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On the Iterated Edge-Biclique Operator

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**Abstract**

A biclique of a graph *G* is a maximal induced complete bipartite subgraph of *G*. The edge-biclique graph of *G*, *KBe*(*G*), is the edge-intersection graph of the bicliques of *G*. A graph *G* diverges (resp. converges or is periodic) under an operator *H* whenever lim*k→∞ |V* (*Hk*(*G*))*|* = *∞* (resp. lim*k→∞ Hk*(*G*)= *Hm*(*G*) for

some *m* or *Hk*(*G*)= *Hk*+*s*(*G*) for some *k* and *s ≥* 2). The kth-iterated edge-biclique graph of *G*, *KBk*(*G*), is the graph obtained by applying the edge-biclique operator *k* successive times to *G*. In this paper we study the iterated edge-biclique operator *KBe*. In particular, we give sufficient conditions for a graph to be convergent or divergent under the operator *KBe* and we propose some conjectures on the subject.

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*Keywords:* Bicliques, Edge-biclique graphs, Divergent graphs, Iterated graph operators, Graph dynamics

# Introduction

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. We can mention line graphs (intersection graphs of the edges of a graph), interval graphs (intersection graphs of a family of subpaths of a path), and in particular, clique graphs (intersection graphs of the the family of all cliques of a graph) [[4,](#_bookmark20)[5,](#_bookmark21)[8,](#_bookmark24)[11,](#_bookmark27)[12,](#_bookmark28)[27,](#_bookmark43)[29](#_bookmark45)].

The *clique graph* of *G* is denoted by *K*(*G*). Clique graphs were introduced by Hamelink in [[19](#_bookmark31)] and characterized in [[33](#_bookmark48)]. It was proved in [[1](#_bookmark17)] that the clique graph recognition problem is NP-Complete.

The clique graph can be thought as an operator from *Graphs* into *Graphs*. The *iterated clique graph Kk*(*G*) is the graph obtained by applying the clique operator *k* successive times. It was introduced by Hedetniemi and Slater in [[20](#_bookmark36)]. Much work has

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been done in the field of the iterated clique operator, looking at the possible different behaviors. The goal is to decide whether a given graph converges, diverges, or is periodic under the clique operator when *k* grows to infinity. This question remains open for the general case, moreover, it is not known if it is computable. However, partial characterizations have been given for convergent, divergent and periodic graphs, restricted to some classes of graphs. Some of them lead to polynomial time algorithms to solve the problem.

For the clique-Helly graph class, graphs which are convergent to the trivial graph have been characterized in [[3](#_bookmark19)]. Cographs, *P*4-tidy graphs, and circular-arc graphs are examples of classes where the different behaviors were also characterized [[7,](#_bookmark23)[22](#_bookmark38)]. On the other hand, divergent graphs were considered. For example, in [[31](#_bookmark47)], fam- ilies of divergent graphs are given. Periodic graphs were studied in [[8,26](#_bookmark42)]. It has been proved that for every integer *i*, there are graphs with period *i* and graphs which converge in *i* steps. More results about iterated clique graphs can be found in [[9,](#_bookmark25)[10,](#_bookmark26)[23,](#_bookmark39)[24,](#_bookmark40)[25,](#_bookmark41)[32](#_bookmark49)].

A *biclique* is a maximal bipartite complete induced subgraph. Bicliques have applications in various fields, for example biology: protein-protein interaction net- works [[6](#_bookmark22)], social networks: web community discovery [[21](#_bookmark37)], genetics [[2](#_bookmark18)], medicine [[30](#_bookmark46)], information theory [[18](#_bookmark32)], etc. More applications (including some of these) can be found in [[28](#_bookmark44)]. The *biclique graph* of a graph *G*, denoted by *KB*(*G*), is the inter- section graph of the family of all bicliques of *G*. It was defined and characterized in [[16](#_bookmark33)]. However no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construction can be viewed as an operator between the class of graphs.

The *iterated biclique graph KBk*(*G*) is the graph obtained by applying to *G* the biclique operator *k* times iteratively. It was introduced in [[15](#_bookmark34)] and all possi- ble behaviors were characterized. It was proven that a graph is either divergent or convergent, but never periodic (with period bigger than 1). Also, general char- acterizations for convergent and divergent graphs were given. These results were based on the fact that if a graph *G* contains a clique of size at least 5, then *KB*(*G*) or *KB*2(*G*) contains a clique of larger size. Therefore, in that case *G* diverges. Similarly if *G* contains the *gem* or the *rocket* graphs as an induced subgraph, then *KB*(*G*) contains a clique of size 5, and again *G* diverges. Otherwise it was shown that after removing false-twin vertices of *KB*(*G*), the resulting graph is a clique on at most 4 vertices, in which case *G* converges. Moreover, it was proved that if a graph *G* converges, it converges to the graphs *K*1 or *K*3, and it does so in at most 3 steps. These characterizations led to an *O*(*n*4) time algorithm (later improved to *O*(*n* + *m*) time [[13](#_bookmark29)]) for recognizing convergent or divergent graphs under the biclique operator.

The *edge-biclique graph* of a graph *G*, denoted by *KBe*(*G*), is the edge- intersection graph of the family of all bicliques of *G*. We recall that edge-intersection means that *KBe*(*G*) has a vertex for each biclique of *G* and two vertices are adja- cent in *KBe*(*G*) if their corresponding bicliques in *G* share an edge (and not just a vertex as in *KB*(*G*)). The edge-biclique graph *KBe*(*G*) was defined in [[17](#_bookmark35)] and

studied in [[14](#_bookmark30)], however there is no characterization so far to recognize edge-biclique graphs.

In this work we study edge-biclique graphs not only because of their mathe- matical interest but also because in real-life problems, bicliques often represent the relation between two types of entities (each partition of the biclique) therefore if would make sense to study when two objects (bicliques) share a common relation- ship (an edge) more than just an entity (a vertex). In particular, we focus on the *iterated edge-biclique graph*, denoted by *KBk*(*G*), and defined as the graph obtained by applying the edge-biclique operator *k* successive times to *G*. We give some non- trivial sufficient conditions for a graph to be convergent or divergent under the *KBe* operator and we propose some conjectures that would help to fully characterize the behavior of a graph under the *KBe* operator.

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This work is organized as follows. In Section 2 the necessary notation is given. In Section 3 and Section 4 we present some results about convergent and divergent graphs, respectively. Finally, in Section 5 we state some conjectures on the subject.

# Preliminaries

Along the paper we restrict to undirected simple graphs. Let *G* = (*V, E*) be a graph with vertex set *V* (*G*) and edge set *E*(*G*). A *clique* of *G* is a maximal com- plete induced subgraph, while a *biclique* is a maximal bipartite complete induced subgraph of *G*. The *open neighborhood* of a vertex *v ∈ G*, denoted *N* (*v*), is the set of vertices adjacent to *v* while the *closed neighborhood* of *v*, denoted by *N* [*v*], is *N* (*v*) *∪ {v}*. A *path* (*cycle*) on *k* vertices (*k ≥* 3), denoted by *Pk* (*Ck*), is a set of vertices *v*1*, v*2*, ..., vk ∈ G* such that *vi /*= *vj* for all 1 *≤ i /*= *j ≤ k* and *vi* is adjacent to *vi*+1 for all 1 *≤ i ≤ k −* 1 (and *vk* is adjacent to *v*1). A graph is *connected* if there exists a path between each pair of vertices. The *girth* of *G* is the length of a shortest induced cycle in the graph. We assume that all graphs of this paper are connected.

Given a family of sets *H*, the *intersection graph* of *H* is a graph that has the members of *H* as vertices, and there is an edge between two sets *E, F ∈H* when *E* and *F* have non-empty intersection.

A graph *G* is an *intersection graph* if there exists a family of sets *H* such that *G* is the intersection graph of *H*. We remark that any graph is an intersection graph [[34](#_bookmark50)].

Let *H* be any graph operator and let *G* be a graph. The iterated graph under the operator *H* is defined iteratively as follows: *H*0(*G*)= *G* and for *k ≥* 1, *Hk*(*G*)= *Hk−*1(*H*(*G*)). We say that *G* diverges (resp. converges or is periodic) under the operator *H* whenever lim*k→∞ |V* (*Hk*(*G*))*|* = *∞* (resp. lim*k→∞ Hk*(*G*) = *Hm*(*G*) for some *m* or *Hk*(*G*)= *Hk*+*s*(*G*) for some *k* and *s ≥* 2). The study of the behavior of a graph *G* under the operator *H* consists of deciding if *G* converges, diverges or is periodic under *H*.

We assume that the empty graph is convergent under the operator *KBe*, as it is obtained by appyling the edge-biclique operator to a graph that does not contain

any bicliques.

# Convergence

To start this section we have this first easy result.

**Lemma 3.1** *For n ≥* 2*, the complete graph Kn converges to the empty graph under the operator KBe in two steps.*

**Proof.** Clearly each edge of *Kn* is a biclique that does not edge-intersect with another one. Then *KBe*(*G*) consists of *n*(*n−*1) isolated vertices (and no bicliques), therefore *KB*2(*G*) is the empty graph. *2*

2

*e*

Next we show that graphs without induced cycles of length 3 and 4 are conver- gent.

**Theorem 3.2** *If G has girth at least ﬁve, then the edge-biclique operator applied to G converges towards the graph induced by the union of all the cycles and paths connecting cycles of G.*

**Proof.** If *G* has girth at least five, then every biclique is a star. Moreover *G* has no triangles, so *N* (*v*) is a stable set and thus, for each *v* of degree more than one, *N* [*v*] is a maximal biclique. Notice also that if *u* is adjacent to *v*, *N* [*u*] and *N* [*v*] contain a common edge, therefore the vertices in *KBe*(*G*) corresponding to the bicliques *N* [*u*] and *N* [*v*] will be adjacent. We can conclude that *KBe*(*G*) is exactly the graph induced by all vertices of degree at least two of *G*. For *k* big enough, the only vertices left in *KBk*(*G*) are those which belong to cycles or to paths connecting cycles, that is, *G* converges under the operator *KBe* towards the graph induced by the cycles and paths connecting cycles of *G*. *2*

*e*

As an immediate result of Theorem [3.2](#_bookmark1), we obtain the following corollary.

**Corollary 3.3** *If G has girth at least ﬁve and has no vertices of degree one, then*

*KBe*(*G*)= *G.*

One natural question that arises from Corollary [3.3](#_bookmark2) is: Given a graph *G* such that *KBe*(*G*)= *G*, does *G* have girth at least five and no vertices of degree one? The answer is no, for instance, the graph *C*7 shown in Figure [1](#_bookmark3) satisfies that *KBe*(*G*)= *G* but its girth is three [2](#_bookmark4) .



Fig. 1. The graph *C*7 is the smallest graph satisfying *KBe*(*G*)= *G* with girth less than five.

2 Found using the computer.

From Theorem [3.2](#_bookmark1), we also obtain the following result.

**Corollary 3.4** *For every k ≥* 1*, there is a graph that converges in k steps under* *the operator KBe.*

**Proof.** Just take any induced cycle *Cn*, *n ≥* 5, and join one of its vertices to the endpoint of a simple path *Pk*. Observe that this graph converges to *Cn* in exactly *k* steps. *2*

**Corollary 3.5** *Trees converge to the empty graph under the operator KBe.*

# Divergence

In this section we study the divergence of the operator *KBe*. We start first with the following definition.

**Definition 4.1** Let *G* be a graph and let *C* = *v*0*v*1 *vn—*1 be an induced cycle of

length *n ≥* 5. We say that *C* has *good neighbors* whenever for all vertices *v ∈ G−C*, if *{vi—*1*, vi*+1*} ⊆ N* (*v*) then *vi ∈ N* (*v*), for *i* = 0*,.* *,n −* 1 and all subindices taken

(*mod n*). (see Fig [2](#_bookmark5)).

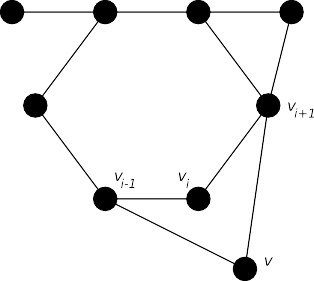
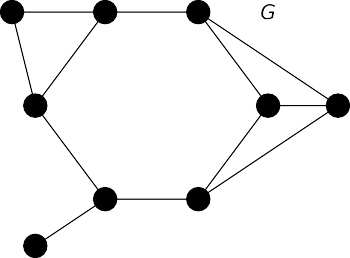


Fig. 2. *G* has a cycle with *good neighbors* while *G′* has not, since *v* is adjacent to *vi−*1 and *vi*+1 but not adjacent to *vi*.

Now we present an important proposition that assures that the good neighbors property is invariant through the iterations of the operator *KBe*.

**Proposition 4.2** *Let G be a graph and let C* = *v*0*v*1 *... vn—*1 *be an induced cycle of length n ≥* 5 *with good neighbors. Let Bi, i* = 0*,...,n −* 1*, be bicliques in G containing the vertices {vi—*1*, vi, vi*+1*}* (*mod n*)*, respectively, Bi ⊆ N* [*vi*]*, and let bi, i* = 0*,...,n −* 1*, be the vertices in KBe*(*G*) *corresponding to the bicliques Bi ∈ G. Then Cj* = *b*0*b*1 *... bn—*1 *is an induced cycle of KBe*(*G*)*. Moreover, Cj has good neighbors.*

**Proof.** As *C* is an induced cycle in *G*, let *Bi*, *i* = 0*,...,n −* 1, be bicliques that contain the vertices *{vi—*1*, vi, vi*+1*}* (*mod n*), respectively. Clearly, each *Bi* intersects *Bi*+1 in the edge *vivi*+1, therefore if we call *bi*, *i* = 0*,...,n −* 1, the corresponding vertices in *KBe*(*G*) to the bicliques *Bi*, then we have that *b*0*b*1 *... bn—*1 form a cycle *Cj* in *KBe*(*G*). Now, let *v ∈ G* be a vertex in *Bi − {vi—*1*, vi, vi*+1*}*. As *Bi* is a

biclique of *G*, either *v* is adjacent to *vi—*1 and *vi*+1 but not adjacent to *vi*, which is not possible because *C* has good neighbors, or *v* is adjacent to *vi*. Therefore, for all *i* = 0*,...,n −* 1, *Bi ⊆ N* [*vi*] and *Cj* is an induced cycle of *KBe*(*G*).

Now, let *b ∈ KBe*(*G*) *− Cj* be a vertex such that *{bi—*1*, bi*+1*}⊆ N* (*b*) for some *i*. If *B* is the biclique of *G* corresponding to the vertex *b ∈ KBe*(*G*), then *B* contains *vi—*1 and *vi*+1, since *Bi—*1 *⊆ N* [*vi—*1] and *Bi*+1 *⊆ N* [*vi*+1]. As *vi—*1 and *vi*+1 are not adjacent in *G*, there exists a vertex *v ∈ B ∩ Bi—*1 *∩ Bi*+1 such that *v* is adjacent to both *vi—*1 and *vi*+1. If *v /*= *vi*, since *C* has good neighbors, *v* must also be adjacent to *vi*, contradicting the fact that *v ∈ Bi—*1 (or *Bi*+1). Therefore, *v* = *vi* and *B* and *Bi* have an edge in common, that is, *b* is adjacent to *bi* in *KBe*(*G*) and thus *Cj* has good neighbors. *2*

Before the main theorem, we define the following family of graphs.

**Definition 4.3** For *n ≥* 3 and *m ≥* 1, the (*n, m*)*−necklace* graph on *n*+*m* vertices consists of an induced cycle *Cn* and a complete graph *Km*, such that for an edge *e ∈ Cn*, every vertex of the *Km* is adjacent to both endpoints of *e*. (see Fig [3](#_bookmark7)).

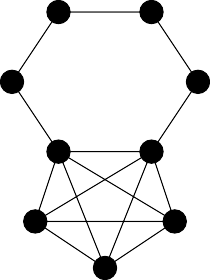




Fig. 3. (5*,* 1) *— necklace* and (6*,* 3) *— necklace* graphs.

Now we present the main theorem of this section.

**Theorem 4.4** *Let G be a graph that contains an induced* (*n, m*) *−necklace, n ≥* 5*,*

*m ≥* 1*, such that its cycle has good neighbors. Then, either KB*2(*G*) *or KB*3(*G*)

*e e*

*contains an induced* (*n, mj*) *− necklace such that its cycle has good neighbors, and*

*mj > m.*

**Proof.** Let *Cn* = *v*0*v*1 *... vn—*1 be the induced cycle and *Km* = *{w*1*,..., wm}* be the complete graph of the (*n, m*) *− necklace*, respectively. Let *vivi*+1, for some *i ∈ {*0*,...,n −* 1*}* (*mod n*), be the edge of the *Cn* such that *wj* is adjacent to *vi* and *vi*+1 for all *j* = 1*,..., m*. Let *Bt*, *t* = 0*,...,n −* 1, be bicliques that contain the vertices *{vt—*1*, vt, vt*+1*}* (*mod n*), respectively, and let *bt*, *t* = 0*,...,n −* 1, be the corresponding vertices in *KBe*(*G*) to the bicliques *Bt*. By Proposition [4.2](#_bookmark6),

*j* = *b*0*b*1 *... bn—*1 is an induced cycle in *KBe*(*G*) with good neighbors.

*C*

*n*

Consider the following two families of bicliques *B*1 = *{B*1 : *{wj, vi, vi—*1*} ⊆*

*j*

*B*1*,j* = 1*,..., m}* and *B*2 = *{B*2 : *{wj, vi*+1*, vi*+2*} ⊆ B*2*,j* = 1*,..., m}*. Clearly, all

*j j j*

these 2*m* bicliques are different and moreover, they are different to the bicliques *Bt*

for *t* = 0*,...,n −* 1 as *Cn* has good neighbors. Now we can see that (T*m B*1) *∩*

*j*=1

*j*

*Bi—*1 *∩ Bi* = *{vi—*1*, vi}* and (T*m B*2) *∩ Bi*+1 *∩ Bi*+2 = *{vi*+1*, vi*+2*}*. Therefore if *b*1

*j*=1 *j j*

and *b*2, *j* = 1*,..., m*, are the corresponding vertices in *KBe*(*G*) to the bicliques *B*1

*j*

and *B*2, we have that in *KBe*(*G*), *K*1

= *{b*1*,..., b*1 *}* and *K*2

*j*

= *{b*2*,..., b*2 *}* are

*j m* 1 *m m* 1 *m*

two complete graphs such that *b*1 is adjacent to *bi—*1 and *bi*, and *b*2 is adjacent to

*j j*

*bi*+1 and *bi*+2, for all *j* = 1*,..., m*. Notice that as *Cn* has good neighbors in *G*, then

in *KBe*(*G*) we have *N* (*b*1) *∩ Cj* = *{bi—*1*, bi}* and *N* (*b*2) *∩ Cj*

= *{bi*+1*, bi*+2*}*, for all

*j n j n*

*j* = 1*,..., m*.

Now, let *B*˜*t*, *t* = 0*,...,n −* 1, be the bicliques of *KBe*(*G*) that contain the vertices *{bt—*1*, bt, bt*+1*}* (*mod n*), respectively, and ˜*bt*, *t* = 0*,...,n −* 1, be the cor-

responding vertices in *KB*2(*G*) to the bicliques *B*˜*t*. Again, by Proposition [4.2](#_bookmark6),

*e*

*Cjj* = ˜*b*0˜*b*1 *...* ˜*bn—*1 is an induced cycle in *KB*2(*G*) with good neighbors.

*n e*

Now for each *b*1, *j* = 1*,..., m*, we have that *{b*1*, bi, bi*+1*}* is contained in a biclique

*j j*

*B*˜1. Similarly, for each *b*2, *j* = 1*,..., m*, *{b*2*, bi, bi*+1*}* is contained in a biclique *B*˜2.

*j*

*j*

*j*

*j*

In the worst case (to minimize the number of bicliques), if there is exactly a perfect

matching between *K*1 and *K*2 , say *b*1 is adjacent to *b*2, for each *j* = 1*,..., m*, then

*m m j j*

*B*˜1 = *B*˜2. We have the following two cases:

*j j*

**Case A**: *There is at least one vertex b*1 *∈ K*1 *not adjacent to any vertex of K*2 *.*

1 *m m*

Clearly, *B*˜1 */*= *B*˜2, for all *j* = 1*,..., m*, and furthermore, these *m* + 1 bicliques are

1

*j*

different to the bicliques *B*˜*t* for *t* = 0*,...,n −* 1. Observe that (T*m B*˜2) *∩ B*1 =

*j*=1

*j*

1

*{bi, bi*+1*}* and moreover, *B*˜*i ∩B*˜*i*+1 = *{bi, bi*+1*}*. Therefore, if ˜*b*1 and ˜*b*2, *j* = 1*,..., m*,

1

*j*

are the corresponding vertices in *KB*2(*G*) to the bicliques *B*˜1 and *B*˜2, respectively,

*e*

1

*j*

we have that in *KB*2(*G*), *{*˜*b*1*,* ˜*b*2*,...,* ˜*b*2 *}* is a complete graph on *m* + 1 vertices such

*e*

1

1

*m*

that, as *Cj* has good neighbors, every vertex of this *Km*+1 is only adjacent to ˜*bi* and

*n*

to ˜*bi*+1 on the cycle *Cjj*. That is, *KB*2(*G*) contains an induced (*n, m* + 1) *−necklace*

*n*

*e*

such that its cycle *Cjj* has good neighbors. See Fig [4](#_bookmark9) for an example where *K*1 and

*n m*

2 have no edge in-between.

*K*

*m*

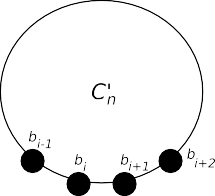


Fig. 4. Graphs *G* containing an (*n, m*) *— necklace*, *KBe*(*G*), and *KB*2(*G*) containing an (*n,* 2*m*) *— necklace*.

*e*

*m*

**Case B**: *Every vertex of K*1

*m*

*is adjacent to at least one vertex of K*2

*(and by*

*symmetry every vertex of K*2 *is adjacent to at least one vertex of K*1 *).* As explained

*m*

*m*

above, the worst case is when there is a perfect matching between *K*1 and *K*2 .

*m*

*m*

Without loss of generality, suppose that *b*1 is adjacent to *b*2 for each *j* = 1*,..., m*,

*j j*

otherwise we would obtain at least *m*+1 bicliques having the edge *bibi*+1 in common

and therefore *KB*2(*G*) will contain an induced (*n, m* + 1) *− necklace* such that its

*e*

*m*

*m*

cycle *Cjj*

*n*

has good neighbors. As there is a matching between *K*1

and *K*2 , let

*B*˜*j*

*j*

be the bicliques that contain the set *{b*1*, bi, bi*+1*, b*2*}* for each *j* = 1*,..., m*.

These bicliques contain the edge *bibi*+1 and they are different to the bicliques *B*˜*t* for *t* = 0*,...,n −* 1. Then, if ˜*bj* , *j* = 1*,..., m*, are the corresponding vertices in

*j*

*j*

*j*

*KB*2(*G*) to the bicliques *B*˜*j* , we have that in *KB*2(*G*), *{*˜*bj ,...,* ˜*bj }* is a complete

*e*

*j*

*e*

1

*m*

graph on *m* vertices such that, as *Cj* has good neighbors, every vertex of this *Km*

*n*

is only adjacent to ˜*bi* and to ˜*bi*+1 on the cycle *Cjj*.

*n*

Now for each *b*1, *j* = 1*,..., m*, we have that *{b*1*, bi—*1*, bi—*2*}* is contained in a

*j j*

biclique *B*˜1. All these *m* bicliques have the edge *bi—*1*bi—*2 in common. In addition,

*j*

they are clearly different to the bicliques *B*˜*t*, *t* = 0*,...,n −* 1 and *B*˜*j* , *j* = 1*,..., m*. Suppose now that there is an edge in common between the bicliques, say *B*˜1 and

*j*

1

*B*˜*j* . Then, there must exist a vertex *b ∈ KBe*(*G*) adjacent to *bi—*2*, b*1 and *bi*+1.

1

1

This implies that in *G*, there must exist a biclique *B* (corresponding to the vertex

*b ∈ KBe*(*G*)) that has edges in common with the bicliques *Bi—*2*, B*1

1

and *Bi*+1.

Therefore, as *Cn* has good neighbors, there must a vertex *v ∈ B* adjacent to the vertices *vi—*2 and *vi*+1. Finally, as *B* has an edge in common with the biclique *B*1, *v* must be adjacent to either to *vi*, or to *vi—*1 and *w*1. In both cases we obtain a contradiction as *B* would contain either the *K*3 = *{v, vi, vi*+1*}* or the *K*3 =

1

*{v, vi—*2*, vi—*1*}* which is not possible if *B* is a biclique. We can conclude then that there are no edges in common between the bicliques *B*˜1 and *B*˜*j* , for all *j* = 1*,..., m*.

*j*

*j*

Now let ˜*b*1 be the vertices in *KB*2(*G*) corresponding to the bicliques *B*˜1 of *KB*(*G*),

*j*

*e*

*j*

for *j* = 1*,...,m* respectively. Then, these vertices form a *Km* in *KB*2(*G*) and they

*e*

are only adjacent to the vertices ˜*bi—*2 and ˜*bi—*1 of the cycle *Cjj*.

*n*

Now, let *βt*, *t* = 0*,...,n −* 1, be bicliques of *KB*2(*G*) that contain the vertices

*e*

*{*˜*bt—*1*,* ˜*bt,* ˜*bt*+1*}* (*mod n*), respectively, and *β*˜*t*, *t* = 0*,...,n −* 1, the corresponding vertices in *KB*3(*G*) to the bicliques *βt*. By Proposition [4.2](#_bookmark6), *Cjjj* = *β*˜0*β*˜1 *... β*˜*n—*1 is

*e*

*n*

an induced cycle in *KB*3(*G*) with good neighbors. To finish, consider the following two families of bicliques: *β*1 = *{β*1 : *{*˜*b*1*,* ˜*bi—*1*,* ˜*bi}⊆ β*1*,j* = 1*,..., m}* and *β*2 = *{β*2 :

*e*

*j*

*j*

*j*

*j*

*{*˜*bj ,* ˜*bi—*1*,* ˜*bi} ⊆ β*2*,j* = 1*,..., m}*. Clearly, all these 2*m* bicliques are different as

*j*

*j*

there are no edges in common between the bicliques *B*˜1 and *B*˜*j* , for all *j* = 1*,..., m*,

*j*

*j*

and moreover, they are different to the bicliques *βt* for *t* = 0*,...,n −* 1 as *Cjj* has good neighbors. Since all these 2*m* bicliques contain the edge ˜*bi—*1˜*bi*, then if *β*˜1 and *β*˜2, *j* = 1*,..., m*, are the corresponding vertices in *KB*3(*G*) to the bicliques *β*1 and

*n*

*j*

*j*

*e*

*j*

*β*2, respectively, we have that in *KB*3(*G*), *{β*˜1*,..., β*˜1 *, β*˜2*,..., β*˜2 *}* is a complete

*j*

*e*

1

*m*

1

*m*

graph on 2*m* vertices such that, as *Cjjj* has good neighbors, every vertex of this *K*2*m*

*n*

is only adjacent to *β*˜*i* and to *β*˜*i—*1 on the cycle *Cjjj*. That is, *KB*3(*G*) contains an

*n*

*e*

induced (*n,* 2*m*) *− necklace* such that its cycle *Cjjj* has good neighbors. See Fig [5](#_bookmark10)

*n*

for a representation of this case. *2*

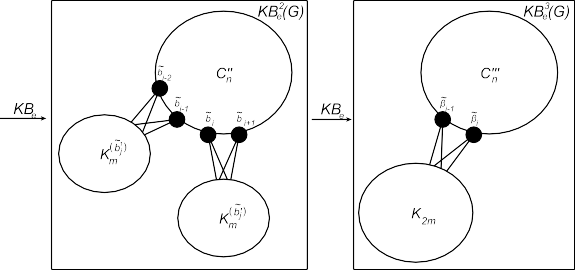
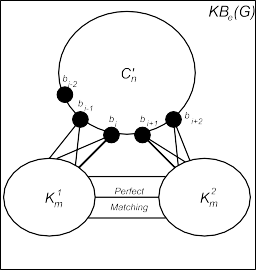


Fig. 5. Graphs *KBe*(*G*) with a perfect matching between *K*1 and *K*2 , *KB*2(*G*) with no edges between

*m m e*

the complete graphs, and *KB*3(*G*) containing an (*n,* 2*m*) *— necklace*.

*e*

As a corollary, we obtain the following divergence theorem.

**Theorem 4.5** *Let G be a graph that contains an induced* (*n, m*) *−necklace, n ≥* 5*, m ≥* 1*, such that its cycle has good neighbors. Then G diverges under the operator KBe.*

**Proof.** Applying Theorem [4.4](#_bookmark8) several times, we obtain that either *KB*2(*G*) or *KB*3(*G*) contains an induced (*n, mj*) *− necklace*, and *mj > m*, then that either *KB*4(*G*), *KB*5(*G*) or *KB*6(*G*) contains an induced (*n, mjj*)*−necklace*, and *mjj > mj*,

*e*

*e*

*e e e*

etc, all having its cycles with good neighbors. Therefore, *G* is divergent under the

operator *KBe* as lim*k→∞ |V* (*KBk*(*G*))*|* = *∞*. *2*

*e*

To finish the section, we obtain a second corollary.

**Corollary 4.6** *Let G be a graph and let Cn be an induced cycle of length n ≥* 5 *with good neighbors. If there is a vertex v ∈ G − Cn such that N* (*v*) *∩ Cn has at least one edge and not all Cn, then G diverges under the operator KBe.*

**Proof (Sketch)** Observe that either *G*, *KBe*(*G*) or *KB*2(*G*), contain an (*n,* 1) *−* *necklace*, *n ≥* 5, such that its cycle has good neighbors. Therefore *G* diverges under the operator *KBe* following Theorem [4.5](#_bookmark11). *2*

*e*

# 5 Open problems

We propose the following conjectures.

**Conjecture 5.1** *A graph G is either divergent or convergent under the KBe oper- ator but never periodic (with period bigger than* 1*).*

**Conjecture 5.2** *G* = *KBe*(*G*) *if and only if either G* = *C*7*, G* = *G*9 *(see Fig.* [*6*](#_bookmark16)*) or G has girth at least ﬁve and has no vertices of degree one.*

Note that Corollary [3.3](#_bookmark2) along with the fact that *KBe*(*C*7)= *C*7, *KBe*(*G*9)= *G*9 prove the “only if” part of Conjecture [5.2](#_bookmark13).

**Conjecture 5.3** *It is computable to decide if a graph diverges or converges under the operator KBe.*

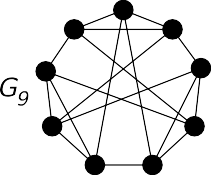




Fig. 6. Graphs *C*7 and *G*9 satisfying *KBe*(*G*)= *G* with girth less than five.

**Conjecture 5.4** *A graph G is divergent under the operator KBe if and only if there exists some k such that KBk*(*G*) *contains an induced* (*n, m*)*−necklace, n ≥* 5*,* *m ≥* 1*, with its cycle having good neighbors.*

*e*

Clearly Theorem [4.5](#_bookmark11) proves the “only if” part of Conjecture [5.4](#_bookmark15) and moreover, the “if” part along with Conjecture [5.1](#_bookmark12) imply Conjecture [5.3](#_bookmark14).

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