problem under intuitionis- tic fuzzy circumstance is regarded as a multi-criteria decision mak- ing (MCDM) problem, and then the TOPSIS method is used to resolve the problem. Nevertheless, traditional TOPSIS method merely provides decision makers a ranking of alternatives which is not completely in line with decision markers’ preferences. There- fore, it is not effective or flexible enough for some portfolio selec- tion problems.

In order to make the methods more flexible to meet the demand of investors with different preferences, the Absolute Positive Ideal Solution (APIS) and the Absolute Negative Ideal Solution (ANIS) are defined for TOPSIS method to represent the returns ceiling and the risk upper limit in portfolio selection, respectively. Compared with PIS and NIS, APIS and ANIS can better achieve the balance between returns and risk. In addition, the closeness coefficient, which deci- des the ranking of alternatives, is also refined to become more effective to provide the results that are consistent with investors’ preferences. Meanwhile, its resistance to extreme data is also improved.

To allocate investment ratios and spread risk, three different kinds of linear programming models based on the distance mea- sure of IFSs are presented. These models can make up for the dis-

advantage of TOPSIS method that it only considers the ranking of investments but neglects the investment ratios. To demonstrate the effectiveness of our models, a numerical example is provided and the results are analyzed. Finally, a sensitivity analysis is held to study the influence of the parameter called preference factor of the weighted closeness coefficient. Compared with the conven- tional one, the analysis also illustrates the advantage of the modi- fied TOPSIS method in the ranking with investors’ preferences.

In the future work, we will study a new kind of intuitionistic fuzzy portfolio selection programming with complex constraints, and then research for a novel algorithm to solve the model more effectively.

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Compliance with ethical standards

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*Conflict of interest:* We declare that we have no conflict of interest.

*Ethical approval:* This article does not contain any studies with human participants or animals performed by any of the authors.

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