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Representation of a Discretely Controlled Continuous System in Tense Arithmetic

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**Abstract**

Specifications for a vehicle control system, as an example of discretely controlled continuous systems, are represented and verified in a formal system called *tense arithmetic* for concurrent programs with rational number time. In this formalism, computational sequences are characterized, or indexed, by *spurs* that are generaliza- tions of program schedulers and that are also temporal propositions. Our formalism describes analysis of the wake-up time of the next action from an observation time, and we obtain the actual rational time value for when the next action will occur. Additionally, we introduce the *continuous variables* with their first- and second- order derivatives, to analyze and verify programs that control discretely certain continuously physical or other external systems.

# Introduction

In this paper we represent and analyze a vehicle control system [6], as an example of discretely controlled continuous systems, in *tense arithmetic* [8] (TA for short) for the verification of concurrent programs involving rational number time. This formalism has three characteristics. First, it deals with rational time, called *tense*, explicitly. Second, it interprets each formula *P* in an extended theory of rationals by its tense, denoted by *[P♩*, that is, the time length between an *observation time*, say *t*, and the time when *P* holds for the first time after *t*. (*∞* is introduced as the abbreviation of *[*false*♩*.) In addition, we introduce the *futurity* operator “ ; ” that forwards the observation time to the future. Third, the observation time *t* is treated as the *present tense* or

*now*, i.e., 0, in the logic, so that we can dispense with the time variable *t*.

In this paper, we extend this formalism to treat continuous systems. To do this, the *continuous variables* with their first- and second-order derivatives will be introduced. Using these variables, we can analyze and verify programs that control discretely certain continuously physical or other external systems.

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Many formalisms to verify realtime systems have been proposed, including the temporal logic [9], TCSP [4], the duration calculi [2], the hybrid automata [1], [3], [14], and so on, as well as our work [6], [7], [11] that is closely related to the present paper. Specifications of an autonomous vehicle control system are found and verified in [13]. The duration calculi and the hybrid automata can treat only first-order differential equations, while our formalism can treat both lower- and higher-order ones.

Section 2 shows the formal system and the semantics of TA, the original version of which can be found in [8]. A vehicle control system is then intro- duced and analyzed in Section 3. In this example, we consider the situation of a car merging with traffic at a T junction. We formulate this problem, then formally analyze and verify the conditions under which the merging car does not crash into others.

# Tense arithmetic

* 1. *Syntax of language*

Let *TQ* be any appropriate first-order theory of rational numbers. Let *V Q* be the set of the free variables of *TQ*. An element of *V Q* is called a *rational variable*.

We extend *TQ* to *TQ* by adding individual constants *J* , *J*1, *J*2, *·· ·*, corre- sponding to program variables over rational numbers, called *special rational constants*; by adding predicate constants *l*, *l*1, *l*2, *·· ·*, corresponding to labels within programs; and by adding predicate constants *γ*˙ , *γ*˙1, *γ*˙2, *·· ·*, correspond- ing to *atomic spurs* by which we construct *spurs*. Those predicate constants are called *special boolean constants*. The spurs are generalizations of program schedulers (see section 2.4). Moreover, we introduce other special rational constants *ξ*, *ξ*1, *ξ*2, *·· ·*, *η*, *η*1, *η*2, *···* called *continuous variables* with their (first- and second-order) derivatives *ξ'*, *ξ'* , *ξ'* , *·· ·*, *η'*, *η'* , *η'* , *·· ·*, *ξ''*, *ξ''*, *ξ''*, *·· ·*,

*C*

1 2 1 2 1 2

*η''*, *η''*, *η''*, *···* to describe continuous systems.

1 2

Terms and formulas of TA are called *tense terms* and *tense formulas*, re-

spectively, in order to distinguish them from those of *TQ*. Let *V T* be the set

*C*

of countably infinite tense variables.

**Definition 2.1** Tense terms are defined as follows:

* + 1. A term *e* of *TQ* is a tense term, which is called a *rational tense term*,

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* + 1. A tense variable *x* is a tense term,
    2. For a formula *P* of *TQ*, *[P♩* is a tense term, called the *tense* of *P* ,

*C*

* + 1. If *s* is a tense term, then *s*; *e*, *s*; *x* and *s*; *[P♩* are tense terms. *✷*

The symbol *[ ♩* will be called the *tense symbol*, while the semicolon “ ; ” as a symbol will be called the *futurity operator*, which is left-associative. *s* is called a *prefix* of *s*; *x*. In general, *s* is called a prefix of *s*; *s*1.

**Definition 2.2** Tense formulas are defined as follows:

1. For a formula *P* of *TQ*, *P* is a tense formula,

*C*

1. If *s* and *s*1 are tense terms, then *s* = *s*1 and *s ≤ s*1 are tense formulas,
2. If *x* is a tense variable, *r* is a rational variable, and *F* and *G* are tense formulas, then *¬F* , *F ∨ G*, *∀xF* and *∀rF* are tense formulas. *✷*

A tense indicates the time relative to the observation time, called *now*. As a tense term, a rational expression *e* whose value is *r* represents the tense *r*. 0 represents now. The tense of *P* denoted by *[P♩* is the earliest time when *P* holds after now. For the futurity operator “ ; ”, here are a few examples:

1. 2*.*0; *[J* = 0*♩* designates the tense when the program variable *J* reaches the value 0 after 2.0 time units, and
2. 2*.*0; *[J* = 0*♩* = *∞* reads “*J* will never come to be 0 after 2*.*0”.
   1. *Sequents and proof system*

**Definition 2.3** For tense formulas *F*1*, ··· , Fm, G*1*, ··· , Gn* (*m, n ≥* 0),

*F*1*, ··· , Fm → G*1*, ··· , Gn*

is a *sequent* (of TA). The case that *m* = 0 is understood as ‘true’ and the case

*n* = 0 as ‘false’. *✷*

A sequent *F*1*, ··· , Fm → G*1*, ··· , Gn* intuitively means that *F*1 *∧· · ·∧Fm ⊃ G*1*∨· · ·∨Gn* holds for any ‘worlds’ (changing with time) and at any observation time. Thus, the sequent expresses the same thing as the formula *✷*(*F*1 *∧· · · ∧ Fm ⊃ G*1 *∨ ··· ∨ Gn*) in temporal logic.

**Definition 2.4** The symbol *∞* is an abbreviation of *[*false*♩*, so that

*∞* = *[*false*♩,*

and “ : ” is defined by

|  |  |  |  |
| --- | --- | --- | --- |
| def  *x* : (*s ≤ s*1) *⇔*  *x* : (*F ∨ G*) *⇔*def | (*x*; *s*) *≤* (*x*; *s*1)*,*  (*x* : *F* ) *∨* (*x* : *G*)*,* | def  *x* : (*s* = *s*1) *⇔*  *x* : (*¬F* ) *⇔*def | (*x*; *s*) = (*x*; *s*1)*,*  *¬*(*x* : *F* )*,* |
| *∀r*(*x* : *F* ) *⇔*def  *x* : *P ⇔*def | *x* : (*∀rF* )*,*  *x* = (*x*; *[P♩*)*.* | *∀y*(*x* : *F* ) *⇔*def | *x* : (*∀yF* )*, and* |

In the definitions of “ : ”, each expansion rule on the left in the table pre- cedes the one to the right, and each upper precedes the lower, if there exists ambiguity to expand *x* : *F* . *✷*

The colon “ : ” as a symbol is called the *coincidental* operator, which intu- itively means that the formula *F* is true at the tense *x*. As in *s*; *[P♩*, *s* or any prefix within *s* is called a *prefix* of *s* : *F* .

**Definition 2.5** The *inference rules* of TA consist of the LK-like rules having the same form as the structural rules and the rules for the logical operators

*¬, ∨* and *∀* of LK, with the only modification being that the rules for *∀x* and

*∀r* are differentiated from each other. Additionally, the *rule for futurity* is adopted as follows:

*→ x* : *F*

*→ F ∧* 0 *≤ s*1 *∧ ··· ∧* 0 *≤ sn*

where *s*1*, ··· , sn* are all of the prefixes of tense terms, which are arguments of equalities or inequalities in *F* . *✷*

**Definition 2.6** A *derivation* of TA consists of sequents arranged in tree form. An *initial sequent*, i.e., a leaf, of the derivation is any of the following:

* + 1. *identic sequent*: *F → F,*
    2. *axiomatic sequent*: *→ A*, for any axiom *A* of TA.

Each *inference* results from one of the inference rules by a usual substitu- tion.

The *end sequent*, i.e., the root of the derivation, *S* is said to be *derivable*

(in TA), and the whole derivation is called a *proof* of *S*. *✷*

**Definition 2.7** If a sequent of the form *→ F* is derivable, we say that *F* is

*provable* (in TA) and *F* is a *theorem* (of TA), which fact will be denoted by

*▶ F* . *✷*

* 1. *Axioms*

We introduce the logical, or, *proper* axioms of TA. It must be noted that *r* and *r*1 are restricted within rationals (not containing *∞*). The symbols = and *≤* are *the same as those used in TQ and TQ*, so that 0 *<* 1, *∀rr*1(*r* + *r*1 = *r*1 + *r*), *J* = *J*1 *⊃ J* +1 = *J*1 + 1, etc., hold, for example.

*C*

* + 1. the ordinal *order axioms* for *≤*.
    2. axioms for tense:
       1. *tense axiom*: *∀x* (*∃r* (*x* = *r*) *∨ x* = *∞*),
       2. *‘Never Land’ (unreachability) axiom*: *∀r*(*r < ∞*),
       3. *truth axiom*: *[*true*♩* = 0,
       4. *futility axiom*: *∀x* (*∞* = *x*; *∞*),
    3. axioms for futurity “ ; ”:
       1. *present axiom*: *∀x* (0 *≤ x ⊃* 0; *x* = *x*),
       2. *passage axiom*: *∀xy* (*x ≤ x*; *y*),
       3. *duration axiom*: *∀rr*1 (0 *≤ r*1 *∧ r*; *r*1 = *r* + *r*1 *∨ r*1 *<* 0 *∧ r*; *r*1 = *r*),
       4. *prefix substitution axiom*: *∀xyz* (*x* = *y ⊃ x*; *z* = *y*; *z*),
    4. axioms for tense symbol *[ ♩*:
       1. *idempotent axiom*: *[P♩* = *[P♩*; *[P♩,*
       2. *precedence axiom*: *∀x* (0 *≤ x ∧ x* : *P ⊃ [P♩≤ x*),
       3. *advance consequence axiom*: *[Q♩ ≤ [P♩*, for any *P* and *Q* such that *P ⊃ Q* is a theorem of *TQ* that includes no continuous variable,

*C*

* + - 1. *monotonicity axiom*: *∀xy* (*x ≤ y ⊃ x*; *[P♩≤ y*; *[P♩*),
    1. theorems of *TQ* as axioms: Any theorem of *TQ* is an axiom of TA.

*C C*

* + 1. axiom for continuous variables:

*ξ'* = *c*1 *∧ ξ* = *c*2 *∧ ∀r* (0 *≤ r < [ξ'' /*= *c♩ ⊃ r* : (*ξ''* = *c*)) *⊃*

*∀r* (0 *≤ r ≤ [ξ'' /*= *c♩ ⊃ r* : (*ξ* = *c r*2 + *c*1*r* + *c*2) *∧ r* : (*ξ'* = *cr* + *c*1))

2

From the axiom (vi), we can treat second-order differential equations in TA. Obviously, our formalism can treat higher-order derivatives when the axioms corresponding to them that are similar to (vi), are introduced.

* 1. *Spurs and program axioms*

Let us suppose that each atomic spur satisfies the condition that it returns to being false after it has become true within the finite time period.

**Definition 2.8** Let *γ*˙ be an atomic spur. A *spur γ* is *[γ*˙ *♩*; *[¬γ*˙ *♩*, which ex-

presses the time when *γ*˙

changes into false after *γ*˙

happens. *✷*

Using spurs, we can represent each *n*-step execution of a process even if no actual time value of execution is given. For example, the tense of a 2- step execution can be written as *γ*; *γ*, which is an abbreviation of *[γ*˙ *♩*; *[¬γ*˙ *♩* ; *[γ*˙ *♩*; *[¬γ*˙ *♩*. From axioms (4c) and (3a) with the rule for futurity, the fact that *γ < γ*; *γ* is guaranteed, while *[γ*˙ *♩* = *[γ*˙ *♩*; *[γ*˙ *♩* by axiom (4a).

We express a program by some *program axioms* in the form of sequents, each of which represents one action step. Each program axiom is used as an initial sequent of a proof. A program axiom may contain a *spur*, a cur- rent program label, some next labels, some conditions and some actions, e.g., assignments. Additionally, we axiomatize the *axiom of conservation* [5], by which the values of the program variables are kept unchanged as long as no action is performed. The spurs are generalizations of schedulers of processes. We consider a multi-CPU parallel program system in which each process has its own CPU. So, we assign distinct spurs *α, β, ···* as the schedulers to the various processes. Program labels are supposed to be exclusive of each other process-wise.

* 1. *Semantics*

Let *Q* be the standard model of rational numbers. An assignment *ρQ* of *TQ* is a function such that *ρQ* : *V Q → Q*. *Q* as a model of *TQ* can be expanded to a model *M* of *TQ* by adding an interpretation of the special constants. Each special constant may have different values in different models.

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We regard the set *Q* + *{∞}* as tense. For *∞* we shall assume *r ∈ Q ⇔*

*r < ∞*, *r* + *∞* = *∞* + *r* = *∞* every *r ∈ Q*, and inf *∅* = *∞*, *∅* denoting the

empty set.

An assignment *ρ* of TA is a pair *⟨ρQ*, *ρT ⟩* of functions, where *ρQ* is an assignment of *TQ* as above and *ρT* is a function, called an *assignment to V T* , such that *ρT* : *Q→ V T → Q* + *{∞}*, sending a tense variable *x* onto *ρT* (*t, x*). *ρQ*(*e*) and *ρQ*(*P* ) denote the expression and the formula obtained from, re- spectively, *e* and *P* by substituting the values specified by the assignment *ρQ* in place of the free variables.

The changes of models are described by a *locus χ* that is a function from a rational time to models, *χ* : *Q→ E*, where *E* is the set of such expansions designated by *M*. For a special rational constant *J* and a special boolean constant *l*, *χ*(*t, J* ) and *χ*(*t, l*) designate the rational value of *J* and the truth value of *l* at *t*(*∈ Q*) on *χ*, respectively. Whenever an assignment *ρQ* of *TQ*,

a locus *χ* and a time *t*(*∈ Q*) are given, the truth of a formula *P* of *TQ* is

*C*

determined. *ρQ, χ*(*t*) *|*= *P* expresses that *P* holds on *ρQ* and *χ*, at *t*, that is,

*χ*(*t, ρQ*(*P* )) = true.

**Definition 2.9** An increasing rational sequence (*ri*)*i*=0*,*1*,*2*,···* is *discrete* if it is either finite or contains arbitrarily large rationals. *✷*

**Definition 2.10** A function *f* : *Q→ S*, where *S* is an arbitrary set, is called a *right-continuous discrete step function* if and only if *f* (*t*) = *xi* for *ri ≤ t < ri*+1 for some *xi* belonging to *S* and a discrete rational sequence (*ri*)*i*=0*,*1*,*2*,···*. *✷*

**Discreteness postulate.** Every locus *χ*, except for continuous variables and their first-order derivatives, and every assignment *ρT* to *V T* must be discrete step functions. *✷*

In order to treat continuous variables and their derivatives, we modify the semantics of the original TA [8]. The main modification is that loci are *right-continuous* step functions, instead of *left-*continuous.

**Postulate for continuous systems.** We suppose that each “variable” in the continuous system that we deal with in this paper, as a function sending a real time in a real value [5], [12], is continuous in realtime and is derivable except for (finite or infinite) discrete rational time points. For each rational time value, the value of the function must be rational. Every locus must reflect the values of the continuous variables corresponding to these functions. Additionally, every first-order derivative must be continuous. *✷*

Given an assignment *ρ* of TA, a locus *χ* and an observation time *t*(*∈ Q*), a tense term *s* is interpreted into a value in *Q* + *{∞}* defined below.

**Definition 2.11** The valuation *∼* of a tense term for given *ρ*, *χ* and *t* is

defined as follows:

*~*

1*. e*

*ρ,χ*

*~*

2*.* *x*

*ρ,χ*

3*. [P♩∼*

*ρ,χ*

(*t*) d=ef

(*t*) d=ef

(*t*) d=ef

*χ*(*t, ρQ*(*e*))*, ρT* (*t, x*)*,*

min*{u|* 0 *≤ u, χ*(*t* + *u, ρQ*(*P* )) = true*},* (min *∅* = *∞*)*,*

4*.* (*s*; *s* )*∼*

(*t*) d=ef

(*s∼*

(*t*) *< ∞→*

1 *ρ,χ*

*ρ,χ*

*~*

*s*

*ρ,χ*

(*t*) + (0 *≤ s*1*∼*

(*t* + *s∼*

(*t*)) *→ s*1*∼*

(*t* + *s∼*

1. )*,* 0)*,*

*∞*)*.*

*ρ,χ*

*ρ,χ*

*ρ,χ*

*ρ,χ*

where *s*1 is either *e*, *x* or *[P♩*. (*a → b, c*) is McCarthy’s operator [10], meaning “if *a* then *b* else *c*.” *✷*

**Definition 2.12** The truth valuation # of a tense formula for given *ρ*, *χ* and

*t* is defined as follows:

* 1. *P* # (*t*) = true

*ρ,χ*

*⇔*def

*~*

*ρ,χ*

*[P♩*

(*t*) = 0*,*

* 1. (*s* = *s* )#

(*t*) = true

*⇔*def

*s∼* (*t*) = *s ∼*

(*t*)*,*

1 *ρ,χ*

*ρ,χ*

1*ρ,χ*

(*s ≤ s* )#

(*t*) = true

*⇔*def

*s∼* (*t*) *≤ s ∼*

(*t*)*,*

1 *ρ,χ*

*ρ,χ*

1*ρ,χ*

* 1. (*¬F* )#

*ρ,χ*

* 1. (*F ∨ G*)#

(*t*) = true

(*t*) = true

*⇔*def

*⇔*def

*F* # (*t*) = false*,*

*F* # (*t*) = true or *G*#

*ρ,χ*

(*t*) = true*,*

*ρ,χ*

* 1. (*∀xF* )#

*ρ,χ*

(*t*) = true

*⇔*def

*ρ,χ*

#

*F*

*⟨ρQ, ρ'T ⟩,χ*

*ρ,χ*

(*t*) = true*,* every *ρ'T*

such that *ρ'T* (*u, y*) = *ρT* (*u, y*)

each *u* and each *y ∈ V T* other than *x,*

* 1. (*∀rF* )#

*ρ,χ*

(*t*) = true

*⇔*def

#

*⟨ρ'Q, ρT ⟩,χ*

*F*

(*t*) = true*,* every *ρ'Q*

such that *ρ'Q*(*v*) = *ρQ*(*v*) each *v ∈ V Q* other than *r.✷*

**Definition 2.13** A locus *χ* is said to *satisfy F* if and only if

*F* # (*t*) = true*,* every *ρ* and *t,*

*ρ,χ*

in which case we write *χ |*= *F* .

Let *P* be the set of loci satisfying the discreteness postulate. A locus *χ ∈P* is said to *satisfy* the sequent *F*1*, ··· , Fm → G*1*, ··· , Gn* if and only if it holds that

*m* + *n >* 0 *and χ |*= *¬F*1 *∨ ··· ∨ ¬Fm ∨ G*1 *∨ ··· ∨ Gn. ✷*

**Definition 2.14** Soundness of derivation. For sequents *S*1, *·· ·*, *Sn* and *S* of TA, a derivation *S* from *S*1, *·· ·*, *Sn* of TA is *sound* if and only if every locus *χ* satisfying all the sequents *S*1, *··· , Sn* also satisfies *S*.

The soundness theorem of the *original* TA is guaranteed in [8]. For the modified system, the following soundness theorem can be easily shown in a similar way.

**Theorem 2.15** *If ▶ F, then χ |*= *F holds for every χ. ✷*

# Analysis of a vehicle control system

Let us consider two roads meeting at a T junction. Many cars *car*1, *car*2, *··· , carn*, *carn*+1, *···* are running one way on the road at a constant speed *v*[m*/*s], and each pair of cars *⟨i, i* + 1*⟩* has a distance within this pair, ∆*i*. A car *car*0 wants to merge between *carn* and *carn*+1 with an initial speed 0 and an acceleration *a* = *a*0 (*>* 0) [m*/*s2] if ∆*n* is less than or equal to *d*. The driver’s decision to merge is delayed 0.5 [s] from the moment at which *carn* reaches a certain “safe” zone, and the beginning of the actual acceleration of the vehicle is delayed 1.5 [s] from the moment of the driver’s decision. Moreover, when the speed of the car is *v*0(*< v*), the driver decides to release the accelerator with 0.2

[s] delay, and the actual effect is delayed by 0.3 [s] after the decision. The car

*carn*+1, on the other hand, will decide to decrease its speed with a deceleration

*−b*=*b*0 (*<* 0) [m*/*s2] after 0.4 [s] from the time when the distance ∆0 between

*car*0 and itself will be less than or equal to *e*. The actual deceleration begins

0.6 [s] after the driver decides to decelerate.

**Question**: How large should *d* and *e* be to avoid a crash between *car*0 and

*carn*+1? (The original problem and its solution are found in [6].)

* 1. *Program axioms*

Let us consider a one-dimensional coordinate along the road, whose origin is at the junction. The positions of *car*0, *carn*, *carn*+1 are expressed by *η*, *ξn*, *ξn*+1, respectively. We suppose *car*0 is waiting at the crossing, that is, *η* = 0, and it finds the interval ∆*n* is greater than or equal to *d* and *carn* will be closer than or equal to *−f* . Additionally, let us suppose *car*0 does not crash into any car when it is at the crossing and that its speed is 0.

The program axioms of *car*0 are as follows:

*Stand-by ∧ ¬Safe ∧ ¬Accel ⊃*

(1)

*[Safe♩ < α ∧ [Safe♩* : (0*.*5 = *[Accel♩* = *α*)*.*

1. *¬Accel ⊃ [Accel♩* : (1*.*5 = *[η''* = *a♩* = *α*)*,*

where, *Stand-by ≡ η* = 0 *∧ η'* = 0 *∧ η''* = 0 and *Safe ≡ −f ≤ ξn*. The axiom

(1) intuitively means that when *car*0 is at the crossing and finds *carn* at *f* , the decision to accelerate (*Accel*), indicated by the spur *α*, is done with 0.5

[s] delay. In other words, if *Stand*-*by* holds, *α* arises as soon as *Accel* holds, and conversely, *α* does not hold before *Accel*. The axiom (2) means the actual acceleration begins 1.5 [s] after the decision *Accel*. Similarly, the deceleration

after the merge is represented as follows:

1. *¬OverSpeed ⊃ [OverSpeed♩* : (0*.*2 = *[ReleaseAccel♩* = *α*)*.*
2. *¬ReleaseAccel ⊃ [ReleaseAccel♩* : (*[η''* = 0) *∧ η'* = *v♩* = *α ≤* 0*.*3)*,*

where *OverSpeed ≡ v*0 *≤ η'*. This represents that when the speed of the car is *v*0(*< v*), the car decides to release the accelerator with 0.2 [s] delay and the actual effect is delayed 0.3 [s] after the decision.

In a similar manner, we represent the axiom of *carn* as

1. *Cruisingn,*

and those of *carn*+1 as

*Cruisingn*+1 *∧ ¬Brake ∧ ¬Dangerous ⊃*

*[Dangerous♩ < β ∧ [Dangerous♩* : (0*.*4 = *[Break♩* = *β*)*.*

*¬Break ⊃ [Break♩* : (0*.*6 = *[ξ''*

= *−b♩* = *β*)*,*

where *Cruisingi ≡ ξ'* = *v ∧ ξ''* = 0(*i* = *n, n* + 1) and *Dangerous ≡* ∆0 *≤ e*.

*n*+1

*i* *i*

* 1. *Analysis*

Let us consider an initial condition in which Θ *≡ Stand-by∧ ¬Safe∧ ¬Accel ∧*

*¬ReleaseAccel∧ ¬Brake ∧ d ≤* ∆*n*.

From (1) and (2) of the axioms of *car*0 and the axiom of conservation, it holds that

Θ *⊃ η* = 0 *∧ η'* = 0 *∧ ∀r* (0 *≤ r < [η''* = *a♩⊃ r* : (*η''* = 0))

*∧ [Safe♩* : (*[η''* = *a♩* = 0*.*5+ 1*.*5)*.*

By the axiom of continuous variables,

1. Θ *⊃ ∀r* (0 *≤ r ≤ [η''* = *a♩⊃ r* : (*η* = 0) *∧ r* : (*η'* = 0))*.*

Thus, (10)

Θ *⊃ [η''* = *a♩* : (*η* = 0 *∧ η'* = 0)*.*

From this fact and from (3),

Θ *⊃*

(11)

*[Safe♩*;2 : (*η* = 0 *∧ η'* = 0 *∧ ∀r* (0 *≤ r < [η''* = 0*♩ ⊃ η''* = *a*))*.*

Therefore, we have

Θ *⊃*

(12)

2

*[Safe♩*;2 : *∀r* (0 *≤ r ≤ [η''* = 0*♩ ⊃ r* : (*η* = *a r*2 *∧ r* : *η'* = *ar*))*.*

from the axiom of continuous variables. This represents the position and speed of *car*0 once it has accelerated.

Furthermore,

*¬OverSpeed ⊃*

(13)

*[OverSpeed ♩* : *∀r*(0 *< r ≤ [η''* = 0*♩ ⊃ η'* = *ar* + *v*0)

*∧ [OverSpeed ♩* : (0*.*2 *< [η''* = 0*♩≤* 0*.*2+ 0*.*3)*.*

by (12), (3) and (4). Therefore, we have *v −* 0*.*5*a ≤ v*0 *< v −* 0*.*2*a* since *η'* = *v*.

On the other hand, after deceleration begins the car drives as Θ *⊃*

(14)

Thus, *c*1

*[η''* = 0*♩* : (*η* = *c*1 *∧ η'* = *v ∧*

*∀r* (0 *≤ r ⊃ r* : (*η* = *vr* + *c*1) *∧ r* : (*η'* = *v*)))*.*

*v*2 holds by (12).

=

2*a*

Similarly, we have the behavior of *carn*+1 as

Θ *∧* ∆*n* = *r*1 *⊃ [Safe♩* : (*ξn*+1 = *−f − r*1 *∧ ξ'*

*n*+1

= *v*)

(15)

*∧ ∀r* (0 *≤ r ≤ [ξ''*

= *−b♩ ⊃*

*r* : (*ξn*+1 = *vr − f − r*1) *∧ r* : (*ξ'*

*n*+1

*n*+1

= *v*))*,*

(16)

Θ *⊃ [Dangerous♩*;1 :

(*ξn*+1 = *c*2 *⊃*

*∀r*(0 *≤ r ⊃ r* : (*ξn*+1 = *−b r*2 + *vr* + *c*2) *∧ r* : (*ξ'*

= *v − br*)))*,*

2 *n*+1

We solve the minimum values of *d* and *e*. If *car*0 decides to accelerate when *f* = 2*v*, and the condition *d* = *e* holds, then these are the minimum values. From (15) and (16), we can get:

Θ *⊃*

(17)

*[Safe♩*;2 + 1 :

*∀r* (0 *≤ r ⊃ r* : (*ξn*+1 = *−b r*2 + *vr* + *v − d*) *∧ r* : (*ξ'*

= *v − br*))*,*

2

which indicates the position and the speed of *carn*+1.

*n*+1

From this formula with (8) under the conditions *η* = *ξn*+1 and *η'* = *ξ'* ,

*n*+1

i.e., *car*0 has just collided with *carn*+1, we have the minimum values of *d* and

*e* whenever the cars do not crash as:

(18)

*d* = *e* =

*ab −* 2*bv* + *v*2

*.*

2(*a − b*)

Finally, we calculate and get the solution when the actual (and reasonable) values are given. For example, suppose *a* = *−b* = 5[m/s2] and *v* = 50*/*3[m/s] (60 km/h). Hence, we have a solution that if *d* = *e >* 755*/*36[m] (about 21m), *carn*+1 does not crash into *car*0.

# Conclusions

We have briefly demonstrated a new formalism TA, for parallel and realtime controlled programs, to analyze two intelligently controlled systems, briefly. In the analysis of a vehicle control system, we have shown, using the axiom of continuous variables, the actual and reasonable values necessary to avoid a collision between two vehicles.

Our future work is that we will make the axiom more sophisticated, in order to describe every *n*–th order derivative in a uniform manner.

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