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On the Relationship between Filter Spaces and Weak Limit Spaces

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**Abstract**

Countably based filter spaces have been suggested in the 1970’s as a model for recursion theory on higher types. Weak limit spaces with a countable limit base are known to be the class of spaces which can be handled by the Type-2 Model of Effectivity (TTE). We prove that the category of countably based filter spaces is equivalent to the category of weak limit spaces with a countable limit base. As a consequence we obtain that filter spaces form yet another category from which the category QCB of quotients of countably based topological spaces inherits its cartesian closed structure.

*Keywords:* Filter Spaces, Weak Limit Spaces, QCB-Spaces, Higher Type Computability

# Introduction

A category of spaces designed for modelling higher type computation should have the property of being cartesian closed. Cartesian closedness means that finite prod- ucts and function spaces can be constructed inside a category. Computation on non-discrete spaces requires to deal with approximations. The usual mathematical tool for modelling approximations are topological spaces. Unfortunately, the cate- gory Top of topological spaces lacks the property of being cartesian closed. However, there exist several relevant cartesian closed subcategories and supercategories of Top which can be used as an alternative. Examples of cartesian closed supercategories of Top are the category of equilogical spaces [[4](#_bookmark19)] and the category of *ﬁlter spaces* [[9](#_bookmark24)].

In the 1970’s, M. Hyland established the relevance of filter spaces for computa- tion on the continuous functionals over N, see [[9](#_bookmark24)]. At least three notions of *ﬁlter spaces* exist in the literature. All notions endow a set *X* with a convergence relation between filters on *X* and points of *X* subject to certain axioms (cf. Section [2.1](#_bookmark1)).

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The most general notion is the one considered in [[9](#_bookmark24)]. In this paper we are interested in the slightly less general notion of *canonical* filter spaces: They enjoy the property that convergence of a filter to a point *x* depends solely on those sets in the filter that contain *x*. This matches with the intuition that a filter should converge to *x* if it is viewed to contain enough properties of *x* providing sufficient information about *x*. A third notion of filter spaces and a comparison of the three notions can be found in [[7](#_bookmark23)]. Note that some authors call filter spaces *convergence spaces*. We prefer the name “filter space”, because the name “convergence space” is also used for other kinds of spaces.

From the perspective of computability theory, we are particularly interested in filter spaces with a countable basis. In these spaces, a countable set of properties suffice to describe the elements of the space. The presence of a countable basis provides a handle to define computable functions between countably based filter spaces by means of enumeration operators.

Weak limit space [[12](#_bookmark27)] are a generalisation of limit spaces [[10](#_bookmark25),[11](#_bookmark26)] and thus of sequential topological spaces [[5](#_bookmark21)]. The constitutive structure of a weak limit space is a convergence relation between sequences and points (cf. Section [2.2](#_bookmark3)). Weak limit spaces play an important role in Weihrauch’s representation-based approach

to Computable Analysis, the *Type-2 Theory of Effectivity* (TTE) [[16](#_bookmark30)]: The class of

spaces admitting an *admissible* (i.e. continuously well-behaved) multi-representation is exactly the class of weak limit space with a countable limit base (cf. [[13](#_bookmark28),[12](#_bookmark27)]).

In Section [3](#_bookmark6) we construct an embedding of the weak limit spaces into the category of canonical filter spaces which preserves countable products. In Section [4](#_bookmark11) we present and prove our main result. It states that the category *ω*WLim of weak limit spaces with a countable limit base is equivalent to the category *ω*CFil of countably based canonical filter spaces. Thus countably based canonical filter spaces are already characterised by the apparently simpler concept of sequence convergence: they are simply weak limit spaces in different guise. The existence of countable basis is crucial for this result.

An important cartesian closed subcategory of Top is the category QCB0 of qcb- spaces satisfying the *T*0-property. A *qcb-space* is defined to be a *q* uotient of a *c*ountably *b*ased topological space. Remarkably, QCB0 has the property of inheriting its cartesian closed structure from many interesting cartesian closed supercategories relevant to higher type computability. Examples are the categories of: equilogical spaces, compactly generated spaces, weak limit spaces, Baire space representations, cf. [[3](#_bookmark20),[6](#_bookmark22),[11](#_bookmark26),[12](#_bookmark27)]. The category of countably based equilogical spaces exemplifies the domain-theoretic approach to Computable Analysis, whereas the category of Baire space representations describes the TTE approach. Hence QCB0 qualifies as a con- venient category for modelling higher type computation [[2](#_bookmark18)]. Our main theorem implies that the aforementioned categories of filter spaces belong to this list. This answers positively a question in [[2](#_bookmark18),[14](#_bookmark29)]. Moreover, at least in topological terms, the approach to higher type computation via countably based (canonical) filter spaces agrees with the TTE approach.

Since this is an extended abstract, most of the proofs are omitted.

# Preliminaries

In this section we repeat the definitions of filter spaces and weak limit spaces to- gether with some known facts about these concepts.

* 1. *Filter Spaces and Related Notions*

Let *X* and *Y* be non-empty sets. By *P*(*X*) we denote the powerset of *X*. A *ﬁlter F* on *X* is a non-empty family of non-empty subsets of *X* which is closed under finite intersection and extension to supersets. For *x ∈ X*, we write [*x*] for the *principal* filter *{A ⊆ X | x ∈ A}*. A *ﬁlter base* Φ is a non-empty subset of *P*(*X*) *\ {∅}* such that for all *A, B ∈* Φ there is some *C ∈* Φ with *∅ /*= *C ⊆ A ∩ B*. For a filter base Φ we denote by [Φ] the smallest filter containing Φ, i.e. *{A ⊆ X | ∃B ∈* Φ*.B ⊆ A}*. Given a function *f* : *X → Y* , *f* +Φ denotes the filter base *{f* (*A*) *| A ∈* Φ*}*. Note that

*f* +[Φ] is equal to [*f* +Φ].

There are several notions of filter spaces in the literature (which are sometimes also called *convergence spaces*). Generally, a *ﬁlter space* is a pair (*X, ↓*), where *X* is a set and *↓* is a *convergence relation* between the filters on *X* and the points of *X* satisfying certain axioms. If *F ↓ x* holds, then one says that the filter *F converges to x* or that *x* is a *limit of F*. For a filter base Φ, it is convenient to write Φ *↓ x* for [Φ] *↓ x* and to say also in this case that Φ converges to *x*. Since any filter *F* is equal to [*F*], this convention does not cause ambiguity.

The least restrictive notion of filter spaces [[9](#_bookmark24)] requires a filter space (*X, ↓*) to fulfil the axioms (F1) and (F2):

(F1) [*x*] *↓ x*;

(F2) if *F ↓ x* and *F ⊆ G* then *G ↓ x*,

where *F, G* are filters on *X* and *x ∈ X*.

A function *f* : X *→* Y between filter spaces X and Y is called *ﬁlter-continuous*, if *F ↓ x* in X implies *f* +*F ↓ f* (*x*) in Y for every filter *F* of X and every *x ∈* X. Obvi- ously, composition of functions preserves filter continuity. We denote the category of filter spaces satisfying Axioms (F1), (F2) and of filter-continuous functions by Fil. In the following, we shall use Gothic letters X*,* Y*,...* for filter spaces. The carrier set of a space X will often be denoted by X as well. As long as no confusion can occur, we shall use the symbol ‘*↓*’ without decorations to denote the convergence relation of any filter space.

In this paper, we are only interested in *canonical* filter spaces (*X, ↓*). They are defined to satisfy the axioms (F1), (F2) and

(F3) if *F ↓ x* then *F ∩* [*x*] *↓ x*.

In the following, we will often use the term “filter space” to mean spaces satisfying Axioms (F1), (F2), (F3). We denote the category whose objects are the canoncial filter spaces and whose morphisms are the filter-continuous functions by CFil.

The idea behind canonical filter spaces is the following: A filter *F* is defined to converge to a point *x*, if *F* is deemed to contain enough properties of *x*, where a

*property* of *x* is any set containing *x*. The fact that our definition allows a converging filter to contain properties that do not belong to the limit slightly mismatches with this idea. A more fitting definition of filter spaces is set up by the Axioms (F1), (F2’), (F3’):

(F2’) if *F ↓ x* and *F ⊆ G ⊆* [*x*] then *G ↓ x*; (F3’) if *F ↓ x* then *F ⊆* [*x*].

However, the corresponding category PFil can easily be verified to be equivalent to CFil.

The category CFil has countable products and is cartesian closed (see [[7](#_bookmark23)], where CFil is denoted by FIL*b*). Given a sequence of filter spaces (X*i*)*i*, the carrier set of the product *i∈*N X*i* is the cartesian product of the carrier sets and its convergence relation is given by

*F ↓ x ⇐⇒ ∀i ∈* N*.* pr+*F ↓* pr*i*(*x*) in X*i ,*

*i*

where pr*i* denotes the respective set-theoretic projection function. The exponential YX in CFil for canonical filter spaces X and Y has the set *C*(X*,* Y) of filter-continuous functions as its carrier set and its convergence relation is defined as follows: a filter *F* on *C*(X*,* Y) converges to a function *f ∈ C*(X*,* Y) iff

*A↓ x* implies [*{F · A | F ∈F ∩* [*f* ]*, A ∈ A}*] *↓ f* (*x*)

for every filter *A* on X and every *x ∈ X*, where *F · A* := *{g*(*a*) *| g ∈ F, a ∈ A}*. Note that the function space construction of CFil is different from the construction for Hyland’s larger category Fil. However, for CFil-spaces satisfying the Merging Axiom

(F4) if *F ↓ x* and *G ↓ x* then *F ∩ G ↓ x*

the function space constructions in Fil and CFil agree, cf. [[7](#_bookmark23)]. We denote the full subcategory of filter spaces satisfying Axioms (F1)–(F4) by MFil. R. Heckmann analyses in [[7](#_bookmark23)] the relationship between equilogical spaces (which is a cartesian closed supercategory of Top) and the three notions of filter spaces.

The importance of filter spaces lies in the fact that CFil presents a cartesian closed supercategory of the non cartesian closed category Top of topological spaces [[7](#_bookmark23),[9](#_bookmark24)]. The corresponding embedding functor *I*Top : Top *‹→* MFil maps a topological space *Z* to the filter space which has the same carrier set and whose convergence relation is given by: *F ↓ z* iff *F* contains the neighbourhood filter [*{U* open *| z ∈ U}*] of *z*.

A family *B* of subsets of X is called a *basis* for a filter space X, if for every filter *F* converging to some *x ∈* X the family *F ∩ B* is a filter base converging to

*x*. Note that in contrast to topological bases, filter space bases do not characterise filter spaces; indeed, the powerset *P*(*Y* ) of *Y* is a basis for every filter space with carrier set *Y* . By *ω*CFil, we denote the category of countably based canonical filter

spaces as objects and filter-continuous functions as morphisms, and by *ω*MFil the

full subcategory of those filter space that satisfy the Merging Axiom (F4). Filter spaces bases relate to topological bases in the following way:

**Lemma 2.1** *Any topological base for a topological space Z is a ﬁlter space basis for*

*I*Top(*Z*)*. If B is a basis for I*Top(*Z*)*, then {*Int(*B*) *| B ∈ B} is a topological base for*

*Z, where* Int(*B*) *denotes the interior of B.*

**Proof.** Omitted.

The categories *ω*CFil and *ω*MFil are cartesian closed as well, because forming countable products and forming expontentials in CFil preserve the existence of a countable basis. Given bases *A* and *B* for canonical filter spaces X and Y, one can show similar to [[9](#_bookmark24)] that

*D* := *A × B* *A ∈ A, B ∈ B*}

is a countable basis for the product X *⊗* Y in CFil and that the closure under finite intersection of the family

*E* := *{f ∈ C*(X*,* Y) *| f* (*A*) *⊆ B}* *A ∈ A, B ∈ B*}

is a countable basis for the exponential YX in CFil.

For functions between countably based filter spaces one can introduce a reason- able notion of computability by considering numberings of the respective bases. We modify slightly the tentitive definition in [[9](#_bookmark24)] and call a triple (*X, ↓, α*) a *coded ﬁlter space*, if (*X, ↓*) is a canonical filter space and *α* is a numbering of a basis for (*X, ↓*). A function *f* between two coded filter spaces (*X, ↓, α*) and (*Y, ↓, β*) is defined to be *computable* iff there is a computable function *g* : *P*(N) *→ P*(N) such that

*x ∈* *αi ∧ {αi | i ∈ I}↓ x* implies *f* (*x*) *∈*

*βj ∧ {βj | j ∈ g*(*I*)*}↓ f* (*x*)

*i∈I j∈g*(*I*)

for all *x ∈ X* and all *I ⊆* N. It is not surprising that computable functions between coded filter spaces are filter-continuous.

* 1. *Weak Limit Spaces and Related Notions*

Let *X* and *Y* be sets. We write sequences *x* : N *→ X* over *X* as (*xn*)*n* or (*xn*)*n∈*N and *generalised* sequences *x* : N+ *→ X* as (*xn*)*n≤∞* or (*xn*)*n∈*N+ , where N+ := N *∪ {∞}*. By [(*xn*)*n*] we denote the *Fr´echet ﬁlter M ⊆ X* *∃n ∈* N*.{xn, xn*+1*,... }⊆ M* and by [(*xn*)*n, y*] the filter [(*xn*)*n*] *∩* [*y*] which is a Fr´echet filter as well.

}

A *weak limit space* ([[13](#_bookmark28),[12](#_bookmark27)]) equips a set *X* with a *convergence relation →* be- tween sequences (*xn*)*n* and points *x* of *X*. If (*xn*)*n → x∞*, then we say that (*xn*)*n* converges to *x∞* in the space (*X, →*) or that *x∞* is a limit of the sequence (*xn*)*n*. The convergence relation of a weak limit space (*X, →*) is required to satisfy the following axioms:

(L1) (*x*)*n → x*;

(L4) if (*xn*)*n → x∞* and (*ξn*)*n →∞* in (N+*, τ*N+ ) then (*xξn* )*n → x∞*; (L5) if (*xn*+1)*n → x∞* and *x*0 *∈ X* then (*xn*)*n → x∞*.

Here *τ*N+ denotes the standard topology *U ⊆* N+ *∞∈ U* =*⇒ U ∈* [(*n*)*n*]} on N+.

Weak limit spaces are a generalisation of Kuratowski’s *limit spaces* [[10](#_bookmark25)] (called *L-spaces* in [[9](#_bookmark24)]). The convergence relation of a limit space (*X, →*) is subject to the axioms (L1), (L2), (L3):

(L2) if (*xn*)*n → x∞*, then (*yn*)*n → x∞* for every subsequence (*yn*)*n* of (*xn*)*n*;

(L3) if (*xn*)*n /→ x∞*, then (*xn*)*n* has a subsequence (*yn*)*n* such that (*zn*)*n /→ x∞* for every subsequence (*zn*)*n* of (*yn*)*n*.

Any weak limit space fulfils Axiom (L2), but not necessarily Axiom (L3). The convergence relation of a topological space *Z* satisfies the axioms of a limit space. We denote the corresponding weak limit space by *L*Top(*Z*). A weak limit space is called *topological*, if it lies the image of *L*Top.

A function *f* between two weak limit spaces X and Y is called *sequentially con- tinuous*, if it preserves convergent sequences [2](#_bookmark5) . By WLim we denote the category whose objects are the weak limit spaces and whose morphisms are the sequentially continuous functions.

Given a weak limit space X, we call a family Φ *⊆ У*(X) a *witness of convergence* for an element *x*, if Φ *⊆* [*x*] and, for every sequence (*yn*)*n*, Φ *⊆* [(*yn*)*n*] implies (*yn*)*n → x* in X. For example, any neighbourhood base of a point *z* in a topological space is a witness of convergence for *z*.

**Lemma 2.2** *In a weak limit space, a sequence* (*xn*)*n converges to an element x∞*

*if and only if the Fr´echet ﬁlter* [(*xn*)*n, x∞*] *is a witness of convergence for x∞.*

**Proof.** This is an easy consequence of the axioms of weak limit spaces.

One can show that a relation ~ *⊆ Y* N *× Y* satisfies the axioms of a weak limit space if and only if it satisfies Axiom (L1) and the equivalence in Lemma [2.2](#_bookmark4).

A *limit base* for a weak limit space X is a family *У ⊆ У*(X) such that for every element *x ∈* X and every sequence (*yn*)*n* converging to *x*, *У* contains a witness of

convergence Φ for *x* such that Φ *⊆* [(*yn*)*n*]. This implies that for every convergent sequence (*xn*)*n≤∞* of X and for every sequence (*zn*)*n* that does not converge to *x∞*

there is some *B ∈ У* such that *x∞ ∈ B*, *xn ∈ B* for almost all *n* and *zm ∈/ B* for infinitely many *m*. Similar to bases of filter spaces, the powerset of a set *Y* is a limit base for every weak limit space with carrier set *Y* . By *ω*WLim we denote the

full subcategory of WLim consisting of all weak limit spaces admitting a countable

limit base.

Both categories WLim and *ω*WLim have countable products and are cartesian closed [[12](#_bookmark27),[13](#_bookmark28)]. The product *i∈*N X*i* of a sequence (X*i*)*i* of weak limit spaces is constructed as one expects. The exponential YX is obtained by equipping the set *C*(X*,* Y) of sequentially continuous functions from X to Y with the convergence rela- tion $ of *continuous convergence* defined by: (*fn*)*n* $ *f∞* iff (*fξn* (*xn*))*n → f* (*x∞*) holds in Y for all (*xn*)*n → x∞* in X and all (*ξn*)*n → ∞* in (N+*, τ*N+ ). Given count-

able limit bases *Уi* for weak limit spaces X*i* closed unter finite intersection, countable

limit bases for the product *i∈*N X*i* and the exponential X are constructed by

X1

2

*Ð* := *i∈*N*Bi* *k ∈* N*, Bk ∈ Уk, 6i /*= *k. Bi* = X*i*} (1)

2 i.e., (*xn*) *→ x∞* in X implies (*f* (*xn*))*n → f* (*x∞*) in Y

and

*E* := *{f ∈ C*(X1*,* X2) *| f* ( *k Ai*) *⊆ B}* *{A*1*,..., Ak}⊆ У*1*, B ∈ У*2} (2)

*i*=1

respectively. Proofs can be found in [[12](#_bookmark27),[13](#_bookmark28)].

# Embedding Weak Limit Spaces into Filter Spaces

We define an embedding *I* of the category of weak limit spaces WLim into the category CFil of canonical filter spaces. It preserves countable products and maps countably based weak limit spaces to countably based filter spaces.

Let X = (*X, →*) be a weak limit space. We define a filter convergence relation *↓*X

on *X* by

*F ↓*X *x* :*⇐⇒ F ∩* [*x*] contains a countable witness of convergence for *x*.

Clearly, *I*(X) := (*X, ↓*X) is a canonical filter space. We call *I*(X) the filter space *associated to* X. From Lemma [2.2](#_bookmark4) we can deduce the following characterisation of convergence of sequences in a weak limit space in terms of filter convergence in the associated filter space.

**Lemma 3.1** *Let* X *be a weak limit space. Then a sequence* (*yn*)*n converges to a point x in* X *if and only if the Fr´echet ﬁlter* [(*yn*)*n*] *converges to x in I*(X)*.*

By setting *I*(*f* ) := *f* for every morphism *f* in WLim, we obtain an embedding functor from WLim to CFil.

**Proposition 3.2** *Let* X *and* Y *be weak limit spaces. Then a function f* : X *→* Y *is sequentially continuous if and only if f is a ﬁlter-continuous function from I*(X) *to I*(Y)*.*

**Proof.** *Only-if-part:* Let *f* be sequentially continuous. Let *F* be a filter that converges to some *x∞* in *I*(X). There is a sequence (*Fi*)*i* of sets in *F ∩* [*x∞*] such

that *{Fi | i ∈* N*}* is a witness of convergence for *x*. We define *Gj* := *f* ( *j Fi*) *∈*

*i*=0

[*f* +*F*] *∩* [*f* (*x∞*)]. Let (*yn*)*n* be a sequence with *{Gj | j ∈* N*} ⊆* [(*yn*)*n*]. Then there

are natural numbers *m*0 *< m*1 *< m*2 *...* such that *yn ∈ Gj* for all *j ∈* N and

*n ≥ mj*. For *j ∈* N and *n ∈ {mj,..., mj*+1 *−* 1*}* we choose some *xn ∈* *j Fi* with

*i*=0

*f* (*xn*) = *yn*. Then (*xn*)*n* converges *x∞*, because we have *{Fi | i ∈* N*} ⊆* [(*xn*)*n*]. By sequential continuity, (*yn*)*n* converges to *f* (*x∞*). Hence the family *{Gj | j ∈* N*}* constitutes a countable witness of convergence for *f* (*x∞*) contained in *f* +*F*,

implying *f* +*F ↓*Y *f* (*x∞*). Therefore *f* is filter-continuous.

*If-part:* Let *f* be filter-continuous. Let (*xn*)*n≤∞* be a convergent sequence of X. By Lemma [3.1](#_bookmark7) the filter *F* := [(*xn*)*n*] *∩* [*x∞*] converges to *x∞* in *I*(X). Hence *f* +*F* converges to *f* (*x∞*). Since *f* +*F ⊆* [(*f* (*xn*))*n*], the Fr´echet filter [(*f* (*xn*))*n*] converges to *f* (*x∞*) by Axiom (F2). Hence (*f* (*xn*))*n* converges to *f* (*x∞*) by Lemma [3.1](#_bookmark7).

The functor *I* gives rise to an alternative embedding of the topological spaces into CFil, namely by the functor *I ◦ L*Top. We characterise the class of topological spaces on which *I◦L*Top agrees with the standard embedding *I*Top from Section [2.1](#_bookmark1).

**Proposition 3.3** *A topological space Z satisﬁes IL*Top(*Z*) = *I*Top(*Z*) *if and only if Z is ﬁrst-countable (i.e. every element has a countable neighbourhood base).*

**Proof.** Omitted.

From Lemma [2.1](#_bookmark2) we know:

**Corollary 3.4** *A topological space Z satisﬁes I*Top(*Z*) *∈ ω*CFil *if and only if Z has a countable topological base.*

By Proposition [3.5](#_bookmark9) the functor *I* preserves the existence of a countable base.

We denote the arising functor from *ω*WLim to *ω*CFil by *Iω*.

**Proposition 3.5** *Any countable limit base У for a weak limit space* X *is a basis for the ﬁlter space I*(X)*, provided that У ∪ {∅} is closed unter ﬁnite intersection.*

**Proof.** Omitted.

We obtain from Propositions [3.2](#_bookmark8) and [3.5](#_bookmark9):

**Theorem 3.6** *The functor I is embeds* WLim *into* CFil*. Its restriction Iω embeds*

*ω*WLim *into ω*CFil*.*

The functor *I* preserves countable products, but not exponentials. By contrast, the restriction *Iω* preserves exponentials, as we will see in Section [4](#_bookmark11).

**Proposition 3.7** *For a sequence* (X*i*)*i of weak limit spaces, we have* *i∈*N *I*(X*i*) =

*I*( *i∈*N X*i*)*.*

**Proof.** Omitted.

**Example 3.8** The functor *I* does not preserve exponentials. As an example we consider an uncountable discrete [3](#_bookmark12) space D *∈* WLim as domain space and as codomain space the two point discrete space 2 with carrier set *{*0*,* 1*}*. Since [*x*] is the only filter to converge to *x* in *I*(D), the filter

*F* := *F ⊆ C*(D*,* 2) *∃E ⊆* X finite*. 6f ∈ F. E ⊆ f−*1*{*0*}*}

converges in *I*(2)*I*(D) to the constant zero function **0**. Assume that *F* contains a countable witness for **0** in D. Then for every there is a finite set

*{Fi | i* *∈* N*}* 2 *i ∈* N

*Ei* with *f* : D *→ {*0*,* 1*}* *E ⊆ f−*1*{*0*}*} *⊆ Fi*. We define the sequentially continuous,

hence filter-continuous function *fn* : D *→ {*0*,* 1*}* by *fn*(*x*) = 0 :*⇐⇒ x ∈* *n Ei*.

*i*=0

Then *{Fi | i ∈* N*} ⊆* [(*fn*)*n*]. By uncountability of D, there exists an element *x ∈* D *\* *i∈*N *Ei*. Since (*fn*(*x*))*n /→* **0**(*x*) in 2, (*fn*)*n* does not converge to **0** in 2D, a contradiction. Therefore *F* does not converge to **0** in *I*(2D).

1. **Equivalence of** *ω*WLim **and** *ω*CFil

We prove in this section our main result that the categories *ω*CFil and *ω*WLim

are equivalent. Actually we prove the stronger result that the object-part of *Iω*

3 i.e., (*xn*)*n → x∞* if and only if *xn* = *x∞* for almost all *n ∈* N

is a bijection between the weak limit spaces with a countable limit base and the countably based canonical filter spaces.

In order to obtain the inverse of *Iω*, we define at first a retraction *W* from the filter spaces back to the weak limit spaces. Any filter space X = (*X, ↓*) induces a natural convergence relation *→*X on *X*. It is defined by

(*xn*)*n →*X *x∞* :*⇐⇒* [(*xn*)*n*] *∩* [*x∞*] converges to *x∞* in X *.*

By the next proposition, *→*X satisfies the axioms of a weak limit space. Furthermore any filter-continuous function is sequentially continuous w.r.t. the induced conver- gence relations. So setting *W*(X) := (*X, →*X) and *W*(*f* ) := *f* for any morphism in CFil defines a functor from the category of canonical filter spaces to the category of weak limit spaces.

**Proposition 4.1** *For any canonical ﬁlter space* X*, W*(X) *is a weak limit space. Moreover, any ﬁlter-continuous function f from* X *to a canonical ﬁlter space* Y *is a sequentially continuous function from W*(X) *to W*(Y)*.*

**Proof.** Omitted.

By the next lemma, *W* preserves the existence of a countable basis. We denote the arising functor from *ω*CFil to *ω*WLim by *Wω*.

**Lemma 4.2** *Any basis У for a canoncial ﬁlter space* X *is a limit base for the weak limit space W*(X)*.*

**Proof.** Omitted.

Lemma [3.1](#_bookmark7) implies that *W* is a left-inverse to *I*.

**Lemma 4.3** *Every weak limit space* X *satisﬁes WI*(X) = X*.*

By contrast, canonical filter spaces X do not necessarily satisfy *IW*(X) = X. The filter space *I*(2)*I*(D), where D and 2 are the discrete weak limit spaces from Example [3.8](#_bookmark10), yields a counterexample. Surprisingly, *IW*(X) = X does hold true, if X has countable basis.

**Proposition 4.4** *Any canonical ﬁlter space* X *with a countable basis satisﬁes*

*IW*(X) = X*.*

**Proof.** Let *У* be a countable of basis of X. Let *F* be a filter on X and *x ∈* X. First, let *F* converge to *x* in X. Then the countable filter base Φ := *F ∩* [*x*] *∩ У* converges to *x* in X. By Axiom (F2), every sequence (*yn*)*n* with Φ *⊆* [(*yn*)*n*] converges to *x* in *W*(X). Hence Φ is a witness of convergence for *x* contained in *F*. Therefore *F* converges to *x* in *IW*(X).

Conversely, let *F* converge to *x* in *IW*(X). Then *F* contains a countable witness of

convergence *{Fi | i ∈* N*}* for *x* in *W*(X). Let *Ð* := *{∅}∪ {B ∈У | 6n.* *n Fi* ¢ *B}*,

*i*=0

and let *i '→ βi* be a numbering of *Ð* with *6i ∈* N*.∃j > i. βj* = *βi*. We choose for

every *n* some *yn ∈* *n Fi \ βn*. Since *{Fi | i ∈* N*} ⊆* [(*yn*)*n*], (*yn*)*n* converges to

*i*=0

*x* in *W*(X). Hence the filter [(*yn*)*n*] *∩* [*x*] and the filter base Φ := [(*yn*)*n*] *∩* [*x*] *∩У*

converge to *x* in X. Since *Ð∩* [(*yn*)*n*] = *∅*, for every *B ∈* Φ there is some *n ∈* N with

*n i*=0

*Fi ⊆ B*, implying [Φ] *⊆ F*. Hence *F* converges to *x* in X by Axiom (F2).

By Proposition [4.4](#_bookmark14) and Lemma [4.3](#_bookmark13), the object-part of the functor *Iω* is a bi- jection between the countably based weak limit spaces and the countably based canonical filter spaces. By Proposition [3.2](#_bookmark8), for all X*,* Y *∈ ω*WLim the morphism part of *Iω* constitutes a bijection between the morphisms from X to Y and the morphisms between from *Iω*(X) and *Iω*(Y). We obtain our main result:

**Theorem 4.5** *The category ω*CFil *of countably based canonical ﬁlter spaces is equiv- alent to the category ω*WLim *of weak limit spaces with a countable limit base.*

From [[13](#_bookmark28)] we know that *ω*WLim is locally cartesian closed and has all countable limits and all countable colimits. Hence:

**Corollary 4.6** *The category ω*CFil *is locally cartesian closed and has all countable limits and all countable colimits.*

It is not difficult to verify that *Iω* maps a space X *∈ ω*WLim to a filter space satisfying the Merging Axiom (F4) if and only if X fulfils the Merging Axiom for weak limit spaces:

(L6) if (*y*2*n*)*n → x* and (*y*2*n*+1)*n → x* then (*yn*)*n → x*. We obtain:

**Theorem 4.7** *The category ω*MFil *is equivalent to the category of countably based weak limit spaces satisfying Axiom (L6).*

A *qcb-space* [[14](#_bookmark29)] is a topological space *Z* that is a topological quotient of a countably based topological space [4](#_bookmark17) . The category QCB0 of qcb-spaces satisfying the *T*0-property and of continuous functions as morphisms is cartesian closed [[13](#_bookmark28)]. It is a full subcategory of several categories in such a way that the respective em- beddings preserve finite products and exponentials. Examples are the categories of: equilogical spaces (as an example of a cartesian closed supercategory of Top), sequential spaces, compactly generated spaces, quotients of core compact spaces (as examples of cartesian closed subcategories of Top), Baire space representations (as an example of an effective category), limit spaces and weak limit spaces with a countable limit base. The corresponding proofs are contributed by several authors, see [[3](#_bookmark20),[6](#_bookmark22),[11](#_bookmark26),[12](#_bookmark27)]. Together with the fact that topological spaces satisfy the Merging Axiom (L6), Theorems [4.5](#_bookmark15) and [4.7](#_bookmark16) imply that QCB0 also lives inside the filter space categories CFil and MFil and inherits the respective constructions of products and exponentials. The same holds true for the filter space category Fil in [[9](#_bookmark24)], because functions spaces in MFil are constructed as in Fil, cf. [[7](#_bookmark23)]. This gives the expected positive answer to a question in [[2](#_bookmark18),[14](#_bookmark29)].

4 i.e., there is a surjection *q* : *A → Z* from a countably based space *A* onto *Z* satisfying: *V* open in *Z ⇐⇒*

*q−*1[*V* ] open in *A*.

# Discussion

We have seen that countably based canonical filter spaces are basically the same mathematical objects as weak limit spaces. In [[13](#_bookmark28)] computability for functions be- tween countably based weak limits spaces is introduced by endowing the spaces with multi-representations obtained from numberings of the respective limit bases. One can show that the induced computability notion is equivalent to the one gener- ated by coded filter spaces (see Section [2.1](#_bookmark1)). However, the corresponding category of coded filter spaces as objects and computable functions as morphisms does not have all finite colimits. In [[13](#_bookmark28)], an effective cartesian closed category, EffWeakLim, with finite limits and colimits is defined by imposing an effectivity condition on the used multi-representations. It would be interesting to know whether this effectivity notion has connections to effectivity properties on filter space bases similar to the ones considered in [[9](#_bookmark24)].

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