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On SI2-continuous Spaces

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**Abstract**

In this paper, we construct a new way below relation from any given *T*0 space by making use of the cut operator, and introduce the concepts of SI2-continuous spaces and SI2-quasicontinuous spaces. The main results are: (1) a space (*X, τ* ) is SI2-continuous iff the set *X* equipped with the SI2-topology *τSI*2 is a

C-space; (2) a space (*X, τ* ) is SI2-quasicontinuous iff (*X, τSI*2 ) is a locally hypercompact space; (3) a space is SI2-continuous iff it is a meet SI2-continuous and SI2-quasicontinuous space.

*Keywords:* SI2-continuous space, SI2-quasicontinuous space, meet SI2-continuous space, irreducible set

# Introduction

The theory of continuous domains, due to its strong background in computer science, general topology and logic, has been extensively studied by people from various areas (see [[1,](#_bookmark29)[7,](#_bookmark35)[8](#_bookmark36)]). An important direction in the study of continuous domains is to extend the theory of continuous domains to that of posets as much as possible, and a lot of work has been done in this area (see [[10,](#_bookmark38)[11,](#_bookmark39)[12,](#_bookmark40)[13,14,](#_bookmark42)[16](#_bookmark44)]), but it is still rather restrictive, taking into consideration only the case of existing a join. In

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[[4,](#_bookmark31)[6](#_bookmark34)], Ern*e*´ introduced *s*2-continuous posets and precontinuous posets respectively by making use of the cut operator instead of joins. The notions of s2-continuity and precontinuity generalize the important characterizations of continuity from dcpos to arbitrary posets and have the advantage that not even the existence of directed joins is required. Recently, based on Ern´e’s work, many interesting continuity of posets were investigated (see [[17,](#_bookmark45)[21,](#_bookmark48)[22,](#_bookmark49)[23,](#_bookmark50)[24,](#_bookmark51)[25](#_bookmark52)]).

As we all known, the non-empty irreducible subsets of a poset with respect to the Alexandroff topology are exactly the directed sets, and the Scott topology *σ*(*P* ) ona poset *P* is defined by directed suprema, i.e., *U ∈ σ*(*P* ) if and only if *U* =*↑ U* and for any directed set *D*, *∨D ∈ U* implies *D∩U /*= *∅* whenever *∨D* exists. In [[20](#_bookmark46)], Zhao and Ho gave a new method of deriving a new topology *τSI* out of a given *T*0 space which makes use of the replacement of directed sets with irreducible sets, and proved that (*X, τ* ) is SI-continuous if and only if *X* equipped with irreducible-derived topology *τSI* is a C-space. Furthermore, a deep study concerning the replacement of directed sets with irreducible sets can be found in [[2,3](#_bookmark32)].

Motivated by the works of Zhao and Ho, we continue to develop domain theory in *T*0 spaces. In this paper, we define a new way below relation in *T*0 spaces by making use of the cut operator and introduce the concept of SI2-continuous spaces. It is proved that a space (*X, τ* ) is SI2-continuous if and only if the set *X* equipped with the SI2-topology *τSI*2 is a C-space. As a common generalization of both SI2- continuous spaces and quasicontinuous domains, we introduce the concept of SI2- quasicontinuous spaces and prove that a space (*X, τ* ) is SI2-quasicontinuous if and only if (*X, τSI*2 ) is a locally hypercompact space. Finally, we introduce the concept of meet SI2-continuous spaces and prove that a space is SI2-continuous if and only if it is a meet SI2-continuous and SI2-quasicontinuous space.

We would like to thank the referee for informing us that some properties of SI2-topology are also presented in [[18](#_bookmark47)]. It is also proved that a space (*X, τ* ) is SI2-continuous if and only if (*X, τSI*2 ) is a C-space independently in [[18](#_bookmark47)].

# Preliminaries

In this section, we recall some basic definitions and notations needed in this paper; more detail can be found in [[8,](#_bookmark36)[20](#_bookmark46)]. For a poset *P* , *x ∈ P* and *A ⊆ P* , let *↓x* = *{y ∈*

*P* : *y ≤ x}*, *↓ A* = S*{↓ x* : *x ∈ A}*; *↑ x* and *↑ A* are defined dually. A subset *A* is

called an *upper set* if *A* =*↑ A* and a *lower set* is defined dually. *A↑* and *A↓* denote the sets of all upper and lower bounds of *A*, respectively. Let *Aδ* = (*A↑*)*↓*. We put *P* (*<ω*) = *{F ⊆ P* : *F* is finite*}* and Fin*P* = *{↑ F* : *F ∈ P* (*<ω*)*}*.

For a poset *P* , the topology generated by the collection of sets *P\ ↓x*(as subbasic open subsets) is called the *upper topology* and denoted by *υ*(*P* ); the *lower topology ω*(*P* ) on *P* is defined dually. The *Alexandroff topology A*(*P* ) on a poset *P* is the topology consisting of all its upper subsets. A subset *U* of a poset *P* is called *Scott open* if *U* =*↑ U* and *D ∩ U /*= *∅* for all directed sets *D ⊆ P* with *∨D ∈ U* whenever

*∨D* exists. The topology formed by all the Scott open sets of *P* is called the *Scott topology*, written as *σ*(*P* ). For a poset *P* and *x, y ∈ P* , we say that *x* is way-below

*y* and write *x y* if for every directed set *D ⊆ P* with *∨D* exists, *x ≤ ∨D* implies *x ≤ d* for some *d ∈ D*. A poset *P* is called *continuous* if *{y ∈ P* : *x y}* is directed and *x* = *∨{y ∈ P* : *x y}* for each *x ∈ P* . We order the collection of nonempty subset of a poset *P* by *G ≤ H* if *H ⊆↑ G*. We say that a family of sets is directed if given *F*1*, F*2 in the family, there exists *F* in the family such that *F*1*, F*2 *≤ F* , i.e., *F ⊆↑ F*1*∩↑ F*2.

Let (*X, τ* ) be a topological space. A non-empty subset *F* of (*X, τ* ) is *irreducible* if whenever *F ⊆ A ∪ B* for closed sets *A* and *B*, then *F ⊆ A* or *F ⊆ B*. The set of all irreducible subsets of (*X, τ* ) is denoted by *Irrτ* (*X*). A space (*X, τ* ) is called a *C-space* if for any *U ∈ τ* , *x ∈ U* , there exists *u ∈ U* such that *x ∈ intτ ↑ u*. A space (*X, τ* ) is called *locally hypercompact* if for any *x* and any open set *U* containing *x*, there exists a finite set *E* such that *x ∈ intτ ↑ E ⊆↑ E ⊆ U* . Obviously, a C-space is locally hypercompact.

Give a topological space (*X, τ* ), denote the interior of a subset *A ⊆ X* by *intτ A* and the closure of *A* by *clτ A*. For a *T*0 topological space (*X, τ* ), the *specialization order ≤τ* on (*X, τ* ) is defined by *x ≤τ y* if and only if *x ∈ clτ* (*y*). Unless otherwise stated, throughout the paper, whenever an order-theoretic concepts is mentioned, it is to be interpreted with respect to the specialization order on (*X, τ* ).

**Definition 2.1** ([[7,15](#_bookmark43)]) Let *P* be a poset and *x, y ∈ P* .

* 1. Define a relation *≺* on *P* by *x ≺ y* iff *y ∈ intυ*(*P* ) *↑x*.
  2. *P* is called *hypercontinuous* if for all *x ∈ P* , *x* = *{u ∈ P* : *u ≺ x}* and

*{u ∈ P* : *u ≺ x}* is directed.

* 1. *P* is called *quasi-hypercontinuous* if for all *x ∈ P* and *U ∈ υ*(*P* ) with *x ∈ U* , there exists *H ∈ P* (*<ω*) such that *x ∈ intυ*(*P* ) *↑H ⊆↑H ⊆ U* .

it is proven in [[5,](#_bookmark33)[15](#_bookmark43)] thata poset *P* is hypercontinuous if and only if for all *x ∈ P* and *U ∈ υ*(*P* ) with *x ∈ U* , there exists *y ∈ P* such that *x ∈ intυ*(*P* ) *↑ y ⊆↑ y ⊆ U* , that is, a poset *P* is hypercontinuous iff *P* equipped with the upper topology *υ*(*P* )) is a C-space.

**Definition 2.2** ([[4](#_bookmark31)]) Let *P* be a poset.

1. For any *x, y ∈ P* , define2 on *P* by *x* 2 *y* if for all directed sets *D ⊆ P* with *y ∈ Dδ*, there exists *d ∈ D* such that *x ≤ d*. The set *{y ∈ P* : *y* 2 *x}* will be denoted by *⇓x* and *{y ∈ P* : *x y}* denoted by *⇑x*.
2. *P* is called *s*2-continuous if for all *x ∈ P* , *x ∈* (*⇓x*)*δ* and *⇓x* is directed.

**Definition 2.3** ([[4,5](#_bookmark33)]) Let *P* be a poset. A subset *U ⊆ P* is called weak Scott open if it satisfies

1. *U* =*↑U* ;
2. for all directed sets *D ⊆ P* , *Dδ ∩ U /*= *∅* implies *D ∩ U /*= *∅*.

The collection of all weak Scott open subsets of *P* forms a topology. This topology will be called the weak Scott topology of *P* and will be denoted by *σ*2(*P* ). Obviously, *σ*2(*P* ) *⊆ σ*(*P* ) and *σ*2(*P* )= *σ*(*P* ) if *P* is a dcpo.

**Theorem 2.4 ([**[**4**](#_bookmark31)**])** *Let P be a poset. Then the following statements are equivalent.*

1. *P is s*2*-continuous;*
2. (*P, σ*2(*P* )) *is a C-space.*

**Definition 2.5** ([[21](#_bookmark48)]) Let *P* be a poset and *G, H ⊆ P* .

1. Define *G* 2 *H* if for all directed sets *D ⊆ P* , *↑H ∩Dδ /*= *∅* implies *↑G∩D /*= *∅*. We write *G* 2 *x* for *G* 2 *{x}* and *y* 2 *H* for *{y}* 2 *H*.
2. *P* is called *s*2-*quasicontinuous* if for each *x ∈ P* , *w*(*x*)= *{F ⊆ P* : *F ∈ P* (*<ω*)

and *F* 2 *x}* is directed and *↑x* = *{↑F* : *F ∈ w*(*x*) *}*.

**Theorem 2.6 ([**[**21**](#_bookmark48)**])** *Let P be a poset. Then the following statements are equiva- lent.*

1. *P is s*2*-quasicontinuous;*
2. (*P, σ*2(*P* )) *is locally hypercompact.*

**Definition 2.7** ([[20](#_bookmark46)]) Let (*X, τ* ) be a *T*0 space. A subset *U* of *X* is called *SI-open*

if the following conditions are satisfied:

1. *U ∈ τ* ;
2. For any *F ∈ Irrτ* (*X*), *∨F ∈ U* implies *F ∩ U /*= *∅* whenever *∨F* exists.

The collection of all SI-open sets of (*X, τ* ) is denoted by *τSI* . Obviously, *τSI ⊆ τ* .

**Definition 2.8** ([[20](#_bookmark46)]) Let (*X, τ* ) be a *T*0 space.

1. Define *x SI y* if for any irreducible set, *y ≤ ∨F* implies *x ∈↓ F* whenever *∨F* exists. Denote the set *{x ∈ X* : *x SI a}* by *⇓SI a* and the set *{x ∈ X* : *a SI x}* by *⇑SI a*.
2. (*X, τ* ) is called *SI-continuous* if for any *x ∈ X*, the following conditions hold:
   1. *⇑SI x* is open in (*X, τ* ).
   2. *⇓SI x* is directed and *x* = *∨ ⇓SI x*.

It is proven in [[2](#_bookmark30)] that the condition ”directed” in Definition [2.8](#_bookmark3)(2) can be replaced by ”irreducible” condition and one still has an equivalent continuous.

**Theorem 2.9 ([**[**20**](#_bookmark46)**])** *Let* (*X, τ* ) *be a T*0 *space. Then the following statements are equivalent.*

1. (*X, τSI* ) *is a C-space;*
2. (*X, τ* ) *is SI-continuous.*

**Lemma 2.10 ([**[**9**](#_bookmark37)**])** *(Order Rudin Lemma) Let P be a preorder and F a directed family of ﬁnitary upper sets of P. Any lower set L that meets all members of F has a directed lower subset D that still meets all members of F.*

# SI2-continuous spaces

In this section, we introduce a new concept of SI2-continuous spaces and prove that a space (*X, τ* ) is SI2-continuous if and only if the set *X* equipped with the SI2-topology *τSI*2 is a C-space.

**Definition 3.1** Let (*X, τ* ) be a *T*0 space and *x, y ∈ X*.

*δ*

* 1. Define *x SI*2 *y* if for any *F ∈ Irrτ* (*X*), *y ∈ F* implies *x ∈↓ F* . Denote the

set *{x ∈ X* : *x SI*2 *a}* by *⇓SI*2 *a*, and the set *{x ∈ X* : *a SI*2 *x}* by *⇑SI*2 *a*.

* 1. (*X, τ* ) is called *SI*2*-continuous* if for each *x ∈ X*, the following conditions hold:
     1. *⇑SI*2 *x* is open in (*X, τ* ).
     2. *⇓SI*2 *x* is directed and *x* = *∨ ⇓SI*2 *x*.

In fact, we have *x* = *∨ ⇓SI*2

*x* iff *x ∈* (*⇓SI*2

*x*)*δ* since *⇓SI*

*x ⊆↓x*.

**Proposition 3.2** *Let* (*X, τ* ) *be a T*0 *space and x, y, u, v ∈ X. Then*

2

1. *If x SI*2 *y, then x ≤ y.*
2. *If u ≤ x SI*2 *y ≤ v, then u SI*2 *v.*
3. *If a smallest element ⊥ exists, then ⊥ SI*2 *x.*
4. *x SI*2 *y implies x SI y.*
5. *If* (*X, τ* ) *is an SI*2*-continuous space, then x SI y ⇔ x SI*2 *y.*
6. *If* (*X, τ* ) *is an SI*2*-continuous space, then* (*X, τ* ) *is SI-continuous.*

**Proof.** The conditions (1)-(4) are straightforward.

* 1. Suppose *x SI y*. Since (*X, τ* ) is an SI2-continuous, *y* = *∨ ⇓SI*2 *y* and *⇓SI*2 *y* is directed. Since every directed set is irreducible, by the definition of *SI* , we have that *x ∈↓* (*⇓SI*2 *y*) =*⇓SI*2 *y*. Thus *x SI*2 *y*.
  2. It is straightforward from (5) *2*

**Definition 3.3** Let (*X, τ* ) bea *T*0 space. A subset *U* of *X* is called *SI*2*-open* if the following conditions are satisfied:

1. *U ∈ τ* ;
2. For any *F ∈ Irrτ* (*X*), *Fδ ∩ U /*= *∅* implies *F ∩ U /*= *∅*.

The collection of all SI2-open subsets of (*X, τ* ) forms a topology. This topology will be called SI2-topology and denoted by *τSI*2 . The complement of an SI2-open set is called *SI*2*-closed*. Recall that an upper set *F* in a poset *P* is a filter if every finite subset of *F* has a lower bounded in *F* . Let SOFilt*τ* (*X*) denote the collection of all SI2-open filters in (*X, τ* ).

**Proposition 3.4** *Let* (*X, τ* ) *be a T*0 *space. Then the following conditions hold.*

1. *For any x ∈ X, clτ {x}* = *clτSI {x}.*

2

1. *The specialization orders of spaces* (*X, τ* ) *and* (*X, τSI*2 ) *coincide.*
2. *A closed set C in* (*X, τ* ) *is SI*2*-closed if and only if for any F ∈ Irrτ* (*X*)*,*

*F ⊆ C implies Fδ ⊆ C.*

1. *An open set U in* (*X, τ* ) *is SI*2*-open if and only if for any F ∈ Irrτ* (*X*)*,*

*Fδ ∩ U /*= *∅ implies F ∩ U /*= *∅.*

1. *τSI*2 *⊆ τSI ⊆ τ.*
2. *A set U is clopen in* (*X, τ* ) *if and only if it is clopen in* (*X, τSI*2 )*.*
3. *U is co-prime in τSI*2 *if and only if U ∈ SOFiltτ* (*X*)*.*
4. *If y ∈ intτSI ↑ x, then x SI*2 *y.*

2

**Proof.** The conditions (1)-(5) are easy to obtained.

* 1. Obviously, if *U* is clopen in (*X, τSI*2 ), then *U* is clopen in (*X, τ* ). Without loss of generality, assume that *U* is a non-trivial clopen set in (*X, τ* ). Let *F ∈ Irrτ* (*X*) with *U ∩ Fδ /*= *∅*. If *U ∩ F* = *∅*, then *F ⊆ X\U* . Since *X\U* is clopen, *Fδ ⊆ X\U* , a contradiction. Hence *U ∩ F /*= *∅*. So *U* is SI2-open. Similarly, we can deduce that *X\U* is SI2-open.
  2. Let *U* is co-prime in *τSI*2 . It is suffices to show that *U* is a filter. Suppose

S

*x, y ∈ U* . Then *X\ ↓ x* and *X\ ↓ y* are SI -open and *U* ¢ (*X\ ↓ x*) (*X\ ↓*

2

*y*) = *X\*(*↓ x* *↓ y*) since *U* is co-prime in *τSI* . So there exists *z ∈ U* such that

2

*z /∈ X\*(*↓ x ↓ y*), that is, *z ≤ x, y*. Thus *U* is a filter. Conversely, suppose that *U*

is not a co-prime in *τSI* , then there exist *V, W ∈ τSI* such that *U ⊆ V* S *W* with

2 2

*U* ¢ *V* and *U* ¢ *W* . Choose *x ∈ U\V* and *y ∈ U\W* . Since *U* is a filter, there is a *z ∈ U* such that *z ≤ x* and *z ≤ y*. Then we have *z /∈ V* S *W* , a contradiction.

Hence, (7) holds.

* 1. Let *y ∈ intτ*

*SI*2

*↑ x* and *F ∈ Irrτ* (*X*). If *y ∈ Fδ*, then *intτ*

*SI*2

*↑ x ∩ Fδ /*= *∅*.

By Definition [3.3](#_bookmark6), *intτSI ↑ x ∩ F /*= *∅*. Thus *x ∈↓ F* and *x SI*2 *y*. *2*

2

**Lemma 3.5** *Let* (*X, τ* ) *be a locally hypercompact space. If A is an irreducible set in* (*X, τ* )*, then there exists a directed subset D ⊆↓ A such that D↑* = *A↑. Furthermore, we have Dδ* = *Aδ, clτ D* = *clτ A.*

**Proof.** Let *A ∈ Irrτ* (*X*). Consider the collection *F* = *{↑ F ∈* Fin*P* : *A ∩ intτ ↑ F /*= *∅}*. Let *H, G ∈ F*. Then *intτ* (*↑ H*) *∩ intτ* (*↑ G*) *∩ A /*= *∅* since *A* is irreducible. Pick *x* in this intersection. Since (*X, τ* ) is locally hypercompact, there exists *E ∈ X*(*<ω*) such that *x ∈ intτ* (*↑ E*) *⊆↑ E ⊆ intτ* (*↑ H*) *∩ intτ* (*↑ G*). Then *↑ E ∈ F* and *E ⊆↑ H∩ ↑ G*. Thus *F* is directed and *↑ F∩ ↓ A /*= *∅* for any *↑ F ∈ F*. By Lemma [2.10](#_bookmark5), there is a directed set *D ⊆↓ A* such that *D∩ ↑ F /*= *∅* for every

*↑ F ∈ F*. Obviously, *A↑ ⊆ D↑* since *D ⊆↓ A*. Let *y* be an upper bound of *D*. Assume that *y /∈ A↑*, then there exists *x ∈ A* such that *x* ¢ *y*. Since (*X, τ* ) is a locally hypercompact space and *x ∈ X\ ↓ y ∈ τ* , there is a finite subset *F* of *X* such that *x ∈ intτ ↑ F ⊆↑ F ⊆ X\ ↓ y*. Thus *↑ F ∈ F*. So *D∩ ↑ F /*= *∅*. Thus *D ∩* (*X\↓ y*) */*= *∅*, that is, there is a *d ∈ D* such that *d* ¢ *y*, a contradiction. Hence *D↑ ⊆ A↑*. Therefore, *D↑* = *A↑*, then we have *Dδ* = *Aδ*.

Now we show that cl*τ A* = cl*τ D*. Obviously, cl*τ D ⊆* cl*τ A*. Let *x ∈* cl*τ A*, *U ∈ τ*

and *x ∈ U* . Since (*X, τ* ) is locally hypercompact, there exists *G ∈ P* (*<ω*) such that

*x ∈* int*τ ↑ G ⊆↑ G ⊆ U* . Note that *x ∈* cl*τ A*, thus *A ∩* int*τ ↑ G /*= *∅*. So *↑ G ∈ F*, which implies *D∩↑ G /*= *∅* and *D∩U /*= *∅*. Thus *x ∈* cl*τ D*. Therefore cl*τ D* = cl*τ A*.*2*

Since a C-space is locally hypercompact, by Lemma [3.5](#_bookmark8), we have the following result.

**Lemma 3.6 ([**[**20**](#_bookmark46)**])** *If F is an irreducible subset of a C-space* (*X, τ* )*, then there is a directed subset D ⊆↓ F such that D↑* = *F↑. In particular, ∨D* = *∨F, if either exists.*

**Lemma 3.7** *Let P be a poset. Then the following conditions hold.*

1. *υ*(*P* )*SI*2 = *υ*(*P* )*SI* = *υ*(*P* )*.*
2. *A*(*P* )*SI*2 = *σ*2(*P* )*.*
3. *If P is an s*2*-continuous poset, then σ*2(*P* )*SI*2 = *σ*2(*P* )*SI* = *σ*2(*P* )*.*

**Proof.** (1) By Proposition [3.4](#_bookmark7)(5), *υ*(*P* )*SI*2 *⊆ υ*(*P* )*SI ⊆ υ*(*P* ). For any *x ∈ P* , let *F* be an irreducible set with *Fδ ∩* (*P\↓ x*) */*= *∅*. Then there exists *z ∈ Fδ* such that *z* ¢ *x*. Suppose *F ∩* (*P\↓ x*)= *∅*. Then *x ∈ F↑*, so *z ≤ x*, a contradiction. Hence

1. holds as desired.
2. Because a non-empty subset *F ⊆ P* is irreducible with respect to the Alexandoff topology *A*(*P* ) iff it is a directed set.
3. Obviously, *σ*2(*P* )*SI*2 *⊆ σ*2(*P* )*SI ⊆ σ*2(*P* ). Let *P* be an *s*2-continuous poset. By Theorem [2.4](#_bookmark1), (*P, σ*2(*P* )) is a C-space. Let *U ∈ σ*2(*P* ) and *F ∈ Irrτ* (*X*). If *Fδ ∩ U /*= *∅*, by Lemma [3.6](#_bookmark9), there is a directed set *D ⊆↓ F* such that *Dδ* = *Fδ*. Thus we have that *D ∩ U /*= *∅*, which implies *F ∩ U /*= *∅*. Hence *U ∈ σ*2(*P* )*SI*2 . Therefore, *σ*2(*P* )*SI*2 = *σ*2(*P* )*SI* = *σ*2(*P* ). *2*

The following example shows that an SI-open set need not be SI2-open.

**Example 3.8** ([[4](#_bookmark31)]) Consider three disjoint countable sets *A* = *{an* : *n ∈* **N0***},B* =

*{bn* : *n ∈* **N0***},C* = *{cn* : *n ∈* **N***}*, and the order *≤* on *P* = *A ∪ B ∪ C* is defined as follows:

*↓ a*0 = *{a*0*}∪ B*,

*↓ an* = *{bm* : *m < n}*(*n ∈* **N***,n /*= 1*,* 2),

*↓ a*1 = *{b*0*}∪ C*,

*↓ a*2 = *{b*0*, b*1*}∪ C*,

*↓ bn* = *{bn}*(*n ∈* **N0**),

*↓ cn* = *{cm* : *m ≤ n}*(*n ∈* **N**),

*x ≤ y ⇔ x ∈↓ y*.

Then *↑ b*0 is open in *σ*(*P* ) but not in *σ*2(*P* ) since *C* = *{cn* : *n ∈* **N***}* is a directed lower set with *b*0 *∈ Cδ∩↑ b*0 */*= *∅* while *C∩↑ b*0 = *∅*. Thus *↑ b*0 *∈ A*(*P* )*SI* = *σ*(*P* ), but *↑ b*0 */∈ A*(*P* )*SI*2 = *σ*2(*P* ). Therefore, *↑ b*0 is an SI-open set but not SI2-open in (*P, A*(*P* )).

The following theorem exhibits an important property of relation *SI*2 on SI2- continuous spaces, i.e., the interpolation property.

**Theorem 3.9** *Let* (*X, τ* ) *be an SI*2*-continuous. Then the following conditions hold.*

1. *The relation SI*2 *satisﬁes interpolation property, i.e., x SI*2 *z implies*

*x SI*2 *y SI*2 *z for some y ∈ X.*

1. *If x SI*2

*z and z ∈ Fδ for an irreducible set F in* (*X, τ* )*, then x SI*

*y for*

*some element y ∈ F.*

2

**Proof.** (1) Let *x SI z*. Since (*X, τ* ) is SI2-continuous, *z ∈* (*⇓SI z*)*δ ⊆* (S*{*(*⇓SI*

*δ δ* 2S *δ* 2 2

*y*) : *y ∈⇓SI*2 *z}*) = ( *{⇓S*S*I*2 *y* : *y ∈⇓SI*2 *z}*) . As the union of a directed family

of directed sets is directed, *{⇓SI*2 *y* : *y ∈⇓SI*2 *z}* is directed. By the definition of

*SI*2 , there exist *y ∈⇓SI*2 *z* and *w ∈⇓SI*2 *y* such that *x ≤ w*. Thus *x SI*2 *y SI*2 *z*.

(2) It is straightforward from (1). *2*

**Lemma 3.10** *If a space* (*X, τ* ) *is SI*2*-continuous, then all sets ⇑SI*2 *x for x ∈ X*

*are SI*2*-open.*

**Proof.** Since (*X, τ* ) is SI2-continuous, *⇑SI*2 *x ∈ τ* . Let *F* be an irreducible subset

2

with *Fδ∩ ⇑SI*

2

*x /*= *∅*. Then there exists *z ∈ Fδ* such that *x SI*

*z*. By Theorem

[3.9](#_bookmark11)(2), there is a *y ∈ F* such that *x SI*2 *y*. Thus *F∩ ⇑SI*2 *x /*= *∅*. Hence *⇑SI*2 *x* is SI2-open. *2*

**Proposition 3.11** *Let* (*X, τ* ) *be an SI*2*-continuous space and x ∈ X. Then*

1. *An upper set U is SI*2*-open iff for every x ∈ U, there is a u ∈ U such that*

*u SI*2 *x.*

1. *The sets of the form ⇑SI*2 *x, x ∈ X form a basis for the SI*2*-topology.*
2. *intτSI ↑ x* =*⇑SI*2 *x.*

2

2

1. *For any subset A ⊆ X, intτ*

*SI*2

*A* = S*{⇑SI*

*u* :*⇑SI*2

*u ⊆ A}.*

**Proof.** (1)Let *U* be an SI2-open and *x ∈ U* . Since (*X, τ* ) is SI2-continuous, *x ∈*

(*⇓SI*2

2

1. *δ* and *⇓SI*

*x* is directed. Thus *U ∩* (*⇓SI*2

1. *δ /*= *∅*. It follows that there exists

*u ∈ U* such that *u SI*2 *x*. Conversely, if for any *x ∈ U* , there is a *u ∈ U* such that

*u SI*2 *x*, then *U* = *{⇑SI*2 *u* : *u ∈ U}*, which is SI2-open by Lemma [3.10](#_bookmark12). Thus *U*

is SI2-open.

* 1. It is immediate consequence of (1).
  2. By Proposition [3.4](#_bookmark7)(8), *intτSI ↑ x ⊆⇑SI*2 *x*. By Lemma [3.10](#_bookmark12), *⇑SI*2 *x* is

2

SI2-open and *⇑SI*2 *x ⊆↑ x*. Thus *intτSI ↑ x* =*⇑SI*2 *x*.

2

* 1. This follows directly from (2). *2*

**Lemma 3.12** *In an SI*2*-continuous space* (*X, τ* ) *the following hold.*

1. *If x SI*2 *y, then there is an SI*2*-open ﬁlter U with y ∈ U ⊆⇑SI*2 *x.*
2. *If y /≤ z, then there is an SI*2*-open ﬁlter U containing y but not z.*

**Proof.** (1) By the interpolation property, we construct inductively a decreasing

sequence of elements *yn* with *x SI*2 *... SI*2 *yn SI*2 *yn−*1 *SI*2 *... SI*2 *y*1 = *y*.

Set *U* = S*{⇑SI yn* : *n* = 1*,* 2*, ...}*. Clearly, *y ∈ U* and *U ⊆⇑SI x*. Now we show

2

2

that *U* is an SI2-open filter. Clearly, *U* is an upper set. If *x*1*, x*2 *∈ U* , then there

are *yn*1 *, yn*2 such that *x*1 *∈⇑SI*2 *yn*1 *, x*2 *∈⇑SI*2 *yn*2 . Without loss of generality, we assume that *n*1 *≤ n*2, then *yn*2 *SI*2 *yn*1 *SI*2 *x*1. Thus *yn*2 *≤ x*1*, x*2. Note that *yn*2 *∈ U* . Thus *U* is a filter. It is from Proposition [3.11](#_bookmark13) that *U* is SI2-open.

(2) Suppose that *y /≤ z*. Since (*X, τ* ) is SI2-continuous, *y* = *∨ ⇓SI*2 *y*. It follows

2

that *z /∈* (*⇓SI*2

1. *↑*. Thus there exists *x ∈⇓SI*

*y* such that *x /≤ z*. By the condition

* 1. , there is an SI2-open filter *U* such that *y ∈ U ⊆⇑SI*2 *x*, but *z /∈ U* . *2*

**Theorem 3.13** *Let* (*X, τ* ) *be a T*0 *space. Then the following conditions are equiv- alent.*

1. (*X, τ* ) *is SI*2*-continuous;*
2. *each ⇑SI x is SI*2*-open, and if U ∈ τSI , then U* = S*{⇑SI x* : *x ∈ U};*

2 2 2

1. *SOFiltτ* (*X*) *is a basis of τSI*2 *and* (*τSI*2 *, ⊆*) *is a continuous lattice;*
2. *τSI*2 *has enough co-primes and* (*τSI*2 *, ⊆*) *is a continuous lattices;*
3. (*X, τSI*2 ) *is a C-space.*

**Proof.** (1)*⇒*(2) By Lemma [3.10](#_bookmark12) and Proposition [3.11](#_bookmark13).

* 1. *⇒*(3) Obviously, *⇑SI*2 *x ∈ τ* is open. By Proposition [3.11](#_bookmark13) and Lemma [3.12](#_bookmark14), SOFilt*τ* (*X*) isa basis of *τSI*2 . In order to prove the continuity of *τSI*2 , let *U ∈ τSI*2 . For any *x ∈ U* , there is a *y ∈ U* such that *y SI*2 *x* by (2). Then *x ∈⇑SI*2 *y ∈ τSI*2 ,

and we claim that *⇑SI*2 *y U* . Indeed, if *D* is a directed family of SI2-open sets

with *U ⊆* S *D*, then there exists *W ∈D* such that *y ∈ W* . Thus *⇑SI y ⊆↑ y ⊆ W* .

S

2

Thus *U* = *{V* : *V U}*.

* 1. *⇔* (4) Consequence of Proposition [3.4](#_bookmark7)(7).

(3)*⇒*(5) Let *x ∈ U ∈ τSI*2 . Since (*τSI*2 *, ⊆*) is continuous, there exists *V ∈ τSI*2 such that *x ∈ V U* . Since SOFilt*τ* (*X*) is a basis of *τSI*2 , there exists *F ∈* SOFilt*τ* (*X*) such that *x ∈ F ⊆ V* . Now we show that there exists *y ∈ U* such that *x ∈ F ⊆↑ y*. If not, then for any *y ∈ U* , *F* ¢*↑ y*. Thus *y ∈ X\ ↓ zy* for some *zy ∈ F* , and thus there exists *Fy ∈* SOFilt*τ* (*X*) such that *y ∈ Fy ⊆ X\↓ zy*. Hence

S

*U ⊆ {Fy* : *y ∈ U}*. Since *V U* , there exists a finite set *{yi* : *i* = 1*,* 2*, ..., n}*

such that *V ⊆* S*{Fy ∈* SOFilt*τ* (*X*) : *i* = 1*,* 2*, ..., n}*. Let *zy* = *zi*. Then *zi ∈ F* .

*i*

*i*

Since *F* is a filter, there exists *z ∈ F* such that *z ≤ zi* for all *i*. Notice that

*z ∈ F ⊆ V ⊆* S*{Fy ∈* SOFilt*τ* (*X*) : *i* = 1*,* 2*, ..., n}*. Then there is a *Fy ∈*

*i*

*k*

SOFilt*τ* (*X*) such that *z ∈ Fk ⊆ X\ ↓ zk*, which contradicts *z ≤ zi*. Thus there

exists *y ∈ U* such that *x ∈ F ⊆↑ y*. Since *F* is SI2-open, *x ∈ intτSI*

2

*↑ y ⊆↑ y ⊆ U* .

Hence (*X, τSI*2 ) is a C-space.

(5)*⇒*(1) Let (*X, τSI*2 ) be a C-space. For any *x ∈ X*, let *Dx* = *{y ∈ X* :

*x ∈ intτSI ↑ y}*. By Proposition [3.4](#_bookmark7)(8), *Dx ⊆⇓SI*2 *x*. First we show that *Dx* is

2

directed and *x* = *∨Dx*. For any *d*1*, d*2 *∈ Dx*, *x ∈ intτSI*

2

2

2

*↑ d*1 *∩ intτSI*

*↑ d*2 *∈ τSI*2 .

Since (*X, τSI*2 ) is a C-space, there is a *d ∈ intτSI*

2

*↑ d*1 *∩ intτSI*

*↑ d*2 such that

*x ∈ intτSI*

2

*↑ d ⊆↑ d ⊆ intτSI*

*↑ d*1 *∩intτSI*

*↑ d*2 *⊆↑ d*1*∩↑ d*2. It follows that *d ∈ Dx*

and *d*1*, d*2 *≤ d*. Thus *Dx* is directed. Obviously, *x* is an upper bound of *Dx*. Let *y* be

2

2

any upper bound of *Dx* and *P* = (*X, ≤τ* ). If *x* ¢ *y*, then *x ∈ P\↓ y ∈ τSI*2 . Since

(*X, τSI*2 ) is a C-space, there is a *w ∈ P* such that *x ∈ intτSI*

2

*↑ w ⊆↑ w ⊆ P\ ↓ y*.

Thus *w ∈ Dx* and *w* ¢ *y*, contradicting to the assumption that *y* is an upper bound

of *Dx*. Thus *x* = *∨Dx*.

Since *Dx ⊆⇓SI*2 *x*, we have *x* = *∨ ⇓SI*2 *x*. Now we show that *⇓SI*2 *x* is directed.

*x*

For any *y*1*, y*2 *∈⇓SI*2

*x*, *y*1 *SI*2

*x, y*2 *SI*2

*x*. By the definition of *SI*2

and *x ∈ Dδ* ,

there are *d*1*, d*2 *∈ Dx* such that *y*1 *≤ d*1*, y*2 *≤ d*2. Since *Dx* is directed, there exists

*d ∈ Dx* such that *d*1*, d*2 *≤ d*. Thus *y*1*, y*2 *≤ d*. Note that *Dx ⊆⇓SI*2 *x*, so *⇓SI*2 *x* is

directed.

Finally, we can directly check that *⇑SI*2

*x* = S

*z∈↑x*

*intτ*

*SI*2

*y*

(*↑ z*). In fact, for any

*y ∈⇑SI*2

*x*, *x SI*2

*y*. From the above argument we can see that *y ∈ Dδ* and *Dy* is

directed, so there is a *z ∈ Dy* such that *x ≤ z*. Follows from the definition of *Dy*,

we have *y ∈ int*

S

*SI*2

*τ*

*↑ z* and *z ∈↑ x*. Thus *y ∈* S

*z∈↑x*

*intτSI*

(*↑ z*), i.e., *⇑SI*2 *x ⊆*

*z∈↑x intτSI*2 (*↑ z*). To prove the inverse inclusion, let *y ∈ z∈↑x intτSI*2 (*↑ z*). Then

S 2

there is a *z ∈↑ x* such that *y ∈ intτSI* (*↑ z*), so *y ∈ intτSI* (*↑ x*). By Proposition

[3.4](#_bookmark7)(8), *x*

*y*, i.e., *y ∈⇑*

2 *⇑ x* = S 2

(*↑ z*). Therefore,

*SI*2

*SI*2 *x*. Thus

*SI*2

*z∈↑x intτSI*2

*⇑SI*2 *x* is open in (*X, τ* ). All there show that (*X, τ* ) is SI2-continuous. *2*

By Theorem [2.9,](#_bookmark4) Lemma [3.7](#_bookmark10) and Theorem [3.13](#_bookmark15), we have the following corollary.

**Corollary 3.14** *Let P be a poset. Then the following conditions are equivalent.*

1. *P is hypercontinuous;*
2. (*P, υ*(*P* )) *is SI-continuous;*
3. (*P, υ*(*P* )) *is SI*2*-continuous.*

**Corollary 3.15** *Let P be a poset. Then the following conditions are equivalent.*

1. *P is an s*2*-continuous poset;*
2. (*P, σ*2(*P* )) *is a C-space;*
3. (*P, σ*2(*P* )) *is SI*2*-continuous ;*
4. (*P, A*(*P* )) *is SI*2*-continuous.*

**Proof.** (1) *⇔* (2) By Theorem [2.4](#_bookmark1).

* 1. *⇒* (3) Let (*P, σ*2(*P* )) be a C-space. By Lemma [3.7](#_bookmark10), *σ*2(*P* )*SI*2 = *σ*2(*P* ). Thus we have that (*P, σ*2(*P* )*SI*2 ) is a C-space. From Theorem [3.13](#_bookmark15), it follows that (*P, σ*2(*P* )) is SI2-continuous.
  2. *⇒* (1) Suppose that (*P, σ*2(*P* )) is SI2-continuous, then for any *x ∈ P* , *x ∈*

(*⇓SI*2

2

*x*)*δ* and *⇓SI*

*x* is directed. Note that *⇓SI*2

*x ⊆ {y ∈ P* : *y* 2 *x}*. Thus *P* is

s2-continuous.

(2) *⇔* (4) From Lemma [3.7](#_bookmark10) and Theorem [3.13](#_bookmark15), it follows that the condition (3)

and (4) are equivalent. *2*

The following example shows that an SI-continuous space need not be SI2- continuous.

**Example 3.16** ([[4](#_bookmark31)]) Consider the Euclidean plane *P* = R*×*R under the usual order, then *P* is a continuous poset, so (*P, σ*(*P* )) is a C-space, which implies that (*P, A*(*P* )) is an SI-continuous space (By Lemma 5.2 and Theorem 6.4 in [[20](#_bookmark46)]). Because every lower half-plane

*Ea* = *{*(*x, y*) *∈* R *×* R : *y ≤ a}*

is a directed lower set with *Eδ* = R *×* R, while *{Ea* : *a ∈* R*}* = *∅*, thusis empty. Hence *P* is not *s*2-continuous. By Corollary [3.15](#_bookmark16), (*P, A*(*P* )) is not an SI2-continuous space.

*a*

# SI2-quasicontinuous spaces

In this section, we introduce the concept of SI2-quasicontinuous spaces and prove that a space (*X, τ* ) is SI2-quasicontinuous if and only if (*X, τSI*2 ) is a locally hyper- compact space.

**Definition 4.1** Let (*X, τ* ) be a *T*0 space and *F ∈ Irrτ* (*X*). *G, H ⊆ X*. Define

*δ*

*G SI*2 *H* if *↑ H ∩ F /*= *∅* implies *↑ G ∩ F /*= *∅*.

Write *G SI*2 *x* for *G SI*2 *{x}* and *y SI*2 *H* for *{y} SI*2 *H*. The set

*{x ∈ X* : *G SI*2 *x}* will be denoted by *⇑SI*2 *G* and *{x ∈ X* : *x SI*2 *H}* denoted

2

by *⇓SI*2

*H*. Let fin(*x*)= *{E ∈ X*(*<ω*) : *E SI*

*x}*.

**Definition 4.2** A *T*0 space (*X, τ* ) is called *SI*2*-quasicontinuous* if for any *x ∈ X*, the following conditions hold:

1. for any *E ∈ X*(*<ω*), *⇑SI*

2

*E* is open in (*X, τ* );

1. fin(*x*) is directed;
2. *↑ x* = *{↑ E* : *E ∈* fin(*x*)*}*.

**Remark 4.3** It is verify that the condition (3) in above definition is equivalent to (3*j*) for any *x, y ∈ X*, if *x* ¢ *y*, then there exists *E ∈* fin(*x*) such that *y /∈↑ E*.

It is easy to get the following proposition and we omit the proof.

**Proposition 4.4** *Let* (*X, τ* ) *be a T*0 *space and G, H ⊆ X. Then*

1. *G SI*2 *H iff G SI*2 *x for all x ∈ H.*
2. *G SI*2 *H ⇒ G ≤ H.*
3. *A ≤ G SI*2 *H ≤ B ⇒ A SI*2 *B.*
4. *If x ∈ intτSI ↑ H, then H SI*2 *x.*

2

**Lemma 4.5 ([**[**8**](#_bookmark36)**])** *(Rudin’s Lemma) Let F be a directed family of nonempty ﬁnite*

*subsets of a poset P. Then there exists a directed set D ⊆* S *F such that*

*F∈7*

*D ∩ F /*= *∅ for all F ∈ F.*

**Lemma 4.6** *Let H be a directed family of nonempty ﬁnite sets in a T*0 *space. If*

*G SI*2

*x and*

*H∈H*

*↑ H ⊆↑ x, then H ⊆↑ G for some H ∈ H.*

**Proof.** Suppose not. Then the collection *{H\ ↑ G* : *H ∈ H}* is a directed family

of nonempty finite sets. By Lemma [4.5](#_bookmark18), there exists a directed set *D ⊆* S*{H\ ↑*

*G* : *H ∈ H}* such that *D ∩* (*H\ ↑ G*) */*= *∅* for all *H ∈ H*. Then *D† ⊆* *↑*

*d∈D*

*δ*

*d ⊆ H∈H ↑* (*H\ ↑ G*) *⊆ H∈H ↑ H ⊆↑ x*. Thus *x ∈ D* . Since every directed

set is irreducible and *G SI*2 *x*, there exists *d ∈ D* such that *d ∈↑ G*. But this contradicts *d ∈ H\↑ G* for some *H*. *2*

We now derive the interpolation property for SI2-quasicontinuous spaces.

**Theorem 4.7** *Let X be an SI*2*-quasicontinuous space. If H SI*2 *x, then there*

*exists E ∈ X*(*<ω*) *such that H SI E SI x.*

2

2

**Proof.** The statements has been proved for quasicontinuous domains in [[8](#_bookmark36)], and the similar proof carries over to this setting. *2*

**Proposition 4.8** *Let* (*X, τ* ) *be an SI*2*-quasicontinuous space. Then*

1. *For any nonempty set H ⊆ X, ⇑SI*2 *H* = *intτSI ↑ H.*

2

1. *A subset U of X is SI*2*-open iff U* = S*{⇑SI E* : *E ∈ X*(*<ω*) *and ↑ E ⊆ U}.*

2

(*<ω*)

*The set {⇑SI*2 *E* : *E ∈ X } form a basis of τSI .*

2

**Proof.** (1) From Proposition [4.4](#_bookmark17)(4), we have *intτSI ↑ H ⊆⇑SI*2 *H*. Obviously,

2

*⇑SI*2 *H ⊆↑ H*. Now we only need to show that *⇑SI*2 *H* is SI2-open. Since (*X, τ* ) is

SI2-quasicontinuous, *⇑SI*2

2

*H ∈ τ* . Let *F ∈ Irrτ* (*X*) and *⇑SI*2

*H ∩ Fδ /*= *∅*. Choose

*x ∈⇑SI*2

*H∩Fδ*, i.e., *H SI*

*x* and *x ∈ Fδ*. By Theorem [4.7](#_bookmark20), there exists *E ∈ X*(*<ω*)

such that *H SI*2 *E SI*2 *x*, which implies *↑ E ∩ F /*= *∅*. Notice that *E ⊆⇑SI*2 *H*,

so *⇑SI*2 *H ∩ F /*= *∅*. Thus *⇑SI*2 *H* is SI2-open. Therefore *⇑SI*2 *H* = *intτSI*

2

*↑ H*.

(2) The sufficiency follows from the condition (1). To prove the necessity, let

*U ∈ τSI*2 and *x ∈ U* . From the definition of SI2-topology, we have *U SI*2 *x*.

By Theorem [4.7](#_bookmark20), there exists *E ∈ X*(*<ω*) such that *U SI E SI*

*x*. Thus

S (*<ω*) 2 2

*x ∈ {⇑SI*2 *E* : *E ∈ X* S and *↑ E ⊆ U}*. Obviously, the converse inclusion is

(*<ω*)

always true. Thus *U* = *{⇑SI*2 *E* : *E ∈ X* and *↑ E ⊆ U}*, and thus the set

(*<ω*)

*{⇑SI*2 *E* : *E ∈ X }* form a basis of *τSI* . *2*

2

**Theorem 4.9** *Let* (*X, τ* ) *be a T*0 *space. Then the following conditions are equiva- lent.*

1. (*X, τ* ) *is an SI*2*-quasicontinuous space;*
2. (*X, τSI*2 ) *is a locally hypercompact space.*

**Proof.** (1)*⇒* (2) For any *x* and any SI2-open *U* containing *x*, by Proposition [4.8](#_bookmark21),

there exists *E ∈ X*(*<ω*) such that *x ∈ intτ*

*SI*2

*↑ E* =*⇑SI*2

*E ⊆↑ E ⊆ U* .

(2)*⇒* (1) For any *x ∈ X*, let *H* = *{H ∈ X*(*<ω*) : *x ∈ intτ*

*SI*2

*↑ H}*. First, we

show that *H* is nonempty and *↑ x* = *H∈H ↑ H*. Since *X* is SI2-open, it follows

from (2) that there exists *H ∈ X*(*<ω*) such that *x ∈ intτ*

*SI*2

*↑ H ⊆↑ H ⊆ U* . Then

*H ∈H /*= *∅*. Obviously, *↑ x ⊆ H∈H ↑ H*. If *x* ¢ *y*, then *x ∈ X\↓ y ∈ τSI*2 . By (2),

there exists *H ∈ X*(*<ω*) such that *x ∈ intτ*

*SI*2

*↑ H ⊆↑ H ⊆ X\ ↓ y*. Thus *H ∈ H*

and *y ∈/↑ H*. Thus *↑ x* = *H∈H ↑ H*.

Now we show that *H* is directed. Let *H*1*, H*2 *∈ H*. Then *x ∈ intτSI ↑ H*1 *∩*

2

*intτ*

*SI*2

*↑ H*2. It follows from (2) that there exists *H ∈ X*(*<ω*) such that *x ∈ intτ*

*SI*2 *↑*

*H ⊆↑ H ⊆ intτSI*

2

2

*↑ H*1 *∩ intτSI*

*↑ H*2 *⊆↑ H*1*∩ ↑ H*2, so *H ∈ H* and *H*1*, H*2 *≤ H*.

Thus *H* is directed. Obviously, *H⊆* fin(*x*). Then by Lemma [4.6,](#_bookmark19) it is easy to show

*H∈H*

that fin(*x*) is directed, and *↑ x ⊆*

*H∈*fin(*x*)

*↑ H ⊆*

*↑ H* =*↑ x*.

Finally, we show that *⇑SI*2

*E* is open in (*X, τ* ) for any *E ∈ X*(*<ω*). For any

*x ∈⇑SI*2 *E*, *E SI*2 *x*. Notice that *↑ x* = *H∈H ↑ H* and *H* is directed, by

Lemma [4.6](#_bookmark19), there exists *H ∈ H* such that *H ⊆↑ E*. Thus *x ∈ intτSI*

2

*↑ H ⊆*

*intτSI ↑ E ⊆⇑SI*2 *E*, which implies *⇑SI*2 *E* is open in (*X, τ* ). Therefore, (*X, τ* ) is

2

SI2-quasicontinuous. *2*

By Theorem [3.13](#_bookmark15) and Theorem [4.9](#_bookmark22), we have the following corollary.

**Corollary 4.10** *If* (*X, τ* ) *is SI*2*-continuous, then* (*X, τ* ) *is SI*2*-quasicontinuous.*

**Corollary 4.11** *Let P be a poset. Then the following conditions are equivalent.*

1. *P is a quasi-hypercontinuous poset;*
2. (*P, υ*(*P* )) *is an SI*2*-quasicontinuous space.*

**Proof.** By Lemma [3.7](#_bookmark10) and Theorem [4.9](#_bookmark22).

*2*

**Lemma 4.12** *Let P be a poset. If* (*P, σ*2(*P* )) *is a locally hypercompact space, then*

*σ*2(*P* )*SI*2 = *σ*2(*P* )*.*

**Proof.** Obviously, *σ*2(*P* )*SI*2 *⊆ σ*2(*P* ). Let *U ∈ σ*2(*P* ) and *F ∈ Irrσ*2(*P* )(*P* ), if *Fδ ∩U /*= *∅*. By Lemma [3.5](#_bookmark8), there exists directed subset *D ⊆↓ F* such that *Dδ* = *Fδ*. Thus *Dδ ∩U /*= *∅*. Since *U ∈ σ*2(*P* ), *D ∩U /*= *∅*, which implies *F ∩U /*= *∅*. Therefore *U ∈ σ*2(*P* )*SI*2 . *2*

**Corollary 4.13** *Let P be a poset. Then the following conditions are equivalent.*

1. *P is an s*2*-quasicontinuous poset;*
2. (*P, σ*2(*P* )) *is a locally hypercompact space;*
3. (*P, σ*2(*P* )) *is an SI*2*-quasicontinuous space;*
4. (*P, A*(*P* )) *is an SI*2*-quasicontinuous space.*

**Proof.** (1)*⇔*(2) By Theorem [2.6](#_bookmark2).

* 1. *⇒*(3) Let (*P, σ*2(*P* )) be a locally hypercompact space. By Lemma [4.12](#_bookmark24), we have *σ*2(*P* )*SI*2 = *σ*2(*P* ). Thus (*P, σ*2(*P* )*SI*2 ) is a locally hypercompact space. By Theorem [4.9](#_bookmark22), (*P, σ*2(*P* )) is an SI2-quasicontinuous space.
  2. *⇒*(1) Let (*P, σ*2(*P* )) be an SI2-quasicontinuous space. It is easy to see that fin(*x*) *⊆ {E ∈ P* (*<ω*) : *E* 2 *x}*. Thus *P* is an s2-quasicontinuous poset.

(2)*⇔*(4) By Lemma [3.7](#_bookmark10)(2), we have *A*(*P* )*SI*2 = *σ*2(*P* ). From Theorem [4.9](#_bookmark22), it follows that (2) and (4) are equivalent. *2*

**Example 4.14** ([[4](#_bookmark31)]) Let *P* = *{a}∪{an* : *n ∈ N}*. The partial order on *P* is defined by setting *an < an*+1 for all *n ∈ N* , and *a*1 *< a*. Then *P* is an s2-quasicontinuous poset which is not s2-continuous. By Theorem [3.13](#_bookmark15) and Corollary [4.13](#_bookmark25), This poset *P* equipped with the Alexandroff topology *A*(*P* ) is an SI2-quasicontinuous space, but it is not SI2-continuous.

# Meet SI2-continuous spaces

In this section, we define a meet SI2-continuous space and prove that (*X, τ* ) is SI2-continuous if and only if it is a meet SI2-continuous and SI2-quasicontinuous space.

**Definition 5.1** A *T*0 space (*X, τ* ) is called *meet SI*2*-continuous* if for any *x ∈ X*

and any *F ∈ Irrτ* (*X*) with *x ∈ Fδ*, then *x ∈ clτ* (*↓ x∩↓ F* ).

*SI*2

**Proposition 5.2** *Let* (*X, τ* ) *be a T*0 *space. Considering the following statements.*

* 1. *For any x ∈ X and U ∈ τSI*2 *, ↑* (*U∩↓ x*) *∈ τSI*2 *.*
  2. (*X, τ* ) *is meet SI*2*-continuous.*

*Then* (1)*⇒*(2)*. If* (*X, τ* ) *satisﬁes ↑* (*U∩↓ x*) *∈ τ for any x ∈ X and U ∈ τSI*2 *, then the two conditions are equivalent.*

**Proof.** (1)*⇒*(2) Let *F* be an irreducible set in (*X, τ* ) with *x ∈ Fδ*. Suppose *x /∈*

*clτSI* (*↓ x∩ ↓ F* ). Then there exists *U ∈ τSI*2 containing *x* such that *U ∩* (*↓ x∩ ↓*

2

*F* ) = *∅*, thus *↑* (*U∩ ↓ x*)

*F* = *∅*. By hypothesis *↑* (*U∩ ↓ x*) *∈ τSI*2 , we have

*↑* (*U∩↓ x*) *∩ Fδ* = *∅*. But *x ∈↑* (*U∩↓ x*) *∩ Fδ*, a contradiction. Thus (*X, τ* ) is meet

SI2-continuous.

(2)*⇒*(1) For any *F ∈ Irrτ* (*X*), if *↑* (*U∩↓ x*) *∩ Fδ /*= *∅*, then there exists *y ∈ Fδ*

and *u ∈ U∩ ↓ x* such that *u ≤ y*, which implies *u ∈ Fδ*. By (2), *u ∈ clτ*

*SI*2

(*↓ u∩ ↓*

*F* ). Note that *U ∈ τSI*2 and *u ∈ U* , thus *U ∩* (*↓ u∩ ↓ F* ) */*= *∅*. It follows that

*U ∩* (*↓ x∩ ↓ F* ) */*= *∅*. Hence *↑* (*U∩ ↓ x*) *∩ F /*= *∅*. Since *↑* (*U∩ ↓ x*) *∈ τ* , we have

*↑* (*U∩↓ x*) *∈ τSI*2 . *2*

**Corollary 5.3** *Let P be a poset. The following statements are equivalent.*

1. *For any x ∈ P and any U ∈ σ*2(*P* )*, ↑* (*U∩↓ x*) *∈ σ*2(*P* )*;*
2. *P is meet s*2*-continuous;*
3. (*P, A*(*P* )) *is meet SI*2*-continuous.*

**Proof.** (1)*⇔*(2) See [[5,](#_bookmark33)[21](#_bookmark48)].

* 1. *⇔*(3) Since any upper set is open in *A*(*P* ) and *A*(*P* )*SI*2 = *σ*2(*P* ), by Propo- sition [5.2](#_bookmark26), the condition (1) and (3) are equivalent. *2*

**Lemma 5.4** *If H is a ﬁnite set in a meet SI*2*-continuous space* (*X, τ* )*, then intτSI ↑*

*H ⊆* S*{⇑*

*SI*2

2

*x* : *x ∈ H}.*

**Proof.** Suppose *y ∈ U* := *intτ*

*SI*2

2

*↑ H* but *y /∈* S*{⇑SI*

*x* : *x ∈ H}*. Let *H* =

*{x*1*, x*2*, ..., xn}*. For each *i* there exists *Fi ∈ Irrτ* (*X*) such that *y ∈ Fδ* with *xi /∈↓ Fi*.

*i*

Since (*X, τ* ) is meet SI2-continuous, *y ∈ clτSI* (*↓ y∩ ↓ Fi*). Choose *z*1 *∈ U∩ ↓*

2

*y∩ ↓ F*1 */*= *∅*. Since *y ∈ Fδ* and *z*1 *≤ y*, we have *z*1 *∈ Fδ*, which implies that

2

2

*z*1 *∈ clτSI* (*↓ z*1*∩ ↓ F*2). Choose *z*2 *∈ U∩ ↓ z*1*∩ ↓ F*2 */*= *∅*. Thus we can get *zi*+1 *∈*

2

*U∩↓ z F*

*,i* = 1*,* 2*,...,n−* 1, and *z*

*∈* *n*

*↓ F* . Note that *z*

*∈ U ⊆↑ H*, so

*i∩↓*

*i*+1

*n i*=1 *i n*

there exists *xj ∈ H* such that *xj ≤ zn*. Thus *xj ∈↓ Fj*, a contradiction to *xi /∈↓ Fi*

2

for any *i ∈ {*1*,* 2*,..., n}*. Hence *intτ*

*SI*2

*↑ H ⊆* S*{⇑SI*

*x* : *x ∈ H}*. *2*

**Theorem 5.5** *Let* (*X, τ* ) *be a T*0 *space. Then the following conditions are equiva- lent.*

1. (*X, τ* ) *is an SI*2*-continuous space;*
2. (*X, τ* ) *is an SI*2*-quasicontinuous and meet SI*2*-continuous space;*
3. (*X, τ* ) *is a meet SI*2*-continuous space, for any x ∈ X, ⇓SI*2 *x is directed with*

*⇑SI*2 *x ∈ τ and whenever x* ¢ *y in X, then there are U ∈ τSI*2 *and V ∈ ω*(*P* )

*such that x ∈ U, y ∈ V , and U ∩ V* = *∅, where P* = (*X, ≤τ* )*.*

**Proof.** (1)*⇒*(2) By Corollary [4.10](#_bookmark23), (*X, τ* ) is SI2-quasicontinuous. Now we prove that (*X, τ* ) is meet SI2-continuous. Consider any *x ∈ X* and any *F ∈ Irrτ* (*X*)

with *x ∈ Fδ*. For any *U ∈ τSI*

2

with *x ∈ U* , by SI2-continuity of (*X, τ* ), *x ∈*

*U ∩* (*⇓SI*2

*x*)*δ /*= *∅*, which implies *U∩ ⇓SI*

*x /*= *∅*, that is, there exists *u ∈ U* such

that *u SI*2 *x*. Thus there exists *e ∈ F* such that *u ≤ e*, so *u ∈ U∩ ↓ x∩ ↓ F* .

2

Hence *x ∈ clτSI* (*↓ x∩↓ F* ).

2

* 1. *⇒*(3) First, we show that *⇓SI*2 *x* is nonempty for any *x ∈ X*. If not, then

*x /∈* S*{⇑SI*

2

2

*y* : *y ∈ X} ⊆* S*{⇑SI*

*E* : *E ∈ X*(*<ω*)*}*. By Proposition [4.8](#_bookmark21)(1) and

S

Lemma [5.4](#_bookmark27), *⇑SI*2 *E* = *intSI*2 *↑ E ⊆ {⇑SI*2 *y* : *y ∈ E}*. From Proposition [4.8](#_bookmark21)(2),

it is follows that *X* = S*{⇑SI E* : *E ∈ X*(*<ω*)*} ⊆* S*{*S *⇑SI y* : *E ∈ X*(*<ω*) and

S 2 2

*y ∈ E}* = *{⇑SI*2 *y* : *y ∈ X}*, which implies *x /∈ X*, a contradiction. Thus *⇓SI*2 *x* is

nonempty.

Now we show that *⇓SI*2 *x* is directed and *⇑SI*2 *x ∈ τ* . For any *x ∈ X*, let *u, v ∈⇓SI*2 *x*. Since (*X, τ* ) is SI2-quasicontinuous, by Theorem [4.7](#_bookmark20), there are

*E*1*, E*2 *∈ X*(*<ω*) such that *u SI*

2

*E*1 *SI*2

*x, v SI*2

*E*2 *SI*2

1. Thus *E*1*, E*2 *∈*

fin(*x*) and *E*1 *⊆↑ u, E*2 *⊆↑ v*. Since fin(*x*) is directed, there exists *E ∈* fin(*x*) such

that *E*1*, E*2 *≤ E*, i.e., *E ⊆↑ E*1*∩ ↑ E*2. Thus *x ∈⇑SI*2 *E ⊆↑ E ⊆↑ E*1*∩ ↑ E*2. By

Proposition [4.8](#_bookmark21)(1) and Lemma [5.4](#_bookmark27), we have *⇑SI*2 *E* = *intSI*2 *↑ E ⊆* S*{⇑SI y* : *y ∈*

2

*E}*. So there is a *y ∈ E* such that *x ∈⇑SI*2 *y*, i.e., *y ∈⇓SI*2 *x* and *u, v ≤ y*. Thus

*⇓SI*2 *x* is directed. Since (*X, τ* ) is SI2-quasicontinuous, *⇑SI*2 *E ∈ τ* for any finite *E*. Therefore, *⇑SI*2 *x ∈ τ* .

Suppose that *x* ¢ *y* in *X*. Let *P* = (*X, ≤τ* ). Then *x ∈ P\ ↓ y ∈ τSI*2 . By

Theorem [4.9](#_bookmark22), there exists *E ∈ X*(*<ω*) such that *x ∈ intτ*

*SI*2

*↑ E ⊆↑ E ⊆ P\ ↓ y*.

Let *U* = *intτSI ↑ E* and *V* = *P\ ↑ E*. Then *x ∈ U ∈ τSI*2 , *y ∈ V ∈ ω*(*P* ) and

2

*U ∩ V* = *∅*.

* 1. *⇒*(1) We only have to check that *x* = *∨ ⇓SI*2 *x* for all *x ∈ X*. Let *y* be any upper bound of *⇓SI*2 *x*. Assume *x* ¢ *y*. By the condition (3), there are *U ∈ τSI*2 and *V ∈ ω*(*P* ) such that *x ∈ U, y ∈ V* and *U ∩ V* = *∅*. We may assume that *V*

is a basic *ω*-open set, i.e., there exists *H ∈ X*(*<ω*) such that *V* = *P\ ↑ H*. Thus

2

*U ⊆↑ H*. By Lemma [5.4](#_bookmark27), *x ∈ U ⊆ intτ*

*SI*2

*↑ H ⊆* S*{⇑SI* : *x ∈ H}*. Hence, there

is a *z ∈ H* such that *z SI*2 *x*. Thus *z ∈⇓SI*2 *x ⊆↓ y*, which implies *y ∈↑ H*, a

contradiction. Thus (1) holds. *2*

Let *P* be a poset. By Corollary [3.15](#_bookmark16), Corollary [4.13](#_bookmark25) and Theorem [5.5](#_bookmark28), we have the following corollary.

**Corollary 5.6 ([**[**21**](#_bookmark48)**])** *Let P be a poset. Then the following conditions are equiva-*

*lent.*

* 1. *P is an s*2*-continuous poset;*
  2. *P is an s*2*-quasicontinuous and meet s*2*-continuous poset;*
  3. *P is a meet s*2*-continuous space, ⇓ x is directed for any x ∈ P, and whenever x* ¢ *y in X, then there are U ∈ σ*2(*P* ) *and V ∈ ω*(*P* ) *such that x ∈ U, y ∈ V , and U ∩ V* = *∅.*

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# References

1. Abramsky, S., and A.Jung, ”Domain theory”, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science 3. Oxford University Press, Oxford, 1994.
2. Andradi, H., C. Shen, W. K. Ho, and D. S. Zhao, *A New Convergence Inducing the SI-Topology*, Filomat, **32** (2018), 6017C6029.
3. Andradi, H., and W. K. Ho, *A Topological Scott Convergence Theorem*, Logical Methods in Computer Science, **15** (2019), 29:1–29:14.
4. Ern´e, M., *Scott convergence and Scott topology on partially ordered sets II*, In: B. Banaschewski and R.-

E. Hoffman, eds., Continuous Lattices, Bremen 1979, Lecture Notes in Math. vol.871, Springer-Verlag, Berlin-Heidelberg-New York, 1981,61–96.

1. Ern´e, M., *Infinite distributive laws versus local connectedness and compactness properties*, Topology and its Applications, **156** (2009), 2054–2069.
2. Ern´e, M., *A completion-invariant extension of the concept of continuous lattices*, In: Banaschewski,

B. Hoffmann, R.-E. (eds.)Continuous Lattices, Proceedings,Bremen 1979. Lecture Notes in Mathematics,vol.871, Springer,Berlin,1981, 43–60.

1. Gierz, G., and J. D. Lawon, *Gneralized continuous and hypercontinuous lattices*, Rocky Mountain J.Math., **11** (1981), 217–296.
2. Gierz, G., Hofmann, K., Keimel, K., Lawson, J. D., Mislove, M., Scott, D., ”Continuous Lattices and Domains”, Encyclopedia of Mathematics and its Applications, **93**, Cambridge University Press, 2003.
3. Heckmann, R., and K. Keimel, *Quasicontinuous Domains and the Smyth Powerdomain*, Electronic Notes in Theoretical Computer Science, **298** (2013), 215–232.
4. Huang, M. Q., Q. G. Li, and J. B. Li, *Generalized Continuous Posets and a New Cartesian Closed Category*, Applied Categorical Structures, **17** (2009), 29–42.
5. Lawson, J. D.,and L. S. Xu, *Posets having continuous intervals*, Theoretical Computer Science, **316**

(2004), 89–103.

1. Mislove, M., *Local DCPOs, local CPOs and local completions*, Electronic Notes in Theoretical Computer Science, **20** (1999), 1–14.
2. Mao, X. X., and L. S. Xu, *Quasicontinuity of Posets via Scott Topology and Sobrification*, Order, **23**

(2006), 359–369.

1. Xu,X. Q., and M. K. Luo, *Quasi Z-continuous domains and Z-meet continuous domains*, Acta Mathematica Sinica. Chinese Series, **48** (2005), 221–234.
2. Xu, X. Q., ”Relational representations of complete lattices and their applications”, Ph.D. thesis, Sichuan University, China, 2004.
3. Xu, L. S., *Continuity of posets via Scott topology and sobrification*, Topology and its Applications, **153**

(2006), 1886–1894.

1. Ruan, X. J., and X. Q. Xu, *A Completion-Invariant Extension of the Concept of Quasi C-continuous Lattices*, Filomat, **31** (2017), 2345–2353.
2. Shen, C., H. Andradi, D. Zhao, and F. Shi, *SI*2*-topology on T*0 *spaces*, Houston Journal of Mathematics (To appear).
3. Zhao, D. S., and T. Fan, *Dcpo-completion of posets*, Theoretical Computer Science, **411** (2010), 2167– 2173.
4. Zhao, D. S., and W. K. Ho, *On topologies defined by irreducible sets*, Journal of Logical and Algebraic Methods in Programming, **84** (2015), 185–195.
5. Zhang, W. F., and X. Q. Xu, *s*2*-qussiconinuous posets*, Theoretical Computer Science, **574** (2015), 78–85.
6. Zhang, W. F., and X. Q. Xu, *Frink quasicontinuous posets*, Semigroup Forum, Springer US. **94** (2015), 6–16.
7. Zhang, W. F., and X. Q. Xu, *Completely precontinuous posets*, Electronic Notes in Theoretical Computer Science, **301** (2014), 169–178.
8. Zhang, W. F., and X. Q. Xu, *Meet precontinuous posets*, Electronic Notes in Theoretical Computer Science, **301**(2014), 179–188.
9. Zhang, W. F., and X. Q. Xu, *Hypercontinuous Posets*, Chinese Annals of Mathematics, **36B** (2015), 195–200.