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Open-independent, Open-locating-dominating Sets in Complementary Prism Graphs

M´arcia R. Cappelle Erika M. M. Coelho Les R. Foulds Humberto J. Longo [1](#_bookmark0)*,*[2](#_bookmark0)

*Instituto de Inform´atica Universidade Federal de Goi´as Goiˆania–GO, Brazil*

**Abstract**

For a finite, simple, undirected graph *G* = (*V* (*G*)*,E*(*G*)), an open-dominating set *S ⊆ V* (*G*) is such that every vertex in *G* has at least one neighbor in *S*. An open-independent, open-locating-dominating set *S ⊆ V* (*G*) (*OLDOIND*-set for short) is such that no two vertices in *G* have the same set of neighbors in *S* and each vertex in *S* is open-dominated exactly once by *S*. The problem of deciding whether or not a given graph has an *OLDOIND*-set is known to be *NP*-complete. The complementary prism of *G* is the graph *GG*¯, formed from the disjoint union of *G* and its complement *G*¯ by adding the edges of a perfect matching between the corresponding vertices of *G* and *G*¯. We provided a logarithmic lower bound on the size of an *OLDOIND*-set in any graph. Various properties of and bounds on *OLDOIND*-sets in complementary prisms were presented and the cases of cliques, paths and cycles have been completely solved. It has been shown that for any graph with girth at least five, it can be decided in polynomial time whether or not its complementary prism has an OLD-OIND-set (and also the set can be found in polynomial time if it exists).

*Keywords:* open-independent sets, open-locating-dominating sets, complementary prisms of graphs, complexity.

# Introduction

Consider the situation where a graph *G* models a facility or a multiprocessor net- work, with limited-range detection devices placed at chosen vertices of *G*. The purpose of these devices is to detect and precisely identify the location of an in- truder such as a thief, saboteur, fire or faulty processor that may be present at any vertex. Sometimes such a device can determine if an intruder is in its neighborhood but cannot detect if the intruder is at its own location. In this case, it is required to find a so-called, *open-locating-dominating* vertex subset *S*, which is a dominating

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2 Email: *{*marcia,erikamorais,lesfoulds,longo*}*@inf.ufg.br

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set of *G*, such that every vertex in *G* has at least one neighbor in *S*, and no two vertices in *G* have the same set of neighbors in *S*. When a device may be prevented from detecting an intruder at its own location, it is necessary to install another de- vice in its neighborhood. A natural way to analyze such situations is to make use of open neighborhood sets which may have useful additional properties, such as being open-independent, dominating, open-dominating or open-locating-dominating.

Throughout this paper, lg(*x*) denotes log2(*x*), and *G* = (*V* (*G*)*, E*(*G*)) is a finite, simple, undirected graph with vertex set *V* (*G*) and edge set *E*(*G*). Let *S ⊆ V* (*G*). The subgraph of *G* induced by *S* is denoted by *G*[*S*].

The *open neighborhood* of a vertex *v ∈ V* (*G*) is *NG*(*v*) = *{u ∈ V* (*G*) *| vu ∈*

*E*(*G*)*}*, and its *closed neighborhood* is *NG*[*v*]= *NG*(*v*)*∪{v}*. The open neighborhood

of *S* is *NG*(*S*) = S

*v∈S*

*NG*(*v*) and its closed neighborhood is *NG*[*S*] = *NG*(*S*) *∪ S*.

Such a set *S* is termed *dominating* if *NG*[*S*] = *V* (*G*) and *open-dominating* (or

*total dominating) if NG*(*S*)= *V* (*G*)*.* A vertex *v ∈ V* (*G*) is *dominated* by *S* if

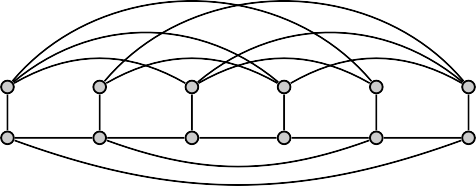
*|N* [*v*] *∩ S|* ≥ 1 and is *open-dominated* by *S* if *|N* (*v*) *∩ S|* ≥ 1. The set *S* is a *locating-dominating set* if it is a dominating set and for every pair of distinct vertices *u, v ∈ V* (*G*)*\S*, *N* (*u*)*∩S /*= *N* (*v*)*∩S*. *S* is an *open-locating-dominating set* (OLD-set for short) if it is an open-dominating set and no two distinct vertices in *G* have the same set of neighbors in *S*, i.e., for all *u, v ∈ V* (*G*), with *u /*= *v*, *N* (*u*)*∩S /*= *N* (*v*)*∩S*. For this last condition, *u* and *v* are said to be *distinguished* by *S*. This is an analogue of the well-studied identifying code problem in the literature [[10](#_bookmark38)]. The parameter *OLD*(*G*) denotes the minimum size of an OLD set *S ⊆ V* (*G*). The concept of an open-locating-dominating set was first considered by Seo and Slater [[12](#_bookmark40)]. The authors showed that to decide if a graph *G* has such a set is an *NP*-complete decision problem and provided some useful results on *OLD*-sets in trees and grid graphs. *S* is *independent* if no two vertices in *S* are adjacent, i.e., *∀ v ∈ S, |N* [*v*] *∩S|* = 1. *S* is an *open-independent set* (OIND-set for short) if every vertex in *S* is open-dominated by *S* at most once, i.e., *∀ v ∈ S, |N* (*v*) *∩ S|* ≤ 1.

If an open-independent, open-locating-dominating set (*OLDOIND*-set for short) exists in a given graph *G*, it is often of interest to establish the minimum size among such sets in *G*, which is denoted by *OLDOIND*(*G*). If *S* is an *OLDOIND*-set for *G*, each component of *G*[*S*] is isomorphic to *K*2 (a complete graph on two vertices). See, for example, the graphs in Figures [1(a)](#_bookmark1) and [1(b)](#_bookmark1), where an *OLDOIND*-set of each graph is represented by the black vertices. Seo and Slater [[13](#_bookmark41)] demonstrated that the problem of deciding whether or not a given graph has an *OLDOIND*-set is *NP*-complete. The authors also presented some results on *OLDOIND*-sets in paths, trees and infinite grid graphs, and characterized *OLDOIND*-sets in graphs with girth at least five.

Haynes et al. [[6](#_bookmark34)] introduced the so-called *complementary product* as a gener- alization of the well-known *Cartesian product* of graphs. As a particular case of complementary products, the authors define the *complementary prism* of a graph *G*, denoted by *GG*¯, as the graph formed from the disjoint union of *G* and its com- plement *G*¯ by adding the edges of the perfect matching between the corresponding vertices of *G* and *G*¯. Note that *V* (*GG*¯)= *V* (*G*)*∪V* (*G*¯). For the purposes of illustra-



(a) *G*. (b) *G*¯.

(c) *GG*¯.



*V* (*G*¯)

*V* (*G*)

Fig. 1. Example of a graph, its complement and the resulting complementary prism.

tion, a graph *G*, its complement *G*¯ and the complementary prism *GG*¯ are depicted in Figure [1](#_bookmark1). To simplify matters, we use *G* and *G*¯ to refer to the subgraph copies of *G* and *G*¯, respectively, in *GG*¯. We term as *primary* the vertices and edges in *G*, as *complementary* the vertices and edges in *G*¯ and as *matching* any edge *vv*¯ that directly connects a vertex *v ∈ V* (*G*) and a vertex *v*¯ *∈ V* (*G*¯. For a set *X ⊆ V* (*G*), let *X*¯ denote the corresponding vertices of *X* in *V* (*G*¯).

Haynes et al. [[6](#_bookmark34)] investigated several graph theoretic properties of complemen-

tary prisms, such as independence, distance and domination. For further study on domination parameters in complementary prisms, see [[1,](#_bookmark29)[3,](#_bookmark30)[4,](#_bookmark32)[7,](#_bookmark35)[8,](#_bookmark36)[9,](#_bookmark37)[11](#_bookmark39)]. Cappelle et al. [[2](#_bookmark31)] described a polynomial time recognition algorithm for complementary prisms and Duarte et al. [[5](#_bookmark33)] studied algorithmic/complexity properties of complementary prisms with respect to cliques, independent sets, domination, and convexity. Here we study open-independent, open-locating-dominating sets in the complementary prisms of graphs. Various properties of and bounds on *OLDOIND*-sets in comple- mentary prisms were presented and the cases of cliques, paths and cycles have been completely solved. Our main result is that if the girth of *G* is at least five, then the

*OLDOIND*-set of *GG*¯, if it exists, can be found in polynomial time.

# Some preliminary results for *OLDOIND*-sets

The following theorem provides necessary and sufficient conditions for the existence of an *OLDOIND*-set in a graph *G* that has girth *g*(*G*) ≥ 5. For general graphs (with arbitrary girth), the conditions stated in Theorem [2.1](#_bookmark2) are necessary but not sufficient, as is stated in Lemma [2.2](#_bookmark3).

**Theorem 2.1 ([**[**13**](#_bookmark41)**])** *If a graph G has girth g*(*G*) ≥ 5 *and S ⊆ V* (*G*)*, then S is an OLDOIND-set if and only if (i) each v ∈ S is open-dominated exactly once, and*

*(ii) each v ∈/ S is open-dominated at least twice.*

**Lemma 2.2** *If S ⊆ V* (*G*) *is an OLDOIND-set of a graph G, then (i) each v ∈ S* *is open-dominated exactly once, and (ii) each v ∈ V* (*G*) *\ S is open-dominated at least twice.*

A corollary to Lemma [2.2](#_bookmark3) is that if *S* exists, the components of *G*[*S*] are of order two. The complexity of deciding whether or not a given graph has an *OLDOIND*-set is established in the following theorem.

**Theorem 2.3 ([**[**13**](#_bookmark41)**])** *Deciding, for a given graph G, whether or not G has an*

*OLDOIND-set is an NP-complete decision problem.*

The next proposition provides a sharp logarithmic lower bound on the size of an *OLDOIND*-set of any graph *G*, if *G* has such a set.

**Proposition 2.4** *For a given graph G of order n, if G has an OLDOIND-set, then*

*OLDOIND*(*G*) ≥ lg(*n* + 1)*.* (1)

**Proof.** Let *S* be an *OLDOIND*-set of *G* with *|S|* = *k* and *A* = *V* (*G*) *\ S*. By Lemma [2.2,](#_bookmark3) for every *v ∈ A*, we have *|N* (*v*) *∩ S|* ≥ 2. Since for any two vertices *u, v ∈ A*, *N* (*u*) *∩ S /*= *N* (*v*) *∩ S*, the size of *A* is bounded by all combinations of at least two vertices taken from *S*. So, *|A|* ≤ 2*k − k −* 1. Hence *n* ≤ 2*k −* 1 and thus *k* ≥ lg(*n* + 1)*.* *2*

It is possible to construct an infinite class of graphs that attains the bound in

([1](#_bookmark5)). Let *k* = 2*p*, where *p* is a positive integer and *n* = 2*k −* 1. For every *i* = 1*,.* *, n*,

let *vi* be a vertex of *G*. For every *j* = 1*,..., p*, let *v*2*j−*1*v*2*j* be an edge of *G*. Let

*X* = *{Xk*+1*,.* *, Xn}* be the set of sets of all combinations of at least two vertices taken from *{v*1*,..., vk}*. Now, for *i* = *k* + 1*,.* *, n*, let *vi* be adjacent to each vertex

of *Xi* and *S* = *{v*1*,..., vk}*. Note that every vertex of *S* is open-dominated exactly once by the vertices of *S* and every vertex of *V* (*G*) *\ S* is open-dominated at least twice. Furthermore, for any two vertices *u, v ∈ V* (*G*) *\ S*, *N* (*u*) *∩ S /*= *N* (*v*) *∩ S*. Therefore, *n* = 2*k −* 1 and *S* is an *OLDOIND*-set of *G* of size *k* = lg(*n* + 1).

# Some results for complementary prisms

Next we consider *OLDOIND*-sets in the complementary prisms of graphs. In any *OLDOIND*-set of a complementary prism there is at most one pair of matched vertices, as proved in Lemma [3.1](#_bookmark7). Moreover, the *OLDOIND*-set has at least one vertex of *G* and at least one vertex of *G*¯, as demonstrated in Lemma [3.2](#_bookmark8).

**Lemma 3.1** *For a given graph G, if GG*¯ *has an OLDOIND-set S* = *S*0 *∪ S*¯1 *with S*0 *⊆ V* (*G*) *and S*¯1 *⊆ V* (*G*¯)*, then there is at most one matching edge uu*¯ *∈ GG*¯[*S*] *with u ∈ S*0 *and u*¯ *∈ S*¯1*.*

**Proof.** Suppose *S* is an *OLDOIND*-set in *GG*¯ as described and assume there are two distinct matching edges *uu*¯ and *vv*¯ say, in *GG*¯[*S*], such that *{u, v} ⊆ S*0 and

*{u*¯*, v*¯*} ⊆ S*¯1. Since *GG*¯

is a complementary prism, either (*i*) edge *uv ∈ G* (and

vertices *u* and *v* open-dominate each other) or else (*ii*) edge *u*¯*v*¯ *∈ G*¯ (and vertices *u*¯

and *v*¯ open-dominate each other). But vertices *u* and *u*¯ open-dominate each other, and vertices *v* and *v*¯ open-dominate each other. Thus, condition (*i*) of Lemma [2.2](#_bookmark3) does not hold, which is a contradiction. *2*

**Lemma 3.2** *For a given graph G, if GG*¯ *has an OLDOIND-set S* = *S*0 *∪ S*¯1 *with*

*S*0 *⊆ V* (*G*) *and S*¯1 *⊆ V* (*G*¯)*, then S*0 */*= *∅ and S*¯1 */*= *∅.*

**Proof.** For a proof by contradiction, suppose *S*0 = *∅*. Let *v ∈ V* (*G*). By Lemma [2.2](#_bookmark3), vertex *v* has to be open-dominated at least twice. Since *v* has at most one neighbor in *S*¯1, we can conclude that *S* is not an *OLDOIND*-set in *GG*¯. The proof for *S*¯1 follows analogously. *2*

In the following Lemma we present some bounds on the sizes of some subsets of vertices of *GG*¯, when *GG*¯ has an *OLDOIND*-set.

**Lemma 3.3** *Let G be a graph of order n such that GG*¯

*has an OLDOIND-set*

*S* = *S*0 *∪ S*¯1 *with S*0 *⊆ V* (*G*)*, S*¯1 *⊆ V* (*G*¯)*, then*

1. *if S contains the endpoints of a matching edge vv*¯*, consider the following sets:*

*A* = *NG*(*v*)*, B* = *V* (*G*) *\* (*A ∪ {v}*)*, A*1 = (*V* (*G*) *∩ N* (*S*¯1)) *\ {v}, A*2 = *A \ A*1*,*

*and B*1 = *B \ S*0*. Then, |A*2*|* ≤ min*{*2*|S*0*|−*1 *−* 1*,* 2*|S*¯1*|−*1 *− |S*¯1*|} and |B*1*|* ≤

min*{*2*|S*¯1*|−*1 *−* 1*,* 2*|S*0*|−*1 *− |S*0*|}.*

1. *if S does not contain the endpoints of a matching edge and r* = min*{|S*0*|, |S*¯1*|}, then n − |S*0*|− |S*¯1*|* ≤ 2*r − r −* 1*.*

**Proof.** Assume that *S* = *S*0 *∪ S*¯1 is an *OLDOIND*-set in *GG*¯

with *S*0 *⊆ V* (*G*),

*S*¯1 *⊆ V* (*G*¯). The reader is referred to Figure [2](#_bookmark12). To prove [(i)](#_bookmark9), let *A*, *B*, *A*1, *A*2,

and *B*1 be the sets as described above and *B*2 = *B \ B*1 = *S*0 *\ {v}*. Observe that *S*0 = *{v}∪ B*2 and *S*¯1 = *{v*¯*}∪ A*¯1. For every *X ∈ {A*1*, A*2*, B*1*, B*2*}* let *X*¯ be the set of corresponding vertices of *X* in *G*¯. Note that the sets *A*, *B* and *{v}* together partition *V* (*G*) and *A*¯, *B*¯ and *{v*¯*}* together partition *V* (*G*¯). The vertices in *A*1 are open-dominated by *v* and distinguished from each other by the vertices in *S*¯1. Note that *NGG*¯ (*A*2) *∩ S*¯1 = *∅*. Since *A*2 *⊆ NGG*¯ (*v*), every *u ∈ A*2 is open- dominated by *v* and distinguished by the vertices in *B*2. So, for every distinct pair *u, x ∈ A*2, the sets (*NG*(*u*) *∩ B*2) and (*NG*(*x*) *∩ B*2) are distinct and non empty.

Thus *|A*2*|* ≤ 2*|B*2*| −* 1 = 2*|S*0*|−*1 *−* 1. Now consider the set *A*¯2. Every *w*¯ *∈ A*¯2

must be open-dominated and distinguished by the vertices in *A*¯1. So, for every

distinct pair *w*¯*, y*¯ *∈ A*2, the sets (*NG*¯ (*w*¯) *∩ B*2) and (*NG*¯ (*y*¯) *∩ B*2) are distinct and non empty. Thus *|A*¯2*|* ≤ 2*|A*¯1*| − |A*¯1*|−* 1, and since *|A*2*|* = *|A*¯2*|* and *|A*¯1*|* = *|S*¯1*|−* 1, we can conclude that *|A*2*|* ≤ min*{*2*|S*0*|−*1 *−* 1*,* 2*|S*¯1*|−*1 *− |S*¯1*|}*. By symmetry, *|B*1*|* ≤ min*{*2*|A*1*| −* 1*,* 2*|B*2*| − |B*2*|}*, i.e. *|B*1*|* ≤ min*{*2*|S*¯1*|−*1 *−* 1*,* 2*|S*0*|−*1 *− |S*0*|}*.

To prove [(ii)](#_bookmark11), assume *r* = min*{|S*0*|, |S*¯1*|}* and *S* does not contain the endpoints of

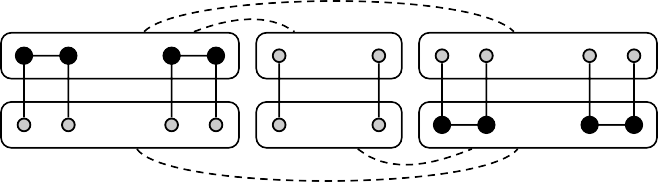
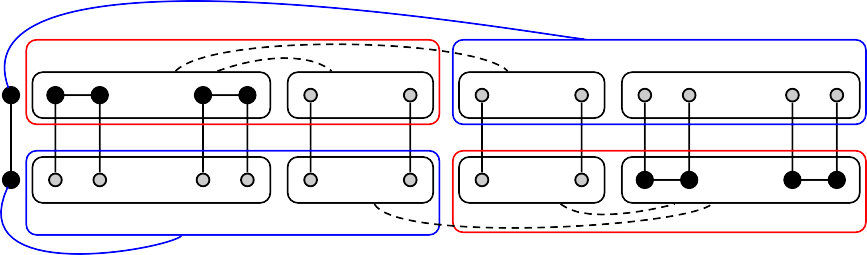
a matching edge. Following Figure [2(b)](#_bookmark12), let *S*1 be the set of corresponding vertices of *S*¯1 in *V* (*G*), *S*¯0 be the set of corresponding vertices of *S*0 in *V* (*G*¯), *C* be the set *V* (*G*)*\*(*S*1*∪S*0) and *C*¯ be the corresponding vertices of *C* in *V* (*G*¯). Note that the sets *S*1, *C* and *S*0 together partition *V* (*G*), and *S*¯0, *C*¯ and *S*¯1 together partition *V* (*G*¯). The vertices in *S*1 are distinguished from each other by *S*¯1 and every vertex in *S*1 is also dominated at least once by the vertices of *S*0. By construction, *N* (*C*) *∩ S*¯1 = *∅*. For every distinct pair *u, x ∈ C*, we have that (*N* (*u*) *∩ S*0) */*= (*N* (*x*) *∩ S*0). So, using the same argument as in the proof of Theorem [2.4,](#_bookmark6) we can conclude that

*|C|* ≤ 2*|S*0*| − |S*0*|−* 1. Analogously, *|C*¯*|* ≤ 2*|S*¯1*| − |S*¯1*|−* 1. Since *|C|* = *|C*¯*|* and

*r* = min*{|S*0*|, |S*¯1*|}*, *|C|* = *n − |S*0*|− |S*¯1*|* is bounded by 2*r − r −* 1, where *n* is the

order of *G*. *2*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *A*¯1 |  | *A*¯2 | *B*¯1 |  | *B*¯2 |
| *v*¯ *...* |  | *...* | *...* |  | *...* |
| *v ...* |  | *...* | *...* |  | *...* |
| *A*1 | *A* | *A*2 | *B*1 | *B* | *B*2 |

*V* (*G*¯)

|  |  |  |  |
| --- | --- | --- | --- |
|  | | (a) Lemma [3.3(i)](#_bookmark9). |  |
| *S*¯1  *...* |  | *C*¯  *...* | *S*¯0  *...* |
| *...* |  | *...* | *...* |
| *S*1 |  | *C* | *S*0 |

*V* (*G*)

(b) Lemma [3.3(ii)](#_bookmark11).

Fig. 2. An illustration of the sets described in Lemma [3.3](#_bookmark10).

*V* (*G*¯)

*V* (*G*)

Next we sharpen the bound presented in Proposition [2.4](#_bookmark6) for complementary prisms.

**Theorem 3.4** *Let G be a graph of order n such that GG*¯

*Then,*

*has an OLDOIND-set.*

2 lg(*n* + 1) *−* 2 ≤ *OLDOIND*(*GG*¯) ≤ *n* + 1*.* (2)

*Furthermore, these bounds are sharp.*

**Proof.** Let *G* be a graph of order *n* such that *GG*¯ has an *OLDOIND*-set *S* = *S*0 *∪S*¯1 with *S*0 *⊆ V* (*G*)*, S*¯1 *⊆ V* (*G*¯). To prove the upper bound in ([2](#_bookmark14)), assume that *OLDOIND*(*GG*¯) ≥ *n* + 2. Then *GG*¯[*S*] has at least two matching edges. But, by Lemma [3.1](#_bookmark7), every *OLDOIND*-set has at most one matching edge, a contradiction, which completes the proof. Note that the upper bound in ([2](#_bookmark14)) is tight, for instance, when *GG*¯ = *P*3*P*¯3 or *C*5*C*¯5.

To begin the proof of the lower bound in ([2](#_bookmark14)), let *|S*0*|* = *k*0 and *|S*¯1*|* = *k*1 with

*k*0*, k*1 ≥ 1. We consider the cases where (*i*) *S* contains the endpoints of a matching edge, and (*ii*) *S* does not contain the endpoints of a matching edge. By symmetry, for each of (*i*) and (*ii*), we can assume that *k*0 ≥ *k*1, and need consider only the subcases (*a*) 2*k*1 ≥ *k*0 and (*b*) *k*0 *>* 2*k*1 .

Case (*i*). By the definition of an *OLDOIND*-set, *k*0 and *k*1 are positive, odd integers. Consider the sets *A*2 and *B*1 defined in Lemma [3.3(i)](#_bookmark9). Since *|A*2*|* ≤ min*{*2*k*0*−*1 *−* 1*,* 2*k*1*−*1 *− k*1*}*, *|B*1*|* ≤ min*{*2*k*1*−*1 *−* 1*,* 2*k*0*−*1 *− k*0*}* and *n* = *k*0 + *k*1 *−* 1+

*|A*2*|* + *|B*1*|* it follows that *n* ≤ *k*0 + *k*1 *−* 1+ 2*k*1*−*1 *− k*1 + 2*k*1*−*1 *−* 1 and thus

*n* +2 ≤ *k*0 + 2*k*1 *.* (3)

Subcase *i*(*a*). From ([3](#_bookmark15)) we have that

*n* +2 ≤ 2*k*1+1 (4)

and thus

lg(*n* + 2) *−* 1 ≤ *k*1*.* (5)

Subcase *i*(*b*). From ([3](#_bookmark15)) we have that

*n* +2 ≤ 2*k*0*.* (6)

and thus

lg(*n* + 2) *−* 1 ≤ lg(*k*0)*.* (7)

Case (*ii*). Here *k*0 and *k*1 are positive, even integers. Consider the set *C* defined in Lemma [3.3(ii)](#_bookmark11). Since *|C|* ≤ min*{*2*k*0 *− k*0 *−* 1*,* 2*k*1 *− k*1 *−* 1*}* and *n* = *k*0 + *k*1 + *|C|* it follows that *n* ≤ *k*0 + *k*1 + 2*k*1 *− k*1 *−* 1 and thus

*n* +1 ≤ *k*0 + 2*k*1 *.* (8)

Subcase *ii*(*a*). From ([8](#_bookmark18)) we have that

*n* +1 ≤ 2*k*1+1 (9)

and thus

lg(*n* + 1) *−* 1 ≤ *k*1*.* (10)

Subcase *ii*(*b*). From ([8](#_bookmark18)) we have that

*n* +1 ≤ 2*k*0*.* (11)

and thus

lg(*n* + 1) *−* 1 ≤ lg(*k*0)*.* (12)

To prove the lower bound in ([2](#_bookmark14)) we use the obvious fact that *|S|* = *k*0 + *k*1. Using

([5](#_bookmark16)) and ([10](#_bookmark19)), this implies that for Cases *i*(*a*) and *ii*(*a*), 2 lg(*n* + 1) *−* 2 ≤ 2*k*1 ≤ *|S|*. Similarly, using ([7](#_bookmark17)) and ([12](#_bookmark20)) implies that for Cases *i*(*b*) and *ii*(*b*), 2 lg(*n* + 1) *−* 2 ≤ 2 lg(*k*0) ≤ *|S|*, and the result follows.

To show that the lower bound in ([2](#_bookmark14)) is tight, let *GG*¯ be the graph constructed as

in Figure [2(b)](#_bookmark12) with 2*k*0 = *n*+1 and 2*k*1+2 = *n*+1. Note that 2*k*1 = *n*+1 *>* lg(*n*+1) = *k*0 *> k*1 and thus Subcase *ii*(*a*) applies. *|C|* = 2*k*1 *−k*1 *−* 1= (*n* + 1)*/*4 *−* lg(*n* + 1)+ 1. Finally, *S*0 *∪ S*¯1 is an *OLDOIND*-set in *GG*¯ with size *k*0 + *k*1 = 2 log(*n* + 1) *−* 2, which complete the proof. *2*

**Lemma 3.5** *For a given, nontrivial graph G, if GG*¯

*has an OLDOIND-set S* =

*S*0 *∪ S*¯1 *with S*0 *⊆ V* (*G*)*, S*¯1 *⊆ V* (*G*¯) *and |S*¯1*|* = 1*, then G is a disconnected graph.*

*Furthermore, G is the disjoint union of an isolated vertex and a collection of l K*2

*subgraphs, where l* ≥ 1*.*

**Proof.** Let *G* and *S* be as described above. If *|S*¯1*|* = 1, *S* contains the endpoints of a matching edge, say *vv*¯, of *GG*¯. Thus *G* consists of an isolated vertex *v* and a collection of *l* ≥ 1 disjoint *K*2 subgraphs and *V* (*G*) *∪ {v*¯*}* is an *OLDOIND*-set for *GG*¯. *2*

* 1. *Complementary prisms of graphs with girth at least ﬁve*

Seo and Slater [[13](#_bookmark41)] presented some results about open-independent, open-locating- dominating sets (*OLDOIND*-sets) in trees. The authors showed that every leaf and its neighbor are contained in any *OLDOIND*-set of any tree *T* , if *T* has such a set. Furthermore, they recursively defined the collection of trees such that each has a unique *OLDOIND*-set. In this section we study *OLDOIND*-sets in the complemen- tary prisms of graphs that have girth at least five, which includes all trees. In particular, for a given graph *G* with girth *g*(*G*) ≥ 5, the following result bounds

the number of vertices of *V* (*G*¯) that can be members of an *OLDOIND*-set of the

complementary prism *GG*¯, of *G*, if *GG*¯ has such a set.

**Proposition 3.6** *For a given connected graph G whose girth satisﬁes g*(*G*) ≥ 5*,*

*if GG*¯ *has an OLDOIND-set S* = *S*0 *∪ S*¯1 *with S*0 *⊆ V* (*G*) *and S*¯1 *⊆ V* (*G*¯)*, then*

2 ≤ *|S*¯1*|* ≤ 3*. Furthermore,*

1. *if |S*¯1*|* = 2*, then S*¯1 *consists of the endpoints of a complementary edge in G*¯*.*
2. *if |S*¯1*|* = 3*, then S*¯1 *consists of three distinct vertices in G*¯*, two of which are the endpoints of a complementary edge in G*¯*, and the three corresponding vertices in G induce a 3-path in G.*

**Proof.** Suppose a connected graph *G* whose girth satisfies *g*(*G*) ≥ 5 is given and

*GG*¯ has an *OLDOIND*-set *S* = *S*0 *∪ S*¯1 with *S*0 *⊆ V* (*G*) and *S*¯1 *⊆ V* (*G*¯).

By Lemma [3.2](#_bookmark8), *S*¯1 */*= *∅* and, since *G* is connected, by Lemma [3.5](#_bookmark21), *|S*¯1*| >* 1. Suppose that *|S*¯1*|* ≥ 4. By Proposition [3.1,](#_bookmark7) there is at most one matching edge induced by the vertices in *S*. Hence if *|S*¯1*|* ≥ 4, there are two distinct complementary

edges *u*¯1*u*¯2 and *v*¯1*v*¯2 *∈ GG*¯, with *{u*¯1*, u*¯2*, v*¯1*, v*¯2*} ⊆ S*¯1. By Lemma [2.2](#_bookmark3), the set

*{u*¯1*, u*¯2*, v*¯1*, v*¯2*}* induces a graph isomorphic to 2*K*2 in *G*, hence *{u*1*, u*2*, v*1*, v*2*}* induces

a cycle of order 4 in *G*. This contradicts the assumption that *g*(*G*) ≥ 5 which completes this part of the proof.

To prove [(i)](#_bookmark22), let *S*¯1 = *{u*¯*, v*¯*}*. Since, by Lemma [3.1,](#_bookmark7) there is at most one matching edge in *G*[*S*] between the vertices in *S*0 and in *S*¯1, we can conclude that *S*¯1 consists

of the endpoints of the complementary edge

*u*¯*v*¯ in *GG*¯. In order to prove [(ii)](#_bookmark24),

let *S*¯1 = *{u*¯*, v*¯*, x*¯*}*. By Lemma [3.1](#_bookmark7) and the definition of an *OLDOIND*-set, *G*[*S*]

must contain at least two components isomorphic to *K*2, one of them containing a matching edge and the other a complementary edge. So *S*¯1 consists of a vertex that is an endpoint of a matching edge in *GG*¯ and the endpoints of a complementary

edge in *G*¯. Suppose that edges in *GG*¯

match *u*¯*, v*¯ and *x*¯ with *u*, *v* and *x* in *G*,

respectively. Without loss of generality, suppose there is an edge *u*¯*v*¯ *∈ E*(*GG*¯).

Since *S* in an *OLDOIND*-set, there is no edge between *u*¯ and *x*¯, nor between *v*¯ and

*x*¯. So *u*, *v* and *x* induce a *P*3 in *G*. *2*

We present some examples of complementary prisms to illustrate Proposition

[3.6](#_bookmark23). As an illustration of [(i)](#_bookmark22), consider the path graph *P*4 = *⟨v*1*, v*2*, v*3*, v*4*⟩*. Then

*{v*¯1*, v*2*, v*3*, v*¯4*}* is an *OLDOIND*-set for *P*4*P*¯4 with *S*¯1 = *{v*¯1*, v*¯4*}* and the comple-

mentary edge

*v*¯1*v*¯4 *∈ E*(*P*¯4) (see Figure [3(a)](#_bookmark25)). As an illustration of [(ii)](#_bookmark24), con-

sider *P*3 = *⟨v*1*, v*2*, v*3*⟩*. Then *{v*¯1*, v*2*, v*¯2*, v*¯3*}* is an *OLDOIND*-set for *P*3*P*¯3 with

*S*¯1 = *{v*¯1*, v*¯2*, v*¯3*}*, the matching edge *v*2*, v*¯2 *∈ E*(*P*3*P*¯3) and the complementary edge *v*¯1*v*¯3 *∈ E*(*P*¯3) (Figure [3(b)](#_bookmark25)). Furthermore, there are instances where neither the graph nor its complement has an *OLDOIND*-set but the complementary prism of the graph does (cf. *P*3 in [(ii)](#_bookmark24) above). Conversely, there are instances where both

a graph and its complement have *OLDOIND*-sets but the complementary prism of the graph does not. An example is the graph *G* of Figure [1(a)](#_bookmark1).

*V* (*P*¯4) *V* (*P*¯3)

*v*1 *v*2 *v*3 *v*4

1. *P*4*P*¯4.

*V* (*P*4)

*v*1 *v*2 *v*3

1. *P*3*P*¯3.

*V* (*P*3)

Fig. 3. *OLDOIND*-sets in complementary prisms of path graphs.

In view of Theorem [2.3](#_bookmark4), another question that arises is, for a given graph *G*, if the problem of deciding whether or not *GG*¯ has an *OLDOIND*-set, remains *NP*- complete. However, the following result establishes that when we have girth *g*(*G*) ≥ 5, the decision problem can be solved in polynomial time.

**Theorem 3.7** *If G is a nontrivial, connected graph with g*(*G*) ≥ 5*, we can decide in polynomial time whether or not GG*¯ *has an OLDOIND-set S. Furthermore, if S exists, it can be determined in polynomial time.*

**Proof.** Let *G* be a nontrivial connected graph whose girth satisfies *g*(*G*) ≥ 5. We begin by assuming that *GG*¯ has an *OLDOIND*-set *S* = *S*0 *∪ S*¯1 with *S*0 *⊆ V* (*G*) and *S*¯1 *⊆ V* (*G*¯). By Proposition [3.6](#_bookmark23), *|S*¯1*| ∈ {*2*,* 3*}*. If *|S*¯1*|* = 2, *S* does not contain the endpoints of a matching edge. Consider the set *C* defined in Lemma [3.3](#_bookmark10) [(ii)](#_bookmark11). Then, *n* = *|S*0*|* + *|S*¯1*|* + *|C|*. By Lemma [3.3](#_bookmark10) [(ii)](#_bookmark11), *|C|* ≤ 1 and it can be concluded that

*|V* (*G*) *\ S*0*|* ≤ 3. If *|S*¯1*|* = 3, *S* contains the endpoints of exactly one matching edge.

Consider the sets *A*2 and *B*1 defined in the proof of item [(i)](#_bookmark9) of Lemma [3.3](#_bookmark10). Then *n* = *|S*0*|* + *|S*¯1*|−* 1+ *|A*2*|* + *|B*1*|*. Since by Lemma [3.3](#_bookmark10) [(i)](#_bookmark9), *|A*2*|* ≤ 1 and *|B*1*|* ≤ 3, it follows that *|V* (*G*) *\ S|* ≤ 7. In both cases, by using a brute force approach, it is possible to verify in polynomial time which vertices of *GG*¯ must correspond to *S*0 and *S*¯1.

Since *S*¯1 is bounded by a constant, if *S* exists, it can be found in polynomial

time and *S*0 can be deduced from it in polynomial time. It must be considered all possible guesses for *S*¯1 and either successfully complete the corresponding solution *S* or decide that such a set is impossible. Assume that every checking step succeeds.

If something within any step goes wrong, then the corresponding combination of guesses cannot lead to a solution.

Note that the algorithm is based on the fact that if graph *GG*¯ has an *OLDOIND*- set *S* with *S*0 *⊆ V* (*G*)*, S*¯1 *⊆ V* (*G*¯), then *|S*¯1*|∈ {*2*,* 3*}*. Since it is possible that more than one case can occur, it is necessary to compare the sizes of *S*0 induced by each

of the cases, assume the best case and proceed with the lower size. *2*

For some graphs *G* it is not possible to bound the size of the set *S*¯1 in *GG*¯ and the size of *S* can be arbitrarily large. Consequently, we present Conjecture [3.8](#_bookmark26) which states that the problem of deciding if a graph has an *OLDOIND*-set remains *NP*-complete even for complementary prisms.

**Conjecture 3.8** *Deciding, for a given graph G, whether or not GG*¯ *has an open- independent, open-locating-dominating set is an NP-complete decision problem.*

* 1. *Complementary prisms of some particular graph classes*

We study *OLDOIND*-sets in the complementary prisms of some particular graph classes. It is easy to see that if *G* is a complete graph *Kn*, with *n* ≥ 1, *GG*¯ has an *OLDOIND*-set if and only if *n* = 1. Next we consider the complementary prism of cycles and paths.

The complementary prisms of the *n*-cycle *Cn* (*n* ≥ 3) and the *n*-path *Pn* (*n* ≥ 1) are denoted by *CnC*¯*n* and *PnP*¯*n*, respectively. When *n* = 5, *CnC*¯*n* is the Petersen graph which has domination number 3 and independence number 4. Henceforth, the vertex set of *Cn* or *Pn* is represented as the set *{v*1*, v*2*,.* *, vn}* and the vertex

set of *C*¯*n* or *P*¯*n* is represented as *{v*¯1*, v*¯2*,.* *, v*¯*n}*. We identify indexes of vertices of

*G* modulo *n*.

As will be seen, only a few cycle graphs have complementary prisms with an *OLDOIND*-set. It can easily be checked that for *n ∈ {*3*,* 4*,* 7*}* the graph *CnC*¯*n* has no *OLDOIND*-set. On the other hand, *OLDOIND*(*C*5*C*¯5)= *OLDOIND*(*C*6*C*¯6)=6 and *OLDOIND*(*C*8*C*¯8)= *OLDOIND*(*C*9*C*¯9) = 8. For *n* ≥ 10, the proof that *CnC*¯*n* does not have an *OLDOIND*-set is given in the next theorem.

**Theorem 3.9** *For n* ≥ 10*, CnC*¯*n does not have an OLDOIND-set.*

**Proof.** For a proof by contradiction, assume *n* ≥ 10 and that *CnC*¯*n* has an

*OLDOIND*-set *S* = *S*0 *∪ S*¯1 such that *S*0 *⊆ V* (*Cn*) and *S*¯1 *⊆ V* (*C*¯*n*).

First we claim that no two vertices in *V* (*Cn*) *\ S*0 are adjacent in *Cn*, that is, if *v ∈ V* (*Cn*) *\ S*0, then *|N* (*v*) *∩S*0*|* = 2. Suppose *|N* (*v*) *∩S*0*|* ≤ 1. So, *v* has a neighbor in *Cn*, say *u ∈ V* (*Cn*) *\S*0. By Lemma [2.2](#_bookmark3), *u* and *v* must be open-dominated at least twice, which implies *{u*¯*, v*¯*}⊂ S*¯1. Since *u*¯ and *v*¯ must be open-dominated and they are not neighbors, *u*¯ has a neighbor in *C*¯*n*, say *u*¯*j*, that belongs to *S*¯1, also *v*¯ has a neighbor in *C*¯*n*, say *v*¯*j*, that belongs to *S*¯1. So *|S*1*|* ≥ 4 and this fact contradicts Proposition [3.6.](#_bookmark23) Thus, we may conclude that *|N* (*v*) *∩ S*0*|* = 2. This implies that *S*0 consists of either (*i*) *n/*3 components of order two (which is possible only if *n*

mod 3 = 0), or (*ii*) *[n/*3*♩* components of order two and exactly one isolated vertex

(which is possible only if *n* mod 3 = 2.) From now on, we assume *n* mod 3 *∈ {*0*,* 2*}*.

Reference to Figure [4](#_bookmark28) may aid the understanding of the following reasoning, where for *C*¯*n*, instead of indicating the edges, we indicate the non-edges by dashed lines. By Proposition [3.6](#_bookmark23), *|S*¯1*| ∈ {*2*,* 3*}*. If *|S*¯1*|* = 2, assume *S*¯1 = *{v*¯*i, v*¯*j}* for some integers *i, j,* 1 ≤ *i < j* ≤ *n*. In order to dominate all vertices in *V* (*C*¯*n*), *i* +3 ≤ *j*. If

*|S*¯1*|* = 3, then *S*¯1 = *{v*¯*i, v*¯*i*+1*, v*¯*i*+2*}*, for some integer *i,* 1 ≤ *i* ≤ *n −* 2.

Now let *W* = *V* (*Cn*) *\ S*0, let *W*¯

denote the corresponding vertices of *W* in

*V* (*C*¯*n*), and let *W*¯ *j* = *W*¯ *\ S*¯1. If *n* mod 3 = 0, we have that *|S*0*|* = 2*n/*3 and

*|S*¯1*|* = 2, and if *n* mod 3 = 2, then *|S*0*|* = 1 + (2*n −* 4)*/*3 and *|S*¯1*|* = 3. In both

cases, since *n* ≥ 10, *|W*¯ *j|* ≥ 2. Let *v*¯*k, v*¯*l ∈ W*¯ *j* for some integers *k, l,* 1 ≤ *k < l* ≤ *n*. Then, (*N* (*v*¯*k*) *∩ S*) = (*N* (*v*¯*l*) *∩ S*) = *S*¯1, implying that *S* is not an *OLDOIND*-set for *CnC*¯*n*. *2*

*v*¯*i*

*v*¯*j*

*v*¯*i*

*v*¯*i*+1 *v*¯*i*+2

*V* (*C*¯)

*V* (*C*)

*...*

*...*

*V* (*C*¯)

*V* (*C*)

Fig. 4. The only two possible configurations for *S*0 *∪ S*¯1 to be an *OLDOIND*-set in *CnC*¯*n*.

Next we consider the path graph *Pn,n* ≥ 2. We can easily check that the graph *P*2*P*¯2 does not have an *OLDOIND*-set and that *OLDOIND*(*P*3*P*¯3) = *OLDOIND*(*P*4*P*¯4) = 4, *OLDOIND*(*P*5*P*¯5) = *OLDOIND*(*P*6*P*¯6)

= 6, *OLDOIND*(*P*7*P*¯7) = *OLDOIND*(*P*8*P*¯8) = *OLDOIND*(*P*9*P*¯9) = 8, and

*OLDOIND*(*P*10*P*¯10) = *OLDOIND*(*P*11*P*¯11) = 10. We prove in the next theorem that *PnP*¯*n* does not have an *OLDOIND*-set when *n* ≥ 12.

**Theorem 3.10** *For n* ≥ 12*, PnP*¯*n does not have an OLDOIND-set. (cf Proposition 3.2 of [*[*13*](#_bookmark41)*] concerning the existence of OLDOIND-sets in path graphs.)*

**Proof.** As *E*(*Pn*)= *E*(*Cn*) *\ {v*1*vn}* and *E*(*P*¯*n*)= *E*(*C*¯*n*) *∪{v*1*vn}*, the proof closely follows that of Theorem [3.9](#_bookmark27). The claim that no two vertices in *V* (*Pn*) *\ S*0 are adjacent in *Pn* mirrors that in Theorem [3.9](#_bookmark27) except when *v* = *v*1 or *vn*. Clearly, in these cases *v* must be open-dominated by its solitary neighbor in *Pn* and *|N* (*v*) *∩*

*S*0*|* = 1, but the claim still holds. Following the notation and reasoning in Theorem [3.9](#_bookmark27), when *n* ≥ 12, *|W|* = *|W*¯ *|* ≥ 4. Since *|W*¯ *j|* ≥ 2 in all cases, the result follows. *2*

# Concluding remarks

A logarithmic lower bound on the size of an *OLDOIND*-set in any graph has been provided. Various properties of and bounds on *OLDOIND*-sets in complementary prisms were presented and the cases of cliques, paths and cycles have been com- pletely solved. It has been shown that for any graph with girth at least five, it can be decided in polynomial time whether or not its complementary prism has an OLD-OIND-set (and also the set can be found in polynomial time if it ex- ists). Furthermore, we conjecture that the problem of deciding if a graph has an *OLDOIND*-set remains *NP*-complete even for complementary prisms and pose the following open questions:

1. Which families of graphs attain the bounds of Theorem [3.4](#_bookmark13)?
2. What are the additional conditions necessary to extend Theorem [2.1](#_bookmark2) for general graphs? and for complementary prisms?

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