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Random Continuous Functions

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Abstract

We investigate notions of algorithmic randomness in the space C(2N) of continuous functions on 2N. A probability measure is given and a version of the Martin-L¨of test for randomness is defined which

allows us to define a class of (Martin-L¨of) random continuous functions. We show that random Δ0 continuous functions exist, but no computable function can be random. We show that a random function maps any computable real to a random real and that the image of a random continuous

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function is always a perfect set and hence uncountable. We show that for any *y* ∈ 2N, there exists a random continuous function *F* with *y* in the image of *F* . Thus the image of a random continuous function need not be a random closed set.

*Keywords:* Martin-Lo¨f test, random real, random closed set, random function.

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# Introduction

The study of algorithmic randomness has been of great interest in recent years. The basic problem is to quantify the randomness of a single real number. Early in the last century, von Mises [[16](#_bookmark20)] suggested that a random real should obey reasonable statistical tests, such as having a roughly equal number of zeroes and ones of the first *n* bits, in the limit. Thus a random real would be *stochastic* in modern parlance. If one considers only *computable* tests, then there are countably many such tests and one can construct a real satisfying all tests.

Martin-Lof [[14](#_bookmark19)] observed that stochastic properties could be viewed as special kinds of measure zero sets and defined a random real as one which avoids certain effectively presented measure 0 sets. That is, a real *x* ∈ 2 is Martin-L¨of random if for any effective sequence *S*1*, S*2*,...* of c.e. open sets with *μ*(*Sn*) ≤ 2−*n*, *x* ∈*/* ∩*nSn*.

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At the same time Kolmogorov [[11](#_bookmark16)] defined a notion of randomness for finite strings based on the concept of *incompressibility*. For infinite words, the stronger notion of prefix-free complexity developed by Levin [[13](#_bookmark18)], G´acs [[9](#_bookmark14)] and Chaitin [[5](#_bookmark8)] is needed. Schnorr later proved that the notions of Martin-Lo¨f randomness and Chaitin randomness are equivalent.

In a recent paper [[2](#_bookmark9)], the notion of (Martin-Lo¨f) randomness was extended to finite-branching trees and effectively closed sets. It was shown that a ran- dom closed set is perfect, has measure 0, and contains no computable elements. In this paper we want to consider algorithmic randomness on the space

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C(2 ) of continuous functions *F* : 2 → 2 .

Some definitions are needed. For a finite string *σ* ∈ {0*,* 1}*n*, let |*σ*| = *n*. For two strings *σ, τ* , say that *τ* extends *σ* and write *σ* ≺ *τ* if |*σ*| *<* |*τ* | and *σ*(*i*) = *τ* (*i*) for *i <* |*σ*|. Similarly *σ* ≺ *x* for *x* ∈ 2N means that *σ*(*i*) = *x*(*i*) for *i <* |*σ*|. Let *σ-τ* denote the concatenation of *σ* and *τ* and let *σ-i* denote *σ* (*i*) for *i* = 0*,* 1. Let *x n* = (*x*(0)*,... , x*(*n* 1)). Two reals *x* and *y* may be coded together into *z* = *x y*, where *z*(2*n*)= *x*(*n*) and *z*(2*n* + 1) = *y*(*n*) for all *n*.

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For a finite string *σ*, let *I*(*σ*) denote *x* 2N : *σ x* . We shall call *I*(*σ*), the *interval* determined by *σ*. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. We let denote the Boolean algebra of clopen sets.

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Now a nonempty closed set *P* may be identified with a tree *TP* ⊆ {0*,* 1}∗ where *TP* = {*σ* : *P* ∩ *I*(*σ*) /= ∅}. Note that *TP* has no dead ends. That is, if *σ* ∈ *TP* , then either *σ-*0 ∈ *TP* or *σ-*1 ∈ *TP* .

For an arbitrary tree *T* ⊆ {0*,* 1}∗, let [*T* ] denote the set of infinite paths

through *T* , that is,

*x* ∈ [*T* ] ⇐⇒ (∀*n*)*x*[*n* ∈ *T.*

It is well-known that *P* 2N is a closed set if and only if *P* = [*T* ] for some tree

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*T* . *P* is a Π0 class, or an effectively closed set, if *P* = [*T* ] for some computable

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tree *T* . *P* is a strong Π0

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class, or a Π0

closed set, if *P* = [*T* ] for some Δ0

tree. The complement of a Π0 class is sometimes called a c.e. open set. We

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remark that if *P* is a Π0 class, then *TP* is a Π0 set, but it is not, in general,

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computable. There is a natural effective enumeration *P*0*, P*1*,...* of the Π0

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classes and, hence, there is a corresponding enumeration of the c.e. open sets. Thus we say that a sequence *S*0*, S*1*,...* of c.e. open sets is *effective* if there is a computable function, *f* , such that *Sn* = 2N *Pƒ*(*n*) for all *n*. For a detailed development of Π0 classes, see [[3](#_bookmark10),[4](#_bookmark11)].

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# Random continuous functions

We will define the notion of a random continuous function along similar lines to the definition of a random closed set in [[2](#_bookmark9)]. The definition of a random (nonempty) closed set *P* = [*T* ] (where *T* = *TP* ) comes from a probability measure *μ*∗ where, given a node *σ* ∈ *T* , each of the following scenarios has

equal probability 1:

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*σ-*0 ∈ *T* and *σ-*1 ∈ *T* ,

*σ-*0 ∈ *T* and *σ-*1 ∈*/ T* , and

*σ-*0 ∈*/ T* and *σ-*1 ∈ *T* .

More formally, we define a measure *μ*∗ on the space C of closed subsets of 2N as follows. Given a closed set *Q* ⊆ 2N, let *T* = *TQ* be the tree without dead ends such that *Q* = [*T* ]. Let *σ*0*, σ*1*,...* enumerate the elements of *T* in order, first by length and then lexicographically. We then define the code *x* = *xQ* = *xT* by recursion such that for each *n*, *x*(*n*) = 2 if both *σn-*0 and *σn-*1 are in *T* , *x*(*n*)=1 if *σn-*0 ∈*/ T* and *σn-*1 ∈ *T* , and *x*(*n*)= 0 if *σn-*0 ∈ *T* and *σn-*1 ∈*/ T* . We then define a measure *μ*∗ on C by setting

*μ*∗(X )= *μ*({*xQ* : *Q* ∈ X }) (1)

for any X ⊆ C and *μ* is the standard measure on {0*,* 1*,* 2}N. Then Brodhead, Cenzer, and Dashti [[2](#_bookmark9)] defined a a closed set *Q* ⊆ 2 to be (Martin-Lo¨f) random if *xQ* is (Martin-L¨of) random.

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A continuous function on 2N is a function with a closed graph. Thus we might simply say that a function *F* is random if the graph *Gr*(*F* ) is a random closed set. Now *Gr*(*F* ) = {*x* ⊕ *y* : *y* = *F* (*x*)}. Thus if [*T* ] is the graph of

a function and *σ* ∈ *T* has even length, then we must have *σ-*0 ∈ *T* and *σ-*1 ∈ *T* . This means that the family of closed sets which are the graphs of functions has measure 0 in the space of closed sets and hence a random closed set will not be the graph of a function. So we need a different measure to define randomness for continuous functions.

A continuous function *F* : 2N → 2N may be represented by a function

*f* : {0*,* 1}∗ → {0*,* 1}∗ such that the following hold for all *σ* ∈ {0*,* 1}∗.

1. |*f* (*σ*)|≤ |*σ*|.
2. *σ*1 ≺ *σ*2 implies *f* (*σ*1) “ *f* (*σ*2).
3. For every *n*, there exists *m* such that for all *σ* ∈ {0*,* 1}*m*, |*f* (*σ*)|≥ *n*.
4. For all *x* ∈ 2N, *F* (*x*)= *f* (*x*[*n*).

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We will define a space of representing functions *f* : 0*,* 1 ∗ 0*,* 1 ∗ to be those which satisfy clauses (1) and (2) above. For such a function *f* , we have *f* (∅)= ∅ by (1). There are three choices for *f* ((0)). If *f* ((0)) = (*i*) where *i* ∈ {0*,* 1}, this means that for all *x* ∈ *I*((0)), *F* (*x*)(0) = *i*. If *f* ((0)) = ∅, we shall take this to mean that there exist *x*0 and *x*1 in *I*((0)) such that *F* (*xi*)(0) = *i* for *i* = 0*,* 1. It will always be the case that *F* (*σ*) “ *τ* , where *τ* is the longest string *τ* with |*τ* |≤ |*σ*| such that *τ* ≺ *F* (*x*) whenever *σ* ≺ *x*.

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We will use the following measure on the set of representing functions to define randomness. Given that *f* (*σ*)= *τ* , we define a measure *μ*∗∗ so that each of the following scenarios has equal probability 1 for *i* = 0*,* 1:

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*f* (*σ-i*)= *τ* ,

*f* (*σ-i*)= *τ-*0, and

*f* (*σ-i*)= *τ-*1.

This can be pictured geometrically as representing the graph of *F* as the intersection of a decreasing sequence of clopen subsets of the unit square. Initially the choice of *f* ((0)) and *f* ((1)) selects from the 4 quadrants. That is, for example, *f* ((0)) = (0) = *f* ((1)) implies that the graph of *F* is included in the lower half of the square. Successive values of *f* restrict the graph of *F* in a similar fashion.

Let be the space of functions *f* : 0*,* 1 ∗ 0*,* 1 ∗ which satisfy clauses

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(1) and (2) above. Then every continuous function *F* has a representative *f* as described above, and, in fact, it has infinitely many representatives. There is a one-to-one correspondence between F and {0*,* 1*,* 2}N defined as fol- lows. Enumerate {0*,* 1}∗ in order, first by length and then lexicographically, as *σ*0*, σ*1*,.. .*. Thus *σ*0 = ∅, *σ*1 = (0)*, σ*2 = (1)*, σ*3 = (00)*,.* Then *r* ∈ {0*,* 1*,* 2}N

corresponds to the function *fr* : {0*,* 1}∗ → {0*,* 1}∗ defined by declarling that

*fr*(∅)= ∅ and that, for any *σn* with |*σn*| ≥ 1,

*f* (*σ*

*r*

)= *fr*(*σk*)*,* if *r*(*n*)= 2;

*r*

*k*

where *k* is such that *σn* = *σk-j* for some *j*. The measure *μ*∗∗ on is then induced by the standard probability measure on 0*,* 1*,* 2 N. We now define a *effectively random continuous function* on 2N to be one which has a represen- tation in which is effectively random. For the rest of this paper, when we say a function, closed set, or real is random, we mean that it is effectively radom.

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*f* (*σ* )*-i,* if *r*(*n*)= *i <* 2*.*

Our first result will be to show that every random function represents a continuous function. To prove such a result, we need to prove the following lemma.

Lemma 2.1 *Let* Σ *be a ﬁnite set and let Q* ⊆ ΣN *be a* Π0 *class of measure 0.*

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*Then no element of Q is Martin-Lo¨f random.*

Proof. We will give a proof only in the case where Σ = {0*,* 1*,* 2} as the general result can be proved in a similar manner. Let *Q* = [*T* ] where *T* ⊆ {0*,* 1*,* 2}∗ is a computable tree (possibly with dead ends). For each *n*, let *Tn* = *T* ∩ {0*,* 1*,* 2}*n* and let

Let *g*(*n*)= *μ*(*Qn*

*Qn* = {*I*(*σ*): *σ* ∈ *Tn*}*.*

)= |*T*n| . Then *g*(*n*) is a computable sequence and

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*limn*→∞*g*(*n*)= *μ*(*Q*)= 0*.*

This Martin-L¨of test shows that *Q* has no random elements. (As observed by Solovay, it is sufficient for a sequence of c.e. open sets {*Sn*}*n*≥0 to be a Martin-L¨of test if *limn*→∞*μ*(*Sn*) = 0 effectively rather than the stricter test with a sequence of measures *μ*(*Sn*) ≤ 2 .)

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Theorem 2.2 *The set of functions in* F *which represent a total continuous function has measure one. Hence every random function represents a contin- uous function.*

Proof. Let *f* ∈ F and suppose that *f* does not represent a total function. Then there is some *x* ∈ 2 and some *τ* ∈ {0*,* 1} such that *f* (*x*[*n*) = *τ* for almost all *n*. Without loss of generality we may assume that *τ* = ∅. Let *A* be the set of functions *f* : {0*,* 1} → {0*,* 1} such that *f* (*σ*)= ∅ for arbitrarily long strings *σ* and let *p* = *μ*∗∗(*A*). Then certainly *p* ≤ 5 since, if *r*(0) and *r*(1) are both in {0*,* 1}, then *fr* ∈*/ A*. Considering the 9 cases for the initial choices

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of *f* ((0)) and *f* ((1)), we see that

*p* = 4 *p* + 1 [1 − (1 − *p*)2]

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so that 1 *p*2 + 1*p* = 0, which implies that *p* = 0. That is, there are 4 cases in

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which *f* ((*i*)) =1 for *i* = 0*,* 1 so that immediately *f / A*, there are 4 cases in

which only one of *f* ((*i*)) = , in which case the remaining function *g*, defined by *g*(*σ*)= *f* (*i-σ*) must be in *A*, and there is one case in which *f* ((*i*)) = for *i* = 0*,* 1, in which case at least one of the remaining functions must be in *A*.

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Observe that *A* is a Π0 class since *fr* ∈ *A* if and only if (∀*n*)(∃*σ* ∈

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{0*,* 1} )*fr*(*σ*) = ∅. It follows from Lemma [2.1](#_bookmark1) that no random function can

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be in *A* and therefore every random function *f* : {0*,* 1}∗ → {0*,* 1}∗ indeed represents a continuous function *F* : 2N → 2N.

Now the set of Martin-L¨of random elements of 0*,* 1*,* 2 N has measure one and there exists a Δ0 Martin-L¨of real. Hence we have the following.

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Theorem 2.3 *There exists a random continuous function which is* Δ0 *com-*

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*putable.*

Next we consider some basic properties of random continuous functions.

Our first result is easy to prove.

Proposition 2.4 *(a) F is a random continuous function if and only if, for every σ* ∈ {0*,* 1}∗*, the function Fσ is random continuous, where*

*Fσ*(*x*)= *F* (*σ-x*)*.*

1. *F is random continuous if and only if both F*(0) *and F*(1) *are random continuous.*

Next we shall show that every random function maps a computable real to a random real. Again, we need a preliminary lemma.

Lemma 2.5 *Let* Σ *be a ﬁnite alphabet where* |Σ| ≥ 3 *and let* Σ1 ⊂ Σ *be a proper subset of* Σ *where* |Σ1| ≥ 2*. If z* ∈ ΣN *is Martin-Lo¨f random and y is the result of removing from z all symbols from* Σ − Σ1*, then y is Martin-Lo¨f*

*random in* ΣN*.*

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Proof. Clearly it is enough to prove the lemma when |Σ|− 1 = |Σ1|. Thus there is no loss in generality in assuming that Σ = {1*,* 2*,... ,n* + 1} and Σ1 = {1*,... , n*} where *n* ≥ 2.

Define the function *G* so that for any *x* with infinitely many values of *x*(*m*) 1*,... , n* , *G*(*x*) is the result of removing from *x* all occurences of *n* + 1.

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Claim 2.6 *For any* Σ0 *subset S of* {1*,... , n*}*N , μ*(*G*−1(*S*)) = *μ*(*S*)*.*

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Proof. [Proof of Claim] Since every Σ0

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class *S* is the effective union of a

disjoint sequence of intervals, that is, there is a computable function *f* such

that *S* = *m I*(*σƒ*(*m*)), it suffices to prove this for intervals *I*(*σ*) ⊆ {1*,... , n*}*N* . The proof is by induction on the length |*σ*|.

For *m* = |*σ*| = 1, we see that

*G*−1(*I*((*i*))) = *I*((*i*)) ∪ *I*((*n* + 1)*-i*)) ∪ *I*((*n* + 1)(*n* + 1)*-i*) ∪ ···

so that

*μ*(*G*−1(*I*((*i*)))) = 1 + 1 + 1 + ···

*n* +1 (*n* + 1)2 (*n* + 1)3

1 1 1

= *n* ( 1 )=  = *μ*(*I*((*i*)))*.*

+1 1 − *n*+1 *n*

Now assume the result to be true for *m* and let *σ* = (*i*)*-σ* where *τ* = *m*. Then it is easy to see that

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*G*−1(*I*(*σ*)) = *i-G*−1(*I*(*τ* ))∪(*n*+1)*i-G*−1(*I*(*τ* ))∪(*n*+1)(*n*+1)*i-G*−1(*I*(*τ* ))∪··· *.*

Thus

*μ*(*G*−1(*I*(*σ*))) = 1 *μ*(*G*−1(*I*(*τ* ))) + 1 *μ*(*G*−1(*I*(*τ* ))) + ···

*n* +1

(*n* + 1)2

= *μ*(*G*−1(*I*(*τ* )))( 1 + 1 + ··· )

*n* +1

= 1 *μ*(*G*−1(*I*(*τ* ))) = 1

(*n* + 1)2

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= *μ*(*I*(*σ*))*.*

*n n nm*

Now let *S*0*, S*1*,...* be an effective sequence of c.e. open sets with *μ*(*Sm*) 2−*m*. Thus there exists a computable function *f* such that

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*Sm* = ∪*iI*(*σƒ* (*i,m*))*.*

Let *Rm* = *G*−1(*Sm*), so that *μ*(*Rm*) = *μ*(*Sm*) by the Claim. It remains to be checked that the sequence *Rm m*∈N is an effective sequence of c.e. open sets. Define the computable function *g* : 1*,... ,n* + 1 ∗ 1*,... , n* ∗ so that *g*(*σ*) is the result of removing from *σ* all occurences of *n* + 1. For each *i, m* define the computable function *h* such that {*σh*(*j,i,m*)}*m*∈N enumerates

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{*τ* : *g*(*τ* )= *σƒ*(*i,m*)}. Then

*Rn* = ∪*j,i,mI*(*σh*(*j,i,m*))*.*

Thus {*Rm*}*m*∈N is a Martin-L¨of test and hence *z* ∈*/ Rm* for some *m*. But then

*y* ∈*/ Sm* so that *y* is random.

Theorem 2.7 *If F is a random continuous function, then, for any com- putable real x, F* (*x*) *is a random real.*

Proof. Suppose that *F* is random with representing function *fr*, let *x* be a computable real and let *y* = *F* (*x*). Define the computable function *g* so that, for each *n*,

*σg*(*n*) = *x*[*n.*

By the Von-Mises–Church–Wald Computable Selection Theorem, the sub- sequence *z*(*n*) = *r*(*g*(*n*)) is random in 0*,* 1*,* 2 N. Now *y* = *F* (*x*) may be computed from *z* by removing the 2’s. Thus *F* (*x*) is random by Lemma [2.5](#_bookmark3).

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In particular, it follows that if *F* is random function and *x* is a computable real, then *F* (*x*) is not a computable or even a c.e. real. Hence, a random function *F* can never be computably continuous and the graph of *F* is not a Π0 class.

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We note that Fouche [[8](#_bookmark13)] has used a different approach to randomness for continuous functions connected with Brownian motion, first presented by Asarin and Prokovsky [[1](#_bookmark7)], and has shown that, under this approach, it is also true that for any random continuous function *F* , *F* (*x*) is not computable for any computable input *x*.

Theorem 2.8 *If F is a random continuous function, then the image F* [2N]

*has no isolated elements.*

Proof. Let *f* be the random representing function for *F* and let *Q* = *F* [2N]. Suppose by way of contradiction that *Q* contains an isolated path *y*. Then there is some finite *τ y* such that *y* is the unique element of *I*(*τ* ) *Q*. Fix *σ* such that *f* (*σ*)= *τ* .

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For each *n*, let *Sn* be the set of all *g* ∈F such that for all *ρ*1*, ρ*2 ∈ {0*,* 1}*n*,

* 1. *g*(*σ-ρ*1) is compatible with *g*(*σ-ρ*2),
  2. *τ* ≺ *g*(*σ-ρ*1), and
  3. *τ* ≺ *g*(*σ-ρ*2).

Then for any each *m < n* and each *ρ* ∈ {0*,* 1}*m*, we are restricted to at most 7 of the 9 possible choices so that in general, *μ*(*Sn*) ≤ ( 7)*n*. Now for each *n*, *Sn* is a clopen set in F and thus the sequence *S*0*, S*1*,...* is a Martin-L¨of test. It follows that for some *n*, *F / Sn*. Thus there are two extensions of *σ* of length *n* which have incompatible images, contradicting the assumption that *y* was the unique element of *Q* ∩ *I*(*τ* ).

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It follows that the image of a random continuous function is perfect and has continuum many elements. There are several natural questions about the image *F* [2N] of a random continuous function *F* . Is the image of *F* a random closed set? What is the measure of the image? Can the function be onto? We will give some partial answers.

It follows from Proposition [2.4](#_bookmark2) that, for any *τ* ∈ {0*,* 1}∗, there is a random continuous function with image *I*(*τ* ). Thus a random continuous function is not necessarily onto.

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Theorem 2.9 *For any σ* 0*,* 1 ∗*, the probability that the image of a con- tinuous function F meets I*(*σ*) *is always >* 3 *.*

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Proof. The proof is by induction on *σ* . Without loss of generality, we may assume that *σ* = 0*n*. For each *n >* 0, let *qn* be the probability that *F* [2N] meets *I*((0*n*)). Let *f* be the representing function for *F* . For *n* = 1, there are 9 equally probable choices for the pair *f* ((0)) and *f* ((1)) which can be broken down into 4 distinct cases.

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Case 1. If *f* ((0)) = (1) = *f* ((1)), then *F* [2N] does not meet *I*((0)). This occurs just once.

Case 2. If *f* ((0)) = (0) or *f* ((1)) = (0), then *F* [2N] meets *I*((0)). This occurs in 5 of the 9 choices.

Case 3. If *f* ((*i*)) = and *f* ((1 *i*)) = (1), then *F* [2N] meets *I*((0)) if and only if *F*(*i*)[2N] meets *I*((0)). This occurs in 2 of the 9 choices, with probability *q*1.

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Case 4. If *f* ((0)) = = *f* ((1)), then *F* [2N] meets *I*((0)) if at least one of *F*(*i*)[2N] meets *I*((0)). This occurs in 1 of the choices, with probability

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1 − (1 − *q*1)2. That is, *F* [2N] *fails to meet I*((0)) if *both F* [2N] *and F* [2N]

(0)

(0)

fail to meet *I*((0)).

Putting these cases together, we see that

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*q* = + *q*

1

9

9

1

1

+ (2*q*

9

1

— *q* )*,*

so that *q*1 satisfies the quadratic equation

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*x*2 + 5*x* − 5= 0*.*

Thus *q*1 is the unique solution in [0,1] of this equation, that is,

*q* = √45 − 5 *,*

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which is indeed *> .*75.

Now let *qn* = *q* and let *qn*+1 = *p*. Once again we consider the 9 initial choices, now breaking down into 6 distinct cases.

Case 1. If *f* ((0)) = (1) = *f* ((1)), then *F* [2N] does not meet *I*((0*n*+1)).

This occurs just once.

Case 2. If *f* ((0)) = (0) = *f* ((1)), then *F* [2N] meets *I*((0*n*+1)) if and only if at least one of *F*(0) and *F*(1) meets *I*((0*n*)). This occurs just once, and with probability 1 − (1 − *q*)2 = 2*q* − *q*2.

Case 3. If *f* ((*i*)) = (0) and *f* ((1 *i*)) = (1), then *F* [2N] meets *I*((0*n*+1)) if and only if *F*(*i*)[2N] meets *I*((0*n*)). This occurs in 2 of the 9 choices, with probability *q*.

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Case 4. If *f* ((*i*)) = and *f* ((1 *i*)) = (1), then *F* [2N] meets *I*((0*n*+1)) if and only if *F*(*i*)[2N] meets *I*((0*n*+1)). This occurs in 2 of the 9 choices, with probability *p*.

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Case 5. If *f* ((0)) = ∅ = *f* ((1)), then *F* [2N] meets *I*((0*n*+1)) if at least one of *F*(*i*)[2N] meets *I*((0*n*+1)). This occurs just once, with probability 1 −(1 − *p*)2.

Case 6. If *f* ((*i*)) = and *f* ((1 *i*)) = (0), then *F* [2N] meets *I*((0*n*+1)) if at least one of the following two things happens. Either *F*(*i*)[2N] meets *I*((0*n*+1)),

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or *F*(1−*i*)

[2N] meets *I*((0*n*)). This occurs in 2 of the 9 choices, with probability

1 − (1 − *p*)(1 − *q*).

Putting these cases together, we see that

*p* = 2 *p* − 1 *p*2 − 2 *pq* + 2 *q* − 1 *q*2*,*

3

9

9

3

9

so that *p* = *qn*+1 satisfies the equation

*p*2 + 3*p* + 2*pq* − 6*q* + *q*2 = 0*.*

We note that for *p* = *q*, the solutions are *p* = *q* = 0 and *p* = *q* = 3. This

4

explains the value 3 in the statement of theorem.

4

Now assume by induction that *q >* 3. Suppose by way of contradiction

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that *p* ≤ 3 . It follows that

4

9 + 9 + 3 *q* − 6*q* + *q*2 ≥ 0*.*

16

4

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Simplifying, this implies that 16*q*2 − 72*q* + 45 ≥ 0. But this factors into (4*q* − 3)(4*q* − 15) and is only ≥ 0 when either *q* ≤ 3 or *q* ≥ 15 . Since the latter

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4

is impossible, we obtain the desired contradiction that *q* ≤ 3 .

4

Corollary 2.10 *For any y* ∈ 2N*, there exists a random continuous function*

*F with y* ∈ *F* [2N]*.*

Proof. Let *Sn* be {*F* ∈ C(2N) : *I*(*y*[*n*) ∩ *F* [2N] /= ∅}. By Theorem [2.9](#_bookmark4),

*μ*(*Sn*) ≥ *.*75 for all *n*.NBut *Sn*+1 ⊆ *Sn* for all *n* and therefore *μ*(∩*nSn*) ≥ *.*75

as well. Thus *y* ∈ *F* [2 ] with probability ≥ *.*75. Since the random continuous

functions have measure 1 in (2N), it follows that some random continuous function has *y* in the image.

C

Corollary 2.11 *The image of a random continuous function need not be a random closed set.*

Proof. It was shown in [[2](#_bookmark9)] that a random closed set has no computable mem- bers. Let *F* be a random continuous function with 0*ω* in the image, as given by Corollary [2.10](#_bookmark5). Then *F* [2N] is not a random closed set.

# *n*-random continuous functions

Our approach also allows us to define the notion of *n*-random continuous functions. That is, recall that

* 1. a Σ0 test is a computable collection {*Vn* : *n* ∈ 2N} of Σ0 classes such that

*n n*

*μ*(*Vk*) ≤ 2−*k* and

* 1. a real *α* is Σ0 random or *n*-random if and only if it passes all Σ0

*n*

*n*

tests,

i.e., if {*Vn* : *n* ∈ 2N} is a computable collection of Σ0 classes such that

*n*

*μ*(*Vk*) ≤ 2−*k*, then *α* ∈*/* ∩*n*≥0*Vn*.

Thus 1-random reals are just Martin-L¨of random reals. See [[6](#_bookmark12)] for details on random and *n*-random reals.

Kurtz [[12](#_bookmark17)] and Kautz [[10](#_bookmark15)] proved the following result. Let ∅(*n*) denote the

*n*-th jump of ∅.

Theorem 3.1 *Let q be a rational number.*

1. *For each* Σ0

*n*

*class S , we can uniformly compute from q and a* Σ0

*index*

*for S, the index of a* Σ∅(n−1) *class U* ⊇ *S such that U is an open* Σ0

*n*

*class*

1 *n*

*and μ*(*U* ) − *μ*(*S*) *< q.*

1. *For each* Π0 *class T , we can uniformly compute from q and a* Π0

*n*

*n*

*index*

*for T , the index of a* Π∅(n−1) *class V* ⊇ *T such that V is a closed* Π0 *class*

1 *n*

*and μ*(*V* ) − *μ*(*T* ) *< q.*

1. *For each* Σ0 *class S, we can uniformly compute from q, and a* Σ0

*n*

*n*

*index*

*for S and an oracle for* ∅(*n*)*, the index of a* Π0 *class V* ⊆ *S such that V*

*n*−1

*is a closed* Π0 *class and μ*(*S*) − *μ*(*V* ) *< q. Moreover, if μ*(*S*) *is a real*

*n*−1

*computable from* ∅(*n*−1)*, then the index for V can be found computably*

*from* ∅(*n*−1)*.*

1. *For each* Π0 *class T , we can uniformly compute from q and* Π0 *index for*

*n*

*n*

*T and an oracle for* ∅(*n*)*, the index of a* Σ0 *class U* ⊆ *T such that U*

*n*−1

*is an open* Σ0 *class and μ*(*T* ) − *μ*(*U* ) *< q. Moreover, if μ*(*S*) *is a real*

*n*−1

*computable from* ∅(*n*−1)*, then the index for U can be found computably*

*from* ∅(*n*−1)*.*

It follows that a real is *n* + 1-random if and only if it is 1-random relative to (*n*). The analogue of Theorem [3.1](#_bookmark6) also holds for 0*,* 1*,* 2 N for our measures *μ*∗ or *μ*∗∗. Thus we can define a closed set *Q* to be *n*-random if and only if is Martin Lo¨f random relative to (*n*) and, similarly, we can define a continuous function *F* : 22N 22N to be *n*-random if and only if it is Martin L¨of random relative to (*n*). One can then easily relativize the results of the previous section to obtain similar results for *n*-random continuous functions.

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∅

∅ { }

# Conclusions and Future Research

In this paper we have proposed a notion of effective randomness for continuous functions on the Cantor space 2N and derived several properties of effectively

random continuous functions. Effectively random Δ0

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continuous functions

exist, but no computable function can be effectively random. In fact, the image

if a computable real under an effectively random function is an effectively random real so that no effectively random function can map a computable real to a computable or even to a c.e. real. We have shown that the image of a random continuous function is always a perfect set and hence uncountable. We have shown that for any *y* ∈ 2N, there exists a random continuous function *F* with *y* in the image of *F* . Thus the image of a random continuous function need not be a random closed set.

We would like to extend the notion of a random continuous function to functions on the real unit interval [0*,* 1] and the real line ঩ by representing functions again in terms of the images of subintervals. We conjecture that a random continuous real function cannot be left or right computable and, in fact, it cannot even be weakly computable. We also conjecture that a random continuous function is nowhere differentiable.

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