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On the Complexity of Gap-[2]-vertex-labellings of Subcubic Bipartite Graphs [1](#_bookmark0)

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**Abstract**

A gap-[*k*]-vertex-labelling of a simple graph *G* = (*V, E*) is a pair (*π, cπ* ) in which *π* : *V* (*G*) *→ {*1*,* 2*,..., k}* is an assignment of labels to the vertices of *G* and *cπ* : *V* (*G*) *→ {*0*,* 1*,..., k}* is a proper vertex-colouring of *G* such that, for every *v ∈ V* (*G*) of degree at least two, *cπ* (*v*) is induced by the largest difference, i.e. the largest gap, between the labels of its neighbours (cases where *d*(*v*) = 1 and *d*(*v*) = 0 are treated separately). Introduced in 2013 by A. Dehghan et al. [[3](#_bookmark17)], they show that deciding whether a bipartite graph admits

a gap-[2]-vertex-labelling is NP-complete and question the computational complexity of deciding whether cubic bipartite graphs admit such a labelling. In this work, we advance the study of the computational complexity for this class, proving that this problem remains NP-complete even when restricted to subcubic bipartite graphs.

*Keywords:* Gap-labellings, proper labellings, graph labellings.

# Introduction

Let *G* be a simple graph with vertex set *V* (*G*) and edge set *E*(*G*). The *elements* of *G* are its vertices and its edges. The degree and the neighbourhood of a vertex *v ∈ V* (*G*) are respectively denoted by *d*(*v*) and *N* (*v*), and the maximum degree of *G* is denoted by Δ(*G*). A *proper colouring* of *G* is an assignment *c* : *V* (*G*) *→ C*, where *C* denotes a set of colours, such that for every edge *uv ∈ E*(*G*), *c*(*u*) */*= *c*(*v*). A *proper vertex-labelling* is a pair (*π, cπ*) where *π* : *V* (*G*) *→ {*1*,* 2*,..., k}* is an assignment of labels to the vertices of *G* and *cπ* is a proper colouring of *G*, *induced*

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*s*13 *s*0



0

2

2

1

1

1

1 1

2 0

1 2

1 0

1 3

0

2

*s* 3

2

2

1

0

2

0

1

1

2

1

2 2

1 2

2

2

0

0

1

1

1

2

2

*s*12

*s*1

*s*11

*s*2

10

*s*

*s*9

*s*4

*s*8 *s*5

2 1

(a)

2

(b)

*s*7 *s*6

(c)

Fig. 1. In (a), a gap-[3]-vertex-labelling of the Petersen Graph; in (b), a gap-[2]-vertex-labelling of a subcubic bipartite graph inducing a 3-colouring; and, in (c), the Heawood Graph, which does not admit a gap-[2]-vertex-labelling. For every vertex, the number inside the box to its lower right corner corresponds to its assigned label

by *π* through some mathematical function over the set of labelled elements. In this work, we discuss proper vertex-labellings induced by gaps of labels.

Most authors trace the origins of proper graph labellings to 1967, when A. Rosa [[7](#_bookmark21)] introduced *β*-valuations. Since then, many different labellings have been studied. Each one varies the elements to which the labels are assigned and how the colouring is induced. In fact, there are several interesting surveys on the topic [[4,6,](#_bookmark20)[10](#_bookmark24)]. Concerning gap-labellings, they were first introduced as a vertex- distinguishing edge-labelling in 2012 by M. Tahraoui et al. [[8](#_bookmark22)], as a generalization of set and multiset labellings [[1](#_bookmark15)]. The vertex-labelling variant we consider in this text is defined as follows.

A *gap-*[*k*]*-vertex-labelling* of a graph *G* is a proper vertex-labelling of *G* over the set *{*1*,* 2*,..., k}* such that *cπ* : *V* (*G*) *→ {*0*,* 1*,..., k}* is a proper colouring of *G* in which, for every vertex *v ∈ V* (*G*): (i) *cπ*(*v*) = 1 if *d*(*v*) = 0; (ii) *cπ*(*v*) = *π*(*u*), *u ∈ N* (*v*), if *d*(*v*) = 1; and (iii) *cπ*(*v*) = max*u∈N* (*v*)*{π*(*u*)*} −* min*u∈N* (*v*)*{π*(*u*)*}* otherwise. Figures [1(a)](#_bookmark1) and [1(b)](#_bookmark1) exemplify gap-[*k*]-vertex-labellings using *k* = 3 and *k* = 2 labels, respectively. Observe in the latter that a gap-[*k*]-vertex-labelling may induce a proper (*k*+1)-colouring if there exists degree-one vertices in the graph.

Gap-[*k*]-vertex-labellings were introduced by A. Dehghan et al. [[3]](#_bookmark17) in 2013. In their article, the authors investigated both the computational complexity of decision problems associated with the gap-[*k*]-vertex-labelling of some families of graphs, as well as the least *k* for which some graphs admit such a labelling. They also show that deciding whether a given graph admits a gap-[*k*]-vertex-labelling is NP-complete for

*k ≥* 3.

For the case *k* = 2, Dehghan et al. show that the problem remains NP-complete for 3-colourable and for bipartite graphs, but if the graph is bipartite and planar, then the problem can be solved in polynomial time. They also show that every *r*-

regular bipartite graph, with *r ≥* 4, admits a gap-[2]-vertex-labelling, and question whether this is also true for cubic bipartite graphs. The dichotomy surrounding bipartite graphs was further investigated by A. Dehghan in 2016 [[2](#_bookmark16)]. The author showed that deciding whether a planar bipartite graph *G* admits a gap-[*k*]-vertex-

labelling such that *cπ* is a proper 2-colouring of *G* is also an NP-complete problem. Recently, C. A. Weffort-Santos [[9](#_bookmark23)] investigated the gap-[*k*]-vertex-labelling of some classical families of graphs; of particular interest to this work, we cite cycles and

unicyclic graphs. For these classes, the author characterized which graphs admit gap-[2]-vertex-labellings. These results imply that the gap-[2]-vertex-labelling deci- sion problem can be solved in linear time for these particular instances.

Although the literature presents “positive” polynomial-time solvability results for *r*-regular bipartite graphs for *r* = 2 and *r ≥* 4, the computational complexity for gap-[2]-vertex-labellings of 3-regular (i.e. *cubic*) bipartite graphs, originally pro- posed by Dehghan et al. [[3](#_bookmark17)] in 2013, remains unknown. In this work, we advance the computational complexity analysis on bipartite graphs, showing that the problem remains NP-complete when restricted to the family of subcubic bipartite graphs.

Our main results are presented in Theorem [1.1](#_bookmark2) and Corollary [1.2](#_bookmark3) and are proved in

the next section. In order to simplify the notation, we denote by G2vl the problem of deciding whether a graph admits a gap-[2]-vertex-labelling.

**Theorem 1.1** *Restricted to subcubic bipartite graphs,* G2vl *is* NP*-complete. 2*

**Corollary 1.2** G2vl *remains* NP*-complete when restricted to the family of subcu-*

*bic bipartite graphs with minimum degree 2.* *2*

# Proof of Theorem [1.1](#_bookmark2)

Given a graph *G* = (*V, E*), the Monochromatic Triangle (MT) problem ques- tions if there exists a partition of *E* into two disjoint sets *E*1, *E*2 such that neither *G*1 = (*V, E*1) nor *G*2 = (*V, E*2) contains a triangle. This problem was proved to be NP-complete by Burr in 1976 (although this result was only published by Garey &

Johnson in 1979 [[5](#_bookmark19)]). It can also be stated as an edge-colouring problem, where the

question is whether *G* admits a *{*red, blue*}*-edge-colouring such that every triangle in *G*, i.e. a subgraph of *G* isomorphic to complete graph *K*3, has at least one red

edge and one blue edge. For an instance *G* of Monochromatic Triangle, we

denote by *T* = (*t*1*, t*2*,..., tp*) an (arbitrary but fixed) ordering of all the *p* triangles in *G*. Also, we abuse notation to say that *ti* = *{ex, ey, ez}* is the triangle of *G* induced by edges *ex*, *ey* and *ez* of *E*(*G*).

We prove Theorem [1.1](#_bookmark2) by reducing Monochromatic Triangle to G2vl in

polynomial time. Equivalently, given an instance *G* of

Monochromatic

Tri-

angle, we construct a subcubic bipartite graph *Gj* such that *G* admits a 2-edge- colouring with no monochromatic triangles if and only if *Gj* admits a gap-[2]-vertex- labelling. The reduction is accomplished with the aid of two gadgets: a *triangle gadget* and a *negation gadget*.

The first, denoted by *GΔ*, is an auxiliary simple bipartite graph with 19 vertices, 20 edges and is described as follows. For a triangle *ti* = *{ex, ey, ez}*, *GΔ* has: a vertex *u* that represents *ti ∈ G* in *Gj*; two adjacent vertices *vj* and *wj* for each edge *ej* in *ti*; and a path *P*12, with *V* (*P*12)= *{q*0*,..., q*11*}*. We also link *u* to every *vj* and add edges *wxq*0, *wyq*4 and *wzq*8. Fig. [2(a)](#_bookmark4) illustrates the triangle gadget for a triangle

*i*

*ui s*3

*vi*

1

*vi*

4

*vi*

7

*wi*

1

*wi*

4

*wi*

7

*s*2

*s*7

*e*

*s*1

*s*8

*s*4 *s*5 *s*6

*qi qi qi*

*qi qi qi*

*qi qi*

*qi qi qi qi*

*v*in

*s*0 *s*13

*s*12

*s*11

*s*10

*v*out *w*

0 1 2

3 4 5

6 7

(a)

8 9 10 11

(b)

Fig. 2. In (a), triangle gadget *GΔ* for triangle *ti* = *{e*1*, e*4*, e*7*}*; and in (b), the negation gadget *G¬*.

*i*

*t*3 = *{e*1*, e*4*, e*7*}*. Note in the image that we index all of the gadget’s vertices with a superscript *i*, as well as the name *GΔ*, with a matching subscript. This is done so as to bind the vertices of triangle gadget *GΔ* to the corresponding triangle *ti* of *G*. We remark that *GΔ* is bipartite since it does not contain any odd cycles. Furthermore, since no vertex has degree greater than 3, *GΔ* is subcubic.

*i*

*x*

*x*

The negation gadget *Gч*

*i,j*

is used to connect vertices *vi*

and *wj* , belonging to

triangle gadgets *GΔ* and *GΔ* (not necessarily distinct). It is an auxiliary simple

*i j*

bipartite graph obtained by removing an edge *e* from the *Heawood Graph*, depicted

in Fig. [1(c)](#_bookmark1), and linking two new vertices, *v*in and *w*, to the ends of *e*. Let *H* be the Heawood Graph, with *V* (*H*)= *{s*0*,..., s*13*}* and *e* = *s*0*s*9. In the proof, we also refer to vertex *s*9 as *v*out. The construction of *Gч* yields a subcubic bipartite graph

with 16 vertices and 22 edges, which is illustrated in Fig. [2(b)](#_bookmark4). Vertices *vi* and *wj*

*x*

*x*

are identified with *v*in and *w* from *Gч* , respectively. Note that, upon performing

*i,j*

*x*

this operation, *vi*

*x*

and *wj*

in the corresponding triangle gadgets have degree 3.

We are now ready to show the reduction, which will be exemplified for graph *G*

presented in Fig. [3(a)](#_bookmark6). For every *ti ∈T* , *ti* = *{ex, ey, ez}*, add a new triangle gadget

*GΔ* to *Gj*. For every 1 *≤ i ≤ p −* 1, add edge *qi qi*+1 to *Gj*. Also, add a copy of

*i*

cycle *C*6, *V* (*C*6)= *{c*0*,..., c*5*}*, and edge *c*0*q*1.

0

11 0

Next, let *px* denote the number of triangles to which an edge *ex ∈ E*(*G*) belongs to; note that *px ≤ p*. Then, each *ex* has *px* different triangle gadgets *GΔ* in *Gj*, each of which has its own pair of vertices *vx* and *wx*. Now, let *tx* denote the *j*-th triangle that contains *ex* in *T* . For instance, consider edge *e*9 from graph *G* in Fig. [3(a)](#_bookmark6), which belongs to *p*9 = 2 triangles: *t*3 and *t*4. Then, *t*9 is the first triangle in *T* containing *e*9, namely *t*3; analogously, *t*9 is the second triangle containing *e*9, i.e.

*j*

1

2

*t*4.

In order to complete our reduction, we refer to vertices *vi*

and *wi*

*x*

belonging

to triangle *tx* as *vx,l* and *wx,l*, respectively. For example, *v*5*,*3 is vertex *vi* in *GΔ*

*x*

*l* 5 *i*

corresponding to the third triangle in *T* to contain edge *e*5, namely *t*3. Then,

connect vertices *vx,j* and *wx,j*+1 cyclically using negation gadget *Gч*

*x,*(*j,j*+1)

, following

the indices of triangles *tx* in *T* . This procedure — and the resulting graph *Gj* — are illustrated in Fig. [3(b).](#_bookmark6) We draw the reader’s attention to edge *e*5, which belongs to three triangles *t*1*, t*2 and *t*3. Thus, vertices *v*5*,*1*, w*5*,*2 are connected with the negation

*j*

*ч*

gadget *G*

5*,*(1*,*2)

, vertices *v*5*,*2 and *w*5*,*3, with *Gч*

and *v*5*,*3 and *w*5*,*1, with *Gч* .

Note that if an edge *ex* belongs to a single triangle *ti* in *G*, then *vi* and *wi*

5*,*(2*,*3)

5*,*(3*,*1)

*x*

*x*

are

in the same triangle gadget *GΔ*, and are connected with negation gadget *Gч*

*i x,*(*i,i*)

as exemplified by edges *e*1*, e*2*, e*3*, e*6*, e*7*, e*8 and *e*9 in the image. This completes the

construction of graph *Gj*, a subcubic bipartite instance of G2vl. We remark that the reduction is accomplished in polynomial time on the size of the input graph *G* since *p* = *O*(*n*3).

The structures used in the construction of *Gj* have important properties that are essential to the reduction and, consequently, to the proof of our main result. We list these properties below and refer the reader to Weffort-Santos’ Master’s Thesis for their detailed proofs [[9](#_bookmark23)].

**Proposition 2.1** *Let GΔ*

*i*

*be a triangle gadget. If GΔ*

*admits a gap-*[2]*-vertex-*

*labelling* (*π, cπ*) *with cπ*(*ui*)= 1*, then π*(*qi* )= *π*(*qi* )= *π*(*qi* )*.*

*i*

0 4 8

**Proof.** Let *GΔ*

*i*

be a triangle gadget, and suppose *GΔ*

admits a gap-[2]-vertex-

labelling such that *cπ*(*ui*) = 1. Since the gadget is bipartite and every vertex *v*

*i*

in *V* (*GΔ*) *\ {qi }* has *d*(*v*) *≥* 2, the only possible colours induced in *v* by *π* are 0

*i* 11

and 1. Now, every vertex *qi ∈ V* (*GΔ*), *l* odd, is in the same part as vertex *u* in

*l* *i*

any bipartition of *GΔ*. Thus, for every vertex *qi*, *l ≤* 9, *cπ*(*qi*) = *cπ*(*ui*) = 1, as

*i l l*

illustrated in Fig. [4.](#_bookmark7)

Let *a ∈ {*1*,* 2*}* be the label assigned to *qi* . Then, since *cπ*(*qi* )=1 and *N* (*qi* )=

0 1 1

*{qi , qi }*, we conclude that *{π*(*qi* )*, π*(*qi* )*}* = *{*1*,* 2*}* and, consequently, that *π*(*qi* )= *b*,

0 2 0 2 2

*b ∈ {*1*,* 2*}\ {a}*. Analogously, consider vertex *qi* and, observing that *cπ*(*qi* )=1 and

3 3

*N* (*qi* ) = *{qi , qi }*, we conclude that *π*(*qi* ) = *a* = *π*(*qi* ). By a similar reasoning, we

3 2 4 4 0

conclude that the same holds for vertex *qi* , and the result follows. *2*

8

*e*6

*e*1

*e*7

*e*9

*e*

*e*2

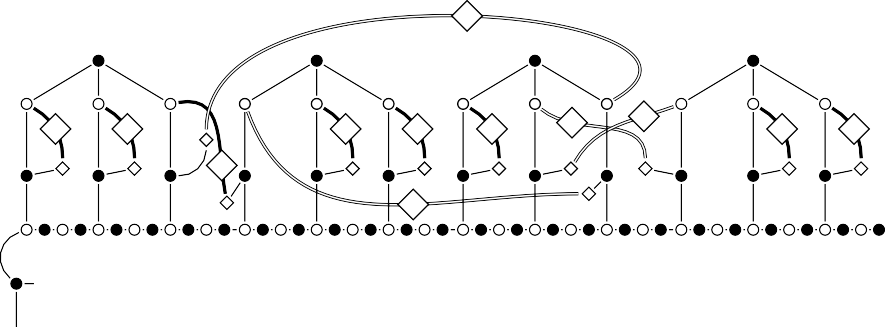
*e*5

8

*e*3

*e*4

(a)



*ч*

*u*1

*u*2

*u*3

*u*4

*v*1

2

*v*1

3

*v*1

5

*v*2

5

*v*2

6

*v*2

7

*v*3

1

*v*3

9

*v*3 *v*4

*v*4 *v*4

*ч ч*

*ч*

5 9

*ч*

8 4

*ч ч*

*ч*

*ч ч*

*w*1

2

*w*1

3

*w*1

5 *ч* 5

*w*2

*w*2

6

*w*2 *w*3

7 1

*w*3

9

*w*3

5

*w*4

9

*w*4

8

*w*4

4

*ч*

*q*1

0

*q*1

4

*q*1

8

*q*2

0

*q*2

4

*q*2 *q*3

8 0

*q*3

4

*q*3

8

*q*4

0

*q*4

4

*q*4

8

(b)

Fig. 3. (a) Instance *G* of Monochromatic Triangle; (b) Graph *G′* obtained by the reduction. Graph *G¬*

*i,j*

is represented by symbol *ч* in doubled lines; the diamond-shaped vertex is *v*out.

*ui*

*i x*

*vi*

*y*

*vi*

*z*

*i x*

*wi*

*y*

*wi*

*z*

*i*

0

*qi qi*

2 4

*qi qi*

6 8

*qi*

10

*v w*

*q*

*a b a b a b*

*q*

*q*

*q*

*i i* *i*

*q*

*q*

*q*

1 3 5

*i i* *i*

7 9 11

Fig. 4. A (partial) labelling of *GΔ* when *cπ* (*ui*) = 1. Vertices with induced colour 1 are filled in black, and

*i*

vertices with colour 0, in white. Vertex *qi* is coloured in orange, implying *cπ*(*q*11) *∈ {*1*,* 2*}*.

11

**Proposition 2.2** *Let Gj be a subcubic bipartite graph with Gч ⊆ Gj and d*(*vin*)=

*i,j*

*d*(*wj* )= 3*. If Gj admits a gap-*[2]*-vertex-labelling* (*π, c*

) *such that c* (*v*

)= 0*, then*

*x π*

*π*(*vin*) */*= *π*(*vout*)*, for vin* = *vi , vout ∈ V* (*Gч* )*.*

*π in*

*x*

**Proof (Sketch)** Let *Gj, Gч*

*i,j*

*i,j*

be as stated in the hypothesis. Suppose *Gj* admits

a gap-[2]-vertex-labelling such that *cπ*(*v*in) = 0. Note that *cπ*(*v*) *∈ {*0*,* 1*}* for every *v ∈ V* (*Gч* ) since *dG′* (*v*) = 3. Also, since *s*0 is adjacent to *v*in, we conclude that *cπ*(*s*0) = 1. Therefore, *cπ*(*sl*)= (*l* +1) mod 2 for every *sl* in the gadget. This implies that *π*(*si*)= *π*(*sj*) for every *i, j* even. The idea of the proof is to verify all possible label assignments to odd-indexed vertices *sl* that induce colouring *cπ*.

*i,j*

Let *{a, b}* = *{*1*,* 2*}*. We begin by considering the label assignments to some neighbours of *s*0, supposing *a* = *π*(*s*1) */*= *π*(*s*13) as illustrated in Fig. [5](#_bookmark11); note that this labelling induces *cπ*(*s*0) = 1 without considering the label assigned to *v*in. Now, suppose *π*(*s*3) = *b*. Since *s*3*, s*13 *∈ N* (*s*4) and *π*(*s*3) = *π*(*s*13) = *b*, we conclude that *π*(*s*5) = *a*. Then, since *s*1*, s*5 *∈ N* (*s*6), *π*(*s*7) = *b*. Analogously, we have *s*7*, s*13 *∈ N* (*s*12) which implies *π*(*s*11)= *a*. The only remaining odd-indexed vertex to be labelled is *s*9 = *v*out. However, if *π*(*v*out)= *a*, then *π*(*v*out)= *π*(*s*5)= *π*(*s*11) which induces *cπ*(*s*10) = 0, contradicting the hypothesis. This allows us to conclude that our last assumption, i.e., *π*(*s*3)= *b /*= *π*(*s*1), is incorrect.

By attempting *π*(*s*3) = *π*(*s*1) and following the same reasoning, we arrive at a similar contradiction, forcing us to backtrack to our first assumption and conclude that *π*(*s*1)= *π*(*s*13) in any such gap-[2]-vertex-labelling of *Gj*. Finally, we suppose *π*(*v*in)= *π*(*v*out) and, with the knowledge that *π*(*s*1)= *π*(*s*13), we need only verify the two possible label assignments for vertex *s*3 and how they affect the induced colouring. Since both labellings arrive at a contradiction, the result follows. *2*

**Proposition 2.3** *Let ex be an edge of G that belongs to px ≥* 2 *triangles and*

*x the j-th triangle in f that contains ex. If Gj admits a gap-*[2]*-vertex-labelling*

*t*

*j*

(*π, cπ*)*, then, for every pair of triangle gadgets GΔ , GΔ*

*representing consecutive*

*triangles tx, tx*

*, π*(*vx,j*)= *π*(*vx,j*+1)*.*

*x,j*

*x,j*+1

*j j*+1

**Proof (Sketch)** Let *G*, *Gj*, (*π, cπ*) and *ex* be as stated in the hypothesis. Now, consider the *px* triangle gadgets, *GΔ* , which have vertices *vx,j* corresponding to edges *ex ∈ tj*. Then, every vertex *vx,j* is connected to *wx,j*+1 through the use of a negation gadget, and vertex *wx,j*, to *vx,j—*1. Also, recall that every vertex *wx,j* is

*x,j*

adjacent to vertices *v*out, *vx,j* and some vertex *qx,j*, with *l ≡* 0 (mod 4); furthermore, every *qx,j* is assigned the same label *a ∈ {*1*,* 2*}* by Proposition [2.1](#_bookmark5).

*l*

*l*

Suppose that there exist vertices *vx,j* and *vx,j*+1 for which *{π*(*vx,j*)*, π*(*vx,j*+1)*}* =

*{a, b}*. Adjust notation so that *π*(*vx,j*) = *b*. By Proposition [2.2](#_bookmark8), *b* = *π*(*vx,j*) = *π*(*v*in) */*= *π*(*v*out) for *v*out connecting *vx,j* and *wx,j*+1. Therefore, every vertex in *N* (*wx,j*+1) has been assigned the same label *a*, which induces *cπ*(*wx,j*+1)= 0. This is a contradiction since *wx,j*+1 is adjacent to *vx,j*+1, whose induced colour is 0 by hypothesis. We conclude that in a gap-[2]-vertex-labelling of *Gj*, every *vx,j* corresponding to an edge *ex ∈ E*(*G*) has received the same label, and the result follows. *2*

**Proposition 2.4** *Let C*6 *⊆ Gj as constructed in the reduction. If Gj admits a gap-*

[2]*-vertex-labelling* (*π, cπ*)*, then π*(*c*3)= *a and π*(*c*1)= *π*(*c*5)= *b, {a, b}* = *{*1*,* 2*}.*

**Proof.** Let *Gj* as stated in the hypothesis and suppose *Gj* admits a gap-[2]-vertex- labelling (*π, cπ*). We prove the result by contradiction. Suppose *cπ*(*c*3) = 1. This implies that *π*(*c*4) */*= *π*(*c*2). By the symmetry of the cycle, we assume, without loss of generality, that *π*(*c*2) = 1. This implies that *π*(*c*0) = 2 since *cπ*(*c*1) =

1. Analogously, considering *N* (*c*5), we conclude that *π*(*c*4) = 1, a contradiction. Therefore, *cπ*(*c*3) = 0, which implies that vertices *c*0, *c*2 and *c*4 all have induced colour 1. By letting *π*(*c*3)= *a*, for *a ∈ {*1*,* 2*}*, we conclude that *π*(*c*1)= *π*(*c*5)= *b*,

*s*3 *s*4 *s*5 *s*6

*b a*

*s*2 *s*7

*s*1 *s*8

*a*

*s*3 *s*4 *s*5 *s*6

*b a*

*s*2 *s*7

*b*

*s*1 *s*8

*a*

*s*3 *s*4 *s*5 *s*6

*b a*

*s*2 *s*7

*b*

*s*1 *s*8

*a*

*v*in *s*0

*b*

13 12

*s s*

*s*11

*s*10

*v*out *w*

*v*in *s*0

*b*

13 12

*s s*

*s*11

*s*10

*v*out *w*

*v*in *s*0

*b*

13 12

*s s*

*a*

*s*11 *s*10

*v*out *w*

(a)

?

(b)

(c)

Fig. 5. Case *π*(*s*1) */*= *π*(*s*13). In (a), (b), and (c), vertex *s*3 has been assigned label *b*, which determines the assignment of labels to vertices *s*5, *s*7 and *s*11, respectively. In all figures, edges and labels highlighted in orange are those that force labels, which are highlighted in red. Also, we denote black vertices as those with induced colour 1 and white vertices, those with colour 0.

*x*

*t*

1

*vx,*1

*x*

*j—*1

*t*

*vx,j−*1

*tx*

*vx,j*

*j*

*x*

*j*+1

*t*

*vx,j*+1

*x*



 *px*

*vx,px*

*t*



*a*

*wx,j*+1

*v*out

*b*

*p*

*v*out

*l a*

*a*

*wx,j−*1

*l a*

*a*

*ч*

*b*

*ч*

*a*

*v*out

*b*

*x*

*... ...*

*x*

*v*out

*qx,j*

*a*

*qx,j*+1

*b*

*v*out

*a*

*ч*

*wx,j*

*wx,*1

*w*

Fig. 6. Two vertices *vx,j* and *vx,j*+1 labelled with *b* and *a*, respectively. The contradiction is reached when observing vertex *wx,j*+1 which, in this labelling, would have induced colour 0.

*b ∈ {*1*,* 2*}\ {a}*. Both cases are illustrated in Fig. [7.](#_bookmark12) *2*

*c*4 *c*2 *c*4 *c*2



*c*0

2

*c*5

*c*1

2

*c*0

*c*5

*b*

*c*1

*b*

*c*3

*a*

1

*c*3

(a)

(b)

Fig. 7. In (a), *c*3 with induced colour 1; and, in (b), colour 0. Black vertices have induced colour 1 and white vertices, colour 0.

In order to prove that

Monochromatic

Triangle *≤p*

G2vl

for subcubic

bipartite graphs, we prove the following claims.

**Claim 2.5** *If G admits an edge-colouring c* : *E*(*G*) *→ {red, blue} such that no triangle is monochromatic, then Gj admits a gap-*[2]*-vertex-labelling.*

**Proof (Sketch)** Let *G, Gj* as stated and suppose *G* admits a *{*red, blue*}*-edge- colouring *c* without monochromatic triangles. Also, let *{A, B}* be the bipartition of

*Gj* such that vertices *vi*

*x*

*∈ A*. Figures [2](#_bookmark4) and [3](#_bookmark6) exemplify vertices in parts *A* and *B*

as coloured in white and black, respectively.

Define a labelling *π* : *V* (*G*) *→ {*1*,* 2*}* of *G* as follows. First, assign label 1 to every

*i*

vertex in part *B*. Now, consider the vertices in part *A*. For every vertex *vi*

*x*

*∈ GΔ*

(corresponding to edge *ex ∈ E*(*G*)), assign *π*(*vx*)=1 if *c*(*ex*) = red, and *π*(*vx*)= 2, otherwise. For negation gadgets *Gч* , assign labels: (*a, b, b, b, a, a, a*) to vertices

*i,j*

(*s*1*, s*3*,..., s*13), with *b* = *π*(*vi* ) and *a ∈ {*1*,* 2*}\ {b}*. For vertices *qi ∈ P*12 *⊆ GΔ*,

*x l* *i*

let *π*(*qi*) = 1 if *l ≡* 0 (mod 4) and *π*(*qi*) = 2 if *l ≡* 2 (mod 4). Finally, assign

*l l*

*π*(*c*1)= *π*(*c*5)=2 and *π*(*c*3) = 1. Define colouring *cπ* as usual.

In order to prove that (*π, cπ*) is a gap-[2]-vertex-labelling of *Gj*, it suffices to show that *cπ* is a proper vertex-colouring of *Gj*. First, since every vertex in part *B* received the same label and *Gj* is connected, every vertex *v ∈ A* has induced colour *cπ*(*v*) = 0. Thus, it remains to consider the induced colours of vertices in *B*. It is possible to inspect the induced colourings of cycle *C*6, paths *P*12 *⊆ GΔ* and

*i*

negation gadgets *Gч*

*i,j*

11

to observe that colours 0 and 1 alternate in the vertices in these

structures (with the exception of vertex *qp*

11

in *GΔ*, whose colour is *cπ*

(*qp* ) *∈ {*1*,* 2*}*).

Furthermore, by properties [2.2](#_bookmark8) and [2.3](#_bookmark9), every vertex *wi*

*p*

*x*

*i*

*∈ GΔ* has (at least) two

neighbours, namely *vx* and *v*out *∈ V* (*Gч*), with labels *π*(*vx*) */*= *π*(*v*out). Therefore,

*i* *i*

*cπ*(*wi* )= 1.

*x*

To complete the proof, note that, since every triangle in *G* is not mono- chromatic, the edges of *ti* = *{ex, ey, ez}* are coloured such that *{c*(*ex*)*, c*(*ey*)*, c*(*ez*)*}* = *{*red*,* blue*}*. This implies that *{π*(*vi* )*, π*(*vi* )*, π*(*vi* )*}* = *{*1*,* 2*}* in every tri-

*x y z*

angle gadget *GΔ ∈ Gj* which, in turn, induces *cπ*(*ui*) = 1. We conclude that *cπ* is a

*i*

proper colouring of *G*, and the result follows. *2*

**Claim 2.6** *If Gj admits a gap-*[2]*-vertex-labelling, then G admits an edge-colouring*

*c* : *E*(*G*) *→ {red, blue} such that no triangle is monochromatic.*

**Proof (Sketch)** Let *Gj,* (*π, cπ*) as stated in the hypothesis and let *{A, B}* be a

bipartition of *V* (*Gj*) with *vi ∈ A*. By Proposition [2.4](#_bookmark10), cycle *C*6 *⊆ Gj* has a unique

*x*

gap-[2]-vertex-labelling which, in turn, defines the induced colour of every vertex in *Gj* with degree at least two: vertices in *A* have induced colour 0 and vertices in *B \ {qp }*, colour 1.

11

Now, since *ui ∈ GΔ* belongs to *B*, we know that *cπ*(*ui*) = 1. Moreover, since

*i*

*N* (*ui*) = *{vi , vi , vi }*, we have *{π*(*vi* )*, π*(*vi* )*, π*(*vi* )*}* = *{*1*,* 2*}*. Then, define a *{*red,

*x y z x y z*

blue*}*-edge-colouring of *G* as follows. For every edge *ex*, assign *c*(*ex*) = red if

*π*(*vx*) = 1 and *c*(*ex*) = blue, otherwise. By Proposition [2.3](#_bookmark9), every *vi*

*x*

receives

the same label and, therefore, no edge *ex* is assigned two colours. Thus, we have

*{c*(*ex*)*, c*(*ey*)*, c*(*ez*)*}* = *{*red*,* blue*}* in every triangle *ti* = *{ex, ey, ez}*, and every *ti* has at least one edge coloured with red and one with blue. *2*

By combining claims [2.5](#_bookmark13) and [2.6](#_bookmark14), we conclude that MT *≤p* G2vl (restricted to subcubic graphs), completing the proof of Theorem [1.1](#_bookmark2).

Now, consider the following modification to the previously described reduction.

Remove vertices *qp*, *qp* and *qp* from *GΔ*, i.e. the last three vertices in path *P*

9 10 11 *p* 12

in the last triangle gadget of *Gj* according to *f* . Call the resulting graph *Gjj*.

Note that this graph is subcubic having *δ*(*Gjj*) = 2. However, this modification to the constructed graph does not alter any of the properties of *Gj*, used to prove Theorem [1.1](#_bookmark2). Therefore, Corollary [1.2](#_bookmark3), below, holds.

**Corollary 1.2** G2vl *remains* NP*-complete when restricted to the family of subcu-*

*bic bipartite graphs with minimum degree 2.* *2*

# Concluding remarks and open problems

The gap-[*k*]-vertex-labelling problem is fairly recent in the field of graph labellings. It is currently being studied under both the computational complexity and algorith- mic points of view. In the former, the problem has been proven to be NP-complete

for graphs in general when *k ≥* 3. The latter approach has shown that, for certain

families of graphs (e.g. cycles, trees, crowns, wheels, unicyclic graphs and some families of snarks), it is possible to decide if they admit such a labelling in linear time. Moreover, there are known results on *optimal* gap-[*k*]-vertex-labellings for these families, i.e. labellings using the least number *k* of labels so as to induce a proper colouring of the graph. There have also been discoveries of classes of graphs that do not admit any gap-[*k*]-vertex-labelling for any *k ∈* N, such as complete graphs of order *n ≥* 4 and some powers of paths and cycles.

In particular, for *k* = 2, the family of bipartite graphs has proven to be quite challenging. It is known that deciding whether a bipartite graph admits a gap-[2]- vertex-labelling is NP-complete in general, but can be solved in polynomial time for

some subfamilies, such as planar bipartite and *r*-regular graphs with *r ≥* 4. Our main result states that the problem remains NP-complete even when restricted to the families of subcubic bipartite graphs. Furthermore, if the minimum degree of

these graphs is two, the problem remains NP-complete. As a consequence, the ex- istence (or lack thereof) of degree-one vertices does not seem to facilitate deciding whether subcubic bipartite graphs admit gap-[2]-vertex-labellings. This result con-

trasts with known results for gap-vertex-labelable graphs since, in many cases, the presence of degree-one vertices helps the gap-[*k*]-vertex-labelling of certain graphs, e.g., every bipartite crown admits a gap-[2]-vertex-labelling whereas this is not the case for every even-length cycle.

In conclusion, we raise the following questions. Why do vertices of degree one

have (apparently) no impact on the hardness of G2vl? Furthermore, although

the problem remains NP-complete for subcubic bipartite graphs, can the structural properties of cubic graphs be used to decide if and when these graphs admit gap- [2]-vertex-labellings?

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