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Proving Termination by Policy Iteration

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**Abstract**

In this paper, we explore the adaptation of policy iteration techniques to compute greatest fixpoint ap- proximations, in order to find sufficient conditions for program termination. Restricting ourselves to affine programs and the abstract domain of template constraint matrices, we show that the abstract greatest fixpoint can be computed exactly using linear programming, and that strategies are related to the tem- plate constraint matrix used. We also present a first result on the relationships between this approach and methods which use ranking functions.

*Keywords:* Abstract interpretation, policy iteration, template constraint matrices, termination analysis.

# Introduction

Abstract interpretation [[5](#_bookmark19)] is a powerful framework to develop program analyses. Most analyses require the computation of approximation of fixpoints on an ab- stract domain, either least fixpoints (*lfp*) or greatest fixpoints (*gfp*). The traditional method to compute approximations of fixpoints uses widening and narrowing op- erators. Widening operators are designed to get beyond the fixpoint, which makes them useful to compute overapproximations of *lfp*, or (dually) underapproximations of *gfp* [[6](#_bookmark20)]. However, used on state abstractions, these approximations can only be used to check safety properties. Liveness properties (and especially termination) can be proved by computing underapproximations of *lfp* or overapproximations of *gfp*.

More recently, other approaches have been developed to compute abstract fix- points: abstract acceleration [[12](#_bookmark23)] and policy iteration [[3](#_bookmark17),[9](#_bookmark24)]. Both methods are designed in order to compute the exact abstract fixpoint for specific transfer func- tions. They have been used to compute reachability analyses (which involves the computation of an *lfp*), and since the abstract domains used were overapproximating

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domains, they were applied to prove safety properties, providing a greater precision than widenings. Using these approaches to compute greatest fixpoints would enable the discovery of sufficient conditions for program termination, and as a particular case proving termination for all inputs.

This paper describes the use of policy iteration techniques to discover sufficient termination conditions. As a first work in this direction, we restrict ourselves to affine programs and to the template constraint matrices abstract domain [[14](#_bookmark28)], a sub-domain of polyhedra. Policy iteration techniques were already used in this framework to approximate the set of reachable states [[10](#_bookmark25)], hence we need to adapt these results to greatest fixpoint computations.

We first present the relationships between approximating fixpoints and proving termination. Then we give an overview of the policy iteration approach. In section [4](#_bookmark5), we explore the extension of these approaches to the computation of an abstract backward semantics designed to prove termination properties. Finally we give a first result on the relationships between our approach and ranking function synthesis, showing that programs admitting a linear ranking function can be treated with our approach.

# Program termination and fixpoint approximation

In this section, we recall a few results on the relationships between termination and fixpoint approximation. A program *P* is defined as a transition system (Σ*,τ* ), Σ being an (infinite) set of states and *τ ⊆* Σ *×* Σ a transition relation. Furthermore, we consider *S*0 as the set of initial states.

The trace semantics of a program *P* is the set of finite and infinite execution traces of *P* . The program is said to (definitely) terminate from *S*0 if any execution trace starting from *s*0 *∈ S*0 is finite. Broadly, three approaches can be used to show this property.

*Variant abstraction analysis.* Many methods use a kind of variant abstraction anal- ysis [[7](#_bookmark21)], where one finds a mapping *r* from the set of reachable states (from *S*0) toa well-founded set (*O, <*), such that for any transition *⟨σ, σj⟩∈ τ* we have *r*(*σj*) *< r*(*σ*). Once the class of variant functions (or the variant abstraction) is fixed, the analysis can be expressed as a safety analysis.

*Least ﬁxpoint underapproximation.* An alternative approach is to prove that *S*0 is included in the set of states which could only terminate, that is:

(1) *S*0 *⊆* lfp *λX.*p˜re(*X*)

where pre(*X*)= *{y ∈* Σ *| ∀x ∈* Σ*, ⟨y, x⟩∈ τ ⇒ x ∈ X}*.

˜

This property requires to *underapproximate* the least fixpoint. As noted in [[7](#_bookmark21)], underapproximations are not much used in practice (most abstract domains, in particular numerical abstract domains, are designed to handle overapproximations). Furthermore, one cannot use a the “classical” fixpoint induction techniques (with widenings) to underapproximate least fixpoint.

*Greatest ﬁxpoint overapproximation.* Similarly, we can show that *S*0 is disjoint from

the set of states that are potentially non-terminating, i.e.:

(2) *S*0 *∩* gfp *λX.*pre(*X*)= *∅*

where pre(*X*)= *{y ∈* Σ *| ∃x ∈ X,⟨y, x⟩∈ τ}*.

Proving this property can be done by *overapproximating* the greatest fixpoint. Compared to the (formally equivalent) previous approach, using overapproximations has the advantage of being compatible with most abstract domains. However, we still cannot use widenings to approximate the fixpoint.

The three approaches are related: with a ranking function, one can prove for- mulas ([1](#_bookmark1)) and ([2](#_bookmark2)). Reciprocally, proving formula ([1](#_bookmark1)) or ([2](#_bookmark2)) proves that a ranking function exists. In fact, some lower fixpoint induction methods (e.g. in [[4](#_bookmark18), Sect. 11]) directly use some kind of ranking function. However, if the approximation is proved with other methods, it may be difficult to make the ranking function ex- plicit. Hence, techniques which compute directly a fixpoint appear as interesting alternatives to infer termination properties.

# Precise fixpoint approximation with policy iterations

The use of policy iterations (also called strategy iterations) to compute the least fixpoint of a self-map *f* in static analysis was first introduced in [[3](#_bookmark17)]. The principle of this techniques is to describe *f* as the minimum (or the maximum) of a set *S* of simpler maps. A *strategy* (or a *policy* ) is a selection of an element of *S*. The least fixpoint of this element is computed. If this fixpoint is a fixpoint of *f* , the algorithm terminates, otherwise a new strategy is selected during the *strategy improvement* step, and the algorithm iterates.

Two different approaches have been proposed to compute least fixpoints: the first one [[3](#_bookmark17),[8](#_bookmark22),[1](#_bookmark15)] uses min-strategy iteration, approaches the least fixpoint from above, and does not guarantee to return it in the general case [2](#_bookmark3) . The second one [[9](#_bookmark24),[10](#_bookmark25),[11](#_bookmark26)] uses max-strategy iteration, approaches the least fixpoint from below, and guarantees to return the least (abstract) fixpoint.

Since our goal is to overapproximate greatest fixpoints, it seems more natural to approach them from above, hence to use a dual version of the second approach. In this paper, we mainly follow the method presented in [[10](#_bookmark25)] and restrict ourselves to affine programs and template constraint matrix domains. Before summarizing the method, we introduce a few notations.

* 1. *Notations*

In the following, *X* = *{x*1*,..., xk}* denotes a tuple of variables. An *assignment ρ* on *X* is defined as a mapping from *X* to R = R *∪ {−∞,* +*∞}*. When there is no ambiguity, *ρ* may be represented as an element of R*k*. The order relation *≤* (and its strict version *<*) on R is extended component-wise to R*k*. We denote by *∨* and *∧*

2 However, it does return the least fixpoint in the case of nonexpansive self-maps.

the minimum and maximum operators on R (and their component-wise extension to R*k*). Hence R*k* has a complete lattice structure.

If *ρ* is an assignment on *X*, we denote by [*ρ*]*f* (resp. [*ρ*]*∞*, [*ρ*]*—∞*) the set of *x* in

*X* such that *ρ*(*x*) is finite (resp. equal to +*∞*, *−∞*).

If *m* is a function from *X →* R to R, dom(*m*) represents the set of assignments *ρ* such that *m*(*ρ*) is finite, and fdom(*m*)= dom(*m*) *∩* R*X* . The function *m* is said to be order-convex (resp. order-concave) iff fdom(*m*) is convex [3](#_bookmark4) and for all comparable *ρ, ρj* in fdom(*m*) and *λ ∈* [0*,* 1], *λm*(*ρ*)+ (1 *− λ*)*m*(*ρj*) *≥ m*(*λρ* + (1 *− λ*)*ρj*) (resp. *λm*(*ρ*)+ (1 *− λ*)*m*(*ρj*) *≤ m*(*λρ* + (1 *− λ*)*ρj*).

Let *D* be a monotonic function from *X →* R to *X →* R. A *preﬁxpoint* (resp. *postﬁxpoint* ) of *D* is an assignment *ρ* such that *D*(*ρ*) *≥ ρ* (resp. *D*(*ρ*) *≤ ρ*). If *ρ* is a prefixpoint (resp. postfixpoint) of *D*, we denote by lfp*≥ρD* (resp. gfp*≤ρD*) the least (resp. greatest) fixpoint of *D* greater than (resp. lower than) *ρ*.

Finally, an *equation system E* on *X* is a *k*-tuple of equations (*x*1 := *e*1*,..., xk* := *ek*) where *ei* are expressions using the variables *X*. If J*·*) represents the semantics of expressions (such that J*ei*) *∈* (*X →* R) *→* R), the semantics of *E* is defined as:

J*E* ) : (*X →* R) *→* (*X →* R)

J*E* ) *ρ* : *xi ⊆→* J*ei*) *ρ*

A solution (resp. postsolution, presolution) of *E* is a fixpoint (resp. postfixpoint, prefixpoint) of J*E* ).

* 1. *Computing the least solution of a system of equations*

We consider an equation system *E* on *X* where the expressions are defined by the grammar:

*e* ::= *a | xi | e* + *e | b · e | e ∨ e | e ∧ e*

where *a ∈* R, *b ∈* R*>*0, *∨* is the max operator and *∧* is the min operator. The semantics of *e* is straightforward:

J*a*) *ρ* = *a* J*e*1 + *e*2) *ρ* = J*e*1) *ρ* + J*e*2) *ρ* J*b · e*) *ρ* = *b* J*e*) *ρ*

J*xi*) *ρ* = *ρ*(*xi*) J*e*1 *∨ e*2) *ρ* = J*e*1) *ρ ∨* J*e*2) *ρ* J*e*1 *∧ e*2) *ρ* = J*e*1) *ρ ∧* J*e*2) *ρ*

The least solution of *E* can be computed using max strategy iteration [[10](#_bookmark25)]:

* a max-strategy *π* is a function mapping every expression *e*1 *∨e*2 to a subexpression *e*1 or *e*2; applying *π* to *E* gives a system of *conjunctive equations* (without the *∨* operator) *Eπ*;
* the least solution *μπ* greater than a current presolution of *Eπ* is computed by solving two linear programs extracted from the system in linear time;
* the computation terminates if *μπ* is a solution *E* , otherwise a new strategy *πj* is selected (such that J*E* ) (*μπ*)= J*Eπ′* ) (*μπ*)), and the computation loops.

3 For all *ρ, ρ′* in fdom(*m*) and *λ ∈* [0*,* 1], *λρ* + (1 *— λ*)*ρ ∈* fdom(*m*).

In [[10](#_bookmark25)] and [[11](#_bookmark26)], the notions of *consistent presolution* and *feasible presolution* are defined to ensure the validity and the termination of this approach. In section [4](#_bookmark5), the dual notions will be used to compute greatest fixpoints.

* 1. *Systems of rational equations with linear programs*

Equations with linear programs (LPs) are defined by adding *LPA,b*(*e,..., e*) in the grammar of expressions, with *A ∈* R*m×n* and *b ∈* R*n*. The semantics of *LPA,b* is defined as:

J*LPA,b*(*e*1*,..., em*)) *ρ* = *{bT x | x ∈* R*n, Ax ≤* (J*e*1) *ρ,...,* J*em*) *ρ*)*}*

Rational equations with linear programs are used to express the abstract semantics of affine programs in the *template constraint matrix* domain [[14](#_bookmark28)].

In [[10](#_bookmark25)], Gawlitza and Seidl show that LP subexpressions can be handled during the resolution of the system of conjunctive equations by adding new variables and inequations. In [[11](#_bookmark26)], this result is generalized to *order-concave equations*, using the fact that the operator *∧* and LP expressions are order-concave. Since the backward semantics of programs also use LP expressions, this result cannot be applied directly to compute overapproximations of greatest fixpoints: one would need order-convex expressions.

# Computation of the backward semantics

* 1. *Backward semantics of the program*

We consider affine programs as a triple (*N, E,* **st**) where *N* is a finite set of program points, *E ⊆ N ×* **Stmt** *× N* is a finite set of transitions labeled by *statements*, and **st** is the *start program point*. A statement is a pair (*g*; *a*) where *g* is an affine *guard A***x** + *b ≥* 0 on the set of program (real) variables **x** = (*x*1*,..., xn*) and *a* is an affine *assignment* **x** := *A***x** + *b*.

The *backward collecting semantics* of a statement *s* = (*A***x** *≤ b*; **x** := *C***x** + *d*) is defined as:

J*s*) : *℘* (R*n*) *→ ℘* (R*n*)

J*s*) (*X*)= *{x ∈* R*n | Ax* + *b ≥* 0 *∧ Cx* + *d ∈ X}*

The backward transformer pre on *N → ℘* (R*n*) is defined as pre(*X*)(*u*) = (*u,s,v*)*∈E* J*s*) (*v*). Our goal is to overapproximate *B* = gfp pre. We use the frame- work of abstract interpretation, and our approach is closely related to the abstract

domain used (we want to compute exactly the abstract fixpoint).

The abstract domain (first introduced in [[14](#_bookmark28)]) is relative to a *template constraint matrix T ∈* R*m×n*. Each row of *T* represents a linear combination of program

variables. The matrix *T* defines an abstraction from R*n* to *TT*

Galois connection R*n →−γ−T−− T* :

*−−α−−→ T*

*T*

= R*m* with the

*γT* (*ρ*)= *{x ∈* R*n | Tx ≤ ρ} αT* (*X*)= *∨{ρ|γT* (*ρ*) *⊆ X}*

The functions *γT* and *αT* are extended component-wise to *N →* R*n* and *N → TT* .

*m*

An element of *γT* (R ) is said to be *canonical*. The best abstract backward semantics

of a statement *s* in this domain is defined as J*s*)= *αT ○* J*s*) *○ γT* .

**Lemma 4.1** *Let s* = (*A***x** + *b ≥* 0; **x** := *C***x** + *d*) *be a statement, and T a non-empty template matrix. Let ρ be an abstract value on the domain TT .*

*Let Aj, bj and ρj be deﬁned as:*

*j* ⎛ *−A* ⎞ *j* ⎛ *−b* ⎞ *j* ⎛ 0 ⎞

⎝

⎠

⎝

⎠

⎝

⎠

(3)

*A* = *b* = *ρ* =

*TC Td ρ*

*Then* J*s*) *satisﬁes:*

⎧⎨ *−∞ if {x | Ajx* + *bj − ρj ≤* 0*}* = *∅*

J*s*)*i* (*e*)=

⎩

*{*(*ρj − bj*)*T λ | λ ≥* 0 *∧ AjT λ* = *Ti} otherwise*

**Remark 4.2** If *{x | Ajx* + *bj − ρj ≤* 0*} /*= *∅* and *{λ | λ ≥* 0 *∧ AjT λ* = *Ti}* = *∅*, then J*s*)(*ρ*)= *∞*. Furthermore, *{x | Ajx* + *bj − ρj ≤* 0*} /*= *∅* implies min*{*(*ρj − bj*)*T λ | λ ≥* 0 *∧ AjT λ* = *Ti} > −∞* (but the converse is not true).

*i*

**Example 4.3** With only one variable *x*1, *s* = (0 *≥* 0; *x*1 = 0) and

⎛

*T* = 1 ⎞, we have *{*(*ρj − bj*)*T λ | λ ≥* 0 *∧ AjT λ* = *T }* = *∞* for all *i* and all

⎝ *−*1 ⎠ *i*

*ρ* = (*ρ*1*, ρ*2). However, if *ρ*1 *<* 0 or *ρ*2 *<* 0, then *{x | Ajx* + *bj − ρj ≤* 0*}* = *∅*. Thus,

by defining (*ρj , ρj* )= J*s*)(*ρ*1*, ρ*2), we have *ρj* = *ρj*

= *−∞* if *ρ*1 *<* 0 or *ρ*2 *<* 0, and

1 2 1 2

*j* = *ρj*

*ρ*

2

1

= *∞* otherwise.

The abstract domain for a program (*N, E,* **st**) is *N → TT* [4](#_bookmark7) . With *X ∈ N → TT* , the abstract backward transformer pre= *αT ○* pre *○ γT* is given by:

*i*

(pre(*X*)(*u*))*i* =

(*u,s,v*)*∈E*

J*s*)(*X*(*v*))

**Lemma 4.4** *The abstract semantics* gfp pre *satisﬁes γT* (gfp pre) *⊇ B.*

To compute the greatest fixpoint of pre, we describe the function as a system of semantic equations of the form *xi* = pre(*x*) when preappears as the maximum of

*i* *i*

(one or) several J*s*). Following lemma [4.1](#_bookmark6), J*s*)can be expressed as the minimum

*i* *i*

of two expressions *φs* (independent of *i*) and *ψi* :

*s*

⎧ *∞* if *{x | Ajx* + *bj − ρj ≤* 0*} /*= *∅*

⎨

* *φs*(*ρ*)= .

⎩ *−∞* if *{x | Ajx* + *bj − ρj ≤* 0*}* = *∅*

* *ψi* (*ρ*)= *{*(*ρj − bj*)*T λ | λ ≥* 0 *∧ AjT λ* = *Ti}*

*s*

4 For the sake of simplicity, we consider only one global template constraint matrix.

*φ* = *alltrue*(*{Cx* + *C−x ≥* 0, *Cy* + *C−y ≥* 0, *Cx*+*y* + *C−x−y ≥* 0, *Cx−y* + *C−x*+*y ≥* 0, *Cx* + *Cy* + *C−x−y ≥* 0, *C−x* + *C−y* + *Cx*+*y ≥* 0, *Cx* + *C−y* + *C−x*+*y ≥* 0, *C−x* + *Cy* + *Cx−y ≥* 0, 2*Cx* + *C−x−y* + *C−x*+*y ≥* 0, 2*C−x* + *Cx*+*y* + *Cx−y ≥* 0, 2*Cy* + *C−x−y* + *Cx−y ≥* 0,

2*C−y* +*Cx*+*y* +*C−x*+*y ≥* 0, 2*C−y* +*C−x* +26 *≥* 0, *Cx* +*C−x−y* +26 *≥* 0, 3*C−x* +2*Cx−y* +

26 *≥* 0, *C−y* + *C−x−y* + 26 *≥* 0, 3*C−y* + 2*C−x*+*y* + 26 *≥* 0, *C−x−y* + *Cx−y* + 52 *≥* 0*}*)*.*

*Cx* := *φ ∧* 10 + *Cx/*2 *∧* 23 + *C−x−y ∧* (17 + 2*C−x*+*y*)*/*3

*∧ Cy —* 3 *∧ Cx*+*y* + *C−x —* 3 *∧ C−x*+*y* + *Cx —* 3

*C−x* := *φ ∧ C−y* +3 *∧ C−x−y* + *Cx* +3 *∧ Cx−y* + *C−x* +3

*Cy* := *φ ∧ C−x/*2 *∧* (*C−x−y* + *Cy*)*/*2 *∧* (*C−x*+*y* + *C−y*)*/*2

*C−y* := *φ ∧ Cx/*2 *∧* (*Cx*+*y* + *C−y*)*/*2 *∧* (*Cx−y* + *Cy*)*/*2 *∧* 13 + *C−y ∧* (13 + *Cx−y*)*/*3 *Cx*+*y* := *φ ∧ C−x/*2+ *Cy —* 3 *∧* 3*C−x/*2+ *Cx*+*y —* 3 *∧* (3*Cy* + *C−x−y*)*/*2 *—* 3

*∧* (*Cy* + *C−x*+*y*)*/*2 *—* 3 *∧* (*Cx*+*y* + 3*C−x*+*y*)*/*4 *—* 3 *∧* 10 + *C−x ∧* 36 + 2*C−x−y*

*∧* (4 + 2*C−x*+*y*)*/*3

*C−x−y* := *φ ∧ Cx/*2+ *C−y* +3 *∧* 3*Cx/*2+ *C−x−y* +3 *∧* (3*C−y* + *Cx*+*y*)*/*2+3

*∧* (*C−y* + *Cx−y*)*/*2+3 *∧* (*C−x−y* + 3*Cx−y*)*/*4+3 *∧* 16 + 2*C−y*

*Cx−y* := *φ ∧* 10 *∧ Cx/*2+ *Cy —* 3 *∧* 3*Cx/*2+ *C−x*+*y —* 3 *∧ C−x/*2+ *Cx*+*y* +3

*∧* (*Cy* + *Cx*+*y*)*/*2 *—* 3 *∧* (3*Cy* + *Cx−y*)*/*2 *—* 3

*C−x*+*y* := *φ ∧ C−x/*2+ *C−y* +3 *∧* 3*C−x/*2+ *Cx−y* +3 *∧ C−x/*2+ *Cx*+*y* +3

*∧* (*C−y* + *C−x−y*)*/*2+3 *∧* (3*C−y* + *C−x*+*y*)*/*2+3

Fig. 1. Backward semantics of the transition *s* = (*x — y ≤* 10; *{x* := *—*2*y, y* := *x* + 3*}*) in the octagon domain, described as a system of equations. Each variable *C*exp represents the maximum of exp in the abstract element. We denote by *alltrue* the function which maps a set of constraints to *∞* if all constraints are satisfiable, and *—∞* otherwise.

One can see that *φs* is monotonic, order-concave and order-convex (since fdom(*φs*) = *∅*). Using the vertex principle of linear programming, we can express *ψi* as the minimum of a finite number of linear expressions:

*s*

**Lemma 4.5** *There exists a ﬁnite (possibly empty) number of tuples* (*λ*1*,..., λk*)

*such that λj ≥* 0 *and AjT λj* = *Ti for all j and, for all ρ > −∞:*

*k*

*ψ* (*ρ*)= (*ρ − b* )*λj*

*i j j*

*s*

*j*=1

This equality is also satisfied when some components of *ρ* are equal to +*∞*. In this case, the matching component of *λ* must be equal to 0.

The number of linear expressions can be exponential in the number of variables. Rather than computing all of them, we plan to lazily compute only the relevant expressions during the selection of the strategy (see remark [4.9](#_bookmark10)).

**Example 4.6** With two variables *x* and *y*, *s* = (*x−y ≤* 10; *{x* := *−*2*y, y* := *x* + 3*}*) and the octagon template matrix [[13](#_bookmark27)], J*s*)is represented Figure [1](#_bookmark8). This example shows that the number of *∧* operators is related to the template domain, both in order to deal with the potential non-canonicity of *e* and to ensure the canonicity of J*s*) (*e*). For example, if the initial assignment is canonical, we have directly *C—y ≤ C—x—y* + *Cx* and *C—y ≤ Cx—y* + *C—x*, hence the equation of *C—x* is equivalent to *C—x* := *φ ∧ C—y* + 3.

From the previous lemma, we deduce:

**Proposition 4.7** *The backward abstract semantics of an affine program* (*N, E,* **st**)

*in a template matrix domain can be expressed as the greatest ﬁxpoint of a system of*

*equations of the form:*

*x* := *U*1 *∨ U*2 *∨ ... ∨ Uk with Ui* := *φi ∧ u*1 *∧ ... ∧ ul*

*i* *i*

*where φi is a monotonic function whose image is included in {−∞,* +*∞} and uj*

*i*

*are linear expressions.*

**Remark 4.8** If an overapproximation of the reachable states has been computed, it can be included in the system of equations (if the abstract forward analysis returns *x* = *a*, the equation becomes *x* := *a ∧* (*U*1 *∨ ... ∨ Uk*)). This combination increases the precision of the backward analysis [[6](#_bookmark20)].

**Remark 4.9** Since the explicit computation of the system is too costly, we express each *Ui* as *φi ∧ ψi* where *ψi* is a linear program. During the strategy selection phase, an optimal *uj* is constructed by solving the linear program with the current affectation *ρ*: if there is an optimal solution *λ*, then *λ*(*ρj − bj*) is used (if it is an improvement compared with the current strategy), otherwise the strategy returns

*i*

+*∞*. The number of basic feasible solutions may be high, but most are related to the canonicity of the abstract elements, so we can expect the number of selected strategies to remain acceptable.

Similarly, we do not expect to compute explicitly *φi* as a set of constraints on *ρ*.

Rather, we check the feasibility of the domain at each strategy iteration.

* 1. *Solving the system of equations*

Following the policy iteration principle, we consider a strategy associating each expression *φi ∧ u*1 *∧ ... ∧ ul* (or rather *φi ∧ ψi*) to either *φi* or a linear expression

*i* *i*

*uk*. If *ρ* is the current assignment, the strategy *πρ* must satisfy:

*i*

(4) *πρ*(*φi ∧ u*1 *∧ ... ∧ ul*)= min(*φi*(*ρ*)*, u*1(*ρ*)*,..., ul*(*ρ*))

*i i i* *i*

Since the image of *φi* is included in *{−∞,* +*∞}*, we can ensure that *πρ*(*φi ∧.. .*)= *φi* only when *φi*(*ρ*)= *−∞*. Furthermore, since *φi* is monotonic and we compute a decreasing sequence, once *φi*(*ρ*)= *−∞* the whole expression can be replaced by *−∞*. Thus, the application of *πρ* gives a system of equations of the form (*x* := *u*1*∨.. .∨uk*) where each *ui* is either *−∞* (which can be ignored) or a linear expression. This system is a system of disjunctive equations.

Given a postsolution *ρ* of a disjunctive system *E* , we want to compute gfp*≤ρ* J*E* ). Since our approach is exactly the dual of the method proposed in [[10](#_bookmark25)], we just give a definition of consistency and the final theorem here.

**Definition 4.10** Given a disjunctive system *E* , a finite solution *ν* of *E* is said to be *feasible* iff there exists *ρ > ν* such that J*E* ) (*ρ*) *< ρ*. A finite postsolution *ρ* is *feasible* iff gfp*≤ρ* J*E* ) is finite and feasible. A disjunctive system is *feasible* iff it admits a feasible solution.

**Definition 4.11** Given a disjunctive system *E* , a postsolution *ρ* is said to be *con- sistent* iff the following conditions are satisfied:

* Jexp) *ρ* = *∞* implies exp = *∞* for every expression exp occurring in *E* ;
  + with *ν* = [gfp*≤ρ* J*E* )]*f* , the system *Ej* on *νf* defined by replacing in the equations of *E* any variable *x ∈ ν∞* by *∞* and *x ∈ ν—∞* by *−∞* is feasible, and *ρ|νf* is a feasible postsolution of *Ej*.

**Theorem 4.12 ([**[**10**](#_bookmark25)**, Thm 3, dual])** *Given a consistent postsolution ρ of a dis- junctive system E,* gfp*≤ρ* J*E* ) *can be computed by solving two LPs extractable from E in linear time.*

* 1. *Strategy improvement*

We still need to prove that the strategy improvement operator preserves consistency. First we rewrite all equations *x* := *U* as *x* := +*∞∧ U* . The initial strategy *π∞* associates each equation to +*∞*. In the associated system, (*xi ⊆→* +*∞*) is consistent. Consistency is preserved under three conditions:

**Lemma 4.13** *Let π be a strategy, ρ an assignment and πj an improved strategy satisfying:*

* + 1. *for each maximum of order-convex expressions ∧U, if π*(*∧U* )(*ρ*)= *πj*(*∧U* )(*ρ*)*, then π*(*U* )= *πj*(*U* )*;*
    2. *if π*(*∧U* ) */*= *πj*(*∧U* )*, then πj*(*∧U* )(*ρ*) *< π*(*∧U* )(*ρ*)*;*
    3. *if πj*(*∧U* )(*ρ*) *< ∞, then for all ρj ≥ ρ with* [*ρj*]*f* = [*ρ*]*f , πj*(*∧U* )(*ρ*) *< ∞. Then any consistent solution of π*(*E* ) *is a consistent postsolution of πj*(*E* )*.*

Conditions ([i](#_bookmark11)) and ([ii](#_bookmark12)) are consequences of the principle of strategy improvement. The third one is satisfied because we use linear expressions. The termination of the computation is guaranteed by the finite number of strategies. As mentioned in remark [4.9](#_bookmark10), we expect the number of iterations to remain low, but more experiments are needed to validate this hypothesis.

Finally, we can state the general result on whole programs:

**Theorem 4.14** *Given an affine program* (*N, E,* **st**) *and a template matrix T, the algorithm terminates and returns the abstract semantics* gfp pre*.*

**Example 4.15** We consider a program (*N, E, i*) with only one program point *N* =

*{i}* and *E* = *{*(*i, s, i*)*}* with *s* defined as in example [4.6](#_bookmark9). Figure [2](#_bookmark14) gives the sequences of strategies (as systems of equations) and the fixpoints (as constraints on *x* and *y*). The set of non-terminating states is included in each fixpoint (in this example, the last fixpoint is exactly the set of non-terminating states). Thus, from any initial state except (*x* = *−*2; *y* = 1), the program terminates.

# Relationships with variant analysis

To compare our method with variant analysis, we search a correspondence between the provability of termination with policy iteration and the kind of ranking functions which can be used to prove the termination of the program.

|  |  |  |
| --- | --- | --- |
| # | Current strategy | Fixpoint |
| 1 | *Cx* := +*∞*, *C−x* := +*∞*, *Cy* := +*∞*, *C−y* := +*∞*,  *Cx*+*y* := +*∞*, *C−x−y* := +*∞*, *Cx−y* := 10, *C−x*+*y* := +*∞* | *x — y ≤* 10 |
| 2 | *Cx* := +*∞*, *C−x* := +*∞*, *Cy* := +*∞*, *C−y* := (13 + *Cx−y*)*/*3, *Cx*+*y* := +*∞*, *C−x−y* := +*∞*, *Cx−y* := 10, *C−x*+*y* := +*∞* | *x — y ≤* 10, *—*23*/*3 *≤ y* |
| 3 | *Cx* := +*∞*, *C−x* := 3 + *C−y*, *Cy* := +*∞*,  *C−y* := (13 + *Cx−y*)*/*3, *Cx*+*y* := +*∞*,  *C−x−y* := 3 + (*Cx−y* + *C−y*)*/*2, *Cx−y* := 10, *C−x*+*y* := +*∞* | *—*32*/*3 *≤ x*, *x — y ≤* 10,  *—*23*/*3 *≤ y*,  *—*71*/*6 *≤ x* + *y* |
| 4 | *Cx* := 10 + *C−x/*2, *C−x* := 3 + *C−y*, *Cy* := *C−x/*2, *C−y* := (13 + *Cx−y*)*/*3, *Cx*+*y* := 10 + *C−x*,  *C−x−y* := 3 + (*Cx−y* + *C−y*)*/*2, *Cx−y* := 10,  *C−x*+*y* := 3 + (*C−x−y* + *C−y*)*/*2 | *—*32*/*3 *≤ x ≤* 46*/*3,  *—*51*/*4 *≤ x — y ≤* 10,  *—*23*/*3 *≤ y ≤* 16*/*3,  *—*71*/*6 *≤ x* + *y ≤* 62*/*3 |
| 5 | *Cx* := *—*3+ *Cy*, *C−x* := 3 + *C−y*, *Cy* := *C−x/*2,  *C−y* := (13 + *Cx−y*)*/*3, *Cx*+*y* := *—*3+ (*C−x*+*y* + *Cy*)*/*2, *C−x−y* := 3 + (*Cx−y* + *C−y*)*/*2, *Cx−y* := 10,  *C−x*+*y* := 3 + (*C−x−y* + *C−y*)*/*2 | *—*32*/*3 *≤ x ≤* 7*/*3,  *—*51*/*4 *≤ x — y ≤* 10,  *—*23*/*3 *≤ y ≤* 16*/*3,  *—*71*/*6 *≤ x* + *y ≤* 145*/*24 |
| 6 | *Cx* := *—*3+ *Cy*, *C−x* := 3 + *C−y*, *Cy* := *C−x/*2, *C−y* := *Cx/*2, *Cx*+*y* := *—*3+ (*C−x*+*y* + *Cy*)*/*2, *C−x−y* := 3 + (*Cx−y* + *C−y*)*/*2, *Cx−y* := *—*3+ (*Cx*+*y* + *Cy*)*/*2, *C−x*+*y* := 3 + (*C−x−y* + *C−y*)*/*2 | *x* = *—*2, *x — y* = *—*3,  *y* = 1, *x* + *y* = *—*1 |

Fig. 2. Computation of the abstract semantics for a single state program with one transition (*x — y ≤* 10; *{x* := *—*2*y, y* := *x* + 3*}*) in the octagon domain. The initial strategy (+*∞*) is omitted.

When the program is given, the result of the policy iteration-based analysis depends only on the abstract domain, since it computes exactly gfp prewhere pre= *α ○* pre *○ γ*. First, we can see that a ranking function can be constructed from the iteration sequence of pre.

**Lemma 5.1** *Let* Σ = *N → ℘* (R*n*) *be the set of states of the program, and τ the transition relation between elements of* Σ*. Let* (*Wi*) *be the iteration sequence of* pre *starting from* (*x ⊆→* +*∞*) *and O an ordinal such that WO—*1 = gfp pre*. Then the function v deﬁned from Z* =Σ *\ γT* (gfp pre) *to O as:*

*v*(*σ*)= min*{i | σ ∈/ γT* (*Wi*)*}*

*is a ranking function on* (*Z, τ* )*.*

Furthermore, for all *n ∈ O*, *v—*1(*{nj ∈ O | nj ≥ n}*) *∪ γT* (gfp pre) is canonical. This property can be reversed to deduce a condition on gfp prefrom the existence of a ranking function:

**Proposition 5.2** *Let Z ⊆* Σ *and Y* =Σ *\ Z. Then γT* (gfp *pre*) *⊆ Y if and only if there exists a function v from Z to an ordinal O such that:*

*∀s ∈ Z, ∀sj ∈* Σ*, ⟨s, sj⟩∈ τ ⇒* (*sj ∈ Z ∧ v*(*s*) *> v*(*sj*))

*∀n ∈ O, v—*1(*{nj | nj ≥ n}*) *∪ Y ∈ γT* (*N → TT* )

Hence, the abstract semantics would prove the termination for all inputs (i.e. *Z* = Σ) if there exists a ranking function *v* for which the successive preimages *v—*1(*O*), *v—*1(*O \ {*0*}*), *v—*1(*O \ {*0*,* 1*}*), *...* for every program point are of the form *TX ≤ B*.

**Corollary 5.3** *If an affine program can be proved to terminate with a linear ranking function X ⊆→ RX, our method proves the termination of the program if −R is a row of T .*

However, the program given in example [4.15](#_bookmark13) does not admit a linear ranking function. Thus our method is not limited to linear ranking functions, and a global characterization of the ranking functions remains to be stated.

# Conclusion and future work

This paper is intended to show how policy iteration techniques can be applied to termination analysis. We performed only a few experiments, and more comparisons with related work needs to be done. Recently, Bozga et al. [[2](#_bookmark16)] presented several results on the decision of conditional termination. Their framework is more restric- tive (as they restrict themselves to integer variables), but may give more precise results as it is not limited by the precision of the abstract domain.

To improve the analysis, we can extend previous work on policy iterations for other abstract domains (e.g. quadratic zones [[11](#_bookmark26)]). Although we dealt only with affine programs, we can also add linearization and non-determinism to extend the framework. Finally, an interesting feature of greatest fixpoint overapproximation, compared with least fixpoint overapproximation, is that the abstract domain can be refined during the computation. To refine the precision of the analysis, we can add new template constraints at any time, using for example decreasing iterations on the domain of polyhedra.

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