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Quadratically adjustable robust linear optimization with inexact data via generalized S-lemma: Exact second-order cone program reformulations[☆](#_bookmark2)

V. Jeyakumar[∗](#_bookmark3), G. Li, D. Woolnough

*Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia*

a r t i c l e i n f o a b s t r a c t

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Adjustable robust optimization allows for some variables to depend upon the uncertain data after its realization. However, the uncertainty is often not revealed exactly. Incorporating inexactness of the revealed data in the construction of ellipsoidal uncertainty sets, we present an exact second-order cone program reformulation for robust linear optimization problems with inexact data and quadratically adjustable variables. This is achieved by establishing a generalization of the celebrated S-lemma for a separable quadratic inequality system with at most one non-homogeneous function. It allows us to reformulate the resulting separable quadratic constraints over an intersection of two ellipsoids in terms of second-order cone constraints. We illustrate our results via numerical experiments on adjustable robust lot-sizing problems with demand uncertainty, showing improvements over corresponding problems with aﬃnely adjustable variables as well as with exactly revealed data.

# Introduction

Adjustable robust optimization (ARO) has proved to be a powerful deterministic methodology to handle dynamic decision-making prob- lems, where the decision-maker is able to adjust her strategy to in- [formation revealed in stages (see Ben-Tal et al., 2009; Delage and Iancu, 2015). It allows for some decision variables to depend upon](#_bookmark24) the uncertain data after realization. However, in many applications, the uncertainty is not revealed exactly, due to, for example, measure- ment error. The construction of uncertainty sets,that incorporates in-

dynamic decision-making problems of medicine and health sciences, such as the two-stage robust optimization models with time-dependent uncertainty sets that appear in radiation therapy planning problems ([Nohadani and Roy, 2017](#_bookmark32)). Here, inexactly revealed data can be incor- porated in the construction of the uncertainty sets, because a patient’s condition or health care needs can change during the course of a treat- ment ([Eikelder et al., 2019; Nohadani and Roy, 2017](#_bookmark41)), but estimates of their condition can be taken and considered mid-treatment.

In this paper, we consider the two-stage robust linear optimization problem with adjustable variables and inexactly revealed data:

exactness of revealed data, has been shown to give high quality so- lutions for practical dynamic decision-making problems. For example,

(*𝐼𝑃* ) min

*𝑥,𝑦*(⋅)

*𝑐𝑇 𝑥*

[de Ruiter et al. (2017)](#_bookmark35) demonstrated that for the multi-stage inventory production problem, ignoring inexactness of revealed demand can lead to violation of the robust constraints in up to 80% of simulations.

subject to *𝐴*(*𝑧*)*𝑥* + *𝐵𝑦*(*𝑧̂*) ≤ *𝑑*(*𝑧*)*,* for all (*𝑧, 𝑧̂*) ∈ Ę *⊂* ℝ*𝑞* × ℝ*𝑞,*

where *𝑐* ∈ ℝ*𝑛*, *𝐵* ∈ ℝ*𝑝*×*𝑠*,

Such a scheme has recently been developed for the case when the

adjustable decisions take on an Aﬃne Decision Rule (ADR), and the

uncertain data together with the inexactly revealed data and the es-

(*𝑎*1 + *𝐴*1*𝑧*)*𝑇*

*𝐴*(*𝑧*) = ⋮

⎡

⎢

(*𝑎*

+ *𝐴 𝑧*)*𝑇*

⎤⎥ ∈ ℝ

*𝑝*×*𝑛*

*𝜈* + *𝑣𝑇 𝑧*

⋮ ℝ*𝑝*

⎢

*𝑑 𝑧*

⎤

and ( ) = ⎡ 1 1 ∈

⎥

*𝜈*

+ *𝑣𝑇 𝑧*

(1)

timation error lie in closed and convex sets ([de Ruiter et al., 2017](#_bookmark35)).

⎣ *𝑝*

*𝑝* ⎦

⎣ *𝑝*

*𝑝* ⎦

Applications of the methods have included multi-stage-inventory prob- [lems (](#_bookmark30)[Ben-Tal et al., 2009](#_bookmark24)[), planning and scheduling problems (Ning and You, 2017) and treatment-length optimization problems in radiation](#_bookmark30) therapy ([Eikelder et al., 2019](#_bookmark41)). In particular, robust optimization tech- niques that employ inexactly revealed data are of great interest in

for some *𝑎*1*,* … *, 𝑎𝑝* ∈ ℝ*𝑛*, *𝐴*1*,* … *, 𝐴𝑝* ∈ ℝ*𝑛*×*𝑞*, *𝜈*1*,* … *, 𝜈𝑝* ∈ ℝ and *𝑣*1*,* … *, 𝑣𝑝* ∈ ℝ*𝑞*. The vector *𝑥* ∈ ℝ*𝑛* is the first-stage “here and now” decision vector and *𝑦*(⋅) is the second-stage “wait and see” adjustable decision function

*𝑦* ∶ ℝ*𝑞* → ℝ*𝑠*. The uncertainty set of (*𝐼𝑃* ) is defined as

Ę = {(*𝑧, 𝑧̂*) ∈ ℝ*𝑞* × ℝ*𝑞* ∶ ‖*𝑧̂* − *𝑢*‖2 ≤ *𝛾*2*,* ‖*𝑧* − *𝑧̂*‖2 ≤ *𝛾*2}*,* (2)

1

2

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∗ Corresponding author.

*E-mail addresses:* [v.jeyakumar@unsw.edu.au](mailto:v.jeyakumar@unsw.edu.au) (V. Jeyakumar), [g.li@unsw.edu.au](mailto:g.li@unsw.edu.au) (G. Li), [daniel.woolnough@unsw.edu.au](mailto:daniel.woolnough@unsw.edu.au) (D. Woolnough).

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where *𝛾*1 *, 𝛾*2 *>* 0 and *𝑢* ∈ ℝ*𝑞* are given. That is, the adjustable deci- sions *𝑦*(⋅) depend on an estimate *𝑧̂* ∈ ℝ*𝑞* of the actual uncertain data

*𝑧* ∈ ℝ*𝑞* , with the *estimation error 𝑤* ∶= *𝑧* − *𝑧̂* bounded in  = {*𝑤* ∈ ℝ*𝑞* ∶

*𝑤* ≤ *𝛾*2}, and *𝑧̂* bounded in the *estimation range* F = {*𝑧̂* ∈ ℝ*𝑞* ∶ *𝑧̂*

2

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[‖ ‖](#_bookmark35) ‖

*𝑢* 2 ≤ *𝛾*2}, with given scalars *𝛾*1 *, 𝛾*2 *>* 0*,* and known vector *𝑢* ∈ ℝ*𝑞* , see

2

[‖](#_bookmark35)

1

[de Ruiter et al. (2017)](#_bookmark35).

As shown in Section 3, the uncertainty set Ę can equivalently be

space, ℝ2*𝑞* . It is indeed a closed, bounded and convex uncertainty set. described as an intersection of two ellipsoids in the higher-dimensional

not specify *𝑧* ∈ F. Our uncertainty set is therefore more general, and Note that our definition of Ę differs from that used in [8] since we do

is useful in cases where the estimation range is known, but the true uncertainty set is not known. An example of such a case is discussed in Section 4.

We allow the adjustable decisions *𝑦*(⋅) to admit a parameterized sep-

arable quadratic decision rule ([Woolnough et al., 2021](#_bookmark38)) of the form

*𝑧̂𝑇 𝑄 𝑧̂*

⎞

system that admits a convex epigraphical set (see [Section 3](#_bookmark12) for de- tails). We then prove that the convex epigraphical set, which is hid- den in many known classes of fully homogeneous quadratic systems ([Ben-Tal and Hertog, 2014; Dines, 1941; Polyak, 1998](#_bookmark26)), is also hid- den in a separable quadratic inequality systems involving at most one non-homogeneous function. This results in a form of S-lemma for the corresponding separable quadratic inequality system, allow- ing us to handle the, not necessarily convex, separable quadratic constraints over the intersection of ellipsoidal uncertainty sets that

appear in the reformulation of (*𝐼𝑃* ). Consequently, we also deduce

a form of S-lemma for a simultaneously diagonalizable quadratic in-

equality system involving at most one non-homogeneous function. We also note that suﬃcient conditions which guarantee epigraphi- cal convexity have been given for specially structured quadratic sys- tems, such as the ones that appear in extended trust-region problems (see [Jeyakumar et al., 2009b; Jeyakumar and Li, 2013](#_bookmark24)). Related con- vexifiability conditions in terms of epigraphical set have been used

*𝑦*(*𝑧̂*) = *𝜃*(*𝑦*0 + *𝑃 𝑧̂*) + (1 − *𝜃*)⎛

⎜

⎝

where *𝜃* ∈ (0*,* 1] is a user-specified parameter, *𝑦*0 ∈ ℝ*𝑠*, *𝑃* ∈ ℝ*𝑠*×*𝑞* and

1

⋮ *,* (3)

*𝑧̂𝑇 𝑄 𝑧̂*

⎟

*𝑠* ⎠

in [Chieu et al. (2019, 2020)](#_bookmark33) for studying duality properties of various classes of non-convex quadratic optimization problems.

(iii) Finally, we illustrate our results via numerical experiments on ad-

justable robust lot-sizing problems with demand uncertainty. Our re-

*𝑄𝑟* ∈ *𝑆 𝑞* , *𝑟* = 1*,* … *, 𝑠*, are diagonal matrices with *𝑄𝑟* = diag(*𝜉*(1)*,* … *, 𝜉*(*𝑞*)). This covers the commonly used aﬃne decision rule where *𝜃* = 1 (see

*𝑟 𝑟*

[Avraamidou and Pistikopoulos, 2020; Ben-Tal et al., 2009; Ben-Tal et al., 2004; Chen and Zhang, 2009; de Ruiter et al., 2017; Yanikoglu et al., 2019) and non-homogeneous separable quadratic decision rule where](#_bookmark23)

0 *< 𝜃 <* 1 (see [Ben-Tal et al., 2009; Xu and Hanasusanto, 2018](#_bookmark24)).

The quadratic decision rules, including a separable quadratic deci-

sion rule, were first introduced by [Ben-Tal et al. (2009)](#_bookmark24) and these rules were used to examine numerically tractable safe approximations to vari- ous classes of intractable adjustable robust optimization problems. They include classes of robust problems where an uncertainty set is allowed to be an intersection of ellipsoids. The interplay between the quadratic optimization and adjustable robust optimization in the case of quadratic decision rules was elegantly discussed in [Bomze and Gabi (2021)](#_bookmark30). These quadratic decision rules were also employed to study the links between [copositive optimization and multi-stage robust optimization in Xu and Hanasusanto (2018). Recently, the parameterized separable quadratic](#_bookmark39) rule [(3)](#_bookmark4) has been shown by the authors ([Woolnough et al., 2021](#_bookmark38)) to ad- mit exact second-order cone programming reformulations for adjustable robust linear optimization problems under single ellipsoidal uncertainty based on exact revealed data. More general nonlinear decision rules have also been examined for adjustable robust optimization problems in [Ben-Tal et al. (2009)](#_bookmark24) and [Yanikoglu et al. (2019)](#_bookmark41).

We make the following technical contributions, in this paper, to two- stage robust optimization.

tion holds for (*𝐼𝑃* ) with the parameterized quadratic decision rule (i) We show that an equivalent second-order cone program reformula-

sults show that the current inexact QDR approach improves over the recent exact QDR scheme by the authors ([Woolnough et al., 2021](#_bookmark38))

actly revealed data, where *𝜃* = 1 in [(3)](#_bookmark4) and other references therein. as well as the commonly used aﬃne decision rule method with inex-

Related results with aﬃne decision rules and exactly revealed data may also be found in [Chuong and Jeyakumar (2020)](#_bookmark37).

The outline of the paper is as follows. [Section 2](#_bookmark5) presents general-

reformulation of (*𝐼𝑃* ). [Section 4](#_bookmark14) describes the numerical experiments izations of S-lemma. [Section 3](#_bookmark12) establishes a second-order cone program

and their outcomes for robust lot-sizing problems with demand uncer- tainty. [Section 5](#_bookmark21) concludes with a discussion on possible future work.

# Generalized S-lemma

In this Section, we present generalizations of S-Lemma for systems involving more than two quadratic functions. We begin by fixing some

preliminaries. The notation ℝ*𝑛* signifies the Euclidean space for each *𝑛* ∈

ℕ ∶= {1*,* 2*,* …} and *𝑆 𝑙* is the space of all real *𝑙* × *𝑙* symmetric matrices. As usual, the symbol *𝐼𝑛* stands for the identity (*𝑛* × *𝑛*) matrix, while ℝ+ ∶= [0*,* +∞) *⊂* ℝ*.* The inner product in ℝ*𝑛* is defined by *𝑥, 𝑦* ∶= *𝑥𝑇 𝑦* for all

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*𝑥, 𝑦* ∈ ℝ*𝑛.* We use *𝑥* to denote the standard Euclidean norm of *𝑥*. A

symmetric (*𝑛* × *𝑛*) matrix *𝐴* is said to be positive semi-definite, denoted by *𝐴* ⪰ 0, whenever *𝑥𝑇 𝐴𝑥* ≥ 0 for all *𝑥* ∈ ℝ*𝑛*. The notation 0*𝑚*×*𝑛* denotes the matrix of all zeros of dimensions *𝑚* × *𝑛*.

For *𝑧* ∈ ℝ*𝑞* , consider the following quadratic functions:

*𝑓* (*𝑧*) = *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* and *𝑔𝑖* (*𝑧*) = *𝑧𝑇 𝑊𝑖𝑧* + *𝑢𝑇 𝑧* + *𝛽𝑖 , 𝑖* = 1*,* … *, 𝑚,*

*𝑖*

where *𝑊* and *𝑊* are symmetric (*𝑞* × *𝑞*) matrices, *𝑤, 𝑢* ∈ ℝ*𝑞* and *𝛼, 𝛽* ∈

(QDR) and inexactly revealed data. In contrast to the conic program *𝑖*

*𝑖 𝑖*

reformulation of adjustable robust linear optimization with the pa- rameterized QDR and *exact* revealed data, called exact QDR, estab- lished by the authors recently in [Woolnough et al. (2021)](#_bookmark38) with the aid of S-lemma ([Ben-Tal and Nemirovski, 2001](#_bookmark28)), the present refor-

mulation for (*𝐼𝑃* ) with the QDR and inexactly revealed data, called

inexact QDR, requires a generalization of S-lemma to handle uncer-

tainty sets involving the intersection of two ellipsoids and a non- convex and nonhomogeneous quadratic function. It is known that generalizations of S-lemma for non-convex quadratic systems involv- ing more than two quadratic functions are often not possible un- less the system enjoys some hidden convexity (see [Example 2.1](#_bookmark6)). For a survey of S-lemma and its generalizations, see [Polik (2007)](#_bookmark33), [Jeyakumar et al. (2009a)](#_bookmark43) and [Derinkuyu and Pinar (2006)](#_bookmark40).

(ii) Using the standard hyperplane separation arguments and the con- vexity of the associated epigraphical set, we first show that a general form of S-lemma holds for a non-homogeneous quadratic inequality

ℝ, *𝑖* = 1*,* … *, 𝑚*. The epigraphical set *𝑈* (*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) associated with the

functions *𝑓* and *𝑔*1 *,* … *, 𝑔𝑚* , is given by

*𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) = {(*𝑣*1 *,* … *, 𝑣𝑚 , 𝑟*) ∶ *𝑣𝑖* ≥ *𝑔𝑖* (*𝑧*) and *𝑟* ≥ *𝑓* (*𝑧*)

for some *𝑧* ∈ ℝ*𝑞* }*.*

Recall that the epigraphical set *𝑈* (*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ), which plays a key role later in establishing generalizations of the S-lemma for systems involv-

value function of the quadratic optimization problem (*𝑄𝑃𝑣* ): ing more than two quadratic functions, is related to the epigraph of the

(*𝑄𝑃𝑣* ) min*𝑧*∈ℝ*𝑞 𝑓* (*𝑧*)

s.t. *𝑔𝑖* (*𝑧*) ≤ *𝑣𝑖 , 𝑖* = 1*,* … *, 𝑚,*

where *𝑣* = (*𝑣*1 *,* … *, 𝑣𝑚* ) ∈ ℝ*𝑚*. The set *𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) is intrinsically re- lated to the epigraph of the optimal value function of (*𝑄𝑃𝑣* ). In- deed, if the optimal value function of (*𝑄𝑃𝑣* ) is *𝜙*, that is, *𝜙*(*𝑣*) =

inf (*𝑄𝑃𝑣* ), then a direct verification shows that *𝑈* (*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) *⊆* epi*𝜙⊆* theorem [(Rockafellar, 1970, Theorem 11.3)](#_bookmark36)there exists (*𝜇*1*,* … *, 𝜇𝑚, 𝑡*) ∈

cl *𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ). Note that, for a subset *𝐴* of ℝ*𝑛*, cl *𝐴* denotes the closure ℝ*𝑚*+1∖{0} such that ∑*𝑚 𝜇𝑘 𝛾𝑘* + *𝑡* ⋅ *𝑟* ≥ 0 for all (*𝛾*1 *,* … *, 𝛾𝑚 , 𝑟*) ∈ *𝑈.* Note

*𝑘*=1

of the set *𝐴*.

+

that *𝑈* + ℝ*𝑚*+1 = *𝑈* . It follows that (*𝜇*1 *,* … *, 𝜇𝑚, 𝑡*) ∈ ℝ*𝑚*+1∖{0}. More-

Moreover, the epigraphical set can be written as the Minkowski sum

∑

*𝑘*=1

+

of the non-negative orthant ℝ*𝑚*+1 and the joint range set

over, as (*𝑔*1 (*𝑧*)*,* … *, 𝑔𝑚* (*𝑧*)*, 𝑓* (*𝑧*)) ∈ *𝑈* for all *𝑧* ∈ ℝ*𝑞* , ∑*𝑚*

*𝑧*

ℝ *.*

*𝑡*

*𝜇*1 *,*

*, 𝜇𝑚*

+

*𝜇𝑘 𝑔𝑘* (*𝑧*) + *𝑡𝑓* (*𝑧*) ≥

0 for all

∈

*𝑞*

If

= 0, then, (

…

) ∈

*𝑚*

*𝑚*

( )

ℝ+ ∖{0} and

0 for all *𝑧* ∈ ℝ*𝑞 .* This is impossible as *𝑔𝑘* (*𝑧*) *<* 0 for all *𝑘* = 1*,* … *, 𝑚*. So,

*𝑘*=1 *𝜇𝑖 𝑔𝑘 𝑧* ≥

*𝑅*(*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) = {(*𝑣*1 *,* … *, 𝑣𝑚 , 𝑟*) ∶ *𝑣𝑖* = *𝑔𝑖* (*𝑧*) and *𝑟* = *𝑓* (*𝑧*)

for some *𝑧* ∈ ℝ*𝑞* }

Therefore, it is easy to see that if the joint range set *𝑅*(*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) is a convex set, then the epigraphical set *𝑈* (*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) is also a convex set.

of quadratic systems. The case where *𝑚* = 1 and both *𝑓* and *𝑔*1 are homo- The convexity of the joint range set is known for various special classes

whereas the case where *𝑚* = 2, and *𝑓, 𝑔*1*, 𝑔*2 are homogeneous quadratic geneous quadratic functions is given in Dine’s theorem ([Dines, 1941](#_bookmark42)),

functions which satisfy a positive definite condition is given in Polyak’s convexity theorem ([Polyak, 1998](#_bookmark34)). An extension of Dine’s theorem may be found in [Jeyakumar et al. (2009b)](#_bookmark24).

In the case where *𝑓, 𝑔𝑖* are all homogeneous and the matrices

*𝑊 , 𝑊*1*,* … *, 𝑊𝑚* are pairwise commutative ([Polik, 2007](#_bookmark33)) (for example,

when they are diagonal matrices), the joint range set reduces to a poly-

hedral set which is, in particular, convex.

On the other hand, if *𝑓, 𝑔𝑖* are nonhomogeneous quadratic func-

tions, then the epigraphical set, in general, is a nonconvex set. How-

ever, the convexity of the epigraphical set can be satisfied by many specially structured systems of non-homogeneous quadratic functions.

For instance, as shown in [Jeyakumar and Li (2013)](#_bookmark25), if *𝑓* (*𝑧*) = *𝑧𝑇 𝑊 𝑧* +

*𝑤𝑇 𝑧* + *𝛼*, *𝑔* (*𝑧*) = *𝑧* − *𝑧* 2 − *𝛽* and *𝑔* (*𝑧*) = *𝑏𝑇 𝑧* − *𝛽 , 𝑖* = 1*,* 2*,* … *, 𝑚* and if

‖

‖

*𝑡 >* 0. Thus, dividing by *𝑡* on both sides, we have *𝑓* (*𝑧*) + *𝑚 𝜆𝑘 𝑔𝑘* (*𝑧*) ≥

0 for all *𝑧* ∈ ℝ*𝑞 ,* where *𝜆𝑘* = *𝜇𝑘* ∕*𝑡*. Then the statement (b) holds. □

*𝑘*=1

∑

In passing observe that [Lemma 2.1](#_bookmark7) includes the celebrated homo- geneous S-lemma and Farkas’ lemma for convex quadratic inequality

systems. Indeed, if *𝑓* and *𝑔*1 are homogeneous quadratic functions, then

it is known by Dine’s theorem that the joint range *𝑅*(*𝑓, 𝑔*1) is convex, and so, the epigraphical set *𝑈* (*𝑓, 𝑔*1) is also convex in this case. Thus, [Lemma 2.1](#_bookmark7) reduces to the homogeneous S-lemma. Moreover, if *𝑓, 𝑔𝑖* ,

*𝑖* = 1*,* … *, 𝑚*, are all convex quadratic functions, then clearly the epi-

graphical set *𝑈* (*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) is also a convex set.

It is also worth noting that a robust separable quadratic inequality

over an intersection of ellipsoids of the form [(2)](#_bookmark1), that appears in the formulations of AROs with inexact data, requires a form of S-lemma for a separable quadratic inequality system involving both homogeneous and non-homogeneous functions with a total of at least three inequali- ties. Although the epigraphical convexity is known to be hidden in sev- eral structured quadratic inequality systems, the following simple one- dimensional example shows that the epigraphical convexity assumption (and also [Lemma 2.1](#_bookmark7)) may fail, in general, for a separable system of three inequalities with two non-homogeneous quadratic functions.

0

dim Ker( −

*𝑊*

*𝜆*min *𝑊 𝐼𝑛*

≥ *𝑠* + 1 with dim span{*𝑏*1*,*

0 0 *𝑖*

( ) )

*𝑖 𝑖*

… } =

*, 𝑏𝑚*

*𝑠*, then the

two non-homogeneous inequalities) Let *𝑔*1 (*𝑧*) = *𝑧*, *𝑔*2 (*𝑧*) = 1 − *𝑧*2 and

**Example 2.1.** (Failure of epigraphical convexity & [Lemma 2.1](#_bookmark7) with

epigraphical set *𝑈* (*𝑓, 𝑔*0 *, 𝑔*1 *,* … *, 𝑔𝑚* ) = {(*𝑣*0 *, 𝑣*1 *,* … *, 𝑣𝑚 , 𝑟*) ∶ ∃*𝑧* ∈ ℝ*𝑞 , 𝑣𝑖* ≥

*𝑔* (*𝑧*)*, 𝑖* = 0*,* 1*,* 2*,* … *, 𝑚, 𝑟* ≥ *𝑓* (*𝑧*)} is a convex set, where *𝐴* ∈ *𝑆 𝑞* , *𝑢, 𝑧 , 𝑏* ∈

*𝑓* (*𝑧*) = *𝑧*2 − *𝑧* − 2. Then, *𝑔*1 and *𝑓* are two non-homogeneous separable

*𝑖* 0 *𝑖*

ℝ*𝑞* and *𝛼, 𝛽𝑖* ∈ ℝ. An extensive discussion on the relationship between

the epigraphical convexity and S-lemma can be found in the recent pa-

[per (](#_bookmark27)[Bomze and Gabi, 2021](#_bookmark30)[). For related results, see Jeyakumar and Li (2018) and](#_bookmark27) [Jeyakumar](#_bookmark24) [et al. (2009b).](#_bookmark27)

As we see later in this Section, the epigraphical convexity is also hid- den in certain systems of non-homogeneous separable quadratic func- tions. We first establish an abstract generalization of S-lemma under epigraphical convexity of a general non-homogeneous quadratic system.

***equality systems)*** *Let 𝑊 , 𝑊𝑖* ∈ *𝑆 𝑞 , 𝑖* = 1*,* … *, 𝑚, be symmetric matrices. Let* **Lemma 2.1. *(Generalized S-Lemma for nonconvex quadratic in-***

*𝑓* (*𝑧*) = *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼 and 𝑔𝑖* (*𝑧*) = *𝑧𝑇 𝑊𝑖𝑧* + *𝑢𝑇 𝑧* + *𝛽𝑖 , 𝑖* = 1*,* … *, 𝑚,*

*where 𝛼, 𝛽𝑖* ∈ ℝ*, 𝑤, 𝑢𝑖* ∈ ℝ*𝑞 . Suppose that the epigraphical set*

*𝑖*

*𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) = {(*𝑣*1 *,* … *, 𝑣𝑚 , 𝑟*) ∶ ∃*𝑧* ∈ ℝ*𝑞 , 𝑣𝑖* ≥ *𝑔𝑖* (*𝑧*)*, 𝑖* = 1*,* 2*,* … *, 𝑚, 𝑟* ≥

quadratic functions, and

*𝑔*1 (*𝑧*) ≤ 0*, 𝑔*2 (*𝑧*) ≤ 0 ⇒ *𝑧* ≤ −1 ⇒ *𝑓* (*𝑧*) = *𝑧*2 − *𝑧* − 2 ≥ 0*.*

[Lemma 2.1](#_bookmark7) fails because there does not exist *𝜆*1*, 𝜆*2 ≥ 0 such that So, statement (a) of [Lemma 2.1](#_bookmark7) holds, but statement (b) of

*𝑓*1 (*𝑧*) + *𝜆*1 *𝑔*1 (*𝑧*) + *𝜆*2 *𝑔*2 (*𝑧*) ≥ 0 for all *𝑧* ∈ ℝ*.*

To see this, suppose on the contrary that, for some *𝜆*1 ≥ 0 and *𝜆*2 ≥ 0,

*𝑓*1 (*𝑧*) + *𝜆*1 *𝑔*1 (*𝑧*) + *𝜆*2 *𝑔*2 (*𝑧*) = (1 − *𝜆*2 )*𝑧*2 + (*𝜆*1 − 1)*𝑧* + (*𝜆*2 − 2) ≥ 0

for all *𝑧* ∈ ℝ*.*

This gives us that 1 − *𝜆*2 ≥ 0 and *𝜆*2 − 2 ≥ 0, which is not possible.

Note that the epigraphical set *𝑈* (*𝑓, 𝑔*1 *, 𝑔*2 ) = {(*𝑠*1 *, 𝑠*2 *, 𝑟*) ∶ ∃ *𝑧* ∈

*𝑓* (*𝑧*)} *is a convex set. If there exists 𝑧* ∈ ℝ*𝑛 such that 𝑔𝑖* (*𝑧*) *<* 0*, 𝑖* = 1*,* … *, 𝑚,*

*then, the following statements are equivalent*

1. *𝑧𝑇 𝑊𝑖𝑧* + *𝑢𝑇 𝑧* + *𝛽𝑖* ≤ 0*, 𝑖* = 1*,* … *, 𝑚* ⇒ *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* ≥ 0;
2. *There exist 𝜆𝑖* ≥ 0*, 𝑖* = 1*,* … *, 𝑚, such that, for all 𝑧* ∈ ℝ*𝑞 ,*

*𝑖*

*𝑚 𝑚 𝑚*

∑ ∑ ∑

*𝑧𝑇* (*𝑊* + *𝜆𝑖𝑊𝑖* )*𝑧* + (*𝑤* + *𝜆𝑖 𝑢𝑖* )*𝑇 𝑧* + *𝛼* + *𝜆𝑖 𝛽𝑖* ≥ 0*.*

*𝑖*=1

*𝑖*=1

*𝑖*=1

1. *There exist 𝜆𝑖* ≥ 0*, 𝑖* = 1*,* … *, 𝑚 such that*

ℝ*, 𝑔𝑖* (*𝑧*) ≤ *𝑠𝑖 , 𝑓* (*𝑧*) ≤ *𝑟, 𝑖* = 1*,* 2} is not a convex set. To see this, take *𝑥*0 =

(−1*,* 0*,* 0) and *𝑦*0 = (0*,* 1*,* −2). Then, *𝑥*0 *, 𝑦*0 ∈ *𝑈* (*𝑓, 𝑔*1 *, 𝑔*2 ), because *𝑔*1 (−1) =

−1, *𝑔*2 (−1) = 0 and *𝑓* (−1) = 0; *𝑔*1 (0) = 0, *𝑔*2 (0) = 1 and *𝑓* (0) = −2. We now see that their mid point 1 (*𝑥*0 + *𝑦*0 ) = (−1∕2*,* 1∕2*,* −1) ∉ *𝑈* (*𝑓, 𝑔*1 *, 𝑔*2 ). Oth- erwise, we have (−1∕2*,* 1∕2*,* −1) ∈ *𝑈* (*𝑓, 𝑔*1*, 𝑔*2), and so, there exists *𝑧* ∈ ℝ

2

such that

*𝑧* ≤ −1∕2*,* 1 − *𝑧*2 ≤ 1∕2 and *𝑧*2 − *𝑧* − 2 ≤ −1*.*

The first two relations imply that *𝑧* ≤ − √2 . The last relation implies that

2

⎛ ∑*𝑚* ∑*𝑚* ⎞

−√5+1 ≤ *𝑧* ≤ √5+1 , which is impossible. So, *𝑈* (*𝑓, 𝑔*1 *, 𝑔*2 ) is not a convex

*𝑊* +

⎜

*𝑖*=1

*𝜆𝑖𝑊𝑖* (*𝑤* +

*𝑖*=1

*𝜆𝑖 𝑢𝑖* )∕2

⎟ ⪰ 0

*.* (4)

2 2

set.

⎜(*𝑤* + ∑ *𝜆𝑖𝑢𝑖*)*𝑇* ∕2 *𝛼* + ∑ *𝜆𝑖𝛽𝑖* ⎟

*𝑚*

*𝑚*

⎝

*𝑖*=1

*𝑖*=1

⎠

tem of separable quadratic functions with at most one non-homogeneous function. It is proved by first establishing the convexity of the epigraph-

The following theorem shows that a form of S-lemma holds for a sys-

**Proof.** Clearly, (b) and (c) are equivalent. Moreover, by construction,

Let *𝐶* = {(*𝛾*1 *,* … *, 𝛾𝑚 , 𝑟*) ∶ *𝛾𝑘* ≤ 0 and *𝑟* ≤ 0} and let *𝑈* ∶= *𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ), (b) implies (a). We show that (a) implies (b). Suppose that (a) holds. for simplicity. Then, from (a), int*𝐶* ∩ *𝑈* = ∅ (Otherwise, there exists (*𝛾*1 *,* … *, 𝛾𝑚 , 𝑟*) ∈ *𝑈* with *𝛾𝑘 <* 0 and *𝑟 <* 0. This shows that there exists *𝑧* ∈ ℝ*𝑞* such that 0 *> 𝛾𝑘* ≥ *𝑔𝑘* (*𝑧*) and 0 *> 𝑟* ≥ *𝑓* (*𝑧*). This contradicts (a).) Since the epigraphical set *𝑈* is a convex set, it follows by the convex separation

ical set of the associated quadratic system.

***homogeneous function)*** *Let 𝑊 , 𝑊𝑖* ∈ *𝑆 𝑞 , 𝑖* = 1*,* 2 … *, 𝑚 be diagonal ma-* **Theorem 2.1. *(Generalized S-Lemma with at most one non-*** *trices, 𝛼, 𝑟𝑖* ∈ ℝ*, 𝑖* = 1*,* 2*,* … *, 𝑚 and 𝑤* ∈ ℝ*𝑞 . Suppose that there exists 𝑧* ∈ ℝ*𝑞 such that 𝑧𝑇 𝑊𝑖𝑧 < 𝑟𝑖 , 𝑖* = 1*,* … *, 𝑚. Then, the following statements are equiv-*

*alent*

1. *𝑧𝑇 𝑊𝑖𝑧* ≤ *𝑟𝑖 , 𝑖* = 1*,* 2*,* … *, 𝑚* ⇒ *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* ≥ 0;
2. *There exist 𝜆𝑖* ≥ 0*, 𝑖* = 1*,* … *, 𝑚 such that, for all 𝑧* ∈ ℝ*𝑞 ,*

*𝑗*

= *𝑔̂𝑖* (*𝜆𝑦*(1) + (1 − *𝜆*)*𝑦*(2)) ≤ *𝜆𝑣*(1) + (1 − *𝜆*)*𝑣*(2)*,*

*𝑚*

∑

*𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* + *𝜆𝑖* (*𝑧𝑇 𝑊𝑖𝑧* − *𝑟𝑖* ) ≥ 0;

*𝑖*=1

1. *There exist 𝜆𝑖* ≥ 0*, 𝑖* = 1*,* … *, 𝑚 and 𝑠𝑗* ≥ 0*, 𝑗* = 1*,* … *, 𝑞 such that*

and

*𝑓* (*𝑧*) =

*𝑖 𝑖*

*𝑞*

*𝑗*=1

∑

*𝑊𝑗𝑗 𝑧*2 +

*𝑞*

*𝑗*=1

∑

*𝑤𝑗 𝑧𝑗* + *𝛼*

⎧*𝑊𝑗𝑗* +

⎪

⎪

*𝑚*

⎪⎨*𝛼* − ∑ *𝜆𝑖 𝑟𝑖* − ∑ *𝑠𝑗* ≥ 0

*𝑚*

*𝑖*=1

∑

*𝜆𝑖* (*𝑊𝑖* )*𝑗𝑗* ≥ 0*, 𝑗* = 1*,* … *, 𝑞*

*𝑞*

*𝑞*

=

∑

*𝑗*=1

= *𝑓̂*(*𝜆𝑦*(1) + (1 − *𝜆*)*𝑦*(2)) ≤ *𝜆𝛾*(1) + (1 − *𝜆*)*𝛾*(2)*.*

*𝑊𝑗𝑗*

(*𝜆𝑦*

(1)

*𝑗*

+ (1 − *𝜆*)*𝑦*

(2)

*𝑗*

) −

*𝑞*

*𝑗*=1

∑

|*𝑤*

*𝑗* | ⋅

√*𝜆𝑦*

(1)

*𝑗*

+ (1 − *𝜆*)*𝑦*

(2)

*𝑗*

+ *𝛼*

⎪‖⎛⎜

*𝑖*=1

*𝑗*=1

*𝑤𝑗*

⎞⎟‖

∑*𝑚*

This shows that

*𝜆*(*𝑣*(1)*,* … *, 𝑣*(1)*, 𝛾*(1)) + (1 − *𝜆*)(*𝑣*(2)*,* … *, 𝑣*(2)*, 𝛾*(2)) ∈ *𝑈* (*𝑓, 𝑔 ,* … *, 𝑔* )*.*

⎪‖⎜*𝑠* − *𝑊*

∑

*𝑚*

– *𝜆* (*𝑊* )

≤ *𝑠* + *𝑊* +

⎟‖ *𝑗 𝑗𝑗*

*𝜆𝑖* (*𝑊𝑖* )*𝑗𝑗, 𝑗* = 1*,* … *, 𝑞.*

1 *𝑚*

1 *𝑚*

1 *𝑚*

⎪‖⎜⎝ *𝑗*

⎩

*𝑗𝑗*

*𝑖*

*𝑖*=1

*𝑖 𝑗𝑗* ⎟⎠‖

*𝑖*=1

So, *𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) is a convex set.

The equivalence between (b) and (c) follows if we show that [(4)](#_bookmark8) is

*diagonal matrix 𝑊 and 𝑊𝑖 , 𝑖* = 1*,* … *, 𝑚, respectively.*

*where, for 𝑗* = 1*,* … *, 𝑞, 𝑊𝑗𝑗 and* (*𝑊𝑖* )*𝑗𝑗 are the* (*𝑗, 𝑗*)*th element of the*

**Proof.** Let *𝑔𝑖* (*𝑧*) = *𝑧𝑇 𝑊𝑖𝑧* − *𝑟𝑖* = ∑*𝑞* (*𝑊𝑖* )*𝑗𝑗 𝑧*2 − *𝑟𝑖, 𝑖* = 1*,* … *, 𝑚,* and

*𝑗*=1

*𝑗*

together with the fact that *𝑊* + *𝑚*

following elementary equivalence,

*𝑖*=1

equivalent to (c). This follows by the standard matrix algebra arguments

∑

*𝑊𝑖* is a diagonal matrix and the

*𝑓* (*𝑧*) = *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼.*The equivalence between (a) and (b) follows

from [Lemma 2.1](#_bookmark7) if we show that the epigraphical set

*𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ) = {(*𝑣*1 *,* … *, 𝑣𝑚 , 𝑟*) ∶ *𝑣𝑖* ≥ *𝑔𝑖* (*𝑧*) and *𝑟* ≥ *𝑓* (*𝑧*)

*𝛼 𝑡*∕2 2

*𝑡*∕2 *𝛽*

( ) ⪰ 0 ⇔ *𝑡* ≤ 4*𝛼𝛽, 𝛼, 𝛽* ≥ 0 ⇔ ‖(

□

*𝑡, 𝛼* − *𝛽* ≤ *𝛼* + *𝛽.*

‖

)

for some *𝑧* ∈ ℝ*𝑞* }

is a convex set. Although this equivalence can be proved by verifying the convexity from the definition of a convex set, as done in many known [cases (c.f Dines, 1941; Jeyakumar et al., 2009b; Jeyakumar and Li, 2018; Polik, 2007; Polyak, 1998), we give the details of the arguments here](#_bookmark42) for the purpose of completeness.

If *𝑤* = 0, then *𝑔𝑖* (*𝑧*) = *𝑧𝑇 𝑊𝑖𝑧𝑖* − *𝑟𝑖 , 𝑖* = 1*,* … *, 𝑚,* and *𝑓* (*𝑧*) = *𝑧𝑇 𝑊 𝑧* + *𝛼.*

In this case, it is known that the joint range set *𝑅*(*𝑓, 𝑔*1*,* … *, 𝑔𝑚* ) =

{(*𝑔*1 (*𝑧*)*,* … *, 𝑔𝑚* (*𝑧*)*, 𝑓* (*𝑧*)) ∶ *𝑧* ∈ ℝ*𝑞* } is a convex set ([Polik, 2007](#_bookmark33)). Thus,

Now, we show that the generalised S-lemma continues to hold for quadratic inequality system with a simultaneously diagonalizablity structure. The simultaneous diagonalizability property can be satisfied for quadratic systems that appear in several important quadratic opti- mization problems such as the classical trust region problems and some of their variants ([Ben-Tal and Hertog, 2014; Jeyakumar and Li, 2018](#_bookmark26)).

***able systems)*** *Let 𝑊 , 𝑊𝑖* ∈ *𝑆 𝑞 , 𝑖* = 1*,* 2 … *, 𝑚, 𝛼, 𝑟𝑖* ∈ ℝ*, 𝑖* = 1*,* 2*,* … *, 𝑚 and* **Corollary 2.1. *(Generalized S-Lemma for simultaneously diagonaliz-***

*𝑤* ∈ ℝ*𝑞 . Suppose that 𝑊 and 𝑊 , 𝑖* = 1*,* … *, 𝑚 are simultaneously diagonal-*

*𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔* ) = *𝑅*(*𝑓, 𝑔 ,* … *, 𝑔* ) + ℝ*𝑚*+1 is also convex.

*𝑖*

*𝑞*×*𝑞 𝑇*

*𝑚* 1

*𝑚* +

*izable, that is, there exists an invertible matrix 𝑈* ∈ ℝ

*such that 𝑈*

*𝑊 𝑈*

Therefore, without loss of generality, we assume that *𝑤* ≠ 0. Let

*and 𝑈 𝑇 𝑊𝑖𝑈 , 𝑖* = 1*,* … *, 𝑚, are all diagonal matrices. Suppose further that*

*𝜆* ∈ [0*,* 1], (*𝑣*(1)*,* … *, 𝑣*(1)*, 𝛾*(1)) ∈ *𝑈* (*𝑓, 𝑔 ,* … *, 𝑔* ) and (*𝑣*(2)*,* … *, 𝑣*(2)*, 𝛾*(2)) ∈

*𝑞 𝑇*

1 *𝑚*

1 *𝑚*

(*𝑙*) *𝑞*

1 *𝑚*

*there exists 𝑧* ∈ ℝ

*such that 𝑧 𝑊𝑖𝑧 < 𝑟𝑖 , 𝑖* = 1*,* … *, 𝑚. Then, the following*

*𝑈* (*𝑓, 𝑔*1 *,* … *, 𝑔𝑚* ). Then, there exist *𝑧*

*𝑣 𝑙* ≥ *𝑔𝑖* (*𝑧 𝑙* ) =

(*𝑊𝑖* )*𝑗𝑗* (*𝑧 𝑙* )

– *𝑟𝑖 , 𝑖* = 1*,* … *, 𝑚* (5)

∈ ℝ , *𝑙* = 1*,* 2 such that

*statements are equivalent*

( ) ( )

*𝑖*

∑*𝑞*

( ) 2

and

*𝑗*=1

1. *There exist 𝜆𝑖* ≥ 0*, 𝑖* = 1*,* … *, 𝑚 such that, for all 𝑧* ∈ ℝ*𝑞 ,*

*𝑚*

∑

(a) *𝑧𝑇 𝑊 𝑧* ≤ *𝑟 , 𝑖* = 1*,* 2*,* … *, 𝑚* ⇒ *𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* ≥ 0;

*𝑗*

*𝑖*

*𝑖*

*𝛾*(*𝑙*) ≥ *𝑓* (*𝑧*(*𝑙*)) = (*𝑧*(*𝑙*))*𝑇 𝑊* (*𝑧*(*𝑙*)) + *𝑤𝑇 𝑧*(*𝑙*) + *𝛼.* (6)

Consider the function *𝑔̂𝑖* ∶ ℝ*𝑞* → ℝ, *𝑖* = 1*,* … *, 𝑚* and *𝑓̂* ∶ ℝ*𝑞* → ℝ given

+

*𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* + *𝜆𝑖* (*𝑧𝑇 𝑊𝑖𝑧* − *𝑟𝑖* ) ≥ 0;

*𝑖*=1

by *𝑔̂* (*𝑦*) = ∑*𝑞*

*𝑞*

*𝑖*

*𝑗*=1

+

(*𝑊* )

+

*𝑦* − *𝑟 ,* for all *𝑦* ∈ ℝ *,* and

*𝑖 𝑗𝑗 𝑗*

*𝑖*

+

*𝑗*=1

*𝑗𝑗 𝑗*

*𝑓̂*(*𝑦*) = ∑*𝑞*

*𝑊 𝑦* −

1. *There exist 𝜆𝑖* ≥

0*, 𝑖* = 1*,* … *, 𝑚 and 𝑠𝑗*

≥ 0*, 𝑗* = 1*,* … *, 𝑞 such that*

∑*𝑞*

*𝑗*=1

*𝑗*

*𝑓̂* are all convex functions on ℝ*𝑞* . Let

|*𝑤𝑗* |√*𝑦* + *𝛼,* for all *𝑦* ∈ ℝ*𝑞 .* Direct verification shows that *𝑔̂𝑖* and ⎧

∑*𝑚*

*𝑦*(*𝑙*) =

+

⎪

(*𝑈 𝑇 𝑊 𝑈* )*𝑗𝑗* +

((*𝑧*(*𝑙*))2 *,* … *,* (*𝑧*(*𝑙*))2 )*𝑇* ∈ ℝ*𝑞 .*

⎪ ∑*𝑚*

*𝑖*=1

∑*𝑞*

*𝑖 𝑖 𝑗𝑗* ⎟⎠‖

*𝜆𝑖* (*𝑈 𝑇 𝑊𝑖𝑈* )*𝑗𝑗* ≥ 0*, 𝑗* = 1*,* … *, 𝑞*

1 *𝑞* +

⎪*𝛼* −

*𝜆𝑖 𝑟𝑖* −

*𝑠𝑗* ≥ 0

It then follows from [(5)](#_bookmark10) and [(6)](#_bookmark11) that *𝑔̂* (*𝑦*(*𝑙*)) = *𝑔* (*𝑧*(*𝑙*)) ≤ *𝑣*(*𝑙*) and

⎪⎨ *𝑖*=1

*𝑗*=1

∑*𝑞*

*𝑓̂*(*𝑦*(*𝑙*)) =

*𝑗*=1

∑*𝑞*

*𝑗*

*𝑖 𝑖 𝑖*

( (*𝑙*) ) + *𝛼* ≤ *𝑓* (*𝑧*(*𝑙*)) ≤ *𝛾*(*𝑙*)*.*

⎪‖⎜⎛

(*𝑈 𝑇 𝑤*)*𝑗*

∑*𝑚*

∑

*𝑖*=1

*𝑚*

⎞⎟‖

Then, by the convexity of *𝑔̂* and *𝑓̂* on ℝ*𝑞* ,

|*𝑤𝑗* | ⋅ | *𝑧 𝑗* |

*𝑊𝑗𝑗* (*𝑧*(*𝑙*) )2 −

*𝑗*=1

⎪≤ *𝑠𝑗* + (*𝑈*

*𝑊 𝑈* )*𝑗𝑗* +

*𝑖*=1 *𝜆𝑖* (*𝑈*

*𝑊𝑖𝑈* )*𝑗𝑗, 𝑗* = 1*,* … *, 𝑞.*

*𝑖* +

*𝜆* (*𝑈 𝑇 𝑊 𝑈* )

*𝑇*

*𝑔̂𝑖* (*𝜆𝑦*(1) + (1 − *𝜆*)*𝑦*(2)) ≤ *𝜆𝑣*(1) + (1 − *𝜆*)*𝑣*(2)

*𝑖 𝑖*

⎩*where, for 𝑗* = 1*,* … *, 𝑞,* (*𝑈 𝑇 𝑊 𝑈* )

*𝑗𝑗*

*and* (*𝑈 𝑇 𝑊𝑖𝑈* )*𝑗𝑗*

*are the* (*𝑗, 𝑗*)*th ele-*

and

*𝑓̂*(*𝜆𝑦*

(1)

+ (1 − *𝜆*)*𝑦*

(2)

) ≤ *𝜆𝛾*

(1)

+ (1 − *𝜆*)*𝛾*

*𝑇*

Now, define *𝑧* = (*𝑧*1 *,* … *, 𝑧𝑞* )

where *𝑧𝑗* = −sign(*𝑤𝑗* )

(2) *.*

√ (1)

*𝜆𝑦𝑗* + (1 − *𝜆*)*𝑦𝑗 .*

(2)

*ment of the diagonal matrix 𝑈 𝑇 𝑊 𝑈 and 𝑈 𝑇 𝑊𝑖𝑈 , 𝑖* = 1*,* … *, 𝑚, respec-*

*tively.*

⎪‖⎜⎝ *𝑗 𝑗𝑗*

*𝑠* − (*𝑈 𝑇 𝑊 𝑈* ) −

⎪ *𝑇*

**Proof.** [(a) ⇔ (b)] It suﬃces to show (a) implies (b) as (b) always implies

(a). Let *𝑈* be the invertible matrix such that *𝑈 𝑇 𝑊 𝑈* = *𝐷* and *𝑈 𝑇 𝑊𝑖𝑈* =

Then, one has, for all *𝑖* = 1*,* … *, 𝑚*,

*𝐷𝑖* , *𝑖* = 1*,* … *, 𝑚*, where *𝐷* and *𝐷𝑖* are all diagonal matrices. Letting *𝑦* =

*𝑞 𝑞*

*𝑈* −1*𝑧*, we see that (a) is equivalent to

*𝑔𝑖* (*𝑧*) = ∑(*𝑊𝑖* )*𝑗𝑗 𝑧*2−*𝑟𝑖* = ∑(*𝑊𝑖* )*𝑗𝑗* (*𝜆𝑦*(1) + (1 − *𝜆*)*𝑦*(2)) − *𝑟𝑖*

*𝑗*=1

*𝑗*

*𝑗*=1

*𝑗*

*𝑗*

*𝑦𝑇 𝐷𝑖 𝑦* ≤ *𝑟𝑖 , 𝑖* = 1*,* 2*,* … *, 𝑚* ⇒ *𝑦𝑇 𝐷𝑦* + (*𝑈 𝑇 𝑤*)*𝑇 𝑦* + *𝛼* ≥ 0*.*

Note also that *𝑦𝑇 𝐷 𝑦* = *𝑧𝑇 𝑊 𝑧 < 𝑟* , *𝑖* = 1*,* … *, 𝑚*, where *𝑦* = *𝑈* −1*𝑧*. Apply-

*𝑖*

*𝑖*

*𝑖*

*𝑞*

ing [Theorem 2.1](#_bookmark9), there exist *𝜆𝑖* ≥ 0, *𝑖* = 1*,* … *, 𝑚* such that for all *𝑦* ∈ ℝ ,

*𝑟*

1

∑*𝑠* ( )

*𝑟*=1

*𝑦𝑇 𝐷𝑦* + (*𝑈 𝑇 𝑤*)*𝑇 𝑦* + *𝛼* + ∑ ( )

*𝑚*

−(1 − *𝜃*)

*𝑏𝑖𝑟 𝜉̃*(*𝑗*) + *𝜆 𝑙 , 𝑗* = 1*,* … *, 𝑞, 𝑙* = 1*,* … *, 𝑝,*

*𝜆𝑖 𝑦𝑇 𝐷𝑖 𝑦* − *𝑟𝑖* ≥ 0*.*

‖⎛−(*𝐴𝑇 𝑥* − *𝑣* ) ⎞‖

*𝑖*=1

‖⎜ ⎟‖ ≤ *𝜎* + *𝜆 , 𝑗* = 1*,* … *, 𝑞, 𝑙* = 1*,* … *, 𝑝,*

This shows that for all *𝑧* ∈ ℝ*𝑞* ,

*𝑙 𝑗* (*𝑙*) (*𝑙*)

‖⎜⎝

*𝜎*(*𝑙*) − *𝜆*(*𝑙*)

⎟⎠‖ *𝑗*

∑*𝑚* ( )

*𝑧𝑇 𝑊 𝑧* + *𝑤𝑇 𝑧* + *𝛼* +

*𝜆𝑖 𝑧𝑇 𝑊𝑖𝑧* − *𝑟𝑖*

≥ 0*.*

2

*𝑖*=1

*𝑗* ‖

( ) ( ) ( ) ( )

*𝜆 𝑙 , 𝜆 𝑙 , 𝜇 𝑙 , 𝜎 𝑙* ≥ 0*, 𝑗* = 1*,* … *, 𝑞, 𝑙* = 1*,* … *, 𝑝,*

1

2

*𝑗*

*𝑗*

*𝑙*

2

‖

Thus, (b) holds. So, (a) implies (b), and hence, (a) is equivalent to (b). [(b) ⇔ (c)] As *𝑈* is invertible, we first note that (b) is equivalent to

where *𝜃* ∈ (0*,* 1] is a parameter given in [(3)](#_bookmark4), and *𝑢* ∈ ℝ*𝑞* is given in the

description of the uncertainty set Ę.

Next, we show that (*𝐼𝑃* − *𝑄𝐷𝑅*) and its associated second order cone

*𝑦𝑇* (*𝑈 𝑇 𝑊 𝑈* )*𝑦* + (*𝑈 𝑇 𝑤*)*𝑇 𝑦* + *𝛼* +

∑*𝑚*

*𝜆𝑖* (*𝑦𝑇* (*𝑈 𝑇 𝑊𝑖𝑈* )*𝑦* − *𝑟𝑖* )

≥ 0*,*

lution to (*𝐼𝑃* − *𝑄𝐷𝑅*) can be uniquely determined from an optimal so- programming problem are equivalent in the sense that an optimal so-

for all *𝑦* ∈ ℝ*𝑞 .*

*𝑖*=1

lution to (*𝑆𝑂𝐶𝑃*

*𝐼𝑃*

). We derive this exact second order cone program

Then, the conclusion follows by [Theorem 2.1](#_bookmark9). □

epigraphical set involving non-homogeneous functions with *𝑍*-matrices, **Remark 2.1.** It is worth remarking that the convexity theorems of the

established in [Jeyakumar et al. (2009b)](#_bookmark24), do not yield corresponding S-lemma type results for diagonal matrices due to homogenization of the given non-homogeneous system. Simplified S-lemma results are [known for homogeneous systems involving diagonal matrices in Ben- Tal et al. (2009);](#_bookmark24) [Polik](#_bookmark33) [(2007)](#_bookmark24)

# Exact SOCP relaxations for AROs with inexact data

In this section we present an exact SOCP relaxation problem for a

relaxation by using the new S-lemma for a partially homogeneous sepa- rable quadratic system involving a non-homogeneous quadratic function and more than two homogeneous quadratic inequalities. This extends the approach used in [Woolnough et al. (2021)](#_bookmark38) where a single ellipsoid and the classical S-lemma were used.

*parameterized quadratic decision rule and inexact ellipsoidal data* (*𝐼𝑃* − **Theorem 3.1.** *Consider the adjustable robust linear program with separable*

*𝑄𝐷𝑅*)*, and the second order cone programming problem* (*𝑆𝑂𝐶𝑃𝐼𝑃* )*.*

1. *It holds that* min(*𝐼𝑃* − *𝑄𝐷𝑅*) = min(*𝑆𝑂𝐶𝑃𝐼𝑃* )*, where* min(*𝐼𝑃* − *𝑄𝐷𝑅*) *(resp.* min(*𝑆𝑂𝐶𝑃𝐼𝑃* ) *denotes the optimal value of the problem (IP-QDR) (resp.* (*𝑆𝑂𝐶𝑃𝐼𝑃* )*).*

∗ ( ∗

1. *Moreover,* (*𝑥*∗*, 𝑦*∗*, 𝑃* ∗*, 𝜉*(1) *,* … *, 𝜉 𝑞*) ) *is a solution to* (*𝐼𝑃* − *𝑄𝐷𝑅*) *if and*

0 ( 1 *𝑠*

*only if there exist 𝜆 𝑙*)*, 𝜆*(*𝑙*)*, 𝜇*(*𝑙*)*, 𝜎*(*𝑙*)*, 𝑙* = 1*,* … *, 𝑝, 𝑗* = 1*,* … *, 𝑞 such that*

robust linear program with adjustable decisions that admit a Separable Parameterized Quadratic Decision Rule ([Woolnough et al., 2021](#_bookmark38)), with

1 2 *𝑗 𝑗*

(*𝑥*∗*, 𝑦̃*∗*, 𝑃̃*∗ *, 𝜉̃*(1)∗ *,* … *, 𝜉̃*(*𝑞*)∗ *, 𝜆*(*𝑙*)*, 𝜆*(*𝑙*)*, 𝜇*(1)*,* … *, 𝜇*(*𝑝*)*, 𝜎*(1)*,* … *, 𝜎*(*𝑝*))*,*

inexact data as well as the estimation error contained within ellipsoidal

0 1 *𝑠*

1 2 1

*𝑞* 1 *𝑞*

uncertainty sets.

*is a solution to* (*𝑆𝑂𝐶𝑃* )*, where* (*𝑦̃*∗*, 𝑃̃*∗ *, 𝜉̃*(1)∗ *,* … *, 𝜉̃*(*𝑞*)∗ ) *is given by the*

The two-stage robust linear optimization problem (*𝐼𝑃* ) with separa-

ble parameterized quadratic decision rule given as in [(3)](#_bookmark4) can be written

in the following compact form:

*𝐼𝑃*

*one-to-one correspondence*

(*𝑦̃*∗*, 𝑃̃*∗ *, 𝜉̃*(1)∗ *,* … *, 𝜉̃*(*𝑞*)∗ )

0 1 *𝑠*

*𝑢𝑇 𝑄*∗*𝑢*

=

*𝑦*0 + *𝑃 𝑢* +

⋮

⎜⎝*𝑢𝑇 𝑄*∗*𝑢*⎟⎠

0 1 *𝑠*

2*𝑢𝑇 𝑄*∗

*, 𝑃* +

⋮

*, 𝜉*1 *,* … *, 𝜉𝑠*

⎜⎝2*𝑢𝑇 𝑄*∗⎟⎠

(*𝐼 𝑃* − *𝑄𝐷𝑅*) min

*𝑥,𝑦*0 *,𝑃 ,𝜉𝑗*

*𝑟*

subject to *𝐴*(*𝑧*)*𝑥* + *𝐵𝑦*(*𝑧̂*) ≤ *𝑑*(*𝑧*)*,*

*𝑐𝑇 𝑥*

⎜⎛ ∗

∗ 1 − *𝜃* ⎛⎜

*𝜃*

1 ⎞⎟

*𝑠*

∗ 1 − *𝜃* ⎛⎜

*𝜃*

1⎞⎟

*𝑠*

(1)∗

(*𝑞*)∗ ⎞⎟

for all (*𝑧, 𝑧̂*) ∈ Ę *⊂* ℝ*𝑞* × ℝ*𝑞 ,*

*where 𝑄*∗ = *diag*(*𝜉*(1)∗ *,* … *, 𝜉*(*𝑞*)∗ )*, 𝑟* = 1*,* … *, 𝑠.*

*𝑧̂𝑇 𝑄 𝑧̂*

⎞

⎛

*𝑟 𝑟 𝑟*

1

*𝑦*(*𝑧̂*) = *𝜃*(*𝑦*0 + *𝑃 𝑧̂*) + (1 − *𝜃*)⎜ ⎟

*,*

*𝑠* ⎠

⎝

*𝑧̂𝑇 𝑄 𝑧̂*

⋮

**Proof.** Fix *𝑙* ∈ {1*,* … *, 𝑝*}. Notice that the *𝑙*th constraint of (*𝐼𝑃* − *𝑄𝐷𝑅*)

can be equivalently rewritten as

⎜⎝

⎟⎠

*𝑧̂𝑇 𝑄 𝑧̂*

where *𝑄*

*𝑟*

*𝑟*

*𝑟*

*𝑙*

*𝑙*

*𝑙*

*𝑙*

= diag(*𝜉*(1)*,* … *, 𝜉*(*𝑞*)). Recall that *𝑎 , 𝐴* and *𝜈 , 𝑣* are given as in

*𝑇 𝑇* ⎛⎜

(*𝑎𝑙* + *𝐴𝑙 𝑧*) *𝑥* + *𝑏𝑙*

*𝜃*(*𝑦*0 + *𝑃 𝑧̂*) + (1 − *𝜃*)⎜

⎛ 1 ⎞⎞ *𝑇*

[(1)](#_bookmark0). We also use *𝑏𝑇* to denote the *𝑙*th row of the matrix *𝐵* in (*𝐼𝑃* ) and let

(*𝑏𝑙* )*𝑟*

be the *𝑟*

*𝑙*

th element of

*𝑏𝑙* (that is, the (*𝑙, 𝑟*)th entry of *𝐵*). In passing,

*𝑠*

⎜⎝ ⎜⎝*𝑧̂𝑇 𝑄 𝑧̂*⎟⎠⎟⎠

note that the constraint system of (*𝐼𝑃* − *𝑄𝐷𝑅*) is a non-convex, semi- infinite constraint. We now associate with (*𝐼𝑃* − *𝑄𝐷𝑅*) the following

⋮

(7)

We first apply a linear transformation to our uncertainty set

⎟⎟ ≤ *𝜈𝑙* + *𝑣𝑙 𝑧,* ∀(*𝑧, 𝑧̂*) ∈ Ę*.*

second order cone programming problem:

Ę = {(*𝑧, 𝑧̂*) ∈ ℝ*𝑞* × ℝ*𝑞* ∶

*𝑧̂* − *𝑢* 2 ≤ *𝛾*2*,*

*𝑧* − *𝑧̂* 2 ≤ *𝛾*2}*.*

(*𝑆𝑂𝐶𝑃𝐼𝑃*

) min

*𝑥*∈ℝ*𝑛,𝑦̃*0 ∈ℝ*𝑠*

*𝑐𝑇 𝑥*

Let

‖ ‖ 1 ‖ ‖ 2

*𝑃̃*∈ℝ*𝑠*×*𝑞 ,𝜉̃*(*𝑗* ) ∈ℝ

*𝑟*

(

*𝜆*1 ∈ℝ*,𝜆 𝑙*) ∈ℝ*,*

2

(*𝑙*)

*𝜇*(*𝑙*) ∈ℝ*,𝜎*(*𝑙*) ∈ℝ

*𝜁*

(1)

*𝜁* (2)

*𝜁* (1) = *𝑧̂* − *𝑢, 𝜁* (2) = *𝑧* − *𝑧̂, 𝜁* =

∈ ℝ2*𝑞 .*

*𝑗 𝑗*

*𝑠*

Then

s.t. −(1 − *𝜃*) ∑ *𝑏𝑖𝑟 𝜉̃*(*𝑗*) + *𝜆*(*𝑙*) ≥ 0*, 𝑗* = 1*,* … *, 𝑞, 𝑙* = 1*,* … *, 𝑝,*

∑

1

*𝑧, 𝑧̂*

ℝ *𝑞*

*𝜁*

≤ *𝛾 ,*

*𝜁*

≤ *𝛾*

1

‖

2

( )

( ) ∈

∈ = {

= (*𝜁* (1) ) ∈ 2 ∶

(1) 2

2 (2) 2 2 }

*𝑟*=1

*𝑟*

(*𝑎𝑇 𝑥* + *𝜃𝑏𝑇 𝑦̃*0 − *𝜈𝑙* ) + (*𝐴𝑇 𝑥* − *𝑣𝑙* )*𝑇 𝑢* + *𝜆*(*𝑙*)*𝛾*2 + *𝜆*(*𝑙*)*𝛾*2

*𝑙*

*𝑙*

*𝑙*

1

1

2

2

where

*𝑗*=1

*𝜁* (2)

{

∑

Ę ⟺ *𝜁*

T

*𝜁*

⟺ *𝜁* ∈ T =

*𝜁* ∈ ℝ

2*𝑞*

∶

≤ *𝛾*2*,*

2*𝑞*

*𝑤*1*𝑗 𝜁*

2

*𝑤*2*𝑗 𝜁*

≤ *𝛾*

*,*

*𝑗*=1

‖ ‖ 2*𝑞* ‖ }

∑ (*𝑙*) (*𝑙*)

*𝑞*

+

(*𝜇𝑗* + *𝜎𝑗* ) ≤ 0*, 𝑙* = 1*,* … *, 𝑝,*

*𝑗*=1

*𝑗* 1

*𝑗* 2

⎛ −(*𝐴𝑇 𝑥* + *𝜃𝑃̃𝑇 𝑏𝑙* − *𝑣𝑙* )*𝑗* ⎞‖

*𝑙*

*𝑏 𝜉̃* − *𝜆* ⎟

‖

(*𝑙*)

{1*,* 1 ≤ *𝑗* ≤ *𝑞*

{0*,* 1 ≤ *𝑗* ≤ *𝑞*

‖⎜*𝜇*(*𝑙*) + (1 − *𝜃*) ∑*𝑠*

(*𝑗*) (*𝑙*) ≤ *𝜇𝑗*

‖

*𝑤*1*𝑗* =

0*, 𝑞* + 1 ≤ *𝑗* ≤ 2*𝑞 , 𝑤*2*𝑗* =

1*, 𝑞* + 1 ≤ *𝑗* ≤ 2*𝑞.* (8)

‖⎝ *𝑗*

2

2

*𝑟*=1

*𝑖𝑟 𝑟*

1 ⎠‖

That is, Ę can be represented as an intersection of two ellipsoids of the form consistent with [Theorem 2.1](#_bookmark9), in a higher dimension. Thus, we can rewrite [(7)](#_bookmark13) under the uncertainty set T :

(*𝑎* + *𝐴* (*𝜁* (1) + *𝜁* (2) + *𝑢*))*𝑇 𝑥*

*𝑙*

*𝑙*

**Remark 3.1.** It is worth noting that the authors in [Woolnough et al. (2021)](#_bookmark38) established exact conic programming re- formulations for two-stage robust optimization problems with exactly

revealed data, where *𝑧̂* = *𝑧* in [(2)](#_bookmark1). In this case, our two-staged robust

+ *𝑇* ⎛⎜ (

(*𝜁* (1) + *𝑢*)*𝑇 𝑄* (*𝜁* (1) + *𝑢*)

+ ( (1) +

)) + (1 −

)⎛⎜ ⋮ 1 ⎞⎟⎞⎟

*𝑏𝑙 𝜃 𝑦*0

⎜

⎝

*𝑃 𝜁 𝑢*

⎜⎝(*𝜁* (1) + *𝑢*)*𝑇 𝑄𝑠* (*𝜁* (1) + *𝑢*)⎟⎠⎟⎠

# Adjustable robust lot-sizing problems with inexact data

≤ *𝜈𝑙* + *𝑣𝑇* (*𝜁* (1) + *𝜁* (2) + *𝑢*)*,* ∀(*𝜁* (1)*, 𝜁* (2)) ∈ T *.* (9)

*𝑙*

problem is equivalent to the robust optimization model examined in [Woolnough et al. (2021)](#_bookmark38).

*𝜃*

In the lot-sizing problem, given a network of *𝑛* stores one needs to determine the stock allocation *𝑥𝑖* (in units) at each store *𝑖* = 1*,* … *, 𝑛*, (de-

Define the

(1)

one-to-one correspondence between variables

(*𝑞*

noted by *𝑥* ∶= (*𝑥*1 *,* … *, 𝑥𝑛* )), and the stock *𝑦𝑖𝑗* to be transported from store

(*𝑦 , 𝑃 , 𝜉 ,* … *, 𝜉𝑠* )) and new variables (*𝑦̃ , 𝑃̃, 𝜉̃*(1) *,* … *, 𝜉̃*(*𝑞*) ) by

0 1

1 − ⎛*𝑢𝑇 𝑄*1*𝑢*⎞

*𝑦̃*0 = *𝑦*0 + *𝑃 𝑢* +

*𝜃*

⎜

⋮

*𝑢𝑇 𝑄 𝑢*

∈ ℝ *, 𝑃̃* = *𝑃* +

⎟

*𝜃 𝑠*

0 1 *𝑠*

1 − ⎛2*𝑢𝑇 𝑄*

*𝜃*

⎜

⋮

2*𝑢𝑇 𝑄*

∈ ℝ

⎟

*𝜃*

1 ⎞ *𝑠*×*𝑞*

*𝑖* to *𝑗* (with *𝑦𝑖𝑖* necessarily set to 0), in order to meet the demand *𝑑𝑖* for

that the storage cost in store *𝑖* is *𝑐* with respect to the stock allocation each store over a set period of time (for example, one day). We assume

*𝑖*

*𝑥𝑖* , and that the cost of transporting one unit of stock from store *𝑖* to

and

⎝ *𝑠* ⎠

⎝ *𝑠*⎠

*𝑗* is *𝑡*

*𝑖𝑗*

, *𝑖, 𝑗* = 1*,* … *, 𝑛*, where *𝑡*

*𝑖𝑖*

= 0 for *𝑖* = 1*,* … *, 𝑛*. This problem can be

mathematically described as

*𝜉̃*(*𝑗*) = *𝜉*(*𝑗*)*, 𝑟* = 1*,* … *, 𝑠, 𝑗* = 1*,* … *, 𝑞.*

*𝑛 𝑛*

*𝑟 𝑟* ∑ ∑

(LS ) min

*𝑥*∈ℝ

*𝑐𝑖 𝑥𝑖* +

To accommodate the transformation of (*𝑧, 𝑧̂*) into *𝜁* in the higher dimen-

2*𝑞*

0 *𝑛*

*𝑖*=1

*𝑖,𝑗*=1

*𝑡𝑖𝑗 𝑦𝑖𝑗*

*𝑎̃*

sional space ℝ

, we also define

*𝑦𝑖𝑗* ∈ℝ

= *𝑎* + *𝐴 𝑢* ∈ ℝ*𝑛, 𝐴̃*

= (*𝐴*

*𝐴* ) ∈ ℝ*𝑛*×2*𝑞 ,*

∑*𝑛*

∑*𝑛*

*𝑙 𝑙 𝑙*

*𝑇*

*𝑙 𝑙*

(*𝑣𝑙* )

*𝜈̃𝑙* = *𝜈𝑙* + *𝑣𝑙 𝑢* ∈ ℝ*, 𝑣̃𝑙* =

*𝑣*

*𝑙*

*𝑙*

2*𝑞*

∈ ℝ

s.t. *𝑥𝑖* +

(10)

*𝑗*=1

*𝑦𝑗𝑖* −

*𝑗*=1

*𝑦𝑖𝑗* ≥ *𝑑𝑖 , 𝑖* = 1*,* … *, 𝑛*

Then [(9)](#_bookmark15) is equivalent to [Theorem 2.1](#_bookmark9) (a) with the settings

*𝑥𝑖* ≥ 0*, 𝑦𝑖𝑖* = 0*, 𝑦𝑖𝑗* ≥ 0*, 𝑖* = 1*,* … *, 𝑛, 𝑗* = 1*,* … *, 𝑛, 𝑖* ≠ *𝑗*

*𝑚* = 2*, 𝑧* = *𝜁* ∈ ℝ2*𝑞 , 𝑟*1 = *𝛾*2*, 𝑟*2 = *𝛾*2*, 𝛼* = −*𝑎̃𝑇 𝑥* − *𝜃𝑏𝑇 𝑦̃*0 + *𝜈̃* ∈ ℝ*,*

In practice the true demand *𝑑* = (*𝑑*1 *, 𝑑*2 *,* … *, 𝑑𝑛* )*𝑇* ∈ ℝ*𝑛* is uncertain,

and partially revealed at some point after the initial allocation of stock;

*̃𝑇*

( *𝑃̃𝑇* )

1 2

2*𝑞*

*𝑙 𝑙 𝑙*

for example, by using a projection of current demand until end of day.

*𝑤* = −*𝐴𝑙 𝑥* − *𝜃*

⎛

0

∑*𝑠*

*𝑞*×*𝑠*

*𝑏𝑙* + *𝑣̃𝑙* ∈ ℝ *,*

⎞

⎟

⎛*𝑤*11

* 1. *Experimental design*

⎞⎟ **Problem Set-up**: Suppose a nominal demand *𝑑*

∈ ℝ*𝑛* is chosen for

*𝑊* =

⎜

⎜⎝

−(1 − *𝜃*)

⎜

⎝

*𝑟*=1

(*𝑏𝑙* )*𝑟𝑄𝑟*

*, 𝑊*1 =

0 ⎟

*𝑞*×*𝑞* ⎠

⎞⎟

*𝑤*12

⋱

0

the model (using, e.g. historical data). The true demand is known to lie

*,*

⎟ within some neighbourhood of this nominal demand. At the beginning

*𝑤*1(2*𝑞*) ⎟⎠

of the period, an initial stock allocation *𝑥*

*𝑖*

is decided for each store in

*𝑊*2 =

⎛*𝑤*21

⎝

⎜

*𝑤*22

⋱

*𝑤*2(2*𝑞*)

⎟⎠

∑*𝑠*

(*𝑏* ) *𝜉*

= −(1 − *𝜃*)

*𝑟*=1

(*𝑗*)

*̃*

*𝑙 𝑟 𝑟*

(11)

estimate *𝑑̂* ∈ ℝ*𝑛* is taken of the true demand for the entire period (using, the network. Then, at a pre-specified point in time later in the period, an

that *𝑑̂* lies within a ball centred at the nominal demand (the *estimation* for example, satisfied demand up until that time point). We will assume

*range*), and that this is expressed in the form

[Theorem 2.1](#_bookmark9), and transforming (*𝑎̃ , 𝐴̃ , 𝜈̃ , 𝑣̃* ) back to (*𝑎 , 𝐴 , 𝜈 , 𝑣* ),

Note that *𝑊*

*𝑗𝑗*

for *𝑗* = 1*,* … *, 𝑞*. Applying

*𝑙*

*𝑙*

*𝑙*

*𝑙*

*𝑙*

*𝑙*

*𝑙*

*𝑙*

⎜

give us that [(9)](#_bookmark15) is equivalent to the condition that there exist

*𝜆*(*𝑙*)*, 𝜆*(*𝑙*)*, 𝜇*(*𝑙*)*, 𝜎*(*𝑙*) ≥ 0*, 𝑗* = 1*,* … *, 𝑞,* such that

1 2 *𝑗 𝑗*

⎧ ∑*𝑠* ( )

*𝑑* 0 100 0

At this point in time we make a decision to transport some stock *𝑦𝑖𝑗* between stores, based on our estimate *𝑑̂*. In reality the estimate *𝑑̂* will deviate from the true demand *𝑑* with some *estimation error 𝛿*, such that

*̂* − *𝑑*

≤

*𝛼*

*𝑑*

*.*

‖ ‖ ‖ ‖

*𝛿* = *𝑑* − *𝑑̂*. We assume also that this deviation lies in its own ball: *𝛿* ≤ *𝛽*

‖ ‖

⎪−(1 − *𝜃*) (*𝑏𝑙* )*𝑟𝜉̃*(*𝑗*) + *𝜆 𝑙* ≥ 0*, 𝑗* = 1*,* … *, 𝑞*

*𝑟*=1

⎪

*𝑟* 1

∑*𝑞*

with *𝛽 >* 0.

By also choosing to assign a separable parameterized QDR to each

(−*𝑎𝑇 𝑥* − *𝜃𝑏𝑇 𝑦̃*0 + *𝜈𝑙* ) + (−*𝐴𝑇 𝑥* + *𝑣𝑙* )*𝑇 𝑢* − *𝜆*1 *𝛾*2 − *𝜆*2 *𝛾*2 −

⎪

*𝜇*(*𝑙*)

*𝑦* (*𝑑̂*), we can express the problem as a two-stage adjustable robust linear

*𝑙*

⎪ − ∑*𝑞*

*𝑙 𝑙*

*𝜎*(*𝑙*) ≤ 0

1 2 *𝑗*

*𝑗*=1

*𝑖𝑗*

optimization problem with inexactly-revealed ellipsoidal data:

⎪‖⎛

⎨

*𝑗*=1

(−

*𝑗*

*𝐴𝑇 𝑥*

*𝑙*

– *𝜃𝑃̃𝑇*

*𝑏𝑙*

+ *𝑣𝑙* )*𝑗*

⎞‖ (*𝑙*)

∑*𝑠*

(*𝑙*)

(LS) min

*𝑥*∈ℝ*𝑛, 𝑦𝑖𝑗* ∈ℝ

*𝑛*

*𝑖*=1

∑

*𝑐𝑖 𝑥𝑖* + max

*𝑑̂*∈Q

*𝑛*

*𝑖,𝑗*=1

{ ∑

∑*𝑛*

*𝑡𝑖𝑗 𝑦𝑖𝑗* (*𝑑̂*)}

⎪‖⎜⎜⎝*𝜇*(*𝑙*) + (1 − *𝜃*) ∑*𝑠*

⎠

⎪‖

*𝑗*

*𝑟*=1

*𝑙 𝑟 𝑟*

1

‖

*𝑗* = 1*,* … *, 𝑞*

⎪

(*𝑏* ) *𝜉̃*(*𝑗*) − *𝜆*(*𝑙*)⎟⎟‖ ≤ *𝜇𝑗*

– (1 − *𝜃*)

(*𝑏𝑙* )*𝑟𝜉̃*(*𝑗*) + *𝜆*1 *,*

=1

*𝑟*

*𝑟*

*𝑗*=1

*𝑦𝑗𝑖*

*𝑗*=1

*𝑖𝑗*

*𝑖*

*𝑤𝑖𝑗* ∈ℝ*𝑛, 𝑞𝑖𝑗* ∈ℝ*𝑛*

s.t. *𝑥𝑖*

+

(*𝑑̂*) −

*𝑦*

(*𝑑̂*) ≥ *𝑑 , 𝑖* = 1*,* … *, 𝑛*

⎪‖ ‖

∑*𝑛*

*𝑥𝑖* ≥ 0*, 𝑦𝑖𝑖* (*𝑑̂*) = 0*, 𝑦𝑖𝑗* (*𝑑̂*) ≥ 0*, 𝑖* = 1*,* … *, 𝑛, 𝑗* = 1*,* … *, 𝑛, 𝑖* ≠ *𝑗*

⎪‖⎛(−*𝐴𝑇 𝑥* + *𝑣𝑙* )*𝑗* ⎞‖

( 0 *𝑇* ) *𝑇*

⎪‖⎜

*𝑙*

⎟‖ ≤ *𝜎* + *𝜆 , 𝑗* = 1*,* … *, 𝑞,*

*𝑦𝑖𝑗* (*𝑑̂*) = *𝜃*

*𝑦𝑖𝑗* + *𝑤𝑖𝑗 𝑑̂*

+ (1 − *𝜃*)(*𝑑̂* diag(*𝑞𝑖𝑗* )*𝑑̂*)*,* for all (*𝑑̂, 𝑑*) ∈ Q*,*

⎪‖⎜⎝

2

*𝜎*(*𝑙*) − *𝜆*(*𝑙*)

⎟⎠‖ *𝑗* { }

‖ *𝑗* ‖

(*𝑙*) (*𝑙*)

2

⎩

where Q = (*𝑑̂, 𝑑*) ∈ ℝ*𝑛* × ℝ*𝑛* ∶ *𝑑* − *𝑑̂* 2 ≤ *𝛽*2*, 𝑑̂* − *𝑑* 2 ≤ *𝛼*2 *𝑑* .

‖ 0‖

‖ ‖ ‖ 0‖

2

(12)

10 000

Notice that our uncertainty set Q takes on the same form as Ę in (*𝐼𝑃* ).

where *𝜇𝑗* = *𝑠𝑗* , *𝜎𝑗* = *𝑠𝑗*+*𝑞* , *𝑗* = 1*,* … *, 𝑞*. The result then follows by sub- stitution of [(12)](#_bookmark16) for each of the *𝑙*th constraints of (*𝐼𝑃* − *𝑄𝐷𝑅*), *𝑙* = 1 … *, 𝑝*. □

(*𝑙*)

(*𝑙*)

The formulation aims to satisfy the uncertain true demand, whilst mak- ing decisions based on the estimated uncertainty, with these decisions functionally determined before estimates are taken.

**Numerical Experiment Design**. We will compare the solution to problem (LS) (referred to here on as the “inexact approach”) to its jux- taposition where we ignore the inexactness of revealed data and instead

assume that *𝑑* = *𝑑̂* (here-on referred to as the “exact” approach).

For *𝑛* = 2*,* 4*,* 8*,* 10*,* 12*,* 15, we will choose our parameters as follows:

**Table 1**

Price of Robustness, over all *𝑑* in the neighbourhood of *𝑑̂* .

max

***𝑛*** Approach Mean Max Min Med.

*𝑐𝑖* = 1*, 𝑡𝑖𝑗* = {2 *, 𝑑*0 = (5*,* … *,* 5)*𝑇*

0*, 𝑖* = *𝑗*

*, 𝑖* ≠ *𝑗*

*𝛽* = 1 ( *𝛼* ‖*𝑑*0‖)*, 𝑖, 𝑗* = 1*,* … *, 𝑛.*

2

100

∈ ℝ*𝑛, 𝛼* = 50*,*

2 Inexact QDR 0.82 1.31 0.49 0.79

Inexact ADR 0.84 1.33 0.51 0.81

Exact QDR 0.50 0.90 0.23 0.48

4 Inexact QDR 0.70 1.03 0.43 0.69

Inexact ADR 0.86 1.21 0.56 0.85

Exact QDR 0.33 0.58 0.12 0.32

Our experiments analyse the solutions for each of the approaches by

ters *𝑑* and *𝑑̂*, which we simulate (see below). Recall that after this point, considering the solution after realization of the true uncertain parame-

our adjustable variables *𝑦* (*𝑑̂*) ∈ ℝ reduce to the scalar *𝜃*(*𝑦*0 ∗ + *𝑤*∗ *𝑇 𝑑̂*) +

8 Inexact QDR 1.59 2.22 1.14 1.55

Inexact ADR 2.18 2.96 1.63 2.13

Exact QDR 0.86 1.32 0.54 0.84

10 Inexact QDR 1.83 2.55 1.43 1.81

*𝑖𝑗*

( )

*𝑖𝑗*

*𝑖𝑗*

Inexact ADR 2.63 3.56 2.12 2.61

(1 − *𝜃*) *𝑑̂𝑇* diag(*𝑞*∗ )*𝑑̂* where (*𝑦*0 ∗*, 𝑤*∗ *, 𝑞*∗ ) are the optimal solution found by solving (LS) with the given approach.

*𝑖𝑗*

*𝑖𝑗*

*𝑖𝑗*

*𝑖𝑗*

Our simulations require us to simulate first the demand estimate *𝑑̂*

and then, based on the outcome, simulate the true demand *𝑑*. In our

case this is done by first uniformly generating 200 simulated values for

Exact QDR 0.98 1.48 0.70 0.97

12 Inexact QDR 1.27 1.63 0.97 1.26

Inexact ADR 2.01 2.50 1.62 2.01

Exact QDR 0.54 0.79 0.34 0.54

15 Inexact QDR 1.95 2.40 1.57 1.94

Inexact ADR 3.09 3.72 2.57 3.07

*𝑑̂* within the ellipsoid *̂* −

‖ ‖

*𝛼*

Exact QDR 0.94 1.23 0.69 0.93

‖*𝑑 𝑑*0‖ ≤ 100 ‖*𝑑*0‖, and then, for each of these,

generating 200 simulated values for the true demand *𝑑* within the ellip-

soid

*𝑑* − *𝑑̂*

≤ *𝛽*.

For proper discussion of the performance of our adjustable robust

tion of some true demand *𝑑*, by examining its realized cost: models, we need to investigate how the solution behaves under realiza-

**Table 2**

Price of Robustness, over all *𝑑* in the neighbourhood

min

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| of *𝑑̂* . |  | | | | |
| ***𝑛*** | Approach | Mean | Max | Min | Med. |
| 2 | Inexact QDR | 0.93 | 1.44 | 0.55 | 0.89 |
|  | Inexact ADR | 0.93 | 1.44 | 0.56 | 0.90 |
|  | Exact QDR | 0.59 | 1.02 | 0.28 | 0.57 |
| 4 | Inexact QDR | 1.97 | 3.08 | 1.29 | 1.89 |
|  | Inexact ADR | 2.25 | 3.46 | 1.50 | 2.16 |
|  | Exact QDR | 1.32 | 2.19 | 0.79 | 1.26 |
| 8 | Inexact QDR | 1.28 | 1.65 | 0.91 | 1.27 |
|  | Inexact ADR | 1.81 | 2.27 | 1.35 | 1.80 |
|  | Exact QDR | 0.64 | 0.91 | 0.38 | 0.64 |
| 10 | Inexact QDR | 1.52 | 1.97 | 1.10 | 1.51 |
|  | Inexact ADR | 2.23 | 2.81 | 1.70 | 2.22 |
|  | Exact QDR | 0.76 | 1.08 | 0.47 | 0.76 |
| 12 | Inexact QDR | 1.92 | 2.42 | 1.45 | 1.93 |
|  | Inexact ADR | 2.88 | 3.55 | 2.26 | 2.89 |
|  | Exact QDR | 0.99 | 1.33 | 0.67 | 0.99 |
| 15 | Inexact QDR | 2.01 | 2.51 | 1.59 | 2.01 |
|  | Inexact ADR | 3.17 | 3.86 | 2.59 | 3.17 |
|  | Exact QDR | 0.97 | 1.30 | 0.70 | 0.97 |

*𝑛*

∑

*𝑖*=1

*𝑐𝑖𝑥𝑖* +

*𝑛*

*𝑖,𝑗*=1

∑

*𝑡𝑖𝑗 𝑦𝑖𝑗* (*𝑑̂*)

**Price of Robustness**. We define the price of robustness *𝑃 𝑂𝑅* as the per-

solution (which is the optimal value *𝑐* produced by solving the problem centage difference between the realized cost achieved by the adjustable

(LS)), and the optimal value *𝑐*LS0 of the nominal problem (LS0), relative

to *𝑐*LS0 :

*𝑃 𝑂𝑅* =

*𝑐* − *𝑐*LS

*.*

0

*𝑐*LS0

true demand parameter *𝑑̂* and its associated simulations of the true de- We emphasise that both values are dependent on the same estimated mand *𝑑*. Note that this realized cost for the adjustable solution is the cost

that would be incurred in practice by utilizing the adjustable model, and likewise the optimal cost of the nominal problem is the lowest possi- ble cost achievable by having all uncertain information available before making any decisions.

Our definition of the price of robustness is equivalent to that em- ployed by [Ben-Tal et al. (2004)](#_bookmark25) in discussions of ADR solutions to multi- stage adjustable robust inventory production problems. We have chosen this definition due to its relevance to our experiments, as opposed to, for example, the definition provided by [Bertsimas and Sim (2004)](#_bookmark31) where it is defined as a trade-off between the probability of violating robust constraints and the effect on the resulting objective function, and only within the scope of Robust Optimization.

Our experiments consist of two parts:

1. Firstly, we compare the price of robustness of the two approaches by comparing their (realised) solutions to the nominal solution to

(LS0). For this case we choose *𝜃* = 1 for the exact approach, and both

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| ***𝑛*** | Approach | Total | Max | Min | Med. |
| 2 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 4024 | 104 | 0 | 0 |
|  | Exact ADR | 15,000 | 200 | 0 | 0 |
| 4 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 13,111 | 166 | 0 | 60 |
|  | Exact ADR | 33,600 | 200 | 0 | 200 |
| 8 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 25,369 | 195 | 1 | 136 |
|  | Exact ADR | 38,000 | 200 | 0 | 200 |
| 10 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 29,158 | 192 | 4 | 155 |
|  | Exact ADR | 40,000 | 200 | 200 | 200 |
| 12 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 31,291 | 200 | 6 | 164 |
|  | Exact ADR | 40,000 | 200 | 200 | 200 |
| 15 | Inexact QDR | 0 | 0 | 0 | 0 |
|  | Exact QDR | 34,436 | 198 | 86 | 179 |
|  | Exact ADR | 40,000 | 200 | 200 | 200 |

**Table 3**

Infeasibility results for the inexact and exact approach. Columns defined above. Note that the inexact ap- proach never delivers an infeasible solution.

1 2

*𝜃* = 2 and *𝜃* = 1 (i.e. ADR) for the inexact approach. Results for this

experiment are presented in [Tables 1](#_bookmark17) and [2](#_bookmark18) and [Fig. 1](#_bookmark20). We will not compare to the exact approach with an ADR, as this result has been examined in [Woolnough et al. (2021)](#_bookmark38).

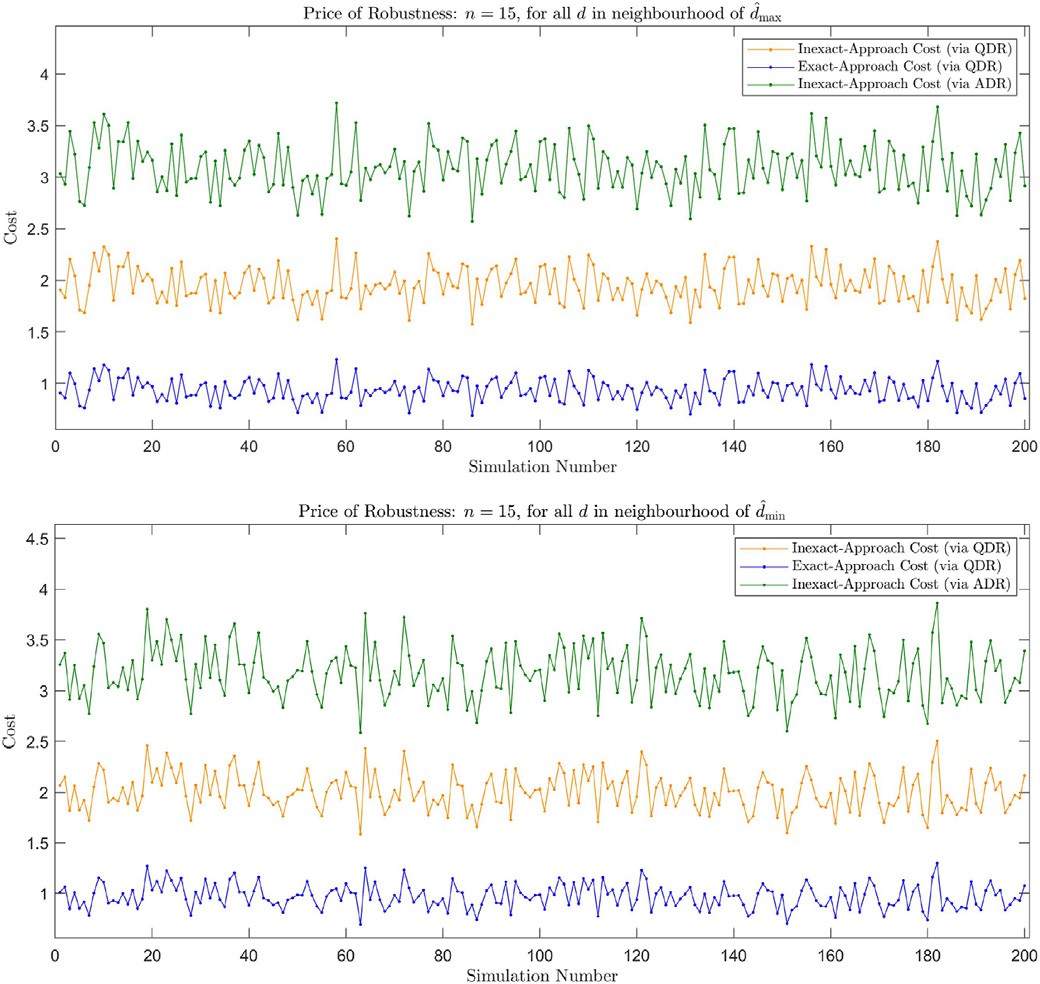
1. Secondly, we compare the exact and inexact approaches (both *𝜃* = 1

and *𝜃* = 1) in terms of the feasibility of their solutions against the

2

adjustable models is considered feasible for a realization *𝑑* if the (inexactly-revealed) true demand. A solution given by one of our constraints of (LS0) hold for this solution and realized *𝑑*. Results for

this experiment are presented in [Table 3](#_bookmark19).

**Fig. 1.** Price of Robustness, for the case *𝑛* = 15. Top: the price of robustness for

* + **Approach**: either the exact or inexact QDR, or the exact ADR.

40 000 generated values for *𝑑 •* **Total**: the total number of infeasible solutions found across all

generated *𝑑̂*, out of 200 (i.e. instance *𝑑̂*max) *•* **Max**: the maximum number of infeasible solutions found for a single

generated *𝑑̂*, out of 200 (i.e. instance *𝑑̂*min) *•* **Min**: the minimum number of infeasible solutions found for a single

generated *𝑑̂*, out of 200. *•* **Med.**: the median number of infeasible solutions found for a single

As expected, the inexact approach never returns infeasible solutions, by design of its uncertainty set to accommodate all possible realisations of the uncertainty. The exact approach, however, ignores inexactness

and, in assuming that *𝑑* = *𝑑̂*, frequently generates adjustable decisions

that are infeasible for the problem. Also notice that the number of in-

feasible solutions increases as the network increases to a larger, more realistic size. This shows a clear advantage for real-world applications, which often contain varying degrees of uncertainty.

lem instances wherein the true demand *𝑑* lay within the estimation Our experiments also reported that there were some simulated prob-

were infeasible for this choice of *𝑑*, despite satisfying all conditions to range, and yet the exact approaches returned realized solutions which

be an optimal solution. This illustrates the true benefit of an inexact approach, which is that it can guarantee that a returned solution will always be feasible for any realized true demand within the estimation

the three approaches, over all simulated *𝑑* in the neighbourhood of *𝑑̂*

max

. Bottom:

error. While this result may appear counter-intuitive to the robustness of

likewise, but over all simulated *𝑑* in the neighbourhood of *𝑑̂* .

min

For the purposes of analysis we cannot present the price of robust-

all 200 simulated *𝑑̂* two realizations: *𝑑̂*min , the estimate that produced ness results for all 40,000 simulations; therefore we have selected from

the *least* number of infeasible solutions when applying the exact QDR

the exact approaches, it is worth noticing that the exact approach is im- mune to uncertainty realization as long as its core assumption - that the

realized value of the uncertainty, *𝑑̂*, is *exact* - holds. This is not the case

for our experiments as we are specifically analysing the case *𝑑̂* ≠ *𝑑*. More explicitly, for solution (*𝑥, 𝑦*(⋅)) returned by the exact approach, there is

no reason that the constraints

approach to all 200 simulated *𝑑* in its neighbourhood; and *𝑑̂*max , the

*𝑥𝑖* +

∑*𝑛*

*𝑦𝑗𝑖*(*𝑑̂*) +

∑*𝑛*

*𝑦𝑖𝑗* (*𝑑̂*) ≥ *𝑑𝑖 ,* ∀*𝑖* = 1*,* … *, 𝑛*

estimate that produced the *most* number of infeasible solutions when

*𝑗*=1

*𝑗*=1

applying the exact QDR approach to all 200 simulated *𝑑* in its neigh-

bourhood. These two choices allow for coverage of both extreme cases

All computations were performed using a 3.2GHz Intel(R) Core(TM) i7-8700 and 16GB of RAM, equipped with MATLAB R2019A. All opti- mization problems were solved via the MOSEK software ([ApS, 2019](#_bookmark22)), handled through the YALMIP interface ([Lofberg, 2004](#_bookmark29)).

* 1. *Results*

culated for each *𝑛* and each generated demand *𝑑* in the neighbourhoods **Price of Robustness Comparison**. The price of robustness was cal- of *𝑑̂*min and *𝑑̂*max . [Tables 1](#_bookmark17) and [2](#_bookmark18) provide statistics for the two cases. The

columns are given by:

* + - ***𝑛***: the number of stores
    - **Approach**: method used: either the inexact QDR approach, inexact

ADR approach, or the exact QDR approach

* + - **Mean**: the mean price of robustness over all (200) relevant *𝑑*
    - **Max**: the maximum price of robustness over all (200) relevant *𝑑*
    - **Min**: the minimum price of robustness over all (200) relevant *𝑑*
    - **Med.**: the median price of robustness over all (200) relevant *𝑑*.

We first note that the QDR (both approaches) outperforms the ADR in all simulations, achieving a lower price of robustness. It can also be seen that the counterpart to infeasible solutions (see below, Infeasibil- ity Comparison) is a lower price of robustness, as would be expected. Interestingly, the infeasible solutions (also included in the below plots) never find a lower cost than the nominal solution (which is a possible

simulation, for all three methods and *𝑛* = 15, are illustrated in [Fig. 1](#_bookmark20). consequence of infeasibility). The price of robustness for each individual

**Infeasibility Comparison**. The results are summarised in [Table 3](#_bookmark19) for which the columns are given by:

* + - ***𝑛***: the number of stores

hold in general for all *𝑑* satisfying *𝑑* − *𝑑̂* ≤ *𝛽*. This is in stark contrast to the inexact approach, which understands that *𝑑̂* can be different to *𝑑*

with deviation up to the estimation error.

‖ ‖

# Conclusion and outlook

We have shown that a second-order cone program reformulation holds for adjustable robust linear optimization problems with a param- eterized quadratic decision rule (QDR) and inexactly revealed data. We achieved this by first establishing a generalization of S-lemma which al- lowed us to transform the semi-infinite non-convex separable quadratic inequality constraint into a second-order cone constraint. We illustrated our results via numerical experiments on adjustable robust lot-sizing problems with demand uncertainty and showed that the inexactly re- vealed data approach with QDRs improves over the infeasibility of the exact QDR approach, whilst maintaining its better performance over an ADR approach in achieving a low price of robustness.

Our adjustable robust optimization approach with inexact data is likely to lead to applications in dynamic decision-making problems of medicine and health sciences, where the conventional static decisions that do not include adjustable (recourse) decisions are often practi- cally meaningless because a patient’s condition or health care needs can [change during the course of a treatment (Eikelder et al., 2019; Nohadani and Roy, 2017), but estimates of their condition can be taken and con-](#_bookmark41) sidered mid-treatment. In particular, our approach may be extended to two-stage robust optimization models with time-dependent uncertainty [sets that appear in radiation therapy planning problems (Nohadani and Roy, 2017).](#_bookmark32)

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