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On Tuza’s Conjecture for Triangulations and Graphs with Small Treewidth

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Abstract

Tuza (1981) conjectured that the cardinality *τ* (*G*) of a minimum set of edges that intersects every triangle of a graph *G* is at most twice the cardinality *ν*(*G*) of a maximum set of edge-disjoint triangles of *G*. In this paper we present three results regarding Tuza’s Conjecture. We verify it for graphs with treewidth

at most 6; and we show that *τ* (*G*) *≤* 3 *ν*(*G*) for every planar triangulation *G* different from *K*4; and that

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*τ* (*G*) *≤* 9 *ν*(*G*)+ 1 if *G* is a maximal graph with treewidth 3.

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*Keywords:* Triangle transversal, triangle packing, treewidth, triangulation.

# Introduction

In this paper, we use the term *graph* to refer to a simple graph and the term *multigraph* to refer to a graph that might contain parallel edges. Other than this, the notation and terminology are standard. A *triangle transversal* of a graph *G* is a set of edges of *G* whose removal results in a triangle-free graph; and a *triangle packing* of *G* is a set of edge-disjoint triangles of *G*. We denote by *τ* (*G*) (resp. *ν*(*G*))

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the cardinality of a minimum triangle transversal (resp. maximum triangle packing) of *G*. Tuza [[14](#_bookmark46)] posed the following conjecture.

**Conjecture 1.1 (Tuza, 1981)** *For every graph G, we have τ* (*G*) *≤* 2 *ν*(*G*)*.*

This conjecture was verified for many classes of graphs. In particular, Tuza [[15]](#_bookmark47) verified it for planar graphs, and Haxell and Kohayakawa [[9](#_bookmark37)] proved that if *G* is a tripartite graph, then *τ* (*G*) *≤* 1*.*956 *ν*(*G*). The reader may refer to [[1,](#_bookmark33)[5,](#_bookmark38)[6,](#_bookmark39)[8,](#_bookmark40)[10,](#_bookmark42)[11](#_bookmark43)] for more results concerning Tuza’s Conjecture. In this paper we present three results regarding Tuza’s Conjecture. We verify it for graphs with treewidth at most 6; and we show that *τ* (*G*) *≤* 3 *ν*(*G*) for every planar triangulation *G* different from *K*4; and that *τ* (*G*) *≤* 9 *ν*(*G*) if *G* is a 3*-tree*, i.e., a graph obtained from *K*3 by successively choosing a triangle in the graph and adding a new vertex adjacent to its three vertices.

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Puleo [[13](#_bookmark45)] introduced a set of tools for dealing with graphs that contain vertices of small degree (Lemma [3.3](#_bookmark7)), and verified Tuza’s Conjecture for graphs with maxi- mum average degree less than 7, i.e., for graphs in which every subgraph has average degree less than 7. In this paper, we extend Puleo’s technique (Lemma [3.4](#_bookmark12)) in order to prove Tuza’s Conjecture for graphs with treewidth at most 6 (Theorem [3.5](#_bookmark14)). Note that there are graphs with treewidth at most 6 and maximum average degree at least 7 (Figure [1](#_bookmark1)).

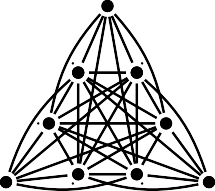


Figure 1. A graph with treewidth 6 and average degree 22*/*3.

In another direction, as suggested in [[9](#_bookmark37)], we show that, for certain classes of graphs, the ratio *τ* (*G*)*/ν*(*G*) can be bounded by a constant smaller than 2. Specifically, we show that, if *G* is a planar triangulation different from *K*4, then *τ* (*G*) *≤* 3 *ν*(*G*) (Theorem [4.5](#_bookmark21)) and, if *G* is a 3-tree, then *τ* (*G*) *≤* 9 *ν*(*G*)+ 1 (Theo-

2 5 5

rem [5.2](#_bookmark27)).

This paper is organized as follows. In Section [2](#_bookmark2), we establish the notation and present some auxiliary results used throughout the paper. In Section [3](#_bookmark5), we verify Tuza’s Conjecture for graphs with treewidth at most 6 and, in Sections [4](#_bookmark15) and [5,](#_bookmark24) we study planar triangulations and 3-trees, respectively. Finally, in Section [6](#_bookmark32) we present some concluding remarks. Due to space limitations, we present only the sketch of some proofs.

# Rooted tree decompositions

In this section we present most of the notation and some auxiliary results regarding tree decompositions. A *rooted tree* is a pair (*T, r*), where *T* is a tree and *r* is a

node of *T* . Given *t ∈ V* (*T* ), if *tj* is a node in the (unique) path in *T* that joins *r* and *t*, then we say that *tj* is an *ancestor* of *t*. Every vertex in *T* that has *t* as its ancestor is called a *descendant* of *t*. If *t /*= *r*, then the *parent* of *t*, denoted by *p*(*t*), is the ancestor of *t* that is adjacent to *t*. The *successors* of *t* are the nodes whose parent is *t*, and we denote the set of successors of *t* by *ST* (*t*). A node of *T* with no successor is a *leaf* of *T* . The *height* of *t*, denoted by *hT* (*t*), is the number of edges of a longest path that joins *t* to a descendant of *t*. When *T* is clear from the context, we simply write *S*(*t*) and *h*(*t*).

**Definition 2.1** A *tree decomposition* of a graph *G* is a pair *D* = (*T, V*) consisting of a tree *T* and a collection *V* = *{Vt ⊆ V* (*G*) : *t ∈ V* (*T* )*}*, satisfying the following conditions:

S

* 1. *{Vt* : *t ∈ V* (*T* )*}* = *V* (*G*);
  2. for every *uv ∈ E*(*G*), there exists a *t ∈ V* (*T* ) such that *u, v ∈ Vt*;
  3. if a vertex *v* is in *Vt*1 *∩ Vt*2 , then *v ∈ Vt* for every *t* in the path of *T* that joins

*t*1 and *t*2.

The *width* of *D* is the number max*{|Vt|−* 1 : *t ∈ V* (*T* )*}*, and the *treewidth tw*(*G*) of *G* is the width of a tree decomposition of *G* with minimum width. Let *G* be a graph with treewidth *k*. If *|Vt|* = *k* +1 for every *t ∈ V* (*T* ), and *|Vt ∩ Vt′ |* = *k* for every *ttj ∈ E*(*T* ), then we say that (*T, V*) is a *full tree decomposition* of *G*.

The following result was proved by Bodlaender [[2](#_bookmark34)] (see also Gross [[7](#_bookmark41)]).

**Proposition 2.2** *Every graph admits a full tree decomposition.*

A triple (*V,T, r*) is a *rooted tree decomposition* of a graph *G* if (*V,T* ) is a full tree decomposition of *G*, (*T, r*) is a rooted tree, and *Vt ∩ Vp*(*t*) */*= *Vt ∩ Vt′* for ev- ery *t ∈ V* (*T* ) *\ {r}* and *tj ∈ S*(*t*). Every full tree decomposition can be modified into a rooted tree decomposition with a root *r*. So the following proposition comes naturally.

**Proposition 2.3** *Every graph admits a rooted tree decomposition.*

Given a rooted tree decomposition (*V,T, r*) of a graph *G* and a node *t ∈ V* (*T* ) *\*

*{r}*, we say that the (unique) vertex in *Vt \ Vp*(*t*) is the *representative* of *t*. We leave undefined the representative of *r*.

We denote by *NG*(*u*) the set of neighbors of *u* for a vertex *u ∈ V* (*G*). When *G* is clear from the context, we simply write *N* (*u*). In what follows, we denote by *N* [*u*] the closed neighborhood *N* (*u*) *∪ {u}* of *u*, by *d*(*u*) the degree of *u*, and by Δ(*G*) the maximum degree of *G*.

**Remark 2.4** If *y* is the representative of a leaf *t* of a rooted tree decomposition of a graph *G*, then *NG*(*y*) *⊆ Vt*.

# Graphs with treewidth at most 6

In this section, we verify Tuza’s Conjecture for graphs with treewidth at most 6 by extending the technique introduced by Puleo [[13](#_bookmark45)], which explores the neighborhood of vertices of small degree in order to decrease the size of the graph studied. For that, we use the following definitions (see also [[13](#_bookmark45)]).

**Definition 3.1** For a graph *G*, a nonempty set *V*0 *⊆ V* (*G*) is called *reducible* if there is a set *X ⊆ E*(*G*) and a set *Y* of edge-disjoint triangles in *G* such that the following conditions hold:

* 1. *|X|≤* 2*|Y |*;
  2. *X ∩ E*(*A*) */*= *∅* for every triangle *A* in *G* that contains a vertex of *V*0; and
  3. if *u, v ∈/ V*0 and *uv ∈ E*(*A*) for some *A ∈ Y* , then *uv ∈ X*.

In this case, we say that (*V*0*, X,Y* ) isa *reducing triple for G*. If *G* has no reducible set, *G* is called *irreducible*.

The following lemma comes naturally (see [[13](#_bookmark45), Lemma 2.2]).

**Lemma 3.2** *Let* (*V*0*, X,Y* ) *be a reducing triple for a graph G, and put Gj* = (*G −*

1. *− V*0*. If τ* (*Gj*) *≤* 2 *ν*(*Gj*)*, then τ* (*G*) *≤* 2 *ν*(*G*)*.*

We say that a graph *G* is *robust* if, for every *v ∈ V* (*G*), each component of *G*[*N* (*v*)] has order at least 5. We often use the following result (see [[13](#_bookmark45), Lemma 2.7]).

**Lemma 3.3** *Let G be an irreducible robust graph and let x, y ∈ V* (*G*)*. The follow- ing statements hold:*

* 1. *if d*(*x*) *≤* 6*, then* Δ(*G*[*N* (*x*)]) *≤* 1 *and |E*(*G*[*N* (*x*)])*| /*= 2*;*
  2. *if d*(*x*) *≤* 6 *and d*(*y*) *≤* 6 *then xy ∈/ E*(*G*)*;*
  3. *if d*(*x*)=7 *and d*(*y*)=6 *then N* [*y*] */⊆ N* [*x*]*; and*
  4. *if d*(*x*) *≤* 8 *and d*(*y*)= 5*, then N* [*y*] */⊆ N* [*x*]*.*

We extend the result above in the following lemma.

**Lemma 3.4** *Let G be an irreducible robust graph and let x, y ∈ V* (*G*)*. If d*(*x*)*, d*(*y*) *≤* 6 *and |N* (*x*) *∪ N* (*y*)*|≤* 7*, then* (i) *d*(*x*)= *d*(*y*)= 5*;* (ii) *|N* (*x*) *∩ N* (*y*)*|* = 3*; and* (iii) *G*[*N* (*x*)] *G*[*N* (*y*)] *K*5*.*

**Proof** The proof consists of the analysis of a series of cases. In each of these cases, we find a reducing triple for *G*, which contradicts the irreducibility of *G*. In this extended abstract, we present only the proof of (i).

By Lemma [3.3](#_bookmark7)([b](#_bookmark8)), we have that *xy ∈/ E*(*G*). Assume, without loss of generality, that *d*(*x*) *≥ d*(*y*). Because *G* is robust, *d*(*y*) *≥* 5. For a contradiction, suppose that *d*(*x*) = 6, and let *N* (*x*)= *{v*1*, v*2*, v*3*, v*4*, v*5*, v*6*}*. In what follows, we analyze the case in which *d*(*x*)= *d*(*y*) = 6; the case in which *d*(*x*) = 6 and *d*(*y*) = 5 is similar.

(a)

*v*2 *v*1

*x*

*y*

*v*3 *v*6

(b)

*v*1 *v*7 (c)

*v*1 *v*7 (d)

*v*1 *v*7

*v*4 *v*5

*x v*2 *v*3

*v*6 *y x v*2

*v*5 *v*3

*v*4

*v*6 *y x v*2

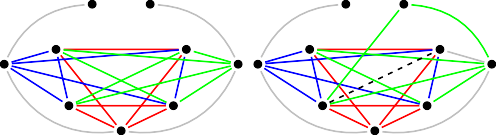
*v*5 *v*3

*v*4

*v*6 *y*

*v*5

*v*4

Figure 2. Illustrations for the proof of Lemma [3.4.](#_bookmark12) The triangles in *Y* containing *x* and *y* are illustrated, respectively, in blue and green, while the triangles in *Y* containing neither *x* nor *y* are illustrated in red. Dashed edges illustrate edges that may not exist.

Because *|N* (*x*) *∪ N* (*y*)*| ≤* 7, we may assume that *N* (*y*) *⊇ {v*2*, v*3*, v*4*, v*5*, v*6*}*. By Lemma [3.3](#_bookmark7)([a](#_bookmark9)), without loss of generality, we may assume that *E*(*G*[*N* (*x*)]) *⊆*

*{v*1*v*4*, v*2*v*5*, v*3*v*6*}*.

**Case 1.** *N* (*y*)= *N* (*x*).

Let *X* = *E*(*G*[*{v*1*, v*2*, v*3*, v*4*, v*5*, v*6*}*]) and *Y* =

*{v*1*v*3*v*5*, v*2*v*4*v*6*, xv*1*v*2*, yv*2*v*3*, xv*3*v*4*, yv*4*v*5*, xv*5*v*6*, yv*1*v*6*}*. Then *|X| ≤* 15 *≤* 16 = 2*|Y |* and (*{x, y}, X,Y* ) is a reducing triple for *G* (Figure [2](#_bookmark13)(a)), a contradiction.

**Case 2.** *N* (*y*) */*= *N* (*x*).

Then *N* (*y*)= *{v*2*, v*3*, v*4*, v*5*, v*6*, v*7*}* for some *v*7 *∈ V* (*G*) *\{v*1*}*. By Lemma [3.3](#_bookmark7)([a](#_bookmark9)), without loss of generality, we may assume that *E*(*G*[*N* (*y*)]) *⊆ {v*2*v*5*, v*3*v*6*, v*4*v*7*}*. Let *H* = *G*[*{v*2*, v*3*, v*4*, v*5*, v*6*}*]. We have three cases.

*Subcase 2.1. |E*(*H*)*|* = 0.

Let *X* = *{xv*1*, yv*7*} ∪ E*(*H*) and *Y* =

*{v*2*v*4*v*6*, v*3*v*4*v*5*, xv*2*v*3*, xv*5*v*6*, yv*2*v*5*, yv*3*v*6*}*. Therefore

*|X|* =2 + *|E*(*H*)*|* = 12= 2*|Y |* and (*{x, y}, X,Y* ) is a reducing triple for *G*

(Figure [2](#_bookmark13)(b)), a contradiction.

*Subcase 2.2. |E*(*H*)*|* = 1.

Without loss of generality, we may assume that *v*2*v*5 *∈ E*(*H*). Let *X* = *{xv*1*, yv*7*, v*3*v*7*}∪ E*(*H*) and *Y* = *{v*2*v*4*v*6*, v*3*v*4*v*5*, xv*2*v*3*, xv*5*v*6*, yv*2*v*5*, yv*3*v*7*}*. Hence *|X|* = 3 + *|E*(*H*)*|* = 12 = 2*|Y |* and (*{x, y}, X,Y* ) is a reducing triple for *G* (Figure [2](#_bookmark13)(c)), a contradiction.

*Subcase 2.3. E*(*H*)= *{v*2*v*5*, v*3*v*6*}*.

Let *X* = *{xv*1*, yv*7*}∪ E*(*H*) and *Y* = *{v*2*v*4*v*6*, xv*2*v*3*, yv*3*v*4*, xv*4*v*5*, yv*5*v*6*}*. So

*|X|* = 2 + *|E*(*H*)*|* = 10 = 2*|Y |* and (*{x, y}, X,Y* ) is a reducing triple for *G* (Fig- ure [2](#_bookmark13)(d)), a contradiction. *2*

Using Lemmas [3.3](#_bookmark7) and [3.4](#_bookmark12), we verify Tuza’s Conjecture for graphs with treewidth at most 6.

**Theorem 3.5** *If G is a graph with treewidth at most* 6*, then τ* (*G*) *≤* 2 *ν*(*G*)*.*

**Proof** Suppose, for a contradiction, that the statement does not hold, and let *G* be a graph with treewidth at most 6 and *τ* (*G*) *>* 2 *ν*(*G*), that minimizes *|V* (*G*)*|*. We claim that *G* is irreducible. Indeed, suppose that *G* has a reducing triple (*V*0*, X,Y* ), and let *Gj* = (*G−X*)*−V*0. Note that *Gj* has treewidth at most 6, and *τ* (*Gj*) *≤* 2 *ν*(*Gj*) by the minimality of *G*. By Lemma [3.2](#_bookmark6), *τ* (*G*) *≤* 2 *ν*(*G*), a contradiction. One can check that *G* is robust by deriving a contradiction to the minimality of *G* if there is

a vertex *v* for which *G*[*N* (*v*)] contains a component with at most four vertices. So

*G* is irreducible and robust.

Suppose that *|V* (*G*)*| ≤* 7. Then Δ(*G*) *≤* 6, and *E*(*G*) = *∅* by Lemma [3.3](#_bookmark7)([b](#_bookmark8)). Hence *τ* (*G*) *≤* 2 *ν*(*G*), a contradiction. Therefore *|V* (*G*)*| ≥* 8. By Proposi- tion [2.3](#_bookmark3), *G* has a rooted tree decomposition (*T, V, r*). Because *|V* (*G*)*| ≥* 8, we have that *|V* (*T* )*| >* 1, and hence there is a node *t ∈ V* (*T* ) with *h*(*t*)= 1. If *t /*= *r*, let *vt* be the representative of *t*; otherwise, let *vt* be an arbitrary vertex of *Vt*.

First, suppose that *S*(*t*)= *{tj}* and let *x* be the representative of *tj*. We have that *d*(*x*) *∈ {*5*,* 6*}* by Remark [2.4](#_bookmark4) and because *G* is robust. Re- mark [2.4](#_bookmark4) applied to *G − x* and *t* implies that *NG—x*(*vt*) *⊆ Vt*, there- fore *d*(*vt*)= *|N* (*vt*)*|≤ |NG—x*(*vt*)*|* +1 *≤ |Vt|≤* 7. If *d*(*vt*) = 7, then *vt* must be adjacent to *x* and every vertex in *Vt \ {vt}*. Because *Vt′ ⊆ Vt ∪ {x}*, we have that *N* [*x*] *⊆ N* [*vt*], a contradiction either to Lemma [3.3](#_bookmark7)([c](#_bookmark10)) or to Lemma [3.3](#_bookmark7)([d](#_bookmark11)). So *d*(*vt*) *≤* 6 and, by Lemma [3.3](#_bookmark7)([b](#_bookmark8)), *vtx ∈/ E*(*G*). Thus *N* (*x*) *∪ N* (*vt*) *⊆ Vt \ {vt}*, and hence *|N* (*x*)*∪N* (*vt*)*|≤ |Vt|−*1 *≤* 6. On the other hand, Lemma [3.4](#_bookmark12) implies that *d*(*x*)= *d*(*vt*)=5 and *|N* (*x*) *∩ N* (*vt*)*|* = 3, which implies that *|N* (*x*) *∪ N* (*vt*)*|* = 7, a contradiction.

Therefore *|S*(*t*)*| >* 1. Now suppose that *|S*(*t*)*|≥* 3, and let *t*1, *t*2, *t*3 *∈ S*(*t*). Let *zi* be the representative of *ti* for *i* = 1*,* 2*,* 3. By Re- mark [2.4](#_bookmark4), *|N* (*zi*) *∪ N* (*zj*)*|≤ |Vt|* = 7, for *i, j ∈ {*1*,* 2*,* 3*}* and *i /*= *j*. Thus, by Lemma [3.4](#_bookmark12), *d*(*zi*)=5 and *G*[*N* (*zi*)] *K*5 for every *i ∈ {*1*,* 2*,* 3*}*, and *|N* (*zi*) *∩ N* (*zj*)*|* = 3, for *i, j ∈ {*1*,* 2*,* 3*}* and *i /*= *j*. Let *N* (*z*1) = *{v*1*, v*2*, v*3*, v*4*, v*5*}*. Be- cause *|N* (*z*1) *∩N* (*z*2)*|* = 3, we may assume, without loss of generality, that *N* (*z*2)=

*{v*3*, v*4*, v*5*, v*6*, v*7*}*. It is not hard to check that *N* (*z*3) contains exactly one vertex in *N* (*z*1) *∩ N* (*z*2), because *|N* (*z*1) *∩ N* (*z*3)*|* = *|N* (*z*2) *∩ N* (*z*3)*|* = 3 and *|N* (*z*3)*|* = 5. So we may assume, without loss of generality, that *N* (*z*3) = *{v*1*, v*2*, v*3*, v*6*, v*7*}*. Let *H* = *G*[*{v*1*, v*2*, v*3*, v*4*, v*5*, v*6*, v*7*}*] and note that every pair of vertices of *H* is contained in at least one *N* (*zi*) for *i ∈ {*1*,* 2*,* 3*}*. Thus, since *G*[*N* (*zi*)] *K*5 for *i ∈ {*1*,* 2*,* 3*}*, we have *H K*7. Let *X* = *E*(*H*) and note that *|X|* = 21. Put

and

*Y*1 = *{z*1*v*1*v*4*, z*1*v*2*v*5*, z*2*v*3*v*4*, z*2*v*5*v*6*, z*3*v*1*v*6*, z*3*v*2*v*3*}*

*Y*2 = *{v*1*v*2*v*7*, v*2*v*4*v*6*, v*3*v*6*v*7*, v*4*v*5*v*7*, v*1*v*3*v*5*},*

and *Y* = *Y*1 *∪ Y*2. Note that *|Y |* = 11 and thus *|X|≤* 2*|Y |*, hence (*{z*1*, z*2*, z*3*}, X,Y* ) is a reducing triple for *G* (Figure [3](#_bookmark16)), a contradiction. We conclude that *|S*(*t*)*|* = 2. Let *t*1*, t*2 *∈ S*(*t*) and let *x, y* be the representatives of *t*1*, t*2, respectively. Again, *d*(*x*)= *d*(*y*)= 5, *|N* (*x*) *∩ N* (*y*)*|* = 3, and *G*[*N* (*x*)]*, G*[*N* (*y*)] *K*5 by Remark [2.4](#_bookmark4) and Lemma [3.4](#_bookmark12). Hence *Vt ⊆ N* (*x*) *∪ N* (*y*). Note that *t* is a leaf of (*T j, Vj, r*),

where *T j* = *T − t*1 *− t*2 and *Vj* = *V\ {Vt*1 *, Vt*2 *}*. Therefore *dG—x—y*(*vt*) *≤* 6 by Re- mark [2.4](#_bookmark4), and *dG*(*vt*) *≤* 8. Note that *vt ∈ N* (*x*) *∪ N* (*y*). Without loss of generality, assume that *vt ∈ N* (*x*). As *G*[*N* (*x*)] is a complete graph, *N* [*x*] *⊆ N* [*vt*], a contra- diction to Lemma [3.3](#_bookmark7)([d](#_bookmark11)). This concludes the proof. *2*

Because a *K*8-free chordal graph has treewidth at most 6, Theorem [3.5](#_bookmark14) general-

izes a result of Tuza regarding chordal graphs [[15](#_bookmark47), Proposition 3].

**Corollary 3.6** *If G is a K*8*-free chordal graph, then τ* (*G*) *≤* 2 *ν*(*G*)*.*

# Planar triangulations

All new definitions in this section are taken from the book of Bondy and Murty [[3,](#_bookmark35) Chapter 10]. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. We refer to such a drawing of a planar graph as a *plane graph*. A plane graph partitions the plane into arcwise-connected open sets, which we call *faces*. The boundary of a face *f* in a plane graph is the boundary of the open set *f* in the usual topological sense. Two faces are adjacent if their boundaries have an edge in common. The *dual graph G∗* of a given plane graph *G* is the multigraph whose vertex set is the set of faces of *G*, and in which two vertices are joined by *k* edges if the boundaries of the corresponding faces have *k* edges in common. Note that if *G∗* has a bridge, then *G* contains a loop. Therefore, the dual of a simple plane graph has no bridges. A simple connected plane graph in which all faces have three edges is called a *plane triangulation*. A *planar triangulation* is a graph which admits a plane triangulation. It is not hard to check that a simple connected plane graph is a triangulation if and only if its dual is a bridgeless cubic multigraph. Moreover, the dual of a triangulation has parallel edges only if the triangulation is a single triangle. Also, every planar triangulation has a unique plane triangulation, thus we will refer to its faces and its dual. A *facial triangle* in a plane graph is a triangle that is the boundary of one of its faces.

In this section, we prove that *τ* (*G*)*/ν*(*G*) *≤* 3*/*2 for every planar triangulation *G* different from *K*4. It is not hard to check that if *G* = *K*5 *−e*, then *τ* (*G*)*/ν*(*G*)= 3*/*2. Therefore, this bound is tight. If *G* is a triangle, then *τ* (*G*)= *ν*(*G*) = 1. So we may assume that *G* is not *K*3. The proof is divided in two parts. In Lemma [4.2](#_bookmark17), we give an upper bound for *τ* (*G*) and, in Lemma [4.4](#_bookmark19), we give a lower bound for *ν*(*G*). For the proof of Lemma [4.2](#_bookmark17), we need the following theorem of Petersen (see [[12](#_bookmark44)]).

**Theorem 4.1 (Petersen, 1981)** *Every bridgeless cubic graph contains a perfect matching.*

**Lemma 4.2** *If G is a planar graph, then τ* (*G*) *≤ n −* 2*.*

*z*1

*v*3

*v*4

*v*2

*v*5

*v*1

*z*2

*v*6 *v*7

*z*3

Figure 3. Illustration for the proof of Theorem [3.5.](#_bookmark14) The triangles in *Y*1 containing *z*1, *z*2, and *z*3 are illustrated, respectively, in blue (dotted), green (dashed), and cyan (dash-dotted), while the triangles in *Y*2 are illustrated in red (solid).

**Proof** We may assume *n ≥* 4. Let *H* be a maximal plane graph containing *G* and note that *τ* (*G*) *≤ τ* (*H*). By Euler’s formula, *H* has 2*n −* 4 faces. Because *H* is a plane triangulation, *H∗* is a bridgeless cubic graph on 2*n −* 4 vertices. Thus, by Theorem [4.1](#_bookmark18), *H∗* contains a perfect matching *M∗*. Let *M* be the edges of *H* corresponding to the edges in *M∗*. Note that *|M|* = *|M∗|* = *n −* 2. Since *H∗ − M∗* contains only vertices with even degree, *H − M* is a bipartite graph. Therefore *H − M* has no triangles, and hence *τ* (*H*) *≤ |M|* = *n −* 2. *2*

To prove a lower bound on *ν*(*G*), we use the well-known Brook’s Theorem (see [[4](#_bookmark36)]). The chromatic number of a graph *G* is denoted by *χ*(*G*).

**Theorem 4.3 (Brooks, 1941)** *If G is a connected graph, then χ*(*G*) *≤* Δ(*G*) *or*

*G is either an odd cycle or a complete graph.*

**Lemma 4.4** *If G is a planar triangulation different from K*4*, then ν*(*G*) *≥* 2 (*n−*2)*.*

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**Proof** We may assume *G /*= *K*3. The dual *G∗* of *G* is a cubic graph different from *K* . By Theorem [4.3](#_bookmark20), there is an independent set *Y ∗* of *G∗* with at least *|V* (*G∗* )*|*

4 3

vertices. Let *Y* be the facial triangles of *G* corresponding to the vertices in *Y ∗*.

As *Y ∗* is an independent set in *G∗*, *Y* is a triangle packing in *G*. Thus *ν*(*G*) *≥*

*|Y |* = *|Y ∗|≥ |V* (*G∗* )*|* = 2 (*n −* 2). *2*

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The main result of this section comes directly from Lemmas [4.2](#_bookmark17) and [4.4](#_bookmark19).

**Theorem 4.5** *If G is a planar triangulation different from K*4*, then τ* (*G*) *≤*

3 *ν*(*G*)*.*

2

We were also able to obtain the following slightly stronger version of Lemma [4.4](#_bookmark19) for a (finite) class of planar 3-trees. This result will be used in the next section. Given a copy *H* of *K*4 in a planar 3-tree *G*, we denote by *g*(*H*) the pair (*i, j*) where *i* (resp. *j*) is the maximum (resp. minimum) number of vertices of *G* inside a face of *H*. We say that a planar 3-tree *G* is *restricted* if *G* contains a copy *H* of *K*4 such that *g*(*H*)= (2*,* 0). In this case, we always consider the face of *H* with no vertices of *G* inside as the external face of *G*, that is, the vertices of *G* not in *H* are drawn in the internal faces of *H*. If, additionally, *H* has two faces with precisely one vertex of *G* inside, then we say that *G* is *super restricted*.

**Proposition 4.6** *If G is a restricted (resp. super restricted) planar* 3*-tree with f faces, then there is a triangle packing P of facial triangles of G containing the external face, and such that |P|≥ [f /*3*| (resp. |P|* = 5*).*

**Proof** Let *v∗* be the vertex of *G∗* corresponding to the external face of *G*, and let *Gj* = *G∗ − N* [*v∗*]. Since *G∗* is a cubic graph, *Gj / K*4. By Theorem [4.3](#_bookmark20), *Gj* contains an independent set *Ij* with *[*(*f−*4)*/*3*|* vertices, and hence the set *P* of facial triangles of *G* corresponding to the vertices in *Ij ∪{v∗}* is a triangle packing of facial triangles of *G* containing the external face, and such that *|P|≥ [*(*f −* 4)*/*3*|* + 1. Hence, if *f −* 1 */≡* 0 (mod 3), then *[*(*f −* 4)*/*3*|* +1 = *[f/*3*|*, and *P* is the desired triangle packing. Thus we may assume that *f −* 1 *≡* 0 (mod 3) and, because *G* is restricted, *G* is one of the ten first graphs in Figure [4](#_bookmark23), which have either 10 faces

and at least 4 independent faces (counting the external face), or 16 faces and at least 6 independent faces (counting the external face). If *G* is super restricted, then *G* is one of the two last graphs in Figure [4.](#_bookmark23) *2*



Figure 4. All 10 restricted planar 3-trees with *f ≡* 1 (mod 3) faces and the two super restricted ones.

In the full version of the paper, we show that Proposition [4.6](#_bookmark22) cannot be extended for general planar 3-trees.

# 3-Trees

In this section, we prove that *τ* (*G*) *≤* 9 *ν*(*G*) + 1

for every 3-tree *G*. Given

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a graph *G*, a transversal *X* and a triangle packing *Y* of *G*, we say that the

pair (*X, Y* ) is a 9 -*TP* of *G* if *|X| ≤* 9 *|Y |* + 1 . Note that if *G* has a 9 -TP,

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then *τ* (*G*) *≤* 9 *ν*(*G*)+ 1 . Let (*T, V, r*) be a rooted tree decomposition of a graph

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*G*. Given a node *t ∈ V* (*T* ) *\ {r}*, we denote by *R*(*t*) the set of all representatives

of all descendants of *t*. Note that the representative of *t* is also in *R*(*t*). Re- call that *S*(*t*) is the set of successors of *t*. For every triple of vertices Δ *⊆ Vt*, let *S*Δ(*t*)= *{tj ∈ S*(*t*): *Vt′ ∩ Vt* = Δ*}*. When *t* is clear from the context, we simply write *S*Δ.

Our proof relies on the analysis of nodes with small height, which guarantees that a minimal counterexample has a particular configuration. For nodes with height 1 we present the following lemma.

**Lemma 5.1** *Let G be a 3-tree such that τ* (*G*) *>* 9 *ν*(*G*)+ 1 *, and G is minimal over*

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*all such graphs. Let* (*T, V, r*) *be a rooted tree decomposition of G of width 3, and*

*let t ∈ V* (*T* ) *\ {r}. If h*(*t*)= 1*, then |S*(*t*)*|* = 1*.*

**Proof** Let *Vt* = *{a, b, c, d}*. Suppose, for a contradiction, that *|S*(*t*)*| >* 1. Without loss of generality, assume that *Vt ∩ Vp*(*t*) = *{b, c, d}*. Let *S*(*t*) = *{t*1*, t*2*,..., tk}*. For every *i ∈* [*k*], let *vi* be the representative of *ti*. Since *h*(*t*) = 1, at least one of Δ in *{abc, abd, acd}* is such that *S*Δ */*= *∅*. Suppose that exactly one triangle Δ in *{abc, abd, acd}*, say Δ = *abc*, is such that *S*Δ */*= *∅*. Let *Gj* = *G − R*(*t*). Note that *Gj* is a 3-tree. By the minimality of *G*, there exists a 9 -TP of *Gj*, say (*Xj,Y j*). Thus, (*Xj ∪ {ab, bc, ac},Y j ∪ {acv*1*, abv*2*}*) is a 9 -TP of *G*, a contradiction. In-

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deed, *τ* (*G*) *≤ τ* (*Gj*)+3 *≤* 9 *ν*(*Gj*)+ 16 *≤* 9 (*ν*(*G*) *−* 2) + 16 *≤* 9 *ν*(*G*)+ 1

(Fig-

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ure [5](#_bookmark26)(a)). Thus, we may assume that at most one triangle Δ in *{abc, abd, acd}* is

such that *S*Δ = *∅*. Suppose that *|S*(*t*)*|* = 2 and assume, without loss of generality, that *t*1 *∈ Sabc* and *t*2 *∈ Sabd*. Let *Gj* = *G − R*(*t*). Again *Gj* is a 3-tree. By the min- imality of *G*, there exists a 9 -TP of *Gj*, say (*Xj,Y j*). Let *e ∈ Xj ∩ E*(*bcd*). Without loss of generality, we have two cases. If *e* = *bc*, then we let *X* = *Xj ∪ {ad, av*1*, bv*2*}*. If *e* = *cd*, then we let *X* = *Xj ∪ {ab, cv*1*, dv*2*}*. In both cases, (*X, Y j ∪ {acv*1*, abv*2*}*) is a 9 -TP of *G*, a contradiction (Figure [5](#_bookmark26)(b)). Finally suppose that *|S*(*t*)*| ≥* 3.

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Without loss of generality, assume that either *t*1 *∈ Sabc* and *t*2*, t*3 *∈ Sabd*, or *t*1 *∈ Sabc*, *t*2 *∈ Sabd*, and *t*3 *∈ Sacd*. Let *Gj* = *G − R*(*t*). By the minimal- ity of *G*, there exists a 9 -TP of *Gj*, say (*Xj,Y j*). Note that *E*(*bcd*) *∩ Xj /*= *∅*. Hence (*Xj ∪ {ab, bc, cd, ac, bd, ad},Y j ∪ {acv*1*, abv*2 *adv*3*}*) isa 9 -TP of *G*,a contra- diction (Figure [5](#_bookmark26)(c)). This concludes the proof. *2*

5

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*b b b*

(c)



(a)

*a*

*c*



(b)

*a*

*c*



*a*

*d d c d*

Figure 5. Illustrations for the proof of Lemma [5.1](#_bookmark25).

In what follows, we prove the main theorem of this section.

**Theorem 5.2** *If G is a 3-tree, then τ* (*G*) *≤* 9 *ν*(*G*)+ 1 *.*

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**Proof** Suppose, for a contradiction, that the statement does not hold. Let *G* be a minimal counterexample over all 3-trees with *τ* (*G*) *>* 9 *ν*(*G*)+ 1 . We claim

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that if (*T, V, r*) is a rooted tree decomposition of *G* of width 3, then *|V* (*T* )*| ≥* 4.

Indeed, if *|V* (*T* )*|* = 1, then *|V* (*G*)*|* = 4, *τ* (*G*) = 2, and *ν*(*G*) = 1. If *|V* (*T* )*|* = 2, then *|V* (*G*)*|* = 5, *τ* (*G*) = 3, and *ν*(*G*) = 2. If *|V* (*T* )*|* = 3, then *|V* (*G*)*|* = 6, *τ* (*G*) *≤* 4, and *ν*(*G*) = 3. In each case, we contradict *τ* (*G*) *>* 9 *ν*(*G*)+ 1 . Due to

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space limitations, we omit the proof of the next claim.

**Claim 5.3** *There exists a rooted tree decomposition* (*T, V, r*) *of G of width 3 with a node t ∈ V* (*T* ) *\ {r} such that h*(*t*)= 2*.*

Let (*T, V, r*) be a rooted tree decomposition of *G* given by Claim [5.3](#_bookmark28), and let *t* be a node in *T* with *h*(*t*) = 2. Let *L* = *{l*1*,..., lm}* be the set of successors of *t* that are leaves, and let *Q* = *S*(*t*) *\ L* = *{q*1*,..., qk}*. For every *i ∈* [*m*], let *ui* be the representative of *li*. By Lemma [5.1](#_bookmark25), *|S*(*qi*)*|* =1 for every *i ∈* [*k*]. For every such *i*, let *S*(*qi*)= *{qj}*, and let *vi* be the representative of *qi* and *vj* be the representative

*i* *i*

of *qj* (Figure [6](#_bookmark29)). Let *Qj* = *{qj ,..., qj }*. Let *Vt* = *{a, b, c, d}* and assume, without

*i* 1 *k*

loss of generality, that *Vt ∩ Vp*(*t*) = *{b, c, d}*.

*t*

(*vj* ) (*vj* ) (*vj* )

*l*1

(*u*1)

*l*2

(*u*2)

*···*

*lm q*1

(*um*) (*v*1)

*q*2

(*v*2)

*···*

*qk*

(*vk*)

*qj*

1

*qj*

*···*

2

*qj*

*k*

1 2 *k*

Figure 6. The subtree of *T* rooted at *t*.

**Claim 5.4** *Let* Δ *be a triple of vertices in Vt with* Δ */*= *bcd. Then* (i) *|S*Δ(*t*)*|≤* 2*; and* (ii) *either S*Δ(*t*) *∩ L* = *∅ or S*Δ(*t*) *∩ Q* = *∅.*

**Proof** Due to space limitations, we omit the proof of (i). For (ii), suppose, for a

contradiction, that Δ = *abd* and *l*1*, q*1 *∈ S*Δ(*t*). Say *Vq*

1

*∩ V ′*

1

*q*

= *{x*0*, x*1*, v*1*}* and

let *Gj* = *G−vj* . Since *G* is minimal, there exists a 9 -TP, say (*Xj,Y j*), of *Gj*. Without

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loss of generality, assume that *yyj ∈ Xj ∩ E*(*x*0*x*1*v*1), let *z ∈ {x*0*, x*1*, v*1*}\ {y, yj}*,

and let *X* = *Xj ∪ {zvj }*. Note that *X* is a transversal of *G*. Suppose *wwj ∈*

1

*E*(*x*0*x*1*v*1) *\ E*(*Y j*). Hence (*X, Y j ∪{wwjvj }*) isa 9 -TP of *G*, a contradiction. Thus,

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every edge of *x*0*x*1*v*1 is in *E*(*Y j*). Since *dG′* (*v*1) = 3, we must have *x*0*x*1*v*1 *∈*

*Y j* and let *x*2*v*1 *∈/ E*(*Y j*). Suppose that both *x*0*u*1 and *x*1*u*1 are not in *E*(*Y j*).

Then (*X, Y j \ {x*0*x*1*v*1*}∪ {x*0*v*1*vj , x*0*x*1*u*1*}*) is a 9 -TP of *G*, a contradiction. So at

1 5

least one between *x*0*u*1 and *x*1*u*1, say *x*0*u*1, is in *E*(*Y j*). Since *dG′* (*u*1) = 3 and

*x*0*x*1*v*1 *∈ Y j*, we have that *x*0*x*2*u*1 *∈ Y j* and that *x*1*u*1 *∈/ E*(*Y j*). But then (*X, Y j \*

*{x*0*x*1*v*1*, x*0*x*2*u*1*}∪ {x*1*v*1*vj , x*0*x*2*v*1*, x*0*x*1*u*1*}*) is a 9 -TP of *G*, a contradiction. *2*

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We proceed with the proof of the theorem. Abusing notation, for every triple of vertices Δ *⊆ Vt* such that Δ */*= *bcd*, let *h*(Δ) = 0 if *S*Δ(*t*) = *∅*, and *h*(Δ) = 1 + max*{h*(*tj*): *tj ∈ S*Δ(*t*)*}* otherwise. Let *kj* = *|{*Δ *⊆ Vt* : *h*(Δ) = 2*}|*, and *mj* = *|{*Δ *⊆ Vt* : *h*(Δ) = 1*}|*. Let *G*+ be the graph obtained from *G*[*Vt ∪ R*(*t*)] by removing, for each triple of vertices Δ *⊆ Vt* such that Δ */*= *bcd* and *|S*Δ(*t*)*|* = 2, the vertices in *R*(*x*), for precisely one *x ∈ S*Δ(*t*). One can check that, by Claim [5.4](#_bookmark30), *G*+ is a restricted planar 3-tree. Also, *G*+ has *f* =4 + 4*kj* + 2*mj* faces. Let *Q*+ be the set *{qi ∈ Q* : *vi ∈/ V* (*G*+)*}*. By Proposition [4.6](#_bookmark22), *G*+ contains a triangle packing *P*+ of facial triangles containing the face *bcd* and such that *|P*+*| ≥ [f /*3*|*, and if *G*+ is super restricted, then *|P*+*|* = 5. For each *qi ∈ Q*+, let *Ti* = *vivjwi*, where *wi* is a vertex adjacent to both *vi* and *vj*, and let *P* = *P*+ *∪ {Ti* : *qi ∈ Q*+*}*. Note

*i*

*i*

that *|P|* = *|P*+*|* + (*k − kj*).

**Claim 5.5** *Let Gj* = *G − R*(*t*)*. If Xj is a transversal of Gj, then* (i) *there exists a transversal of G with size |Xj|* +1+ 2*k* + *m; and* (ii) *there exists a transversal of G with size at most |Xj|* +5+ *k.*

**Proof** (i) Because *Xj* is a transversal of *Gj* and *bcd* is a triangle in *Gj*, *Xj ∩ E*(*bcd*) */*= *∅*. Without loss of generality, assume that *bc ∈ Xj*. Given a 4-clique *K* in *G* and a set *X ⊆ E*(*G*) such that *E*(*K*) *∩ X* = *{e}*, denote by *K ⊗ X* the only edge in *K* that does not share any vertex with *e*. For instance, *Vt ⊗ {bc}* = *ad*. For every *i ∈* [*m*], let *ei* = *Vli ⊗ {bc, ad}*. For ev-

ery *i ∈* [*k*], let *fi* = *Vq*

*⊗ {bc, ad}* and *fj* = *Vq′ ⊗ {bc, ad, fi}*. Note that *X* =

*i i* *i*

*Xj ∪ {ad}∪ {ei* : *i ∈* [*m*]*}∪ {fi* : *i ∈* [*k*]*}∪ {f j* : *i ∈* [*k*]*}* is a transversal of *G* and

*i*

*|X|* = *|Xj|* +1+ 2*k* + *m*. (ii) Let *X* = *Xj ∪ E*(*G*[*Vt*]) *∪ {vivj*

*i*

: *i ∈* [*k*]*}*. Note

that *X* is a transversal of *G*. Because *Xj* is a transversal of *Gj* and *bcd* is a triangle in *Gj*, *Xj ∩ E*(*bcd*) */*= *∅*, and hence *|Xj ∩ E*(*G*[*Vt*])*| ≥* 1. Thus

*|X|* = *|Xj|* + *|E*(*G*[*Vt*])*|− |Xj ∩ E*(*G*[*Vt*])*|* + *k ≤ |Xj|* +5+ *k*. *2*

Let *Gj* = *G − R*(*t*) and let (*Xj,Y j*) be a 9 -TP of *Gj*. The only triangle in *P* containing edges of *Gj* is *bcd*. Therefore *Y* = *Y j ∪* (*P\ {bcd}*) is a triangle packing of *G* of size *|Y j|* + *|P*+*|−* 1+ (*k − kj*). By Claim [5.5](#_bookmark31), there exists a transversal *X* of *G* with size *|Xj|* + min*{*1+ 2*k* + *m,* 5+ *k}*. Note that *kj ≥* 1 because *h*(*t*) = 2.

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Also *kj* + *mj ≤* 3. If *mj* = 2, then *kj* = 1 and *G*+ is super restricted. In this case, because 5(5 + *k*) *<* 9(3 + *k*) we have that *|X|≤ |Xj|* +5+ *k <* 9 (*|Y j|* +3+ *k*)+ 1 =

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9 (*|Y j|* + *|P*+*|−* 1+ (*k − kj*)) + 1 *≤* 9 *|Y |* + 1 . So we may assume that *mj ≤* 1.

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Note that, for all possible values of *kj* and *mj*, we have *[f/*3*|−* 1+ (*k − kj*) =

*[*(4 + 4*kj* + 2*mj*)*/*3*|−* 1+ *k−kj* = 1 + *k* + *[*(*kj −* 2+ 2*mj*)*/*3*|* = 1 + *k* + *[kj/*3*♩* + *[mj/*2*|*. Suppose that *k ≤* 2. Then we have that 10*k ≤* 9*k* +2 and 5*m ≤* 9*[mj/*2*|* +2 because *mj ≤ m ≤* 2*mj ≤* 2, and hence 5(1 + 2*k* + *m*)=5 + 10*k* + 5*m ≤* 5+ 9*k* +

2+ 9*[mj/*2*|* +2 = 9(1 + *k* + *[mj/*2*|*)*.* Moreover, we have

*|X|≤ |Xj|* +1+ 2*k* + *m ≤*

9 *|Y*

5

*j|* +

1 + 9 1+ *k* +

5 5

*mj*

2

9 *|Y |* + 1 *.*

5 5

*≤*

Now suppose that *kj ≤* 2 and *mj* = 0. Since *kj ≥ [k/*2*|*, we have that *k ≤* 4, and hence 10*k ≤* 9*k* + 4. Moreover, we have that *m* = 0 = *[mj/*2*|*, and hence 5(1 + 2*k* + *m*) = 5 + 10*k* + 5*m ≤* 5+ 9*k* +4+ 9*[mj/*2*|* = 9(1 + *k* + *[mj/*2*|*)*.* Analogously to the case above, we have *|X| ≤ |Xj|* +1+ 2*k* + *m ≤* 9 *|Y |* + 1 *.*

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Hence we may assume that *k ≥* 3, and either *kj* =3 or *mj /*= 0. This implies that

5*k* = 9*k −* 4*k ≤* 9*k −* 12, and therefore 5(5 + *k*) = 25 + 5*k ≤* 9*k* + 13 *≤* 9(*k* + 2). Moreover, we have that *[kj/*3*♩* + *[mj/*2*|≥* 1. So, the proof is complete because

*|X|≤ |Xj|* +5+ *k ≤*

5

9 *|Y j|* + 1 + 9 (*k* + 2)

9 *j*

, *kj* ,

*mj* 1 9 1

*≤ |Y*

5

5

5

*|* +1+ *k* +  +

3 2

+ = *|Y |* + *.*

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*2*

# Concluding remarks

In this paper we present three results related to Tuza’s Conjecture. In Section [3,](#_bookmark5) we obtained a lemma (Lemma [3.4](#_bookmark12)) that extends Puleo’s tools [[13](#_bookmark45)], and allowed us to verify Tuza’s Conjecture for graphs with treewidth at most 6. Since any minimal counterexample to Tuza’s Conjecture is an irreducible robust graph, Lemma [3.2](#_bookmark6) may be used in further results regarding this problem. In Sections [4](#_bookmark15) and [5](#_bookmark24), we obtained stronger versions of Tuza’s Conjecture for specific classes of graphs. We believe that the techniques used here may also be useful to deal with other classes of graphs, perhaps by introducing new ingredients.

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