 Electronic Notes in Theoretical Computer Science 203 (2008) 93–107 

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On Products of Transition Systems

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Abstract

For an arbitrary set endofunctor *F* we give a sufficient and necessary criterium for the existence of products of *F* -coalgebras. In the case of transition systems, where *F* = P is the covariant powerset functor, we introduce impeding paths whose existence impedes the existence of the product. Moreover we show, that the product *A⊗A* of a finite transition system *A* exists if and only if the product *A⊗B* for each finite transition system *B* exists.

*Keywords:* Coalgebras, transition systems, products

# Introduction

Given a set functor *F* , it is well known that the category *SetF* of *F* -coalgebras has arbitrary colimits. In fact (see [[4](#_bookmark21)]) the forgetful functor *U* : *SetF → Set* creates and reflects colimits. The situation is different for limits. Even though equalizers and inverse images always exist in *SetF* [[2](#_bookmark18)] they are, in general, not created by the forgetful functor.

General products need not exist at all in *SetF* . In particular, the product over the empty index set, that is the terminal coalgebra need not exist, unless certain assumptions are made about the functor *F* . This is mainly due to Lambeks Lemma

[[3](#_bookmark20)] which states that the structure map of the terminal coalgebra *T* must be a bijection *α* : *T → F* (*T* ). Unless *F* is bounded [[4](#_bookmark21)], this requirement often leads to set theoretical problems.

A classical case of an unbounded functor is given by the power set functor P whose coalgebras are the familiar transition systems. Clearly, the terminal P- coalgebra, i.e. the empty product, does not exist since *|X| < |*P(*X*)*|* for any set *X*.

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doi:10.1016/j.entcs.2008.05.021

Still, there may be some products of transition systems existing. In [[1](#_bookmark19)] examples of finite transition systems were given that show the whole range of situations that may occur. In particular, Gumm and Schr¨oder exhibit pairs *A* and *B* of nonempty finite transition systems so that *A⊗B* does not exist, *A⊗B* exists and is the empty transition system or *A⊗B* exists and is the largest bisimulation *∼A,B*.

In this paper we examine the reasons why a product of two transition system *A* and *B* might exist, or not. Analysing the critical example in [[1](#_bookmark19)], we study certain ”bisimilar paths” in a transition system *A* whose existence allows or impedes the existence of products *A⊗ B*.

As an interesting corollary we obtain the somewhat surprising fact that for any finite transition system *A* we have that *A⊗A* exists in *Set*P if and only if *A⊗B* exists for each finite transition system *B*.

# Categorical products of coalgebras

**Definition 2.1 (Coalgebra)** *Let F* : *Set → Set be a functor. A pair* (*A, α*) *is called F -coalgebra, if A ∈ Set and α* : *A → F* (*A*)*. We call A the base set and α the structure of the coalgebra A.*

*For the remainder of this section let A* = (*A, α*)*, B* = (*B, β*) *be F -coalgebras. A map ϕ* : *A → B is called homomorphism, if Fϕ ◦ α* = *β ◦ ϕ.*

**Definition 2.2 (Bisimilarity)** *A subset R ⊆ A × B is called a bisimulation be- tween A and B, if there exists a structure ρ* : *R → F* (*R*)*, so that πA* : *R→ A, πB* : *R→B are homomorphisms. Then we say R* = (*R, ρ*) *is a* ***bisimulation structure*** *for A and B. Elements a ∈ A, b ∈ B are called* ***bisimilar*** (*a ∼ b*)*, if there exists a bisimulation R with* (*a, b*) *∈ R.*

There always exists a largest bisimulation *∼A,B*= *{*(*a, b*) *| a ∈ A, b ∈ B, a ∼ b}* between *A* and *B*. Note, that bisimilarity is a reflexive and symmetric relation. Moreover it is transitive if the functor *F* preserve weak pullbacks (see [[4](#_bookmark21)], theorem 5.4).

We spell out the categorical product of coalgebras.

**Definition 2.3 (Product)** *Let Q* = (*Q, ξ*) *be an F -coalgebra and ϕA* : *Q → A, ϕB* : *Q→B homomorphisms. Then* **Q** = (*Q, ϕA, ϕB*) *is called an A****-****B****-cone****.*

*The* ***categorical product*** *of A and B is an A-B-cone* **P** = (*P, πA, πB*)*, so that for all A-B-cones* (*Q, ϕA, ϕB*) *there exists exactly one homomorphism ψ* : *Q → P with ϕA* = *πA ◦ ψ and ϕB* = *πB ◦ ψ.*

*Q ,,,*

*ϕA*

*,*

*j*

*A ¸*,*¸ ¸∃*!*ψ*

*π ¸¸¸¸*

*,ϕB*

*,,*v*z*

¸*B,*

*A ¸* J *πB*

*P*

*The base set of the categorical product of A and B need not be the cartesian*

*product A × B, so we write* **P** = *A⊗B for the (categorical) product of A and B.*

Every bisimulation structure *R* together with the projections *πA, πB* yields an

*A*-*B*-cone.

Whenever we speak about *A*-*B*-cones **Q***,* **Q***'*, then let **Q** = (*Q, ϕA, ϕB*)*,* **Q***'* = (*Q', ϕ' , ϕ'* ) and *Q* = (*Q, ξ*)*, Q'* = (*Q', ξ'*) be the associated coalgebras. If the

*A B*

product **P** = *A⊗B* exists, we denote its corresponding coalgebra by *P* = (*P, η*) and the *A*-*B*-cone by **P** = (*P, πA, πB*).

**Definition 2.4** *Let* **Q***,* **Q***' be A-B-cones and q ∈ Q, q' ∈ Q'. We write q ∼*=*A,B q', if there exists an F -coalgebra R* = (*R, ρ*)*, homomorphisms χ* : *R → Q, χ'* : *R → Q' and r ∈ R, so that:*

* + *χ*(*r*) = *q and χ'*(*r*) = *q', i.e. q*1 *∼ q*2

*B*

* + *ϕA ◦ χ* = *ϕ'*

*A*

* *χ' and ϕB ◦ χ* = *ϕ'*
* *χ', i.e. the following diagram commutes:*

*Q, ϕA*  *A*¸ *,*

*ϕB*

*χ*

*,,,,*

*ccc*

*R ¸¸¸¸*

*¸*

*,,, cccc*

*ϕ' cccc ,,,*

*,c,*

*' ¸¸¸ Ac ,,*

*χ ¸*z *cc*

z*z*

*Q'*

*,*

*'*  *B*

*ϕ*

*B*

*We say q, q' are A****-****B****-perspective*** *q A,B q', if there exist n ∈* N*, A-B-cones*

**Q**1*,...,* **Q***n and qi ∈ Qi*(*i* = 1*,..., n*)*, so that q* =*∼ q*1 *∼*= *... ∼*= *qn ∼*= *q', that is,*

*A,B* = (*∼*=*A,B* )*∗.*

The relation *∼*=*A,B* is reflexive and symmetric, therefore the transitive closure

*A,B* of *∼*=*A,B* is an equivalence relation on the class of all elements of *A*-*B*-cones.

We introduce the equivalence classes with respect to *A,B*. Let **Q** be an *A*-*B*- cone and *q ∈ Q*. We define *C***Q***,q* = *{*(**Q***', q'*) *|* **Q***' A*-*B*-cone*, q' ∈ Q', q' A,B q}* the equivalence class of *q ∈ Q*.

**Lemma 2.5** *Assume the product A⊗B to exist. Let* **Q***,* **Q***' be A-B-cones, q ∈ Q, q' ∈ Q' and ψ* : *Q → A ⊗ B, ψ'* : *Q' → A ⊗ B be the unique homomorphisms. Then q ∼*=*A,B q'* =*⇒ ψ*(*q*) = *ψ'*(*q'*)*.*

**Proof** The projections *πA, πB* are jointly mono, otherwise there would be an *A*-*B*- cone **Q** and homomorphisms *ψ, ψ'* : *Q → P* with *ϕA* = *πA ◦ ψ* and *ϕB* = *πB ◦ ψ* but *ψ /*= *ψ'* in contradiction to the uniqueness of the homomorphism *Q→P* in the definition of the product.

Let *q ∼*=*A,B q'*. There exists a coalgebra *R* = (*R, ρ*), *r ∈ R* and homomorphisms

*χ* : *R→ Q, χ'* : *R→ Q'* with *χ*(*r*) = *q* and *χ'*(*r*) = *q'*. Then *R* with the projections

*B*

*ϕA ◦ χ* = *ϕ'*

*A*

* *χ'* and *ϕ ◦ χ* = *ϕ'*
* *χ'* is an *A*-*B*-cone. We compute

*πA ◦ ψ ◦ χ* = *ϕA ◦ χ*

= *ϕ' ◦ χ'*

*A*

= *πA ◦ ψ' ◦ χ' π ◦ ψ ◦ χ* analo=gously *π ◦ ψ' ◦ χ'.*

*B B*

Hence *ψ◦χ* = *ψ'◦χ'* since *πA, πB* are jointly mono. The following diagram commutes.

*,,,,* *A*,*¸,,*

*ϕ,A,,,,,*

*cccc*

*,,,,,,,,*

*Q,,¸¸¸¸ψ*

*ccc'*

*ccc ϕA πA*

*χ*

*,,, ¸¸¸¸¸¸ccc*

*,,,*

*ccc ¸¸¸¸¸¸*

*R*

*¸¸¸*

*ψ◦χ*=*ψ'◦χ'*

*, c*

*ccc ,,,*

z*˛*

*,,,,* *A¸⊗B*

*,c,*

*¸¸¸¸*

*ccc*

*,,,,,,,,*

*χ ¸¸*z *cc,,,,',*

*Q' ¸, ψ*

*'*

*¸¸¸¸*

*,,,,*

*,,,ϕB*

*πB*

*¸¸¸¸¸¸¸¸ ,,,,*

*'*

*B ¸¸¸¸¸,*zJ*z*

*ϕ*

z*˛*

*B*

Therefore *ψ*(*q*) = (*ψ ◦ χ*)(*r*) = (*ψ' ◦ χ'*)(*r*) = *ψ'*(*q*).

**Lemma 2.6** *Assume the product* **P** = *A⊗B exists and let p, p' ∈ P. Then p A,B*

*p' ⇒ p* = *p'.*

**Proof** Assume *p A,B p'*. There exists *A*-*B*-cones **Q***i* and *qi ∈ Qi*(*i* = 1*,..., n*), so that *p ∼*=*A,B q*1 *∼*=*A,B ... ∼*=*A,B qn ∼*=*A,B p'*. Let *ψi* : *Qi → A ⊗ B* be the unique homomorphisms. Then by lemma [2.5](#_bookmark2) *p* = *idP* (*p*) = *ψ*1(*q*1) = *...* = *ψn*(*qn*) =

*idP* (*p'*) = *p'*.

**Lemma 2.7** *Let* **Q***,* **Q***' be A-B-cones, ϑ* : *Q → Q' a homomorphism with ϕA* =

*' ◦ ϑ, ϕB* = *ϕ' ◦ ϑ . Then q ∼*=*A,B ϑ*(*q*) *for all q ∈ Q.*

*ϕ*

*B*

*A*

**Proof** This follows immediately from the diagram.

*,* *Q¸*

*ϕA*  *A ,*

*idQ,,,,,,*

*,,*

*,,*

*,, B*

*, ϕ*

*ccc*

*c*

*c*

*,,,,,*

*Q ¸¸¸¸*

*,,,, ccc*

*ϑ ccc,c,,,*

*¸¸¸¸¸*

*cccc ,,,*

*ϑ ¸¸¸*

*c ϕ' ,,*

*¸*zJ*˛' c A*  z *z*

*Q ' B*

*ϕ*

*B*

Particularly with regard to the product we get for every *A*-*B*-cone **Q** with unique homomorphism *ψ* : *Q → A ⊗ B* that *q ∼*=*A,B ψ*(*q*). We can now prove the following theorem.

**Theorem 2.8** *The product A⊗B exists iff the class*

*M* = *{C***Q***,q |* **Q** *A-B-cone,q ∈ Q}*

*of equivalence classes of A,B is a set.*

**Proof** First assume the product **P** = *A⊗B* exists. We prove that *M* is a set by showing *|M|≤ |P |*. Assume there exists an equivalence class *C***Q***,q* with (**P***, p*) */∈ CQ,q* for all *p ∈ P* . Let *ψ* : *Q → P* be the unique homomorphism with *ϕA* = *πA ◦ψ, ϕB* =

*πB ◦ ψ*. From lemma [2.7](#_bookmark3) follows *p* = *ψ*(*q*) *∼*=*A,B q* in contradiction to *p /∈ C***Q***,q*. Hence *M* = *{C***P***,p | p ∈ P}* and therefore *|M|≤ |P |*.

We assume now that *M* is a set. We shall equip *M* with a coalgebraic structure so that it becomes the product. For any *A*-*B*-cone **Q** we define a map *η***Q** : *Q → M* by *η***Q**(*q*) = *C***Q***,q*. Then we define a structure map *μ* : *M → FM* by *μ*(*C***Q***,q*) = (*Fη ◦ ξ*)(*q*) and denote *M* = (*M, μ*). Note, that then *η***Q** : *Q → M* is a homomorphism. We have to show that *μ* is well-defined. Let **Q***,* **Q***'* be *A*-*B*-cones, *q ∈ Q, q' ∈ Q'* and *C***Q***,q* = *C***Q***',q'* . We may assume *q ∼*=*A,B q'*. Then there exists an *F* -coalgebra *R* = (*R, ρ*), homomorphisms *χ* : *R → Q, χ'* : *R → Q'* and *r ∈ R* with *ϕA ◦ χ* =

*' ◦ χ', ϕB ◦ χ* = *ϕ' ◦ χ'* and *q* = *χ*(*r*)*, q'* = *χ'*(*r*). Let *μ*(*C***Q***,q*) = (*F η***Q** *◦ ξ*)(*q*). We

*ϕ*

*B*

*A*

show *μ*(*C***Q***,q*) = (*F η***Q***' ◦ ξ'*)(*q'*):

(*F η***Q***' ◦ ξ'*)(*q'*)= (*F η***Q***' ◦ ξ' ◦ χ'*)(*r*)

= (*F η***Q***' ◦ F χ' ◦ ρ*)(*r*)

= (*F η***Q** *◦ Fχ ◦ ρ*)(*r*)

= (*F η***Q** *◦ ξ ◦ χ*)(*r*)

= (*μ ◦ η***Q** *◦ χ*)(*r*)

= (*μ ◦ η***Q**)(*q*)

= *μ*(*C***Q***,q*)*.*

The projections *πA* : *M → A, πB* : *M → B* are defined by *πA*(*C***Q***,q*) = *ϕA*(*q*) and

*πB*(*C***Q***,q*) = *ϕB*(*q*). We compute *ϕA*(*q*) = (*ϕA ◦ χ*)(*r*) = (*ϕ' ◦ χ'*)(*r*) = *ϕ'* (*q*), hence

*A*

*A*

the projections are well-defined too. We show, that *πA* is a homomorphisms (*πB* analogously):

*α ◦ πA* = *α ◦ ϕA ◦ η***Q**

= *FϕA ◦ ξ ◦ η***Q**

= *FϕA ◦ F η***Q** *◦ μ*

= *F* (*ϕA ◦ η***Q**) *◦ μ*

= *FπA ◦ μ.*

The uniqueness of the homomorphism *η* : *Q→M* follows from lemma [2.7](#_bookmark3). There- fore *A ⊗ B* = (*M, πA, πB*).

# Transition systems and trees

Theorem [2.8](#_bookmark4) gives an abstract characterisation for the existence of products. If we like to apply this characterisation to coalgebras *A, B*, we have to observe whether

two elements *q ∈* **Q***, q' ∈* **Q***'* of *A*-*B*-cones are *A*-*B*-perspective or not. In this section we will show, that in the case of transition systems we can represent the equivalence classes of *A,B* by roots of trees. This opportunity allows us to give another, more technical characterisation for the existence of products in section [4](#_bookmark7).

**Definition 3.1** *Let* P *be the covariant powerset functor. A* P*-coalgebra* (*A, α*) *is called* ***transition system****, where A is interpreted as a set of states and α* : *A →*

P(*A*) *is the transition function. We write a →α a' instead of a' ∈ α*(*a*)*. If α is clear from the context, we write also a → a'.*

*A transition system* (*A, α*) *is called a* ***tree****, if there exist ωA ∈ A (root of A) with:*

* *∀a ∈ A* : *ωA /∈ α*(*a*)*,*
* *∀a ∈ A* : *a /*= *ωA ⇒ ∃*!*v ∈ A* : *a ∈ α*(*v*)*,*
* *∀a ∈ A.∃n ∈* N*.∃a*0*,..., an ∈ A* : *ωA* = *a*0 *→ a*1 *→ ... → an* = *a.*

In this section let (*A, α*)*,* (*B, β*) be transition systems. We will show, that there is a tree in any equivalence class *CQ,q*. Therefore we can represent the equivalence classes of *A,B* by trees.

**Lemma 3.2** *Let a ∈ A. Then ⟨a⟩* = (*⟨a⟩, η*) *with*

}

*⟨a⟩* = *a*0*a*1 *... an ∈ A*+ *| a* = *a*0*, ∀i* : *ai → ai*+1 *, η*(*a*0 *... an*)= *{a*0 *... anan*+1 *| an → an*+1*}*

*is a tree with root ω⟨a⟩* = *a and there exists a homomorphism ϑ* : *⟨a⟩ → A with*

*ϑ*(*ω⟨a⟩*) = *a.*

**Proof** It is easy to check, that *⟨a⟩* is a tree. We define *ϑ*(*a*0 *... an*) = *an* and show, that it is a homomorphism:

(*α ◦ ϑ*)(*a*0 *... an*)= *α*(*an*)

= *{an*+1 *| an → an*+1*}*

= P*ϑ* (*{a*0 *... anan*+1 *| an → an*+1*}*)

= (P*ϑ ◦ η*)(*a*0 *... an*)*.*

If (*A, α, λ*) is a labeled transition system with label *λ* : *A →* Λ and *a ∈ A*, we can define *κ* = *λ ◦ ϑ* : *⟨a⟩→* Λ. Then *ϑ* is a homomorphism of P( ) *×* Λ-coalgebras. **Corollary 3.3** *Let* **Q** *be an A-B-cone and q ∈ Q. Then* (*⟨q⟩, ϕA ◦ ϑ, ϕB ◦ ϑ*) *is an*

*A-B-cone and q ∼*=*A,B ω⟨q⟩.*

**Proof** Lemma [3.2](#_bookmark5) shows the existence of *⟨q⟩*. Moreover *ϑ*(*ω⟨q⟩*) = *q* and with lemma [2.7](#_bookmark3) *q ∼*=*A,B ω⟨q⟩*.

**Corollary 3.4** *Let a ∈ A, b ∈ B and a ∼ b. Then there exist a tree* (*T, τ* ) *and homomorphisms ϑA* : *T → A, ϑB* : *T → B with ϑA*(*ωT* ) = *a and ϑB*(*ωT* ) = *b.*

**Proof** Because *a, b* are bisimilar, there exist a transition system (*R, ρ*), homomor- phisms *πA* : *R → A, πB* : *R → B* and *r ∈ R* with *a* = *πA*(*r*) and *b* = *πB*(*r*). Then

*⟨r⟩* with the homomorphisms *ϑA* = *πA ◦ ϑR* and *ϑB* = *πB ◦ ϑR* is the wanted tree.

*⟨r⟩ ¸¸¸¸*

*ϑA*

*s*

*A* ¸*,*

*ϑ ¸¸ϑ¸B¸¸*

J *¸¸¸¸*z*˛*

*R*

*πA R πB B*

**Lemma 3.5** *Let* (*T, τ* ) *be a tree and ϑ* : *T → A a homomorphism with ϑ*(*ωT* ) = *a. Then there exist a homomorphism ψ* : *T → ⟨a⟩ with ψ*(*ωT* ) = *ω⟨a⟩* = *a.*

**Proof** We define *ψ* by induction over the construction of *T* :

*'*  *ω⟨a⟩* if *t'* = *ωT*

*ψ*(*t*) *· ϑ*(*t'*) if *t' /*= *ωT* and *t' ∈ τ* (*t*)

*ψ*(*t* ) =

Note, that for *t /*= *ωT* there is a unique element *t'* with *t ∈ τ* (*t'*), because *T* is a tree.

We show, that *ψ* is a homomorphism. Let *s, t ∈ T* with *t ∈ τ* (*s*). (For *t* = *ωT*

let *ψ*(*s*) = *ϵ* be the empty word.)

(*η ◦ ψ*)(*t*)= *η* (*ψ*(*s*) *· ϑ*(*t*))

=  *ψ*(*s*) *· ϑ*(*t*) *· a' | a' ∈ α*(*ϑ*(*t*))}

= *ψ*(*t*) *· a' | a' ∈* (P*ϑ ◦ τ* )(*t*)}

= *ψ*(*t*) *· ϑ*(*t'*) *| t' ∈ τ* (*t*)}

= *ψ*(*t'*) *| t' ∈ τ* (*t*)}

= (P*ψ ◦ τ* )(*t*)*.*

# Impeding paths

In this section we consider P-coalgebras *A* = (*A, α*)*, B* = (*B, β*). We introduce impeding paths and prove, that the existence of such a path impedes the existence of the product *A⊗ B*. First we look at an example of a transition system whose

product with itself exists but is larger than one might expect.

**Example 4.1** Consider the following transition system *A* = (*A, α*):

*,,, a*0 *¸¸¸¸*

*,,,,,,*

*,,,,*

*b*1 *s*

*¸¸¸¸¸¸*

*¸*

*¸¸* z*˛*

*b*2

The largest bisimulation *∼AA* is the least equivalence relation with *b*1 *∼AA b*2. We construct the product *A⊗A* = **P** = (*P, π*1*, π*2). For all *i, j ∈ {*1*,* 2*}* there is *pij ∈ P* with *π*1(*pij*) = *bi*, *π*2(*pij*) = *bj* and *η*(*pij*) = *∅*. For every subset *S ⊆*

*{p*11*, p*12*, p*21*, p*22*}* with *π*1(*S*) = *π*2(*S*) = *{b*1*, b*2*}* we obtain a state *pS ∈ P* with

*η*(*pS*) = *S* and *πi*(*pS*) = *a*0. Note, that for every such *S* the set *Q* = *{pS}∪ S* with the structure *η|Q* and the projections *π*1*|Q, π*2*|Q* yields an *A*-*B*-cone. We leave it to the reader to show that for *S /*= *S'* the states *pS* and *pS'* are not *A*-*A*-perspective. Therefore we get seven states *p ∈ P* with *π*1(*p*) = *π*2(*p*) = *a*0. The following figure shows the product *P*:

¸*p*¸1*,*1,*,*¸*¸*¸*¸¸¸, ¸,¸¸ p{p*11*,p*22*}*

˛ *p* 2*¸* 2,*,¸*,*,¸,*

*ss ,,*

*¸¸¸ ¸ ¸ ¸ ¸ ¸*

*ee ,, ¸¸¸*

*ss , ,*

*¸¸¸¸¸¸e¸e*

*,, ¸¸¸*

*s s*

*,,,*

*eee*

*¸¸¸¸¸,¸, ¸ ¸¸*

*p{p*11*,p*1*,*2*,p*22*} sss*

*,,,*

*eee*

*,,, p{p*11*,p*21*,p*22*}*

*,,,*

*sss*

*,,,,*

*eeee*

*,,,*

*sss*

*,,,sss*

*p , ee*

*,,,sss*

*ss,,,*

*{p*11*,p*12*,p,*21*,p*22*}*

*sss,,*

*sss*

*ss*

*,*

*,,,*

*,,*

*eeee*

*eee*

*,,,,*

*,,,*

*ss ,,*

*ss ,*

*s*

*p{p*11*,p*12*,¸p*21*} ¸,, ee*

*,, ss*

*p{p*12*,p*21*,p*22*}*

*¸¸¸ ¸¸,,¸¸¸¸¸¸ee*

*,, ss*

*¸¸¸¸ ,,*

*eee¸¸¸¸¸¸¸ ¸ ¸ ¸ ¸ ,,,*

*ss*

*¸*z*,*t*"* *ze e ¸¸¸¸¸¸¸¸ ,*z *sz* , *rss*

*p*12 ¸¸*,c*

*p{p*12*,p*21*}*

z *p*2*#* 1

If additionally there is a path *an → ... → a*1 *→ a*0, then the product increases faster than exponentially with respect to *n*. In fact there would be 27 *−* 1 states

*p'* with *π* (*p'*) = *π* (*p'*) = *a* and 227*−*1 *−* 1 states *p''* with *π* (*p''*) = *π* (*p''*) = *a*

1 2 1 1 2 2

in the product. This already suggests that it would be very difficult to construct the product *A⊗A* if there is a loop in the added path. We will see that indeed in this case the product *A⊗A* does not exist. The above consideration motivates the following definition:

## Definition 4.2 (Impeding path)

* *A bisimilar path in A×B is a sequence* (*a ,b* ) *...* (*a ,b* ) *with a*

*→α a ,b β*

*n n* 0 0

*bi and ai ∼ bi bisimilar for all i.*

* *A bisimilar path is called impeding, if*

*i*+1

*i i*+1 *→*

* *there exist* 0 *≤ k < n with* (*ak, bk*) = (*an, bn*)*,*
* *there exist a /*= *a' ∈ α*(*a*0)*,b /*= *b' ∈ β*(*b*0)*, so that {a, a'}× {b, b'} ⊆∼A,B.*
* *An impeding path is called reduced, if* 0 *≤ i < j < n ⇒* (*ai, bi*) */*= (*aj, bj*)*.*

Let *σ* = (*an, bn*) *...* (*a*0*, b*0) be an impeding path. Concatenating any of (*a, b*)*,* (*a, b'*)*,* (*a', b*)*,* (*a', b'*) to *σ* yields bisimilar paths (*an, bn*) *...* (*a*0*, b*0)(*a, b*)*,.. .*:

(*an−*1*, bn−*1) ¸

J*'*

(*a, b*) (*a, b'*)

,*,* ¸*,*

. (*an, bn*) (*a k−*1*, bk−*1) (*a* 0*, b*0)

. ,*,*

*¸¸¸¸¸*

*¸¸¸¸*

z*\_*

(*a*

*k*+1

*, bk*+1)

J

(*a ,b* )

*' '*

( *'*

z*)*

*a , b*)

**Lemma 4.3** *If there exists an impeding path in A × B, then there exists a reduced impeding path in A × B too.*

**Proof** Let (*an, bn*) *...* (*a*0*, b*0) be an impeding path in *A × B* and *j* = min*{j' | ∃i < j'* : (*ai, bi*) = (*aj' , bj'* )*}*. Then (*aj, bj*) *...* (*a*0*, b*0) is a reduced impeding path.

**Theorem 4.4** *If there is an impeding path in A × B then the product A ⊗ B does not exist.*

**Proof** We prove the theorem by contradiction. Assuming that the product *A⊗B* exists we construct an *A*-*B*-cone **Q** which contains pairwise not *A*-*B*-perspective states *qi*(*i ∈* κ) for an ordinal number κ with *|*κ*| > |A ⊗ B|*. This would be a contradiction to theorem [2.8](#_bookmark4). The proof is structured in the following way:

1. Construction of **Q**
2. Showing that **Q** is an *A*-*B*-cone
3. Proof that the states *qi* are pairwise not *A*-*B*-perspective

## Construction of Q

Assume that the product (*A⊗B, η, πA, πB*) exists and that there is an impeding path in *A×B*. By lemma [4.3](#_bookmark8) there exists a reduced impeding path (*an, bn*) *...* (*a*0*, b*0) in *A × B* and

* + *k /*= *n* with (*ak, bk*) = (*an, bn*),
  + *a /*= *a' ∈ α*(*a*0)*,b /*= *b' ∈ β*(*b*0) and *a ∼ a' ∼ b ∼ b'*. For *a*ˆ *∈ A,* ˆ*b ∈ B* with *a*ˆ *∼* ˆ*b* we define

,

*Sa*ˆˆ*b* = *p ∈ A ⊗ B |* (*πA, πB*)(*p*) *∈ α*(*a*ˆ) *× β*(ˆ*b*) *Ta*ˆˆ*b* = *p ∈ A ⊗ B |* (*πA, πB*)(*p*) = (*a*ˆ*,* ˆ*b*)

,

Let κ be an ordinal number with *|*κ*| > |A⊗ B|*. We construct an *A*-*B*-cone **Q** with

*Q* = *{qi | i ∈* κ*}∪{q' |* 0 *≤ j < k}∪A⊗B*. The projections *ϕA* : *Q → A, ϕB* : *Q → B*

*j*

are defined by

⎧⎪

(*πA, πB*)(*q*) if *q ∈ A ⊗ B*

(*a , b* ) if *i < k ∧ q ∈ {q , q'}*

⎪⎨ *i i i i*

(*ϕA, ϕB*)(*q*) = (*ai, bi*) if *k ≤ i < n* and for some *m ∈* N*.* : *q* = *qi*+*m*(*n−k*)

⎪⎩ so that *q* = *qj* for some *j* = κ*'* + (*i − k*)+ *m*(*n − k*)*.*

⎪

(*ai, bi*) if *k ≤ i < n* and there exist a limit ordinal number κ*',*

Before introducing the transition function, we define some helpful sets for all *i ∈* κ

⎧⎪⎨ *aibi ai−*1*bi−*1

*S \ T* if 0 *< i < k* or *k < i < n Pi* = *Sakbk \ Tak−*1*bk−*1 *∪ Tan−*1*bn−*1 if *i* = *k*

⎪⎩

*Pj* if (*ϕA, ϕB*)(*qj*) = (*ϕA, ϕB*)(*qi*)

*Ri* = *{qj | k ≤ j < i* and (*ϕA, ϕB*)(*qj*) = (*ϕA, ϕB*)(*qi*+*n−k−*1)*} .*

Note that *Pi* is uniquely defined because the impeding path is reduced. We can now define the transition function *ξ*:

⎧⎪

*η*(*q*) *q ∈ A ⊗ B*

*Sa*0*b*0 *\* (*Tab' ∪ Ta'b*) *q* = *q*0

⎪

0

*Sa*0*b*0

*\* (*Tab ∪ Ta'b'* ) *q* = *q'*

*{q }∪ P* 0 *< i < k ∧ q* = *q*

*ξ*(*q*) =

*{q*

*'*

*i−*1

*}∪ Pi* 0 *< i < k ∧ q* = *q'*

*{qk−*1*, qn−*1*}∪ Pk q* = *qk*

⎪

⎪⎨ *i−*1 *i* *i*

*i*

⎪

⎪ *k−*1

⎩

*{q'*

*{q'*

⎪

*}∪ Ri ∪ Pk i* = *n* + *m*(*n − k*) *∧ q* = *qi*

*}∪ Ri ∪ Pk i* = κ*'* + *m*(*n − k*) *∧ q* = *qi*

*k−*1

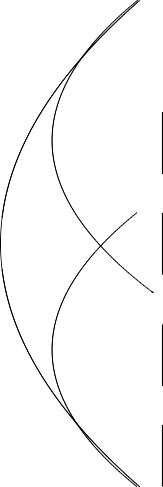
*Ri ∪ Pi* else

For the special case *k* = 0 we define *ξ*(*q*0) = *{qn−*1*}∪ Sa*0*b*0 *\* (*Tab' ∪ Ta'b*) and *ξ*(*qm·n*) = *Rm·n ∪ Sa*0*b*0 *\* (*Tab ∪ Ta'b'* ). The proof for this special case is very similar to the proof for *k >* 0, so we consider only the case *k >* 0. Figure [1](#_bookmark10) shows the

*A*-*B*-cone **Q** without the states from *A⊗ B*.

*q*3*n−*2*k |* (*ak, bk*)

*,,,,*



*,,,*

.J

.

J

*,,,,*

*,,,,*

*,,,*

*,*

*,, '*

*q*2*n−k |* (*ak, bk*)

*¸¸¸¸¸¸¸*

*,,,*

*,*

*,*

(*a, b* )

¸*,*

J *¸¸¸¸¸¸¸ ,,,*

z.

z*˛*z*z*

*q' ,b*

) *· · · * *q' |* (

)  (*a ', b*)

. *k−*1 *|* (*ak−*1

*,* *¸*

*,,,,,,*

*k−*1 0

*a*0*, b*0

J *,,,,,,,*

*qn |* (*ak, bk*)

**z**J*,)*

*qn−*1 *|* (*an−*,1*,¸bn−*1)

.J

.

J

(*a, b*)

¸*,*

*qk |* (*ak, bk*) *qk −*1 *|* (*ak−*1*, bk−*1)  *· · · * *q*0 *|* (*a*0*, b*0)  (*a ', b'*)

Figure 1. Extract of the *A*-*B*-cone **Q** without the transitions to *A⊗B*

(in addition to the states *qi* we give the image (*ϕA, ϕB*)(*qi*))

## Q is an *A*-*B*-cone

We have to show, that *ϕA, ϕB* are homomorphisms. First we consider some properties:

1. The largest bisimulation *∼A,B* between *A* and *B* with a bisimulation structure is an *A*-*B*-cone. Hence for *a*ˆ *∼* ˆ*b* there exists a *p ∈ A⊗B* with (*πA, πB*)(*p*) = (*a*ˆ*,* ˆ*b*).
2. Therefore, for all *q ∈ Q* there exists *p ∈ A ⊗ B* with (*a*ˆ*,* ˆ*b*) = (*ϕA, ϕB*)(*q*) = (*πA, πB*)(*p*). Moreover (P*πA ◦ η*)(*p*) = (*α ◦ πA*)(*p*) = *α*(*a*ˆ), hence *η*(*p*) *⊆ Sa*ˆˆ*b* =

*S*(*ϕA,ϕB*)(*q*) and

*α*(*a*ˆ) = (P*πA ◦ η*)(*p*) *⊆* P*πA* *S*(*ϕA,ϕB* )(*q*) *⊆ α*(*a*ˆ) = (*α ◦ ϕA*)(*q*)*.*

1. From the definition of *Ta*ˆˆ*b* it follows immediately that P*πA*(*Ta*ˆˆ*b*) = *{a*ˆ*}*, therefore P*πA* (*Tab ∪ Ta'b'* ) = P*πA* (*Tab' ∪ Ta'b*) = *{a, a'}* and P*πA* (*Sa b* ) =

0 0

P*πA* (*Sa*0*b*0 *\* (*Tab ∪ Ta'b'* )).

1. For *qi* with *i >* 0 and (*ϕA, ϕB*)(*qi*) = (*aj, bj*) */*= (*ak, bk*) we compute

*{aj−*1*}∪* P*ϕA*(*Pi*) = *{aj−*1*}∪* P*ϕA* *Sajbj \ Taj−*1*bj−*1 = *α*(*aj*) = (*α ◦ ϕA*)(*qi*)*.*

For *qi* with (*ϕA, ϕB*)(*qi*) = (*ak, bk*) we get

*{ak−*1*, an−*1*}∪* P*ϕA*(*Pi*)= *{ak−*1*, an−*1*}∪* P*ϕA Sakbk \ Tak−*1*bk−*1 *∪ Tan−*1*bn−*1

= *α*(*ak*)*.*

1. Let *i > k* and *ϕA*(*qi*) = *aj* for some *j ∈ {k* + 1*,..., n}*. Then *ϕA*(*qi*+*n−k−*1) =

*aj−*1. Furthermore, *j −* 1 *≥ k* and therefore P*ϕA*(*Ri*) = *{aj−*1*}*.

We divide the proof that *ϕA* is a homomorphism into cases as in the definition of *ξ*:

* + *q ∈ A ⊗ B*: (P*ϕA ◦ ξ*)(*q*) = (P*πA ◦ η*)(*q*) = (*α ◦ πA*)(*q*),
  + *q* = *q*0:

(P*ϕA ◦ ξ*)(*q*0)= P*ϕA*(*Sa*0*b*0 *\* (*Tab ∪ Ta'b'* ))

[*i*](#_bookmark12)=[*ii*](#_bookmark12)P*πA*(*S*(*ϕA,ϕB*)(*q*0))

=[*ii*](#_bookmark11)(*α ◦ ϕA*)(*q*0)*,*

* + *q* = *q'* : analogously

0

* + *q* = *qi* for some 0 *< i < k*:

(P*ϕA ◦ ξ*)(*qi*)= P*ϕA*(*{qi−*1*}∪ Pi*)

= *{ϕA*(*qi−*1)*}∪* P*ϕA*(*Pi*)

= *{ai−*1*}∪* P*πA*(*Pi*)

=[*iv*](#_bookmark13)(*α ◦ ϕA*)(*qi*)*,*

* + *q* = *q'* for some 0 *< i < k*: analogously

*i*

* + *q* = *qk*:

(P*ϕA ◦ ξ*)(*qk*)= P*ϕA*(*{qk−*1*, qn−*1*}∪ Pk*)

= *{ak−*1*, an−*1*}∪* P*πA*(*Pk*)

=[*iv*](#_bookmark13)(*α ◦ ϕA*)(*qk*)*,*

* *q* = *qi* and *i* = *n* + *m*(*n — k*) or *i* = *ω'* + *m*(*n — k*): Then (*ϕA, ϕB*)(*qi*) = (*an, bn*).

(P*ϕA ◦ ξ*)(*qi*)= P*ϕA*(*{q' }∪ Ri ∪ Pk*)

*k−*1

= *{ak−*1*}∪* P*ϕA*(*Ri*) *∪* P*ϕA*(*Pk*)

=[*v*](#_bookmark14) *{ak−*1*}∪ {an−*1*}∪* P*ϕA*(*Pk*)

=[*iv*](#_bookmark13)(*α ◦ ϕA*)(*qi*)*,*

* ”else“: Then *q* = *qi* for some *i > k* and (*ϕA, ϕB*)(*qi*) = (*aj, bj*) */*= (*ak, bk*). (P*ϕA ◦ ξ*)(*qi*)= P*ϕA*(*Ri ∪ Pi*)

= P*ϕA*(*Ri*) *∪* P*ϕA*(*Pi*)

=[*v*](#_bookmark14) *{aj−*1*}∪* P*ϕA*(*Pi*)

=[*iv*](#_bookmark13)(*α ◦ ϕA*)(*qi*)*.*

We can prove analogously that *ϕB* is a homomorphism, so **Q** is indeed an *A*-*У*-cone.

## The states *qi ∈ Q* are pairwise not *A*-*У*-respective

Let *ψ* : *Q → A⊗У* be the unique homomorphism. We show by contradiction, that *ψ*(*qi*) */*= *ψ*(*qj*) for all *i /*= *j*, i.e. *qi, qj* are not *A*-*У*-respective (*qi / A,B qj*) by theorem [2.8](#_bookmark4). Assume *ψ*(*qi*) = *ψ*(*qj*) for some *i < j*. We choose *i* minimal, then *ψ*(*qi'* ) */*= *ψ*(*qj*) for all *i' < i* and all *j ∈* N.

* First assume *i < k*. Since the impeding path is reduced (*ϕA, ϕB*)(*qi*) */*= (*ϕA, ϕB*)(*qj*) and therefore *ψ*(*qi*) */*= *ψ*(*qj*) for all *j /*= *i*.
* Assume now *i > k*. *ψ*(*qi*) = *ψ*(*qj*) implies (*ϕA, ϕB*)(*qi*) = (*ϕA, ϕB*)(*qj*). Because *j'* = *i*+ *n— k* = min*{j'' |* (*ϕA, ϕB*)(*qi*) = (*ϕA, ϕB*)(*qj''* )*}* we have *qj'−*1 *∈ Rj*. Then we need *q ∈ ξ*(*qi*) with *ψ*(*q*) = *ψ*(*qj'−*1) since (P*ψ ◦ ξ*)(*qi*) = (*η ◦ ψ*)(*qi*) = (*η ◦*

*ψ*)(*qj*) = (P*ψ◦ξ*)(*qj*). Because (*ϕA, ϕB*)(*q*) = (*ϕA, ϕB*)(*qj'−*1) we get *q* = *qi' ∈ Ri*.

Then *ψ*(*qi'* ) = *ψ*(*qj'−*1) and *i' < i ≤ j' —* 1 in contradiction to *i* being minimal.

* Let *i* = *k*. Then (*ϕA, ϕB*)(*qj*) = (*ak, bk*) and hence *q' ∈ ξ*(*qj*). We need *q ∈*

*k−*1

*ξ*(*qk*) with *ψ*(*q*) = *ψ*(*q'* ). The only possibility is *q* = *qk−*1. Then *ψ*(*q'*) = *ψ*(*qi*)

*k−*1

*i*

for all *i < k* by induction. There exists a state *p ∈ A⊗У* with *p ∈ ξ*(*q*0) and

(*ϕA, ϕB*)(*p*) = (*a, b*). Then we need *p' ∈ ξ*(*q'* ) with *ψ*(*p'*) = *ψ*(*p*). Otherwise

0

there is no state *q' ∈ ψ*(*q'* ) with (*ϕA, ϕB*)(*q'*) = (*a, b*).

0

This proves *ψ*(*qi*) */*= *ψ*(*qj*). Hence *|*κ*| ≤ |A ⊗ У|* in contradiction to the choice of κ. Consequently, the product *A⊗У* does not exist.

**Example 4.5** In [[1](#_bookmark19)] it is shown, that the product *A⊗A* of the following transition

system *A* does not exist.

,*c*

*A* = J,¸˜¸0` *,*z,¸*,* J,˜¸1` z,

We verify this, using theorem [4.4](#_bookmark9). (0*,* 0)(0*,* 0) is an impeding path in *A× A*, since we can extend it with (0*,* 0)*,* (0*,* 1)*,* (1*,* 0)*,* (1*,* 1) and 0 *~* 1. Therefore the product

*A⊗A* does not exist.

**Lemma 4.6** *Let* **Q***,* **Q***' be A-У-cones, q ∈ Q, q' ∈ Q', so that*

* + (*ϕA, ϕB*)(*q*) = (*ϕ' , ϕ'* )(*q'*)

*A B*

* + *for all bisimilar paths* (*a*0*, b*0) *...* (*an, bn*) *with* (*a*0*, b*0) = (*ϕA, ϕB*)(*q*) *there does not exist a /*= *a' ∈ α*(*a*0) *and b /*= *b' ∈ β*(*b*0) *with a ~ a' ~ b ~ b'.*

*Then q A,B q'.*

**Proof** We define a tree (*T, τ* ) with *T ⊆* (*A×B*)+ and projections *ϑA* : *T → A, ϑB* :

*T → B* recursively:

* + *ωT* = (*ϕA, ϕB*)(*q*) = (*ϑA, ϑB*)(*ωT* ),
  + *∀t ∈ T* : *τ* (*t*) = *{t ·* (*a, b*) *| a ∈* (*α ◦ ϑA*)(*t*)*,b ∈* (*β ◦ ϑB*)(*t*)*} ,* (*ϑA, ϑB*)(*t ·* (*a, b*)) = (*a, b*).

We define a map *χ* : *⟨q⟩→ T* recursively. Note, that the definition yields *ϑA ◦ χ* =

*ϕA, ϑB ◦ χ* = *ϕB*.

* + *χ*(*ω⟨q⟩*) = *ωT* ,
  + for *q*2 *∈ ξ*(*q*1):

*χ*(*q*2)= *χ*(*q*1) *·* ((*ϕA, ϕB*)(*q*2))

*∈ {χ*(*q*1) *·* (*a, b*) *| a ∈ α*(*ϕA*(*q*1))*,b ∈ β*(*ϕB*(*q*1))*}* = *τ* (*χ*(*q*1))*.*

Because (*α◦ϕA*)(*q*1) = (P*ϕA◦ξ*)(*q*1) we have *ϕA*(*q*2) *∈ α*(*ϕA*(*q*1)) = (*α◦ϑA*)(*χ*(*q*1)) *⊆*

*τ* (*χ*(*q*1)). Therefore *χ* is well-defined.

The second condition in the lemma yields

*∀p ∈ ⟨q⟩* : *a ∈* P*ϕA*(*ξ*(*p*)) *∧ b ∈* P*ϕB*(*ξ*(*p*)) *∧ a ~ b ⇔* (*a, b*) *∈* (*ϕA, ϕB*)(*ξ*(*p*))*.*

We show, that *χ* is an epimorphism:

(*τ ◦ χ*)(*q*)= *{χ*(*q*) *·* (*a, b*) *| a ∈* (*α ◦ ϑA ◦ χ*)(*q*)*,b ∈* (*β ◦ ϑB ◦ χ*)(*q*)*}*

= *{χ*(*q*) *·* (*a, b*) *| a ∈* (*α ◦ ϕA*)(*q*)*,b ∈* (*β ◦ ϕB*)(*q*)*}*

= *{χ*(*q*) *·* (*a, b*) *| a ∈* (P*ϕA ◦ ξ*)(*q*)*,b ∈* (P*ϕB ◦ ξ*)(*q*)*}*

= *{χ*(*q*) *·* (*ϕA, ϕB*)(*p*) *| p ∈ ξ*(*q*)*}*

= *{χ*(*p*) *| p ∈ ξ*(*q*)*}*

= (P*χ ◦ ξ*)(*q*)*.*

We show, that *ϑA* is a homomorphism (note, that *χ* is surjective): *α ◦ ϑA ◦ χ* =

*α ◦ ϕA* = P*ϕA ◦ ξ* = P*ϑA ◦* P*χ ◦ ξ* = P*ϑA ◦ τ ◦ χ*. We can define *χ'* : *⟨q'⟩ → T* analogously. Hence with the lemma [2.7](#_bookmark3) and corollary [3.3](#_bookmark6) *q ~*=*A,B ω⟨q⟩ ~*=*A,B ωT ~*=*A,B ω⟨q'⟩ ~*=*A,B q'*.

**Lemma 4.7** *Let* **Q***,* **Q***' be A-У-cones, q ∈ Q, q' ∈ Q'. Then*

*q ~*=*A,B q' ⇐⇒ ∀p ∈ ξ*(*q*)*.∃p' ∈ ξ'*(*q'*) : *p ~*=*A,B p' ∧ ∀p' ∈ ξ'*(*q'*)*.∃p ∈ ξ*(*q*) : *p ~*=*A,B p'.*

**Proof** Let *q ~*=*A,B q'* and *R* = (*R, ρ*) be the P-coalgebra from defintion [2.4](#_bookmark1), *χ* : *R→Q* and *χ'* : *R→ Q'* homomorphisms and *r ∈ R* with *χ*(*r*) = *q* and *χ'*(*r*) = *q'*. Let *p ∈ ξ*(*q*), then there exists *s ∈ ρ*(*r*) with *χ*(*s*) = *p* and therefore *p ~*=*A,B χ'*(*s*).

Assume now, that the right side of the equivalence in the lemma holds. Then for all *p ∈ ξ*(*q*)*, p' ∈ ξ*(*q'*) there exists an *A*-*У*-cone **R***pp'* , homomorphisms *χpp'* : *Rpp' →*

*Q, χ' '* : *Rpp' → Q'* and *r ∈ Rpp'* , so that *χpp'* (*r*) = *p, χ'*

*'* (*r*) = *p'*. We define a

*pp*

coalgebra

*R* = (*R, ρ*) = *{r*0*}⊕*

*pp*

*Rpp'* and *ρ*(*r*0) = *r ∈ Rpp' | χpp'* = *p, χ' '* = *p'*} *.*

*pp*

*p∼*=*A,B p'*

Then we can define homomorphisms *χ* : *R → Q, χ'* : *R → Q'* with *χ*(*r*0) = *q, χ'*(*r*0) = *q'* and *χ*(*r*) = *χpp'* (*r*) if *r ∈ Rpp'* . The reader is invited to check, that *χ, χ'* are homomorphisms and that they commute with the projections. Hence *q ~*=*A,B q'*.

**Lemma 4.8** *Let* (*A, α*)*,* (*B, β*) *be ﬁnite transition systems. Assume there is no impeding path in A × B. Then the product A⊗У exists and is ﬁnite.*

**Proof** Since *A, B* are finite, there exists *m ∈* N, so that any bisimilar path of length at least *m* contains a loop. Consider a bisimilar path (*a*1*, b*1) *...* (*an, bn*) of length

*n ≥ m*. Then *a* = *a'* or *b* = *b'* for all *a, a' ∈ α*(*an*)*, b, b' ∈ β*(*b*0) with *a ~ a' ~ b ~ b'*, otherwise the path would be impeding. Then the following situation is impossible:

(*a, b*) (*a, b'*)

,*,* ¸*,*

*.....*

(*a*1*, b*1) (*a* 2*, b*2) (*a n, bn*)

*¸¸¸¸¸¸*

J *¸¸¸¸*z*0*

(*a', b'*) (*a', b*)

We start the construction of the product with elements (*a*1*, b*1) *∈~A,B*, so that no such bisimilar path exists for any *n ∈* N. Then by lemma [4.6](#_bookmark15) there is only one equivalence class *C* with (*πA, πB*)(*C*) = (*a*1*, b*1). Take (*a*1*, b*1) *∈ A × B*, so that for

all *a*2 *∈ α*(*a*1)*, b*2 *∈ β*(*b*1) there exist only finite many classes *C* with (*πA, πB*)(*C*) = (*a*2*, b*2). Then by lemma [4.7](#_bookmark16) there are only finite many classes *C* with (*πA, πB*)(*C*) = (*a*1*, b*1). After at most *m* steps we have considered all bisimilar pairs (*a, b*). Then for every bisimilar pair (*a, b*) there exist only finite many equivalence classes *C* with (*πA, πB*)(*C*) = (*a, b*). Therefore *M* = *{C***Q***,q |* **Q** *A*-*У*-cone*,q ∈ Q}* is finite and with theorem [2.8](#_bookmark4) the product *A⊗У* exists. Furthermore, *M* is the base set of the product and therefore *A⊗У* is finite.

**Corollary 4.9** *Let A* = (*A, α*)*, У* = (*B, β*) *be ﬁnite transition systems. If the product A*2 = *A⊗A exists, then A⊗У exists too.*

**Proof** Assume the product *A⊗У* does not exist. Then there exists an impeding path (*an, bn*) *...* (*a*0*, b*0) in *A× У*. Let (*a, b*)*,* (*a', b'*) with *a ~ a' ~ b ~ b'* be the

possible continuations of the impeding path. Then (*an, an*) *...* (*a*0*, a*0) is a bisimilar path in *A×A* and (*a, a*)*,* (*a, a'*)*,* (*a', a*)*,* (*a', a'*) are possible continuations of this path. Hence (*an, an*) *...* (*a*0*, a*0) is an impeding path in *A×A* and by lemma [4.4](#_bookmark9) *A*2 does not exist.

# Conclusions and Further Work

For arbitrary *F* -coalgebras *A, У* we have introduced an equivalence relation *A,B* on the class of all elements of *A*-*У*-cones. We have seen that, if the class of all equivalence classes is a set, then it is a base set of the product *A⊗ У*. Otherwise the product does not exists (theorem [2.8](#_bookmark4)).

The invention of impeding paths led us to a more technical criterium for the existence of products of transition systems in theorem [4.4](#_bookmark9). It followed in corollary

[4.9](#_bookmark17) that, if the product *A⊗A* exists, then *A⊗У* exists for any transition system

*У*. It would be interesting to generalise this result to arbitrary *F* -coalgebras.

# References

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