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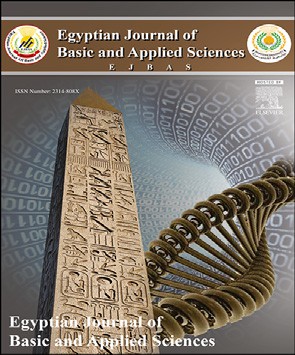
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Full Length Article

Separation of variables in one case of motion of a gyrostat acted upon by gravity and magnetic fields



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## a b s t r a c t

A version of the integrable problem of motion of a dynamically symmetric gyrostat about a fixed point similar to the Kowalevski top, while acted upon by a combination of uniform gravity and magnetic fields is considered. This version is reduced, in general, to hyper- elliptic quadratures. The special case when the gyrostatic momentum is absent is solved in terms of elliptic functions of time.

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# Introduction

The problem of motion of a heavy rigid body about a fixed point has a long history beginning with Euler's work [[1]](#_bookmark39). For most of this history, the main concern of authors was to isolate cases, when the general solution of the equations of motion can be expressed explicitly in terms of functions of time, or, at least, can be reduced to quadratures. This recipe has succeeded in two cases: Euler's case of a body moving by inertia and Lagrange's case of a symmetrical top [[2]](#_bookmark40).

Separation of variables in Euler's case was found by Euler himself, but the solution was expressed by Jacobi in terms of his newly invented elliptic functions. Lagrange reduced the case of axisymmetric top to separation of variables involving elliptic integrals. Explicit expression of the solution in terms of time was initiated by Jacobi and can be found with some variations in [[3](#_bookmark41)e[5]](#_bookmark41).

The following historical turn in rigid body dynamics came in the opposite direction, from the study of the nature of so- lutions of the equations of motion. Kowalevski [[6]](#_bookmark42) isolated the possible cases which share with Euler's and Lagrange's cases

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the property of having their solutions as meromorphic func- tions of time. It turned out that only one more case satisfies this criterion. That case became known as Kowalevski's.

Kowalevski obtained the complementary integral for that case

as a quartic polynomial in velocities. She also integrated the equations of motion in terms of hyperelliptic functions of time. Her solution was simplified by Ko€tter [[7]](#_bookmark43) and reconsid- ered by a series of authors. For a detailed history, see e.g. [[8]](#_bookmark44). When a uniformly rotating rotor has its axis of symmetry fixed in the rigid body, the resulting system is known as a gyrostat. For this system, the generalization of Lagrange's case and its solution was straightforward. Euler's case was gener-

alized by Zhukovsky [[9]](#_bookmark45). The corresponding solution was shown by Volterra to be expressible in terms of sigma func- tions of Weierstrass [[10]](#_bookmark46). In [[11]](#_bookmark47), Wittenburg pointed out another solution in terms of elliptic functions of time. Detailed presentation of the history of general and particular solutions for a heavy gyrostat can be found in [[12]](#_bookmark48).

A century after the discovery of Kowalevski's case, its

generalization to the problem of gyrostat has been found in a different context. The problem of motion of a gyrostat similar to the Kowalevski top and acted upon by two skew uniform fields (gravity and magnetic) was considered in [[13]](#_bookmark31). A general first integral quartic in velocities and generalizing the Kowa- levski integral was found for this generic problem. As, in the

general case, the two force fields problem does not admit a

results for separated systems [[23]](#_bookmark49) give a method to calculate the exact topological invariants of singular points and all regular iso-energy levels. This will give the complete topo- logical analysis of the problem which will be different (as far as special types of motions are concerned) from the corre- sponding Goryachev and Chaplygin cases [[17,24]](#_bookmark35) since the analogy of these problems with the motion of a gyrostat in two fields does not give a global diffeomorphism of the corre- sponding phase spaces.

It will also be interesting to analyze all special cases when the trajectories in reduced systems become periodic. For the Goryachev case the corresponding quadratures are found in [[17]](#_bookmark35). In our problem such quadratures can lead to explicit calculation of the orientation matrix and therefore provide the analytical basis for the geometric interpretation of periodic and two-frequency motions of the considered gyrostat in two fields.

# Equations and integrals

The equation of the motion of a gyrostat acted upon by two homogeneous fields in the general case can be written in the Euler e Poisson form

symmetry group, the additional cyclic integral does not exist. It turned out that such integral still exists in two special cases

M\_ = M × u + c1 × a + c2 × b;

a\_ = a × u; b\_ = b × u.

(1)

[[13]](#_bookmark31). The first case generalizes Kowalevski's case of one field to

the gyrostat motion in one field. The second case does not contain the classical case of Kowalevski, since the intensities of the two fields are proportional and can vanish only simul- taneously. In the last case the cyclic variable is a comple- mentary angle to the sum (or difference) of the two angles of precession and proper rotation.

In the present paper we accomplish separation of variables for a version of the last case, corresponding to a special value of the cyclic constant proportional to the gyrostatic moment and singled out by the condition that the reduced system becomes time-reversible. We give two algebraic separations of variables. In the first one, the cyclic constant is supposed non- zero and the variables of separation are to be determined as functions of time by solving hyperelliptic AbeleJacobi equa- tions. This result is based on the analogy established in [[14]](#_bookmark32) (see also [[15]](#_bookmark33) for further generalizations) of a special class of problems of the gyrostat motion in two fields with the prob- lems of the gyrostat motion in axially symmetric field with zero momentum constant. Thus, the first separation given below corresponds to the algebraic separation [[16,17]](#_bookmark34) found by the method proposed in [[18,19]](#_bookmark36) for the Goryachev case. This separation is not applicable if the gyrostatic moment in the initial problem is zero. Nevertheless, as it is shown in [[14]](#_bookmark32), the equivalent problem of the rigid body motion in an axisym- metric field is the integrable case of Chaplygin [[20]](#_bookmark37). Therefore we give the second separation which transfers the elliptic separation found by Chaplygin to the two-fields problem.

The two types of separation of variables accomplished here make it easy to apply the algorithm of finding the admissible regions for the integral constants and to establish the rough phase topology of the system [[21,22]](#_bookmark38). Moreover, the recent

Here u is the angular velocity, a and b are the characteristic

vectors of the force fields (say, the vectors of the gravity force and of the magnetic field strength), c1; c2 are the vectors pointing from the fixed point *O* to the centers of the fields application. All objects are referred to some moving axes. The

kinetic momentum vector M is connected with the angular velocity by the relation

M = uI + l;

where I and l are the inertia tensor at *O* and the gyrostatic momentum vector. Both I and l are constant in the moving frame. We consider the components of all vectors as rows, thus obtaining the unusual order of the objects in the above expression for M.

It is known [[8]](#_bookmark44) that without changing the plane *O*c1c2 in the body, one can make the pair of the vectors c1; c2 to be ortho- normal. Let us choose the moving frame *O*e1e2e3 of the prin- cipal axes of the inertia tensor. Suppose that the gyrostat is dynamically symmetric e1I•e1 = e2I•e2, l = {0; 0; l} and the centers of the fields application lie in the equatorial plane c1•e3 = 0, c2•e3 = 0. In this case (see [[25,26]](#_bookmark50)) by some linear change of variables one can make the immovable in space vectors a; b to be mutually orthogonal. Then, after the pair

c1; c2 is made orthonormal, the modules of the vectors a; b

contain all scalar information on the interaction of the gyro- stat with the fields (e.g. for the gravity field, the module of the corresponding vector is equal to the product of the gyrostat weight and the distance from the mass center to the fixed point). Therefore, we call a and b the intensities of the force fields. In the case of dynamic symmetry any orthonormal pair in the equatorial plane becomes principal for the inertia tensor, so we take e1 = c1, e2 = c2.

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In what follows, we assume that the inertia tensor satisfies the Kowalevski conditions

I = diag{*I*1, *I*1, *I*3}, *I*1 = *I*2 = 2*I*3 (2)

and the intensities of the orthogonalized fields are equal

a2 = b2 = *a*2, a$b = 0 (*a* > 0). (3) We also choose the dimension units in such a way that

*I*3 = 1.

Note that introducing dimensionless variables we can also obtain *a* = 1. Still, we keep the inessential parameter *a* for a possibility to pass to the limit case *a* = 0, and also for better control on the dimensions of expressions in the obtained formulas.

As shown in [[13]](#_bookmark31), under conditions [(2)](#_bookmark3), [(3)](#_bookmark4) in addition to the energy integral

*H* = u2 + u2 + 1 — a — b

2

3

1

2

u

2

1

2

therefore, *G* is, up to the constant multiplier *a*, the corre- sponding momentum integral.

# Reduction by cyclic coordinate

We introduce the Euler angles q, 4, j (0⩽q⩽p, 0⩽4⩽2p, 0⩽j⩽2p), supposing q is the angle between g and e3:

a1 = *a*(cos4cosj — sin4sinjcosq), a2 = —*a*(sin4cosj + cos4sinjcosq), a3 = *a*sinjsinq,

b1 = *a*(cos4sinj + sin4cosjcosq), b2 = —*a*(sin4sinj — cos4cosjcosq), b3 = —*a*cosjsinq,

g1 = *a*sin4sinq, g2 = *a*cos4sinq, g3 = *a*cosq,

\_

u1 = j\_ sin4sinq + qcos4, u2 = j\_ cos4sinq — q\_ sin4,

u3 = 4\_ + j\_ cosq.

The Lagrange function for the system [(1)](#_bookmark2) is

Equation [(1)](#_bookmark2) have the first integrals

*L* = 1 \_ 2 + \_ 2

1. 3 — cos2q)j\_ 2 + 4\_ j\_ cosq + l 4\_ + j\_ cosq

+

4

(

*K* = u2 — u2 + a

1

2

1

— b 2 + (2u u

4 q

2

2

+ a + b )

2

2

1

+ 2l (u

3

— l) u2 + u2 + 2(a u

+ b u ) ,

(4)

+ *a*cos(4 + j)(1 + cosq).

*G* = 2u1g1 + 2u2g2 + (u3 + l)(g3 — *a*).

2

1

1

2

3

1

3

2

Here the vector g = *a*—1 a × b augments the pair a, b to form a fixed in space orthogonal basis normalized by the value *a*. In particular, the matrix of direction cosines is

a

1 0 1

*Q* = *a* @ b A.

g

Let us consider the action on *SO*(3) of the subgroup {*g*t} of matrices

cost sint 0

0@ 1A

*g*t = —sint cost 0

0 0 1

*Q* 1 *Q*(t)= *g Q g*—1.

t

t

by inner automorphisms

This action is not free; the subgroup {*g*t} itself is a sta- tionary subgroup for each of its elements. Suppose that passing to the quotient space we identify the whole subgroup

Remark 1. The points [(5)](#_bookmark7) correspond to the values q = 0 and

q = p, i.e. to the case

sinq = 0. (6)

We shall consider only those solutions of the system [(1)](#_bookmark2) which do not cross the set of the phase space satisfying [(6)](#_bookmark6).

We will now introduce a change of variables

4 = F — J, j = J, q = 2Q.

Then Q2[0, p/2], and the angles F, J can be taken to vary on the segment [0, 2p], since the change (4, j)1(F, J) is given

by the integer matrix with determinant 1. Obviously, J is a cyclic coordinate corresponding to the integral *G*. In the new variables, this integral is

*G* = *a* = 2*a* (3 + cos2Q)J\_ — F\_ — l sin2Q.

v*L*

vJ\_

It is natural to denote its constant by 2*ag*. From the equa- tion of the cyclic integral we find

F\_ + l sin2Q + *g*

J\_ =

.

*D*sin2Q

the two-dimensional sphere, but this bundle is not locally trivial [[27,28]](#_bookmark51). Therefore, such symmetry is called a singular symmetry. The singular points on the quotient sphere obvi- ously correspond to the cases

{*g*t} to one point. Then we obtain the fiber bundle of *SO*(3) over

g = ±*a*e3. (5)

Here we denote

*D* = 3 + cos2Q = 2 2 — sin2Q .

Eliminating the cyclic coordinate we construct the Routhian

The plus sign corresponds to the points of the subgroup

\_ 2 cos2Q \_ 2

2lcos2Q — *g* \_ *a* 2

{*g*t}. For the minus sign the covering orbit of the action is twice shorter than all close ones.

*R* = Q +

2*D* F +

2 2

2*D* F +

cosFcos Q

2

The integral *G* is a cyclic integral generated by the action of

{*g*t}. Indeed, the instant angular velocity of the rotation *Q*(t) at

t = 0 equals

(*g* + lsin Q) .

4*D*sin2Q

—

Note that the solutions crossing the set [(6)](#_bookmark6) indeed need some special consideration since the values sinQ = 0 and

*Q* (0) 

T *d*

*d*t t=0

*Q*(t

1

g — *a*e

)= *a* (

3) = *a* ( 1

1

g , g , g

2

3

— *a*),

cosQ = 0 cause obvious singularities either in the potential or

in the kinetic energy of the reduced system.

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The term linear in F\_ does not affect the Lagrange equations

1

if

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ2ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

*p*1 =

*k* + *a*l

2

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ2ﬃﬃ

*z*2 — *k* + *a*l ,

0≡ v

— *z*1, *q*1 =

vQ

2lcos2Q — *g*

2l *g*

=—

+

2*D*2

sin2Q,

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ2ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

2

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ2ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

2

(10)

i.e. for the value

*r*1 = q*z*ﬃﬃﬃ1ﬃﬃ ﬃﬃ*z*ﬃﬃ1ﬃﬃﬃﬃ—ﬃﬃﬃﬃlﬃﬃﬃ2ﬃﬃ ﬃﬃﬃ+ﬃﬃﬃﬃﬃ2ﬃﬃlﬃﬃﬃ2ﬃﬃ*h*ﬃﬃﬃ—ﬃﬃﬃﬃﬃ*k*ﬃﬃﬃ,

2*D*

*p*2 =

*k* + *a*l

— *z*2, *q*2 =

*k* — *a*l

— *z*2 ,

*g* = —2l. (7)

*r*2 = q*z*ﬃﬃﬃ2ﬃﬃﬃﬃlﬃﬃ2ﬃﬃﬃ—ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃ2ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃ2ﬃﬃlﬃﬃﬃ2ﬃﬃ*h*ﬃﬃﬃ+ﬃﬃﬃﬃﬃ*k*ﬃﬃﬃ.

Theorem 1. On common levels of the integrals

In the sequel we shall consider only this case. Under con- dition (7), the reduced system is a natural mechanical system

*H*~ = *h*,

*K*~ = *k*

with

\_ 2

cos2Q 2 *a*

1. l2

the angles Q, F are expressed via the variables *z*1, *z*2 by the

formulas

cosQ = 1 2 ,

*R* = Q +

2*D* F\_

+ 2cosFcos Q —

4sin2Q

. (8)

sin

For Q and F, we have the pair of differential equations

l

Q = ,*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃ,

1

2

p*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ

1

2

4tanQ

*aD*sinF

sin F = ,ﬃﬃﬃﬃﬃﬃ

*p*2*q*1*r*1 + *p*1*q*2*r*2

(11)

€F =

,*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃ

*D*

2

F\_ Q\_ — ,

2 l 2*a*(*z*

2

1

2

1

2

1

— *z* ),*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃp*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ ,

€Q sin2Q \_ 2

1. l2 cosQ — *a*cos

sin2Q .

cos F = ,ﬃﬃﬃﬃﬃﬃ

*p*1*q*2*r*1 — *p*2*q*1*r*2 ,

=— 2*D*2

F + 4

sin3Q F

1. l 2*a*(*z*1 — *z*2),*z*ﬃﬃﬃ1ﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃ2ﬃﬃp*z*ﬃﬃﬃ1ﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃ2ﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ

The cyclic variable is obtained by integrating the equation for J\_

\_ F\_ l

and the dependency of *z*1, *z*2 on time is described by the following differential equations

l(*z*2 — *z*1)*z*\_1 = p*Z*ﬃﬃﬃ(ﬃﬃ*z*ﬃﬃﬃ1ﬃﬃ)ﬃ, l(*z*2 — *z*1)*z*\_2 = p*Z*ﬃﬃﬃ(ﬃﬃ*z*ﬃﬃﬃ2ﬃﬃ)ﬃ, (12)

J = *D* — 2sin2Q , (9)

# Separation of variables for non-zero gyrostatic momentum

It is known that in the corresponding problem of the dynamics

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

where *Z*(*z*) is a polynomial of degree six defined as

*Z*(*z*) = h*a*2l4 — *k* — *z*2 2 i *z* *z* — l2 + 2l2*h* — *k* .

Remark 2. The differential equations for the auxiliary variables can obviously be written in the form of AbeleJacobi equations

of a rigid body in one axially symmetric force field [[29]](#_bookmark52) there

*dz*1

*dz*2 0

*z*1*dz*1

*z*2*dz*2

1. *dt*.

exists a separation of variables. One of the possible ways to obtain a separation was pointed out in [[30]](#_bookmark53). Complete real algebraic separation for the Goryachev case was given in [[16,17]](#_bookmark34). Here we give a real separation for the gyrostat in the double field with singular symmetry under condition [(7)](#_bookmark9).

*Z*(*z* ) — *Z*(*z* ) = , *Z*(*z* ) — *Z*(*z* ) = —l

Solutions can be expressed in hyperelliptic functions of time.

1

2

1

2

Remark 3. In the above notation we consider the radicals

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

*z* + *z* — l2 and ,*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃ to be always positive. Indeed, it

1

2

1

2

follows from the definition of *z*1, *z*2 that *z*1 + *z*2⩾l2 > 0. But the

2

It is convenient to pass from *H*, *K* to the “shifted” first

integrals

case *z*1 + *z*2 = l gives cosQ = 0. Such solutions are excluded

~ 1 l2

~ 1 2

according to Remark 1. Thus, these radicals (and obviously the

*H* = 4 *H* + 2 , *K* = 4 *K* + 2l *H*

constant ,2ﬃﬃﬃ*a*ﬃﬃﬃ > 0) do not generate multiple values in [(11)](#_bookmark11). On

We denote by *h* and *k* the corresponding integral constants.

In explicit form we have

the contrary, if we fix some rectangle of oscillation of the separation variables *z*1, *z*2 in such a way that the expressions

[(11)](#_bookmark11) are all real, then some of the radicals [(10)](#_bookmark8) periodically

~ \_ 2

cos2Q \_ 2 *a*

1. l2

change their signs. In formulas [(10)](#_bookmark8), [(11)](#_bookmark11), where for consis-

*H* = Q +

1

1 1 1

2

2 2 2

2*D* F

— 2cosFcos Q + 4sin2Q ,

tency we must choose p*Z*ﬃﬃﬃ(ﬃﬃ*z*ﬃﬃﬃﬃﬃ)ﬃ = *p q r*

and p*Z*ﬃﬃﬃ(ﬃﬃ*z*ﬃﬃﬃﬃﬃ)ﬃ = *p q r* ,

4 sin42Q 4

2sin22Q 2 2

8*a*sinFcosQsin3Q

we can make any formal substitution of the type *pi*/ — *pi* or

*K*~ = 4Q\_ + 4 F\_ +

2

4*D*

*D*

l

2 F\_ Q\_ +

*D* F\_ Q\_

*q* / — *q* , thus obtaining a completely equivalent form of the

l2

+2 sin2 Q + 2*a*cosFsin Q Q

2 \_ 2

2cos Q 2

*D*2

algebraic solution.

4

*i*

*i*

— 2*a*cosFsin Q

2 4

The expression for sinQ, cosQ follow straightforwardly

×F\_

2

l4

+ 4sin4Q

+

+ *a* sin Q.

from the definition of the auxiliary variables, according to

which

l2

Now we formulate the main result. Define the variables

*z*1, *z*2 as the roots of the quadratic equation

*z*1 + *z*2 = sin2Q . (13)

Taking into account that

2 2 2

2 \_ 2

sin22Q

\_ 2 2 l4

*z* sin Q — l *z* + 2l Q + 4*D*2 F — *k*sin Q + 2sin2Q = 0

\_

2l2 2 sin22Q 2 l4

and introduce the following two-valued ramified functions

*z*1*z*2 =

sin2Q Q +

4*D*2 F\_

— *k* —

2sin4Q

of *z*1 and *z*2

we find

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1

sin2Q

Q\_ +

sin22Q 2

2*k z*2 *z*2

= 4l2 . (14)

1 2

— +

Q\_ =

D*t*sinQ cosQ ,

D*t*sin F

F\_ 2 2 . (20)

=

cos F

2

To simplify the calculation let us introduce the variable *V*

2

4*D*2 F\_

by putting

Eliminating Q\_ , F\_ in [(19) and (20)](#_bookmark21) gives the system of equa- tions linear in *z*\_1, *z*\_2,

\_ *V*

(15)

2(*z*1 + *z*2)— l2 *p*1*q*1*r*2 — *p*2*q*2*r*1

Q = 2,*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃ.

1

2

l

*C*1*z*\_1 + *C*2*z*\_2 +

= 0,

Then eliminating Q from ([13), (14)](#_bookmark14), we obtain

l2 — 2 *z*2 + *z*2 p2ﬃﬃﬃ*k*ﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃ(ﬃ*z*ﬃﬃﬃ2ﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃ2ﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃﬃ*V*ﬃﬃﬃ2ﬃﬃ

1 2 1 2

\_

(16)

*z*\_1

Here

+ *z*\_2

*p*1*q*1*r*1 + *p*2*q*2*r*2 0.

l(*z*1 — *z*2)

+ =

F = l,*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃp*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ .

1 2 1 2

Substitution of the found expressions for sinQ, cosQ, Q\_ , F\_

1

*r*1

1

1

2

1

1

2

*p*1*q*1

*C* = *r*2 2*z*

* l2 (*z*

+ *z* )— 2*z* *z* + *z*

* l2 *p*2*q*2 ,

into the equations of the first integrals gives two equations to

2

*r*2

2

1

2

2

1

2

*p*2*q*2

1

2

determine *V* and cosF as the functions of *z* , *z* :

*C* = *r*1 2*z* — l2 (*z* + *z* )— 2*z* *z* + *z* — l2 *p*1*q*1 .

4*a*2l4*V*4 + 4*a*l2 l2 *z*2 + *z*2 — 2*k* + *k* — *z*2 *k* — *z*2 + l4 cosF}*V*2

1

2

1

2

1 2

1

2

1

2

cyclic constant and the gyrostatic momentum we obtain the

+ *k*2 + *a*2l4 + *z*2*z*2 + *a*l2 *z*2 + *z*2 cosF — *k* 2*a*l2cosF + *z*2 + *z*2 2

Solving this system we come to Equation [(12)](#_bookmark13).

Thus, in the reduced system under the condition [(7)](#_bookmark9) on the

= 0, *z*1 + *z*2 — l2 *V*2 + *a*l2cosF + *k* l2 — 2(*z*1 + *z*2)

*z*4 — *z*4

2 *z*3 — *z*3

separation of variables and express the trigonometric func-

+ 2*h*l (*z*1 + *z*2)+ *z* — *z* — l *z* — *z* = 0.

1 2

1 2

2

tions of non-cyclic angles as rational expressions in the basic

Whence,

1 2 1 2

radicals [(10)](#_bookmark8) with coefficients defined by one-valued functions

of the separation variables. For the cyclic coordinate (the angle

J) the generalized velocity can be presented in the same form.

*V*2 = h *z* + *z* — l2 (*z* — *z* )2 (*z* + *z* )i—1 2*p q r p q r* + 2*k*3

1

2

1

2

1

2

1 1 1 2 2 2

Indeed, from [(9)](#_bookmark12) in virtue of the formulas obtained for Q\_ , F\_ we

— *k*2 l2(4*h* — *z*1 — *z*2)+ 3 *z*2 + *z*2 + *k* 3 *z*4 + *z*4 " #

( *z*

get

1

2

1

2 6

2

\_

1

*p*2*q*2*r*1 — *p*1*q*1*r*2

+ 4*h*l *z*1 + *z*2 — 2l *z*1 + *z*2 — 2*a* l + *a* l (4*h* — *z*1 — *z*2)

2

2

2

2

3

3

2 4

J = —2l

*z*1 + *z*2)+

+ *z* — l2 (*z*

.

— *z* )

2 4 2 2 2 5 5

+ *a* l

*z*1 + *z*2

+ l

*z*1 + *z*2 — 2*h z*1 + *z*2

—

*z*1 + *z*2

, cosF

4 4 6 6 }

1 2 1 2

= h *z* + *z* — l2 (*z* — *z* )2 (*z* + *z* i—1 2*p q r p q r* + *a*2l4

1

2

1

2

1

2)

—

1 1 1 2 2 2

right-hand side as a function of time.

After integrating differential Equation [(12)](#_bookmark13), we find the

— *k* — *z*2 *k* — *z*2 2*k* — l2(4*h* — *z*1 — *z*2)— *z*2 + *z*2 }.

1

2

1

2

To conclude this section let us make one interesting

observation. Having in mind the topological analysis of the

Let us calculate qﬃ1ﬃﬃ(ﬃﬃ1ﬃﬃﬃ+ﬃﬃﬃﬃﬃcﬃﬃﬃoﬃﬃsﬃﬃﬃﬃﬃﬃ)ﬃﬃ, q1ﬃﬃﬃ(ﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃcﬃﬃﬃoﬃﬃsﬃﬃﬃﬃﬃﬃ)ﬃﬃ and ,*V*ﬃﬃﬃﬃ2ﬃﬃ. It is

(17)

2

F

2

F

the first integrals. Such set is the part of the discriminant set D

problem, one has to construct the so-called bifurcation set of

possible to avoid double radicals in the resulting formulas by obvious identity

*k*' = *k* — l2 turns into the set bearing the bifurcation diagram

constructed in [[31]](#_bookmark54) for another integrable case of Chaplygin

of the polynomial *Z*(*z*) in the right-hand side of separated Equation [(12)](#_bookmark13) corresponding to real motions. It is easy to show

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ,ﬃﬃﬃﬃﬃﬃ

*B* =

*A* + 2

1. ,ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

,ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

,ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

that the set D after the normalization *a* = 1 and the shift

if *C* is a whole expression (see Chaplygin's techniques in [[20]](#_bookmark37)). Applying this to [(17)](#_bookmark18), we obtain the sine and cosine of the half-

,ﬃ2ﬃﬃ

*A* + *C* +

*A* — *C* , *C* =

*A*2 — 4*B*,

angle in the form [(11)](#_bookmark11), while for *V* we get

also found in [[20]](#_bookmark37) and defined by the Hamiltonian

*H* = 1 u2 + u2 + 1u2 + 1 a2 — a2 + *c*a

(21)

*V* = *p*1*q*1*r*1 + *p*2*q*2*r*2

1

2

1

2

1

2

(18)

2 1 2 2 3 2 1 2 1

2 2 2

(*z* — *z* ),*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃp*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ.

with a = 1 and 2*c* = l . It was shown in [[32]](#_bookmark55) that the bifur-

cation diagram of the axisymmetric problem with the

Note that at this point we can choose arbitrary formal signs

at these values since only cos2Q, *V*2, and cosF are uniquely defined as the functions of *zi*, *pi*, *qi*, *ri*. But the choice of signs now made determine all choices of signs in the consequent expressions (see Remark 3 above).

Substituting [(18)](#_bookmark19) for *V* in [(15)](#_bookmark16), [(16)](#_bookmark17) we find Q\_ , F\_ in terms of

*z*1, *z*2,

\_ *p*1*q*1*r*1 + *p*2*q*2*r*2

Hamiltonian [(21)](#_bookmark20) can be represented as the discriminant set of some polynomial. Still for this Chaplygin's case the complete algebraic separation of variables is not known. It seems likely that such a separation can be obtained based on the obvious analogy between the two problems.

# The case of zero gyrostatic momentum

Q = 2(*z*2 — *z*2 )p*z*ﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃ*z*ﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃlﬃﬃ2ﬃﬃ ,

1

2

1

2

(19)

( 1 + 2)— 1 1 2 — 2 2 1

1. *z z* l2 *p q r p q r*

\_

F = l(*z*2 — *z*2) *z* + *z* — l2 .

1

2

1

2

The above separation, obviously, takes place only for the case

ls0. In particular, there is no analog for this solution for the

two-fields Kowalevski top in the *S*1-symmetric case. Never-

On the other hand, considering the operator

D*t* = *z*\_1v*z*1 + *z*\_2v*z*2 of the complete time derivative of a function of *z*1, *z*2, we calculate from Equations [(11)](#_bookmark11)

theless, when l = 0 (and then also *g* = 0 in virtue of the con-

dition [(7)](#_bookmark9)) the Routhian [(8)](#_bookmark10) and the first integrals do not have singular terms on the (Q, F)-sphere. It is shown in [[14]](#_bookmark32) (see

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 2 ( 201 5 ) 236](http://dx.doi.org/10.1016/j.ejbas.2015.05.001) e[242](http://dx.doi.org/10.1016/j.ejbas.2015.05.001) 241

also [[27]](#_bookmark51)), that such problem has an analogy with Chaplygin's solution of the problem of motion of a body in a liquid under

the Kowalevski condition *I*1 = *I*2 = 2*I*3 [[20]](#_bookmark37). We now transfer the results of Chaplygin's classical work [[20]](#_bookmark37) to the case under

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

The explicit expressions in the Jacobi functions depend on the chosen intervals of oscillation of the variables t1, t2. The possible intervals are defined by the condition that the values

[(24) and (25)](#_bookmark26) should be real.

consideration of two fields.

In [(26)](#_bookmark29), we must choose

p*U*ﬃﬃﬃﬃ1ﬃﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ)ﬃ = *P*1*Q*1*R*1 and

Denote

2 2

U = u1 + u2, 8 =

,ﬃﬃﬃ

*U*2(t2) = *P*2*Q*2*R*2. As in the previous case (see Remark 3), in formulas [(24), (25), and (26)](#_bookmark26), we can make any formal substi-

The latter notation is possible since for l = 0 the integral *K* in [(4)](#_bookmark5) is the sum of two squares and, consequently, the integral *K*~ is non-negative. Let us introduce the variables t1, t2 similar to the separated variables of Chaplygin putting

*k*.

tution of the type *Pi*/ — *Pi* or *Qi*/ — *Qi*, thus obtaining a

completely equivalent form of the algebraic solution. Our choice was made to have the combinations of the basic radi- cals in [(24), (25)](#_bookmark26) similar to those in [(10), (11)](#_bookmark8). This choice gave the minus sign in the first Equation [(26)](#_bookmark29) and at the first integral in [(27)](#_bookmark30).

1

t = (U + 28), t

1

= (U — 28). (22)

PROOF. From [(22)](#_bookmark22),

1 2*a*sin2Q

Let

2 2*a*sin2Q

U = 28 t1 + t2, sin2Q = 28 . (28)

t1 — t2 *a*(t1 — t2)

*P*1 = pﬃtﬃﬃﬃﬃ1ﬃﬃﬃﬃﬃﬃﬃ+ﬃﬃﬃﬃﬃﬃﬃﬃﬃ1ﬃﬃﬃﬃﬃ, *Q*1 = pﬃtﬃﬃﬃﬃ1ﬃﬃﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃﬃﬃﬃﬃ1ﬃﬃﬃﬃﬃ, *R*1 = pﬃ*a*ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ1ﬃﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ8ﬃﬃﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃﬃﬃﬃﬃ2ﬃﬃﬃﬃﬃﬃ*h*ﬃﬃﬃﬃﬃ,

*P*2 =

1 + t2, *Q*2 =

1 — t2, *R*2 =

2*h* — 8 — *a*t2.

(23)

The second equation defines the expressions for

sinQ, cosQ. Calculate the value U in the angle variables,

Theorem 2. On common levels of the integrals

U = 4Q\_ +

sin22Q

2F\_

2

. (29)

2

*H*~ = *h*, *K*~ = 82

(3 + cos2Q)

the angles Q, F are expressed in terms of the variables t1, t2 by the formulas

As before, let us introduce an auxiliary variable *V*,

\_ ,ﬃ8ﬃﬃ*V*

(30)

Q = p2ﬃﬃﬃ*a*ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃ)ﬃﬃ .

1

2

sinQ =

,2ﬃﬃﬃ8ﬃﬃ , cosQ = pﬃﬃ*a*ﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃ ,

Then from [(28), (29)](#_bookmark23) we get

p*a*ﬃﬃﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃ pﬃﬃ*a*ﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃ

\_ 2[*a*(t1 — t2)— 8]p*a*ﬃﬃﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ+ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ*V*ﬃﬃﬃﬃ2ﬃﬃ

sin F = *P*2*Q*1*R*1 + *P*1*Q*2*R*2 ,

(24)

(31)

F =— pﬃﬃ*a*ﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃpﬃﬃ*a*ﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃ .

2 p2ﬃﬃﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃp*a*ﬃﬃﬃ(ﬃﬃtﬃﬃ1ﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃ2ﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃ

1 2 1 2

cos F = *P*1*Q*2*R*1 — *P*2*Q*1*R*2

The equations of the first integrals give

2 p2ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃp*a*ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃ ,

1

2

1

2

[*a*(t1 — t2) — 28] *V*

2

+ *a*cosF

— *a*[*a*(t1 — t2) — 8](t1 + t2)

The corresponding generalized velocities have the form

,ﬃﬃﬃ

8

+ 2*ah*(t1 — t2)= 0,

4*V*4 + 4*a*[(1 + t1t2)cosF — (t1 + t2)]*V*2

(32)

Q\_ = ,ﬃﬃﬃ

(*P Q R*

+ *P Q R* ), 2 2

2(t

1

2

1

2

(25)

— t )p*a*ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃ

1 1 1

2 2 2

+ *a* [(1 + t1t2)— (t1 + t2)cosF]

= 0.

F\_ = 2[*a*(t1 — t2)— 8] (*P Q R*

— *P Q R* ),

Hence we write

(t1 — t2)[*a*(t1 — t2)— 28]

1 1 2

2 2 1

—1 —1

and the dependency of t , t

on time is described by the

cosF = (t1 — t2)

[*a*(t1 — t2)— 28]

{[*a*(t1 + t2)— 4*h*](1 — t1t2)

1 2 —2*P*1*Q*1*R*1*P*2*Q*2*R*2},

1

2

following differential equations

*V*2 = (t1 — t2)—1[*a*(t1 — t2) — 28]—1*a a* t1 t2 — 1

— t2 1 — t2

1

2

1

2

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

*U*1(t1),

*U*2(t2), (26)

t\_ 1 =—

t\_ 2 =

—2*h* t2 + t2 — 2 — 8 t2 — t2 + 2*P*1*Q*1*R*1*P*2*Q*2*R*2}.

(33)

where

*U*1(t1)= t2 — 1 (*a*t1 — 8 — 2*h*), *U*2(t2)= 1 — t2 (2*h* — 8 — *a*t2).

1

2

Having chosen at this moment one of the possible solu- tions of system [(32)](#_bookmark28), we thus have determined the formal

choice of the signs of the two-valued radicals [(23)](#_bookmark24) in the

The expression for J is obtained by integrating the relation

following expressions of the phase variables. The radicals

themselves can have any sign on a trajectory (moreover, some of them change the sign periodically) but the formal expres-

F\_

J\_

= *D* = 2[*a*(t1

*a*

— t )— 28] (*P*1*Q*1*R*2 — *P*2*Q*2*R*1),

2

sions must be consistent. The first equation in [(33)](#_bookmark24) leads to the formulas for the sine and cosine of the half-angle as in [(24)](#_bookmark26).

Remark 4. The general solution of Equation [(26)](#_bookmark29) is obtained

From the second one we find

by inverting the following elliptic integrals

,ﬃﬃﬃ

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

Zt1

Zt2

*a P Q R P Q R*

,tﬃﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃp*a*ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃﬃﬃ,

*V* = ( 1 1 1 + 2 2 2)

1

2

1

2

8

2

2

1

1

1

2

*a*(t1

+ t2)— *V*2

— p(ﬃﬃtﬃﬃﬃ2ﬃﬃﬃ—ﬃﬃﬃﬃﬃ1ﬃﬃ)ﬃﬃ(ﬃﬃ*a*ﬃﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃ8ﬃﬃﬃ—ﬃﬃﬃﬃﬃ2ﬃﬃﬃ*h*ﬃﬃﬃ)ﬃ = =

*d*t1

*t*

1

1

1

2

1

2

p(ﬃﬃ1ﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃ2ﬃﬃ)ﬃﬃ(ﬃﬃ2ﬃﬃﬃ*h*ﬃﬃﬃ—ﬃﬃﬃﬃﬃ8ﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃ*a*ﬃﬃtﬃﬃﬃﬃﬃ)ﬃ .

,ﬃ*a*ﬃﬃ(*P Q R* — *P Q R* )

(27)

*d*t2

2

2

= ,tﬃﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃp*a*ﬃﬃﬃ(ﬃﬃtﬃﬃﬃﬃﬃ—ﬃﬃﬃﬃﬃtﬃﬃﬃﬃﬃ)ﬃﬃﬃ—ﬃﬃﬃﬃ2ﬃﬃﬃ8ﬃﬃ .

242 [e gypti an j o ur nal o f b a sic and a p p l i ed sci e n c e s 2 ( 201 5 ) 236](http://dx.doi.org/10.1016/j.ejbas.2015.05.001) e[242](http://dx.doi.org/10.1016/j.ejbas.2015.05.001)

Whence, according to [(30) and (31)](#_bookmark25), we obtain the expres- sions [(25)](#_bookmark27) for the generalized velocities.

Substituting the values [(24) and (25)](#_bookmark26) in [(20)](#_bookmark15), we come to separated Equation [(26)](#_bookmark29). Let us emphasize that this separation is elliptic with polynomial of degree 3 in the radicals, therefore it is complete in the sense that both Equation [(26)](#_bookmark29) are inte- grated independently of each other. Of course, the solutions can be easily written out in Jacobi functions.

Finally, we find J\_ from [(9)](#_bookmark12) by putting l = 0.

This completes the expressions for Q, Q\_ , F, F\_ , J\_ as elliptic functions of time.

1. [Yehia YM. New integrable cases in the dynamics of rigid](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref13) [bodies. Mech Res Commun 1986;13(3):169](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref13)e[72](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref13).
2. [Yehia HM. Equivalent problems in rigid body dynamics-II.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref14) [Celest Mech 1988;41:289](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref14)e[95](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref14).
3. [Yehia HM. Geometric transformations and new integrable](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref15) [problems of rigid body dynamics. J Phys A Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref15)&[Gen](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref15) [2000;33:4393](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref15)e[9](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref15).
4. [Ryabov PE. Explicit integration and topology of](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref16) [D.N.Goryachev case. Dokl Math 2011;84(1):502](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref16)e[5](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref16).
5. [Ryabov PE. The phase topology of a special case of Goryachev](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref17) [integrability in rigid body dynamics. Sb Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref17) [2014;205(7):1024](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref17)e[44](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref17).
6. [Kharlamov MP. Separation of variables in the generalized 4th](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref18) [Appelrot class. Reg](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref18)&[Chaotic Dyn 2007;12(3):267](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref18)e[80](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref18).

[19] [Kharlamov MP. Separation of variables in the generalized 4th](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref19)

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## references

[Appelrot class. II. Real solutions. Reg](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref19)&[Chaotic Dyn](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref19) [2009;14(6):621](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref19)e[34](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref19).

1. [Chaplygin SA. A new partial solution of the problem of](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref20) [motion of a rigid body in fluid. Tr Otd Fiz Nauk Obsh Liub Est](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref20) [1902;11(2):7](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref20)e[10. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref20).
2. [Kharlamov MP. Topological analysis and Boolean functions.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref21)
   1. [Methods and application to classical systems. Nonlin Dyn](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref21) [2010;6(4):769](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref21)e[805. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref21).

[22] [Kharlamov MP. Topological analysis and Boolean functions.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref22)

* 1. [Application to new algebraic solutions. Nonlin Dyn](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref22)

1. [Euler L. Du mouvemente rotation des corps solides autour](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1) [d’un axe variable. M](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1)e´[moires l’Acad](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1)e´[mie Sci Berl](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1) [1758;14:154](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1)e[93. French](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref1).
2. [Lagrange JL. M](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref2)e´[canique Analitique. Paris: La Veuve DeSaint;](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref2) [1788. French](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref2).
3. [Lottner C. Reduction der Bewegung eines schweren, um](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3) [einen festen Punct rotirenden Revolutionsk](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3)o€[rpers auf die](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3) [elliptischen Transcendenten. Crelle J 1855;50:111](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3)e[25.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3) [German](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref3).
4. [Klein F, Sommerfeld A. U¨ ber die Theorie des Kreisels. Leipzig:](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref4)

[Teubner; 1897](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref4)e[1910. German](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref4).

1. [Wittaker ET. A treatise on analytical dynamics of particles](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref5) [and rigid bodies. Cambridge Univ. Press; 1904](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref5).
2. [Kowalevski S. Sur le probl](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref6)e`[me de la rotation d’un corps](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref6)

[solide autour d’un point fixe. Acta Math 1889;2:177](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref6)e[232.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref6) [French](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref6).

1. [K](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7)o€[tter F. Sur le cas trait](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7)e´ [par M-me Kowalevski de rotation](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7)

[d’un corps solide pesant autor d’un point fixe. Acta Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7) [1893;17:209](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7)e[63. French](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref7).

1. [Borisov AV, Mamaev IS. Rigid body dynamics. Izhevsk:RCD](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref8) [Publ; 2001. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref8).
2. [Zhukovsky NE. On the motion of a rigid body with holes filled](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref9) [with a homogeneous fluid. In: Collected Works,](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref9) [Gostekhizdat:Moscow, vol. 1; 1949. p. 31](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref9)e[152. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref9).
3. [Volterra V. Sur la theorie des variations des latitudes. Acta](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref10) [Math 1899;22:201](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref10)e[358. French](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref10).
4. [Wittenburg J. Dynamics of systems of rigid bodies. Stuttgart:](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref11) [Teubner; 1977](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref11).
5. [Gorr GV, Maznev AV. Dynamics of a gyrostat having a fixed](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref12) [point. Donetsk: IAMM NASU; 2010. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref12).

[2011;7(1):25](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref22)e[51. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref22).

1. [Kharlamov MP. Phase topology of one system with separated](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref23) [variables and singularities of the symplectic structure. J](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref23) [Geom Phys 2015;87:248](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref23)e[65](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref23).
2. [Nikolaenko SS. A topological classification of the Chaplygin](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref24) [systems in the dynamics of a rigid body in a fluid. Sb Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref24) [2014;205(1):224](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref24)e[68](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref24).
3. [Kharlamov MP. Bifurcation diagrams of the Kowalevski top](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref25) [in two constant fields. Reg](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref25)&[Chaotic Dyn 2005;10(4):381](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref25)e[98](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref25).
4. [Kharlamov MP. Bifurcation diagrams and critical subsystems](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref26) [of the Kowalevski gyrostat in two constant fields. Hiroshima](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref26) [Math J 2009;39(3):327](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref26)e[50](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref26).
5. [Borisov AV, Mamaev IS. Nonlinear Poisson brackets and](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref27) [isomorphisms in dynamics. Reg](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref27)&[Chaotic Dyn 1997;2(3/](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref27) [4):72](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref27)e[89](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref27).
6. [Kharlamova II, Savushkin AY. Bifurcation diagrams](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref28) [involving the linear integral of Yehia. J Phys A Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref28)&[Theor](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref28) [2010;43(10520):301](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref28)e[11](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref28).
7. [Goryachev DN. New cases of integrability of Euler's](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref29) [dynamical equations, vol. 3. Warsaw Univ Izvestiya; 1916.](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref29)

[p. 1](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref29)e[13. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref29).

1. [Tsiganov AV. On the generalized Chaplygin system. J Math](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref30) [Sci 2010;168(8):901](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref30)e[11](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref30).
2. [Ryabov PE. Bifurcation sets in an integrable problem on](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref31) [motion of a rigid body in fluid. Reg](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref31)&[Chaotic Dyn](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref31) [1999;4(4):59](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref31)e[76](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref31).
3. [Ryabov PE. The phase topology of the Chaplygin problem of](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref32) [the motion of a rigid body in a liquid. Mekh Tverd Tela](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref32) [2000;30:140](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref32)e[50. Russian](http://refhub.elsevier.com/S2314-808X(15)00025-1/sref32).