

Electronic Notes in Theoretical Computer Science 221 (2008) 17–21

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

On the Effective Existence of Schauder Bases (Extended Abstract)

Volker Bosserhoff[1](#_bookmark0)*,*[2](#_bookmark0)

*Institut fu¨r Theoretische Informatik und Mathematik Fakult¨at fu¨r Informatik*

*Universit¨at der Bundeswehr Munich, Germany*

**Abstract**

We construct a computable Banach space which possesses a Schauder basis, but does not possess any computable Schauder basis.

*Keywords:* Computable analysis, Schauder bases

# Introduction

Let *X* be an infinite-dimensional Banach space over F *∈ {*R*,* C*}*. A sequence (*xi*) of elements of *X* is called a *Schauder basis* (or simply a *basis*) of *X* if for every *x ∈ X* there is a unique sequence (*αi*) of elements of F*ω* such that *x* is the limit of the

norm-convergent series Σ*∞ αixi*. If *X* is a finite-dimensional vector space then a

*i*=0

finite sequence (*x*1*,..., xn*) *∈* F*<ω* is a (Schauder) basis of *X* if for every *x ∈ X*

there are unique *α*1*,..., αn* such that *x* = Σ*n αixi*. A finite or infinite sequence

*i*=1

is called *basic* if it is a basis of the closure of its linear span. (Finite sequences are

hence basic if, and only if, they are linearly independent.)

The theory of Schauder bases is a central area of research and also an important tool in functional analysis. Background information can be found in e.g. [[1](#_bookmark6),[8](#_bookmark13),[10](#_bookmark14),[11](#_bookmark16)]. In computable analysis [[4](#_bookmark9),[13](#_bookmark18)], Brattka and Dillhage [[3](#_bookmark8)] have shown that com- putable versions of a number of classical theorems on compact operators on Banach spaces can be proved under the assumption that the computable Banach spaces under consideration possess computable bases (with certain additional properties).

1 The work was partially supported by DFG grant HE 2489/4-1.

2 Email: [volker.bosserhoff@unibw.de](mailto:volker.bosserhoff@unibw.de)

1571-0661 © 2008 Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2008.12.003

The restriction to spaces with computable bases does not seem to be too costly in terms of generality because virtually all of the separable Banach spaces that are important for applications are known to possess a computable basis.

Complete orthonormal sequences in Hilbert spaces are particularly well-behaved examples of Schauder bases. It is a fundamental fact that every separable Hilbert space contains such a complete orthonormal sequence. It is furthermore known (see [[5](#_bookmark10), Lemma 3.1]) that every computable Hilbert space contains a computable complete orthonormal sequence and hence a computable basis. Can this be gen- eralized to arbitrary computable Banach spaces with bases? More precisely: If a computable Banach space possesses a basis, does it necessarily possess a computable basis? The aim of the present note is to show that the answer is “no” in general.

Our example will be a subspace of the space of zero-convergent sequences whose terms are in *Enflo’s space* – a famous example of a separable Banach space that lacks the *approximation property* (see below). The construction will proceed by

direct diagonalization.

Some remarks on notation: As we will never consider more than one norm on the same linear space, we will denote every norm by  *· *; which norm is meant will be clear from what it is applied to. If *x*1*, x*2*,...* are elements of a Banach space, denote by [*x*1*, x*2*,.. .*] the closure of their linear span; analogously, let [*x*1*,..., xn*] denote the linear span of *x*1*,..., xn*. Whenever we speak of the *rational span* of a set of vectors, we mean all their finite linear combinations with coefficients taken from Q (if F = R) or Q[*i*] (if F = C), respectively. If *X* is a normed space, put *BX* := *{x ∈ X* :  *x * *≤* 1*}*. Denote by *K*(*X*) the hyperspace of compact subsets of *X*.

As far as computable analysis is concerned we shall adopt the notation from [[3](#_bookmark8)]. In particular, see [[3](#_bookmark8)] and the references cited therein for the definition of *com- putable Banach spaces*. Computability of points, sequences, continuous functions, compact subsets, etc. shall always be understood as computability with respect to the representations considered in [[3](#_bookmark8)].

# Some properties of Enflo’s space

The question whether every separable Banach space has a basis was posed by Banach in 1932; forty years later, it was answered in the negative by Enflo [[7](#_bookmark12)]. Enflo in fact constructed a Banach space that lacks the *approximation property* (AP). A Banach space is said to have AP if the identity operator can be approximated uniformly on every compact subset by finite-rank operators (see [[8](#_bookmark13), Definition 3.4.26, Theorem 3.4.32]). A Banach space with a basis necessarily has AP (see [[8](#_bookmark13), Theorem 4.1.33]).

Enflo’s example was simplified by Davie [[6](#_bookmark11)]. It is easy to verify (when looking at Davie’s proof) and not surprising that the space defined by Davie is computable:

**Lemma 2.1** *There is a computable (complex) Banach space* (*Z, * *· ,* (*ei*)) *without AP.*

Associated with every basic sequence (*xi*) of elements of an infinite-dimensional

Banach space *X* is the sequence (*Pn*) of *natural projections Pn* : [*x*0*, x*1*,.. .*] *→*

[*x*0*,..., xn*], which are defined by *Pn*(Σ*∞ αixi*) := Σ*n * *αixi*. The *Pn* are bounded

*i*=0

*i*=0

linear mappings, and it is a fundamental fact that sup*n Pn < ∞* (see [[8](#_bookmark13), Corollary

4.1.17]). The value sup*n Pn * is called the *basis constant* bc((*xi*)) of (*xi*). Basis constants can also be defined (in an analogous way) for finite basic sequences, and hence also if *X* is only finite-dimensional. Finally, the basis constant of a space *X* that possesses a basis is defined as

bc(*X*) := inf*{*bc((*xi*)) : (*xi*) is a basis of *X}.*

A Banach space *X* is said to have *local basis structure* if there is a constant *C* such that for every finite-dimensional subspace *V ⊆ X* there is a finite-dimensional space *W* with *V ⊆ W ⊆ X* and bc(*W* ) *≤ C*. This notion was introduced in [[9](#_bookmark15)] (under a different name). A sufficient criterion given by Szarek [[12](#_bookmark17)] can be used to prove:

**Lemma 2.2** *Z has local basis structure.*

Via exhaustive search techniques, the following effective statement can be de- rived:

**Lemma 2.3** *There is a constant C >* 0*, a computable linearly independend se- quence* (*ci*) *of elements of Z, and a strictly increasing computable function σ* : N *→* N *such that* [*c*0*, c*1*,.. .*] = *Z and* bc([*c*0*,..., cσ*(*n*)]) *< C for every n ∈* N*.*

# The construction

Let *Y* be the Banach space of all zero-convergent sequences of elements of *Z*

equipped with the sup-norm, that is

 (*zi*)  := sup

*i∈*N

 *zi * for every (*zi*) *∈ Y.*

For *m ∈* N let *π*(*m*) : *Y → Z* be the projection *π*(*m*)((*zi*)) := *zm*; on the other hand, let *η*(*m*) : *Z → Y* be the isometric embedding defined by

*π*(*i*)(*η*(*m*)(*z*)) := *z* if *m* = *i,*

0 otherwise.

For all *i, m ∈* N, put *g⟨i,m⟩* := *η*(*m*)(*ei*). It is easy to verify that (*Y, * *· ,* (*gi*)) is a computable Banach space and that (*η*(*m*)) and (*π*(*m*)) are computable sequences of functions.

The facts that *Z* lacks AP and that AP is inherited by complemented subspaces [3](#_bookmark3) yield:

3 Recall that a subspace *F* of *X* is called *complemented* if there is a continuous linear mapping *P* : *X → X* with *P* 2 = *P* and *P* (*X*)= *F* . (Remark: It is known that having a basis is *not* inherited by complemented subspaces in general; see [[12](#_bookmark17)].)

**Lemma 3.1** *Let X be a closed subspace of Y such that there is an m ∈* N *with*

*η*(*m*)(*Z*) *⊆ X. Then X does not have a basis.*

Let (*ci*) and *σ* be as in Lemma [2.3](#_bookmark1). Put *Xm* := [*c*0*,..., cσ*(*m*)]. The idea of the proof of the following lemma is to construct a subspace of *Y* which does not include any space of the form *η*(*m*)(*Z*), but includes infinitely many spaces of the form *η*(*m*)(*Xk*). In view of Lemma [3.1](#_bookmark2) and the fact that clo ( *m Xm*) = *Z*, this means, loosely speaking, that the constructed space is “close to not having a basis”. Using diagonalization, we will ensure that the space is “close enough to not having a basis” such that no computable sequence is a basis of it.

**Lemma 3.2** *There is a computably enumerable set L ⊆* N *such that the following holds:*

* 1. *{k* : *⟨n, j, k⟩∈ L} is ﬁnite for all n, j ∈* N*.*
  2. *Put*

*τ* (*⟨n, j⟩*) := max *{k* : *⟨n, j, k⟩∈ L}∪ {*0*} , n, j ∈* N*.*

*No computable sequence of elements of Y is a basis of the subspace*

*X* := *{*(*zi*) *∈ Y* : *zm ∈ Xτ* (*m*) *for all m ∈* N*}.*

Next, we have to make sure that the space from Lemma [3.2](#_bookmark4) *has* a basis. This can be proved based on the uniform boundedness of the basis constants of the *Xm*:

**Lemma 3.3** *The space X constructed in Lemma* [*3.2*](#_bookmark4) *possesses a basis.*

We are now ready to complete the construction: Let *X* be the Banach space constructed in Lemma [3.2](#_bookmark4). We have just seen that *X* has a basis. Let (*hi*) be a computable enumeration of the set

*{η*(*m*)(*cs*) : *m ∈* N*,* 0 *≤ s ≤ σ*(*τ* (*m*))*}.*

Then [*h*0*, h*1*,.. .*] = *X*. Reference [[2](#_bookmark7), Proposition 3.10] yields that (*X, * *· ,* (*hi*)) is a computable Banach space and that the embedding *X ‹→ Y* is computable. We conclude that (*X, * *· ,* (*hi*)) cannot have a computable basis, because this basis would be a computable sequence in *Y* in contradiction to Lemma [3.2](#_bookmark4). The foregoing results are subsumed under the following theorem:

**Theorem 3.4** *The computable Banach space* (*X, * *· ,* (*hi*)) *as constructed above possesses a basis, but does not possess any computable basis.*

**Remark.** One of the additional properties of bases considered by Brattka and Dillhage [[3](#_bookmark8)] is the property of being *shrinking*. A basis (*xi*) of *X* is shrinking if the following holds for every element *f* of the topological dual *X∗*:

lim

*n→∞*

sup*{|f* (*x*)*|* : *x ∈ B*[*xn*+1*,xn*+2*,...*]*}* = 0*.*

It is well known that the dual *c∗* of *c*0 can be identified with *l*1 (see [[8](#_bookmark13), Example 1.10.4]); here (*qi*) *∈ l*1 applied to (*pi*) *∈ c*0 is defined as Σ*i qipi*. This easily leads to

0

the fact that the unit vector basis ((1*,* 0*,* 0*,.. .*)*,* (0*,* 1*,* 0*,.. .*)*,.. .*) of *c*0 is shrinking. In a completely analogous way, one can show that the basis, constructed in Lemma [3.3](#_bookmark5), of the space *X*, constructed in Lemma [3.2](#_bookmark4), is shrinking: In fact, the dual of *X* can

be identified with the space of all sequences (*fi*) with *fi ∈ X∗* and Σ*i * *fi * *< ∞*,

*τ* (*i*)

where (*fi*) applied to *x* is Σ*∞ fi*(*π*(*i*)(*x*)).

*i*=0

# References

1. Albiac, F. and N. Kalton, “Topics in Banach Space Theory,” Springer, New York, 2006.
2. Brattka, V., *Computability of Banach space principles*, Informatik Berichte 286, FernUniversit¨at Hagen, Hagen (2001).

URL <http://cca-net.de/vasco/publications/banach.html>

1. Brattka, V. and R. Dillhage, *Computability of compact operators on computable Banach spaces with bases*, Math. Logic Quart. **53** (2007), pp. 345–364.
2. Brattka, V., P. Hertling and K. Weihrauch, *A tutorial on computable analysis*, in: S. Cooper, B. L¨owe and A. Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, Springer, New York, 2008 pp. 425–491.
3. Brattka, V. and A. Yoshikawa, *Towards computability of elliptic boundary value problems in variational* *formulation*, J. Complexity **22** (2006), pp. 858 – 880.
4. Davie, A., *The approximation problem for Banach spaces*, Bull. London Math. Soc. **5** (1973), pp. 261– 266.
5. Enflo, P., *A counterexample to the approximation property in Banach spaces*, Acta Math. **130** (1973),

pp. 309–317.

1. Megginson, R., “An Introduction to Banach Space Theory,” Springer, 1998.
2. Pujara, L., *Some local structures in Banach spaces*, Math. Japonicae **20** (1975), pp. 49–54.
3. Singer, I., “Bases in Banach Spaces I,” Springer, Berlin, 1970.
4. Singer, I., “Bases in Banach Spaces II,” Springer, Berlin, 1981.
5. Szarek, S., *A Banach space without a basis which has the bounded approximation property*, Acta Math.

**159** (1987), pp. 81–98.

1. Weihrauch, K., “Computable Analysis: An Introduction,” Springer, Berlin, 2000.