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Remarks on an Edge-coloring Problem

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**Abstract**

We consider a multicolored version of a problem that was originally proposed by Erd˝os and Rothschild. For positive integers *n* and *r*, we look for *n*-vertex graphs that admit the maximum number of *r*-edge-colorings with no copy of a triangle where exactly two colors appear. It turns out that for 2 *≤ r ≤* 12 colors and *n* sufficiently large, the complete bipartite graph on *n* vertices with balanced bipartition (the *n*-vertex

Tura´n graph for the triangle) yields the largest number of such colorings, and this graph is unique with this property.

*Keywords:* Edge-colorings, Tur´an Problem, Erd˝os-Rothschild Problem

# Introduction and main results

This paper is concerned with a multicolored version of a problem that was originally proposed by Erd˝os and Rothschild [[9](#_bookmark23)]. The motivation for their problem lies in the well-known Tur´an problem, where, given an integer *n* and a graph *F* , we look for the maximum number ex(*n, F* ) of edges in an *n*-vertex graph *G* such that *G* does not contain *F* as a subgraph. A graph *G* that does not contain *F* as a subgraph is said to be *F-free* and an *F* -free *n*-vertex graph with ex(*n, F* ) edges is called *F-extremal*. Tur´an [[22](#_bookmark36)] solved this problem for all *n* whenever *F* = *Kl*+1 is a complete graph on (*l* + 1) vertices. He showed that, for all positive integers *n* and *l ≥* 2, any

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*Kl*+1-extremal graph is isomorphic to the *Tur´an graph Tl*(*n*), the complete *l*-partite graph on *n*-vertices whose partition *V* = *{V*1*,..., Vl}* is *balanced*, that is, such that

*|Vi|≤ |Vj|* +1 for all *i, j ∈* [*l*]= *{*1*,..., l}*. In particular, ex(*n, K*3)= *[n/*2*♩· [n/*2*|*. There is a vast literature about the Tur´an problem, we refer to [[12](#_bookmark26)] (and to the references therein) for more information.

The question of Erd˝os and Rothschild involves *r-edge-colorings* of *n*-vertex graphs with the property that *every color class is F-free*. They wondered whether any *n*-vertex graph would admit more such colorings than the corresponding *F* - extremal graph. Note that *n*-vertex *F* -extremal graphs admit *r*ex(*n,F* ) colorings, as their edge set may be colored arbitrarily. Precisely, Erdos and Rothschild conjec- tured that the number of *Kl*+1-free 2-colorings is maximized by *Tl*(*n*). Yuster [[23]](#_bookmark37) verified this conjecture for *l* = 2 and *n ≥* 6. Alon, Balogh, Keevash and Sudakov [[2]](#_bookmark15) showed that, for *r ∈ {*2*,* 3*}* and *n ≥ n*0, where *n*0 is a constant depending on *r* and *l*, the Tura´n graph *Tl*(*n*) is also optimal for the number of *Kl*+1-free *r*-colorings. However, they also provided a construction showing that *Tl*(*n*) is not optimal for any *r ≥* 4, but did not characterize the graphs that achieve extremality. Pikhurko and Yilma [[20](#_bookmark34)] determined the extremal graphs for *r* = 4 and *l ∈ {*2*,* 3*}*. Together with Staden [[19](#_bookmark33)], they also generalized the original Erd˝os-Rothschild problem and showed that it always admits an extremal solution that is a complete multipar- tite graph (however, their proof does not settle whether it is necessarily balanced). Moreover, they defined an optimization problem whose solution produces a complete multipartite graph for which the number of colorings approximates the maximum. Balogh [[3](#_bookmark17)] was the first to consider *r*-colorings that avoid a copy of a graph *F* colored in a non-monochromatic way. A similar problem was investigated by Hoppen and Lefmann [[15](#_bookmark29)] and by Benevides, Hoppen and Sampaio [[6](#_bookmark18)], who considered edge- colorings of a graph avoiding a copy of *F* with a *prescribed pattern*. Given a number *r ≥* 1 of colors and a graph *F* , an *r-pattern P* of *F* is a partition of its edge set into at most *r* classes, and an edge-coloring of a graph *G* is said to be (*F, P* )*-free* if *G* does not contain a copy of *F* in which the partition of the edge set induced by the coloring is isomorphic to *P* . For instance, if *F* = *K*3, there are three possible patterns: the monochromatic pattern *PM* (where all edges lie in the same class), the rainbow pattern *PR* (where each class is a singleton) and the 2-colored pattern *P*2 (where there are two classes, one singleton and one with cardinality two). It is clear that the original Erd˝os-Rothschild problem is precisely the problem of finding

the largest number of (*F, PM* )-free colorings in an *n*-vertex graph.

For a formal statement of this multicolored version of the Erd˝os-Rothschild problem, fix a positive integer *r* and a graph *F* , and let *P* be a pattern of *F* . Let *Cr,F,P* (*G*) be the set of all (*F, P* )-free *r*-colorings of a graph *G*. We write

*cr,F,P* (*n*)= max *{ |Cr,F,P* (*G*)*|* : *|V* (*G*)*|* = *n } ,*

and we say that an *n*-vertex graph *G* is (*F, P* )*-extremal* if *|Cr,F,P* (*G*)*|* = *cr,F,P* (*n*). In this paper, our main objective is to study (*K*3*, P*2)-extremal graphs for the 2-colored pattern *P*2.

Regarding patterns *P* of *K*3, the following is known. As mentioned above, the

Tur´an graph *T*2(*n*) is the single (*K*3*, PM* )-extremal graph for *r* = 2 and *n ≥* 6 (see [[23](#_bookmark37)]) and for *r* = 3 and *n ≥ n*0 (see [[2](#_bookmark15)] and [[14](#_bookmark28)]). Moreover, the graph *T*4(*n*) is the single (*K*3*, PM* )-extremal graph for *r* = 4 and *n ≥ n*0 (see [[20](#_bookmark34)]). To the best of our knowledge, the extremal graphs for *r ≥* 5 are not known. For the rainbow pattern *PR*, the complete graph *Kn* is trivially the single (*K*3*, PR*)-extremal graph for *r* = 2. If *r ≥* 5, Odermann and the current authors [[16](#_bookmark30)] have proved that the Tur´an graph *T*2(*n*) is the single (*K*3*, PR*)-extremal graph for *n ≥ n*0 (and for *r ≥* 10 and *n ≥* 5). Very recently, Balogh and Li [[4](#_bookmark19)] have proved that the complete graph *Kn* and the Tur´an graph *T*2(*n*) are the single (*K*3*, PR*)-extremal graphs for *r* = 3 and *r* = 4, respectively. Approximate results had also been obtained in [[5,](#_bookmark20)[6,](#_bookmark18)[10,](#_bookmark24)[16](#_bookmark30)].

Less is known for the pattern *P*2. The work of [[3](#_bookmark17)] implies that the Turan graph *T*2(*n*) is (*K*3*, P*2)-extremal for *r* = 2 and *n ≥ n*0. Results from [[6](#_bookmark18)] prove that, for any *r ≥* 3, one of the extremal configurations is always a complete multipartite graph (see [[6](#_bookmark18), Theorem 1.1]) and that the Tur´an graph *T*2(*n*) is (*K*3*, P*2)-extremal for *r* =3 and *n ≥ n*0 (see [[6](#_bookmark18), Theorem 1.3]). On the other hand, let *r* = 27 and consider a partition of the set of colors into three sets *C*1*, C*2*, C*3, where *|C*1*|* = *|C*2*|* = *|C*3*|* = 9. We shall color the 4-partite graph *T*4(*n*) whose vertex set is partitioned *V*1 *∪· · ·∪V*4 as follows (for simplicity, assume that *n* is divisible by 4). Edges between *V*1 and *V*2, and between *V*3 and *V*4 are assigned colors in *C*1; edges between *V*1 and *V*3, and between *V*2 and *V*4 are assigned colors in *C*2; edges between *V*1 and *V*4, and between *V*2 and *V*3 are assigned colors in *C*3. Clearly, any triangle in *T*4(*n*) must be rainbow, so that this produces colorings in *C*27*,K*3*,P*2 (*T*4(*n*)). Moreover, the number of colorings produced in this way is equal to

6*· n*2 *n*2 ex(*n,K* )

9 16 = 27 4 = 27 3 *.*

Since there are many other ways of coloring *T*4(*n*) (for instance, changing the choice of the sets *C*1*, C*2*, C*3), we conclude that *cr,K ,P* (*n*) *> r*ex(*n,K*3), so that the Tur´an graph is not (*K*3*, P*2)-extremal for *r* = 27. Actually, a similar analysis shows that *T*4(*n*) admits more (*K*3*, P*2)-free colorings than *T*2(*n*) for all *r ≥* 27. We believe that *T*2(*n*) is (*K*3*, P*2)-extremal for all *r ≤* 26, at least for *n ≥ n*0. In this note, we offer a partial result in this direction by using the regularity lemma combined with a linear programming approach.

3 2

**Theorem 1.1** *Let P*2 *be the pattern of K*3 *with exactly two classes and let* 2 *≤ r ≤*

12*. Then there exists n*0 *such that, for every n ≥ n*0 *and every n-vertex graph G, we have*

*|Cr,K ,P* (*G*)*|≤ r*ex(*n,K*3)*.*

3 2

*Moreover, equality holds in this equation if and only if G is isomorphic to the bi- partite Tur´an graph T*2(*n*)*.*

As we shall see below, in light of [[17](#_bookmark31), Lemma 3.1], to prove Theorem [1.1](#_bookmark1), it suffices to prove the following stability result, which states that any *n*-vertex graph with a ‘large’ number of colorings must be ‘almost bipartite’. For a graph *G* = (*V, E*) and a subset *W ⊂ V* , we write *eG*(*W* ) to denote the number of edges of *G* with

both endpoints in *W* . We simply write *e*(*W* ) if the graph *G* under consideration is obvious from the context.

**Lemma 1.2** *Let* 2 *≤ r ≤* 12 *be ﬁxed. For all δ >* 0*, there exists n*0 *with the following property. If G* = (*V, E*) *is a graph on n > n*0 *vertices which has at least r*ex(*n,K*3) *distinct* (*K*3*, P*2)*-free r-colorings, then there is a partition V* = *W*1 *∪ W*2 *of its vertex set such that et*(*W*1)+ *et*(*W*2) *≤ δn*2*.*

Note that Lemma [1.2](#_bookmark2) immediately implies that *|C* (*G*)*| ≤ r*ex(*n,K*3)+*o*(*n*2)

*r,K*3*,P*2

for *r ∈ {*2*,...,* 12*}*.

In the remainder of the paper, we shall discuss the main ingredients in our proof of Lemma [1.2](#_bookmark2). We should mention that, in the last few years, there has been a lot of activity on the Erd˝os-Rothschild problem for several combinatorial structures, such as set systems, the power lattice and sum-free-sets [[7,](#_bookmark21)[8,](#_bookmark22)[13](#_bookmark27)].

# Main ingredients

We first observe that, because of previous results by Hoppen, Lefmann and Oder- mann [[17](#_bookmark31)], the stability of Lemma [1.2](#_bookmark2) implies Theorem [1.1](#_bookmark1). The authors of [[17]](#_bookmark31) defined the following notion of stability.

**Definition 2.1** Let *F* be a graph with chromatic number *χ*(*F* ) = *l* +1 *≥* 3 and let *P* be a pattern of *F* . The pair (*F, P* ) satisfies the Color Stability Property for a positive integer *r* if, for every *δ >* 0, there exists *n*0 with the following property. If *n > n*0 and *G* is an *n*-vertex graph such that *|Cr,F,P* (*G*)*| ≥ r*ex(*n,F* ), then there

exists a partition *V* (*G*)= *V*1 *∪ ··· ∪ Vl* such that Σ*l et*(*Vi*) *≤ δn*2.

*i*=1

Then they showed that the Tur´an graph *Tl*(*n*) is the only *n*-vertex graph that

maximizes *cr,KÆ*+1*,P* (*n*) for a class of patterns of complete graphs that satisfy the Color Stability Property, namely patterns for which there is a vertex *v* such that all edges incident with *v* lie in different classes. Patterns of this type are called *locally rainbow*. Note that the 2-colored triangle is locally rainbow.

**Lemma 2.2** *[*[*17*](#_bookmark31)*, Lemma 3.1] Let l ≥* 2 *and let P be a locally rainbow pattern of Kl*+1 *such that* (*Kl*+1*,P* ) *satisﬁes the Color Stability Property of Deﬁnition* [*2.1*](#_bookmark3) *for a positive integer r > e*(*l* + 1)*. Then there is n*0 *such that every graph of order n > n*0 *has at most r*ex(*n,KÆ*+1) *distinct* (*Kl*+1*,P* )*-free r-edge colorings. Moreover, the only graph on n vertices for which the number of such colorings is r*ex(*n,KÆ*+1) *is the Tur´an graph Tl*(*n*)*.*

They also remarked that, in the case where the forbidden graph is a triangle, the lower bound *r > e*(*l* + 1) in the statement of this lemma may be replaced by *r ≥* 3. Since Lemma [1.2](#_bookmark2) establishes that the 2-colored triangle satisfies the Color Stability Property for 3 *≤ r ≤* 12, Theorem [1.1](#_bookmark1) follows.

* 1. *Regularity Lemma*

Our proof of Lemma [1.2](#_bookmark2) is based on the Szemer´edi Regularity Lemma [[21](#_bookmark35)]. Let *G* = (*V, E*) be a graph, and let *A* and *B* be two disjoint subsets of *V* (*G*). If *A* and *B* are non-empty, define the edge-density between *A* and *B* by

*d*(*A, B*)=

*e*(*A, B*)

*,*

*|A||B|*

where *e*(*A, B*) is the number of edges with one endpoint in *A* and the other in

*B*. For *ε >* 0 the pair (*A, B*) is called *ε-regular* if, for every *X ⊆ A* and *Y ⊆ B*

satisfying *|X| > ε|A|* and *|Y | > ε|B|*, we have

*|d*(*X, Y* ) *− d*(*A, B*)*| < ε.*

An *equitable partition* of a set *V* is a partition of *V* into pairwise disjoint classes *V*1*,..., Vm* of almost equal size, i.e., *||Vi|− |Vj|| ≤* 1 for all *i, j ∈* [*m*]. An equitable partition of the vertex set *V* of *G* into the classes *V*1*,..., Vm* is called *ε-regular* if

at most *ε* of the pairs (*Vi, Vj*) are not *ε*-regular.

*m*

2

We now state a (colored) version of the Regularity Lemma, which may be found in [[18](#_bookmark32)].

**Lemma 2.3** *For every ε >* 0 *and every integer r, there exists an M* = *M* (*ε, r*) *such that the following property holds. If the edges of a graph G of order n > M are r-colored E*(*G*) = *E*1 *∪ · · · ∪ Er, then there is a partition of the vertex set V* (*G*) = *V*1 *∪· · ·∪ Vm, with* 1*/ε ≤ m ≤ M, which is ε-regular simultaneously with respect to the graphs Gi* = (*V, Ei*) *for all i ∈* [*r*]*.*

A partition *V*1 *∪· · ·∪ Vm* of *V* (*G*) as in Lemma [2.3](#_bookmark4) will be called a *multicolored ε-regular partition*. For *η >* 0, we may define a *multicolored cluster graph H*(*η*) associated with this partition: the vertex set is [*m*] and *e* = *{i, j}* is an edge of *H*(*η*) if *{Vi, Vj}* is a regular pair in *G for every* color *c ∈* [*r*] and is *η*-dense for some color *c ∈* [*r*]. Each edge *e* is assigned the list *Le* containing all colors for which it is *η*-dense, so that *|Le|≥* 1 for every edge in the multicolored cluster graph *H*(*η*).

Given a colored graph *F* , we say that a multicolored cluster graph *H* contains *F* if *H* contains a copy of *F* for which the color of each edge of *F* is contained in the list of the corresponding edge in *H*. More generally, if *F* is a graph with color pattern *P* , we say that *H* contains (*F, P* ) if it contains some colored copy of *F* with pattern isomorphic to *P* . In connection with this definition, we may obtain the following embedding result. The proof of this result follows from arguments such as in the proof of the Key Lemma [[18](#_bookmark32)].

**Lemma 2.4** *For every η >* 0 *and all positive integers k and r, there exist ε* = *ε*(*r, η, k*) *>* 0 *and a positive integer n*0(*r, η, k*) *with the following property. Suppose that G is an r-colored graph on n > n*0 *vertices with a multicolored ε-regular partition*

*V* = *V*1 *∪· · ·∪ Vm which deﬁnes the multicolored cluster graph H* = *H*(*η*)*. Let F be a ﬁxed k-vertex graph with a prescribed color pattern P on t ≤ r classes. If H contains* (*F, P* )*, then the graph G also contains* (*F, P* )*.*

* 1. *Stability*

Another basic tool in our paper are stability results for graphs.

It will be convenient to use the following recent theorem by Fu¨redi [[11](#_bookmark25)].

**Theorem 2.5** *Let G* = (*V, E*) *be a Kk*+1*-free graph on m vertices. If |E|* =

ex(*m, Kk*+1) *− t, then there exists a partition V* = *V*1 *∪ ... ∪ Vk with* Σ*k*

*i*=1

We also use the following simple lemma due to Alon and Yuster [[1](#_bookmark16)].

*e*(*Vi*) *≤ t.*

**Lemma 2.6** *Let G be a bipartite graph on m vertices with partition V* (*G*)= *U*1 *∪U*2 *and with at least* ex(*m, K*3) *− t edges. If we add at least* 3*t new edges to G, then in the resulting graph there is a copy of K*3 *with exactly one new edge, which connects* *two vertices of K*3 *in the same class Ui.*

# Proving Lemma [1.2](#_bookmark2)

In this section, we provide a sketch of the proof of Lemma [1.2](#_bookmark2). Fix the number *r ∈ {*2*,...,* 12*}* of colors and *δ >* 0. To avoid case analysis, we concentrate on the case *r ≥* 6 [4](#_bookmark9) .

With foresight, we consider auxiliary constants *ξ >* 0 and *η >* 0 such that

*ξ <*  *δ , rrη*+*h*(*rη*) *<*

14

*r ξ*

*M* (*r*)

*δ*

and *η <*

2*r*

*,* (1)

where *M* (*r*) is defined in ([11](#_bookmark12)) and *h*(*x*) = *−x* log2 *x −* (1 *− x*) log2(1 *− x*), with

*h*(0) = *h*(1) = 0, is the *entropy function* in [0*,* 1]. It is well known that

*n ≤* 2*H*(*α*)*n*

*αn*

for any 0 *≤ α ≤* 1*/*2.

Let *ε* = *ε*(*r, η,* 3) *>* 0 satisfy the assumption in Lemma [2.4](#_bookmark5), and assume without loss of generality that *ε < η/*2. Fix *M* = *M* (*r, ε*) given by Lemma [2.3](#_bookmark4).

Let Δ be an *r*-edge coloring of *G* = (*V, E*) that contains no 2-colored triangle. By Lemma [2.3](#_bookmark4), there is a multicolored *ε*-regular partition *V* = *V*1 *∪· · ·∪ Vm* of the colored graph, where 1*/ε ≤ m ≤ M* .

For each color, there are at most *ε* *m* irregular pairs with respect to the partition

2

*V* = *V*1 *∪· · · ∪ Vm*, hence at most

*r · ε ·*  *m* *·* *n* 2 *≤ rε · n*2 *≤ rη · n*2 (2)

2

*m*

2

4

edges of *G* are contained in an irregular pair with respect to some color. Moreover, there are at most

*m ·* *n* 2 = *n*

*m*

2

*m*

*≤ εn*2 *≤*

*η · n*2

2

(3)

4 The cases 2 *≤ r ≤* 5 may be proved with similar arguments.

edges with both ends in some class *Vi*, where *m ≥* 1*/ε*. Finally, the number of edges *e* with ends in different classes *Vi* and *Vj* such that the color of *e* has density less than *η* between *Vi* and *Vj* is at most

*r · η ·* *m* *·* *n* 2 *≤ rη · n*2*.* (4)

2

*m*

2

Using ([2](#_bookmark8)), ([3](#_bookmark8)) and ([4](#_bookmark8)) gives at most *rηn*2 edges of these three types, which may be chosen in at most *n*2 ways. Note that this set of edges could be colored in at most *rrηn*2 different ways.

*rηn*2

Let *H* = *H*(*η*) be the multicolored cluster graph associated with the partition

*V* = *V*1*∪· · ·∪Vm*. Let *Ej*(*H*)= *{e ∈ E*(*H*): *|Le|* = *j}* and *ej*(*H*)= *|Ej*(*H*)*|*, *j ∈* [*r*]. The number of *r*-edge colorings of *G* that give rise to the partition *V* = *V*1 *∪· · ·∪Vm* and to the multicolored cluster graph *H* is bounded above by

*r*

*n*2 *rηn*2

*· rrηn*2

*·* ⎛

*jej* (*H*)

⎞( *n* )2

⎠

*m*

*≤* 2*h*(*rη*)*n*2

* *rrηn*2
* ⎛

*jej* (*H*)

⎞( *n* )2

⎠

*m*

*.* (5)

*j*=1

*r*

⎝

Recall that *ξ* is a constant defined in ([1](#_bookmark8)).

*j*=1

**Claim 3.1** *There must be a multicoloured cluster graph H such that*

⎝

*er—*3(*H*)+ *···* + *er*(*H*) *≥* ex(*m, K*3) *− ξm*2*.*

Before proving this claim, we show that it implies the desired result. Let *Hj* be the subgraph of *H* with edge-set *Er—*3 *∪ ··· ∪ Er*. By Theorem [2.5](#_bookmark6) there is a partition *U*1 *∪ U*2 = [*m*] with

*eH′* (*U*1)+ *eH′* (*U*2) *≤ ξm .*

2

Let *H*^ be a bipartite subgraph of *Hj* with bipartition *U*1 *∪ U*2 and the maximum

number of edges. Note that by Theorem [2.5](#_bookmark6) we have

*e*(*H*) *≥* ex(*m, K*3) *−* 2*ξm*2*.*

^

We claim that *e*1(*H*)+ *···* + *er—*4(*H*) *≤* 6*ξm*2. Otherwise, by Lemma [2.6,](#_bookmark7) the graph

obtained by adding the edges of *E*1 *∪ · · · ∪ Er—*4 to *H*^ would contain a triangle

such that exactly one of the edges *f*1 is in *E*1 *∪· · ·∪ Er—*4. Let *f*2 and *f*3 be the

other two edges of the triangle, which lie in *Er—*3 *∪ · · · ∪ Er*. If there is a color

*α ∈ Lf*1 *∩* (*Lf*2 *∪ Lf*3 ), say *α ∈ Lf*1 *∩ Lf*2 , we may choose a color *β /*= *α* in *Lf*3 , as

*|Lf*3 *| ≥ r −* 3 *≥* 3. Otherwise, let *β ∈ Lf*1 and note that *|Lf*2 *∪ Lf*3 *| ≤ r −* 1, while

*|Lf*2 *|* + *|Lf*3 *|≥* 2*r −* 6 *≥ r*. So there is a color *α ∈ Lf*2 *∩ Lf*3 , where *α /*= *β*. In both cases, this would lead to a 2-colored triangle in *G* by Lemma [2.4](#_bookmark5), a contradiction.

As a consequence, the number of edges of *H* with both ends in the same set *Ui* is at most 7*ξm*2. Let *Wi* = *∪j∈Ui Vj* for *i ∈ {*1*,* 2*}*. Then, by our choice of *η* and *ξ*, we have

*et*(*W*1)+ *et*(*W*2) *≤ rηn*2 + (*n/m*)2(*eH* (*U*1)+ *eH* (*U*2)) *< δn*2*,* (6)

as required.

To conclude the proof of Lemma [1.2](#_bookmark2), we need to prove Claim [3.1](#_bookmark10).

**Proof.** (Proof of Claim [3.1](#_bookmark10)) Suppose for a contradiction that any coloring of *G*

avoiding a 2-colored triangle leads to a multicolored cluster graph *H* for which

*er—*3(*H*)+ *···* + *er*(*H*) *<* ex(*m, K*3) *− ξm*2*.* (7)

Given a 2-element set *S ⊂* [*r*] and *j ∈ {*2*,...,r −* 4*}*, let *Ej*(*S, int≥*1; *H*) be the set of all edges *ej ∈ Ej*(*H*) that satisfy *|Le′ ∩ S| ≥* 1, and let *ej*(*S, int≥*1; *H*) =

*|Ej*(*S, int≥*1; *H*)*|*.

**Proposition 3.2** *Consider a multicolored cluster graph H with no* 2*-colored trian- gle.*

1. *For all* 2*-element subsets S ⊆* [*r*] *of colors, the subgraph Hj of the multicolored*

*cluster graph H with edge set* *r—*4 *Ej*(*S, int≥*1; *H*) *∪* *r El*(*H*) *is triangle-*

*free.*

*j*=2

*l*=*r—*3

1. *Moreover, there exists a* 2*-element subset S ⊆* [*r*] *such that*

*r—*4

Σ*r—*4 *r* *−* *r—j*

*j*=2

*Ej*(*S, int≥*1; *H*) *≥*

2

*j*=2

*r* 2 *· |Ej*(*H*)*|.* (8)

2

Before proving Proposition [3.2](#_bookmark12), note that it leads to the following inequality:

*r—*4 *r − r—j*

2 2

Σ

*r*

Σ

*r* *· ej*(*H*)+

*l*=*r—*3

*j*=2

2

*el*(*H*) *≤* ex(*m, K*3)*.* (9)

**Proof.** We first argue that *r—*4 *Ej*(*S, int≥*1; *H*) *∪* *r El*(*H*) is triangle-free.

*j*=2

*l*=*r—*3

For a contradiction suppose that there is a triangle with edges *f*1*, f*2*, f*3. Note that

the lists of these edges have size at least two.

If one of the edges lies in *r El*(*H*), we may sum the sizes of the lists *Lƒ* , *Lƒ*

*l*=*r—*3

1 2

and *Lƒ*3 to obtain at least (*r−*3)+4 = *r* +1. In particular, two of the lists must have

a color *α* in common, and the third list contains *β /*= *α*, which produces a 2-colored triangle, a contradiction. Next, assume that *f*1*, f*2*, f*3 *∈ r—*4 *Ej*(*S, int≥*1; *H*). If we sum the sizes of *Lƒ*1 *∩ S*, *Lƒ*2 *∩ S* and *Lƒ*3 *∩ S*, we obtain at least three, so that two of the lists must contain the same color *α ∈ S*, and the third list contains an element *β /*= *α*, which proves part (a)

*j*=2

For part (b), we claim that

*r—*4

Σ

2

2

Σ

*—*

*|Ej*(*S, int≥*1; *H*)*|* =

Σ*r—*4 *r*

*r − j*

* *ej*(*H*)*.*

*S∈*([*r*]) *j*=2

2

*j*=2

Indeed, for *j* = 2*,...,r −* 4, every edge *e ∈ Ej*(*H*) is counted on the left hand side for all sets *S ∈* [*r*] such that *|S ∩ e|≥* 1, which amounts to *r* *−* *r—j* times.

2

2

2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| *r* | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| *M* (*r*) | 25*/*3 *≈* 3.17 | 37*/*5 *≈* 4.65 | 414*/*11 *≈* 5*.*84 | 56*/*5 *≈* 6*.*90 | 43*/*2 =8 | 455*/*34 *≈* 9*.*42 | 311*/*5 *≈* 11*.*21 |

Table 1 Approximate values of *M* (*r*).

By averaging, as there are *r* distinct 2-element subsets in [*r*], there exists a 2-element subset *S ⊆* [*r*] such that

2

*r—*4

Σ*r—*4 *r* *−* *r—j*

*j*=2

*Ej*(*S, int≥*1; *H*) *≥*

2

*j*=2

*r* 2 *· ej*(*H*)*.*

2

*2*

There are at most *Mn* partitions on *m ≤ M* classes. Thus, using ([5](#_bookmark8)), summing

over all partitions and all corresponding multicolored cluster graphs *H*, the number of *r*-edge-colorings of *G* avoiding a 2-colored triangle is bounded above by

⎛ ⎞( *n* )2

*m*

*r*

*Mn ·* Σ 2*h*(*rη*)*n*2 *· rrηn*2 *·* ⎝ *jej* (*H*)⎠

*H*

*j*=1

*.* (10)

Note that finding the maximum *M* (*r*) of *r jej* (*H*) in this equation is equivalent

*j*=1

to maximizing

*e*2 ln 2 + *e*3 ln 3 + *···* + *er* ln *r,*

which is a linear objective function with respect to the variables *e*2*,..., er ≥* 0. Together with linear constraints in ([14](#_bookmark14)) and ([9](#_bookmark12)), we obtain a linear program as follows. Given *H*, set *ζ*(*H*) = (ex(*m, K*3) *− er—*3(*H*) *−· · · − er*(*H*)) */m*2, so that *ζ*(*H*) *≥ ξ* by ([7](#_bookmark11)). The inequalities ([14](#_bookmark14)) with *j* = 2 and ([9](#_bookmark12)) tell us that to find an upper bound on ([10](#_bookmark12)), we may consider the linear program

max *x*2 ln 2 + *x*3 ln 3 + *···* + *xr—*4 ln (*r −* 4) (11)

Σ

*r—*4 *r − r—j*

2 2

*r* *· xj ≤* 1

*j*=2

2

*x*2*,..., xr—*4 *≥* 0*,*

where *xi* plays the role of *ei*(*H*)*/*(*ζ*(*H*)*m*2). As it turns out, for *r ∈ {*6*,...,* 12*}*, if *y*(*r*) is the optimum of the linear program, the value of *M* (*r*)= *ey*(*r*) is given in Table [1.](#_bookmark13)

Clearly, for any multicolored cluster graph *H*, we have

⎛*r—*4

*r*

*jej* (*H*) = ⎝

*jej* (*H*)⎞ ⎛

⎠ ⎝

*r*

*jej* (*H*)⎞

⎠

*j*=1

⎛

*j*=1

*r—*4

*≤*

⎝

*j*=1

*jej* (*H*)

*j*=*r—*3

⎞ *er−*3(*H*)+*···*+*er*(*H*)

⎠ *r*

as *M* (*r*) *< r*.

*≤ M* (*r*)*ζ*(*H*)*m*2 *r*ex(*m,K*3)*—ζ*(*H*)*m*2 *≤ M* (*r*)*ξm*2 *r*ex(*m,K*3)*—ξm*2 *,* (12)

Since, for each partition, there are at most 2*rM*2*/*2 choices for the multicolored cluster graph *H*, with ([12](#_bookmark13)), equation ([10](#_bookmark12)) is at most

*n h*(*rη*)*n*2 *rηn*2 *rM* 2

*M ·* 2 *· r ·* 2 2

*·*

*M* (*r*)

*r*

*ξn*2

* *r*ex(*m,K*3)

*n* 1 3 *h*(*rη*)*n*2 *rηn*2

*≤* 22 *· r ·*

*M* (*r*)

*r*

*ξn*2

* *r*ex(*m,K*3)

*≤ r*(*rη*+*h*(*rη*))*n*2 *·*

*M* (*r*) *ξn*2

*r*

* *r*ex(*m,K*3)

*n* 1 ex(*m,K*3)

This implies that *G* has fewer than *r*ex(*n,K*3) colorings, a contradiction that proves Claim [3.1](#_bookmark10). *2*

*r .* (13)

# Concluding Remarks

We proved that the Tur´an graph for *K*3 is the unique *n*-vertex graph maximizing the number of *r*-edge-colorings with no copy of a triangle where exactly two colors appear. Of course, one may try to apply the same approach to *r ≥* 13, but the optimum value *M* (*r*) of the linear program ([11](#_bookmark12)) satisfies *M* (*r*) *> r*, so that ([12](#_bookmark13)) fails to hold.

A possible way to circumvent this problem might be to include additional linear constraints to the linear program ([11](#_bookmark12)), in order to decrease the optimum value *M* (*r*).

For instance, for *j* = 2*,..., [r/*3*♩*, let *Hj*

*j*

be the subgraph of the multicolored

cluster graph *H* (defined in the proof of Lemma [1.2](#_bookmark2)) with edge set *Ej ∪···∪ Er—*2*j*,

and fix a bipartite subgraph *Bj*

*j*

*j*

of *Hj*

with the maximum number of edges, so

that *|E*(*Bj* )*| > |E*(*Hj* )*|/*2= (*ej*(*H*)+ *···* + *er—*2*j*(*H*)) */*2 (this is a well-known fact

*j j*

about the maximum cut of a graph). Let *Hjj* be the subgraph of *H* with edge set

*j*

*E*(*Bj* ) *∪ Er—*2*j*+1 *∪ · · · ∪ Er*. Note that *Hjj*

is triangle-free, as any such triangle

*j j*

would have three edges *f*1*, f*2*, f*3 such that *|Lƒi | ≥ j ≥* 2 for all *i* and such that

max*i |Lƒi |≥ r −* 2*j* +1 *≥* 2. By the pigeonhole principle, two of the lists would have a common color *α*, and the third list has a color *β /*= *α*. For *j* = 2*,..., [r/*3*♩*, this implies that

1

2 *·* (*ej*(*H*)+ *···* + *er—*2*j*(*H*)) + *er—*2*j*+1(*H*)+ *···* + *er*(*H*) *≤* ex(*m, K*3)*.* (14)

So far we have not been successful in achieving an optimum that is less than *r* for some *r ≥* 13. It is conceivable that such an approach could be extended for all *r ≤* 26, as we already know that the bipartite Tur´an graph cannot be optimal for *r ≥* 27.

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