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Singular Coverings and Non-Uniform Notions of Closed Set Computability

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**Abstract**

The empty set of course contains no computable point. On the other hand, surprising results due to Zaslavski˘ı, Tse˘ıtin, Kreisel, and Lacombe have asserted the existence of *non-*empty co-r.e. closed sets devoid of computable points: sets which are even ‘large’ in the sense of positive Lebesgue measure.

This leads us to investigate for various classes of computable real subsets whether they always contain a (not necessarily effectively findable) computable point.

*Keywords:* co-r.e. closed sets, non-uniform computability, connected component

# Introduction

A discrete set *A*, for example a subset of *{*0*,* 1*}∗* or N, is naturally called r.e. (i.e. semi-decidable) if a Turing machine can enumerate the members of (equivalently: terminate exactly for inputs from) *A*. The corresponding notions for open subsets of reals [[12](#_bookmark56),[13](#_bookmark57),[21](#_bookmark62)] amount to the following

**Definition 1.1** *Fix a dimension d ∈* N*. An open subset U ⊆* R*d is called* r.e. *if and only if a Turing machine can enumerate rational centers* ***q****n ∈* Q*d and radii rn ∈* Q *of open Euclidean balls B◦*(***q****, r*)= ***x*** *∈* R*d* : ***x*** *−* ***q *** *< r exhausting U.*

}

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*A real vector* ***x*** *∈* R*d is (Cauchy– or ρd–)*computable *if and only if a Turing machine can generate a sequence* ***q****n ∈* Q*d of rational approximations converging to* ***x*** fast *in the sense that ****x*** *−* ***q****n * *≤* 2*−n.*

Notice that an open real subset is r.e. if and only if membership “***x*** *∈ U* ” is semi-decidable with respect to ***x*** given by fast convergent rational approximations; see for instance [[24](#_bookmark68), Lemma 4.1c].

* 1. *Singular Coverings*

A surprising result due to E. Specker implies that the (countable) set Rc of com- putable reals is contained in an r.e. open *proper* subset *U* of R: In his work [[19](#_bookmark63)] he

constructs a computable function *f* : [0*,* 1] *→* [0*,*  1 ] attaining its maximum  1 in

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no computable point; hence *U* := (*−∞,* 0) *∪ f−*1[(*−*1*,*  1 )] *∪* (1*, ∞*) has the claimed properties, see for example [[21](#_bookmark62), Theorem 6.2.4.1]. This was strengthened in [[23](#_bookmark67),[9](#_bookmark53)] to the following

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**Fact 1.2** *For any ϵ >* 0 *there exists an r.e. open set Uє ⊆* R *of Lebesgue measure*

*λ*(*Uє*) *< ϵ containing all computable real numbers.*

**Proof.** See [[11](#_bookmark55), Section 8.1] or [[1](#_bookmark45), Section IV.6] or [[21](#_bookmark62), Theorem 4.2.8].

The significance of this improvement thus lies in the constructed *Uє* intuitively being very ‘small’: it misses many non-computable points. On the other hand it is folklore that a certain smallness is also necessary: Every r.e. open *U* Ç R covering Rc *must* miss uncountably many non-computable points. Put differently, an at most countable non-empty closed real subset must, if its complement is r.e., contain a computable point; see Observation [2.4](#_bookmark8) below.

This leads the present work to study further natural effective classes of closed Euclidean sets with respect to the question whether they contain a computable point. But let us start with reminding of the notion of

# Computability of Closed Subsets

Decidability of a discrete set *A ⊆* N amounts to computability of its characteristic function

**1***A*(*x*) = 1 if *x ∈ A,* **1***A*(*x*) = 0 if *x /∈ A .*

Literal translation to the real number setting fails of course due to the continu- ity requirement; instead, the characteristic function is replaced by the continuous distance function

dist*A*(*x*) = inf *x − a * : *a ∈ A*}

which gives rise to the following natural notions [[3](#_bookmark46)], [[21](#_bookmark62), Corollary 5.1.8]:

**Definition 2.1** *Fix a dimension d ∈* N*. A closed subset A ⊆* R*d is called*

* r.e. *if and only if* dist*A* : R*d →* R *is upper computable;*
* co-r.e. *if and only if* dist*A* : R*d →* R *is lower computable;*
* recursive *if and only if* dist*A* : R*d →* R *is computable.*

Lower computing *f* : R*d →* R amounts to the output, given a sequence (***q****n*) *∈* Q*d* with ***x****−****q****n * *≤* 2*−n*, of a sequence (*pm*) *∈* Q with *f* (***x***)= sup*m pm*. This intuitively means approximating *f* from below and is also known as (*ρd, ρ<*)–computability with respect to standard real representations *ρ* and *ρ<*; confer [[21](#_bookmark62), Section 4.1] or [[22](#_bookmark64)]. A closed set is co-r.e. if and only if its complement (an open set) is r.e. in the sense of Definition [1.1](#_bookmark1) [[21](#_bookmark62), Section 5.1]. Several other reasonable notions of closed set computability have turned out as equivalent to one of the above; see [[3](#_bookmark46)] or [[21](#_bookmark62), Section 5.1]: recursivity for instance is equivalent to *Turing location* [[5](#_bookmark49)] as well as to being simultaneously r.e. and co-r.e. This all has long confirmed Definition [2.1](#_bookmark5) as natural indeed.

* 1. *Non-Empty Co-R.E. Closed Sets without Computable Points*

Like in the discrete case, r.e. and co-r.e. are logically independent also for closed real sets:

**Example 2.2** For *x* := Σ*n∈H* 2*−n* (where *H ⊆* N denotes the Halting Problem), the compact interval *I<* := [0*, x*] *⊆* R is r.e. but not co-r.e.; and *I>* := [*x,* 1] is co-r.e. but not r.e.

Notice that both intervals have continuum cardinality and include lots of com- putable points. As a matter of fact, it is a well-known

**Fact 2.3** *Let A ⊆* R*d be r.e. closed and non-empty. Then A contains a computable point* [[21](#_bookmark62), Exercise 5.1.13b]*.*

More precisely, closed *∅ /*= *A ⊆* R*d* is r.e. if and only if *A* = *{****x***1*,...,* ***x****n,.. .}* for some computable sequence (***x****n*)*n* of real vectors [[21](#_bookmark62), Lemma 5.1.10].

A witness of (one direction of) logical independence stronger than *I>* is thus a non-empty co-r.e. closed set *A* devoid of computable points: *A ⊆* [0*,* 1] *\* Rc. For example every singular covering *Uє* with *ϵ <* 1 from Section [1.1](#_bookmark2) due to [[23](#_bookmark67),[9](#_bookmark53)] gives rise to an instance *Aє* := [0*,* 1] *\ Uє* even of positive Lebesgue measure *λ*(*A*) *>* 1 *− ϵ*, and thus of continuum cardinality. Conversely, it holds

**Observation 2.4** *Every non-empty co-r.e. closed set of cardinality strictly* less

*than that of the continuum* does *contain computable points.*

Notice that this claim also covers putative cardinalities between *ℵ*0 and 2*ℵ*0 = c i.e. does not rely on the Continuum Hypothesis.

In a finite set, every point is isolated; in this case the claim thus follows from the well-known

**Fact 2.5** *a) Let A ⊆* R*d be co-r.e. closed and suppose there exist* ***a****,* ***b*** *∈* Q*d such*

*that A ∩* [***a****,* ***b***]= *{****x****} (where* [***a****,* ***b***] := *d* [*ai, bi*]*). Then,* ***x*** *is computable.*

*i*=1

*b) A perfect subset A ⊆ X (of X* = R*d or of X* = *{*0*,* 1*}ω), i.e. one which*

*coincides with the collection A' of its limit points,*

*A'* := ***x*** *∈ X* *∀n∃****a*** *∈ A* : 0 *< |****a*** *−* ***x****| <* 1*/n*} *,*

*is either empty or of continuum cardinality.*

See for instance [[3](#_bookmark46), Proposition 3.6] and [[8](#_bookmark52), Corollary 6.3].

**Proof (of Observation** [**2.4**](#_bookmark8)**).** Suppose that *A* has cardinality strictly less than that of the continuum. Then *A /*= *A'* by Fact [2.5](#_bookmark9)b). On the other hand, *A* contains *A'* because it is closed. Hence the difference *A\ A' /*= *∅* holds and consists of isolated points which are computable by Fact [2.5](#_bookmark9)a).

So every non-empty co-r.e. closed real set *A ⊆* [0*,* 1] devoid of computable points must necessarily be of continuum cardinality. On the other hand, Fact [1.2](#_bookmark3) yields such sets with positive Lebesgue measure *λ*(*A*) *>* 0. In view of (and in-between) the strict [5](#_bookmark11) chain of implications

nonempty interior a*⇒* positive measure a*⇒* continuum cardinality we make the following [6](#_bookmark12)

**Remark 2.6** *There exists a non-empty co-r.e. closed real subset of measure zero*

*without computable points.*

This is different from [[11](#_bookmark55), Section 8.1] which considers

* coverings of (0*,* 1) having measure *strictly* less than 1
* by disjoint enumerable ‘segments’, that is *closed* intervals [*an, bn*],
* or by enumerable open intervals (*an, bn*) as in Definition [1.1](#_bookmark1), however in terms of the *accumulated* length Σ*n*(*bn − an*), that is counting interval overlaps doubly

[[11](#_bookmark55), Theorem 8.5].

**Proof of (Remark** [**2.6**](#_bookmark10)**).** Take a subset *A* of Cantor space with these properties and consider its image *A*˜ under the canonical embedding

*{*0*,* 1*}ω e* (*bn*) *'→* Σ *bn*2*−n ∈* [0*,* 1] *.*

*n*

Notice that this mapping, restricted to *A*, is indeed injective because only dyadic rationals have a non-unique binary expansion; and in fact two of them, both of which are decidable. Therefore

* *A*˜ has continuum cardinality but, being contained in Cantor’s Middle Third set, has measure zero.

5 Consider for instance the irrational numbers R *\* Q and Cantor’s uncountable Middle Third set, respec- tively.

6 We are grateful to a careful anonymous referee for indicating this simple solution to a question raised in an earlier version of this work.

* + The enumeration of open balls in *{*0*,* 1*}ω* exhausting *A*’s complement translates to one exhausting [0*,* 1] *\ A*˜.
  + Suppose *x ∈ A*˜ were computable. Then *x* has decidable binary expansion [[21](#_bookmark62),

Theorem 4.1.13.2], contradicting that all elements of *A*˜ arise from uncom-

putable binary sequences (*bn*) *∈ A*.

* 1. *Computability on Classes of Closed Sets of Fixed Cardinality*

Observation [2.4](#_bookmark8) and Fact [2.5](#_bookmark9)a) are non-uniform claims: they assert a computable point in *A* to *exist* but not that it can be ‘found’ effectively. Nevertheless, a uniform version of Fact [2.5](#_bookmark9)a) does hold under the additional hypothesis that ***a*** and ***b*** are known; compare [[21](#_bookmark62), Exercise 5.2.3] reported as Lemma [2.8](#_bookmark14)a) below. The present section investigates whether and to what extend this result can be generalized to- wards Observation [2.4](#_bookmark8) and, to this end, considers the following representations for (classes of) closed real sets of fixed cardinality:

**Definition 2.7** *For d ∈* N *and closed A ⊆* R*d,*

* + - *ψd encodes A as a* [*ρd →ρ>*]*–name of* dist*A;*

*<*

* + - *ψd encodes A as a* [*ρd →ρ<*]*–name of* dist*A*

*>*

*in the sense of* [[22](#_bookmark64)]*.*

*Write Ad* := *{A ⊆* [0*,* 1]*d closed* : Card(*A*)= *N} for the hyperspace of compact*

*N*

*sets having cardinality exactly N, where N ≤* c *denotes a cardinal number. Equip*

*Ad d Ad*

*d Ad*

*N with restrictions ψ<| N and ψ>| N of the above representations.*

*If N ≤ ℵ*0*, we furthermore can encode A ⊆* [0*,* 1]*d of cardinality N (closed or not) by the join of the ρd–names of the N elements constituting A, listed in arbitrary order* [7](#_bookmark17) *. This representation shall be denoted as* (*ρd*)*∼N .*

Let us first handle finite cardinalities:

**Lemma 2.8** *Fix d ∈* N*.*

*d*

1. *ψ<* 1 *>* 1
2. *For* 2 *≤ N ∈* N*, it holds ψ<* *N * *> N*

*d Ad*

*≡* (*ρd*)*∼*1 *≡ ψd* *A*

*d Ad*

*≡* (*ρd*)*∼N ψd* *A*

*d*

1. *For N ∈* N*, A ∈ Ad is ψd–computable if and only if it is ψd–computable.*

*N < >*

**Proof** omitted.

In particular, [[21](#_bookmark62), Example 5.1.12.1] generalizes to arbitrary finite sets:

**Corollary 2.9** *A ﬁnite subset A of* R*d is r.e. if and only if A is co-r.e. if and only if every point in A is computable.*

The case of countably infinite closed sets:

**Lemma 2.10** *a) In the deﬁnition of* (*ρd*)*∼ℵ*0 *, it does not matter whether each element* ***x*** *of A is required to occur exactly once or at least once.*

7 see also Lemma [2.10](#_bookmark16)a)

1. *It holds* (*ρd*)*∼ℵ*0 *Ad ψ ℵ .*

*ℵ*0

*d Ad*

*<* 0

1. *There exists a countably inﬁnite r.e. closed set A ⊆* [0*,* 1] *which is neither*

*ρ∼ℵ*0 *–computable nor co-r.e.*

1. *There is a countably inﬁnite co-r.e. but not r.e. closed set B ⊆* [0*,* 1]*.*

**Proof** omitted.

# Closed Sets and Naively Computable Points

A notion of real computability weaker than that of Definition [1.1](#_bookmark1) is given in the following

**Definition 3.1** *A real vector* ***x*** *∈* R*d is* naively computable *(also called* recursively approximable*) if a Turing machine can generate a sequence* ***q****n ∈* Q*d with* ***x*** = lim*n* ***q****n (i.e. converging but not necessarily fast).*

A real point is naively computable if and only if it is Cauchy–computable *relative*

to the Halting oracle *H* = *∅'*, see [[7](#_bookmark51), Theorem 9] or [[26](#_bookmark70)].

Section [2.1](#_bookmark6) asked whether *certain* non-empty co-r.e. closed sets contain a Cauchy–computable element. Regarding naively computable elements, it holds

**Proposition** [8](#_bookmark21) **3.2** Every *non-empty co-r.e. closed set A ⊆* R*d contains a naively computable point* ***x*** *∈ A.*

W.l.o.g. *A* may be presumed compact by proceeding to *A ∩* [***u****,* ***v***] for appropriate ***u****,* ***v*** *∈* Q*d* [[21](#_bookmark62), Theorem 5.1.13.2]. In 1D one can then explicitly choose *x* = max *A* according to [[21](#_bookmark62), Lemma 5.2.6.2]. For higher dimensions we take a more implicit approach and apply Lemma [3.4](#_bookmark20)a) to the following relativization of Fact [2.3](#_bookmark7):

**Scholium** [9](#_bookmark22) **3.3** *Let non-empty A ⊆* R*d be r.e. closed* relative *to O for some oracle*

*O. Then A contains a point computable* relative *to O.*

**Lemma 3.4** *Fix closed A ⊆* R*d.*

1. *If A is co-r.e., then it is also r.e. relative to ∅'.*
2. *If A is r.e., then it is also co-r.e. relative to ∅'.*

These claims may follow from [[2](#_bookmark47),[6](#_bookmark50)]. However for purposes of self-containment we choose to give a direct

**Proof.** Recall [[21](#_bookmark62), Definition 5.1.1] that a *ψd*–name of *A* is an enumeration of all closed rational balls *B* disjoint from *A*; whereas a *ψd*–name enumerates all open

*>*

*<*

8 A simple reduction to the counterpart of this claim for Baire space [[4](#_bookmark48), Theorem 2.6(c)] does not seem feasible because, according to [[21](#_bookmark62), Theorem 4.1.15.1], there exists no *total* (compact or not) representation equivalent to *ρ*.

9 A scholium is “*a note amplifying a proof or course of reasoning, as in mathematics* ” [[17](#_bookmark61)]

rational balls *B◦* intersecting *A*. Observe that

*B◦ ∩ A /*= *∅ ⇔ ∃n ∈* N : *B−*1*/n ∩ A /*= *∅*

*B ∩ A* = *∅ ⇔ ∃n ∈* N : *B◦*

(1)

*∩ A* = *∅*

+1*/n*

where *B±є* means enlarging/shrinking *B* by *ϵ* such that *B◦* = *n B*+1*/n* and *B* =

*n B*

*◦*

*−*1*/n*

. Formally in 1D e.g. (*u, v*)*−є* := (*u*+*ϵ, v−ϵ*) in case *v−u >* 2*ϵ*, (*u, v*)*−є* :=

*{}* otherwise. Under the respective hypothesis of a) and b), the corresponding right hand side of Equation ([1](#_bookmark23)) is obviously decidable relative to *∅'*.

A simpler argument might try to exploit [[7](#_bookmark51), Theorem 9] that every *ρ<*–computable single real *y* is, relative to *∅'*, *ρ>*–computable; and conclude by uniformity that (Definition [2.1](#_bookmark5)) every (*ρ → ρ<*)–computable function *f* : *x '→ f* (*x*) = *y* is, relative to *∅'*, (*ρ → ρ>*)–computable. This conclusion however is wrong in general because even a relatively (*ρ→ρ>*)–computable *f* must be upper semi-continuous whereas a (*ρ→ρ<*)–computable one may be merely lower semi-continuous.

* 1. *(In-)Effective Compactness*

By virtue of the Heine–Borel and Bolzano–Weierstrass Theorems, the following prop- erties of a real subset *A* are equivalent:

* + - *A* is closed and bounded;
    - every open rational cover *n∈*N *B◦*(***q****n, rn*) of *A* contains a finite sub-cover;
    - any sequence (***x****n*) in *A* admits a subsequence (***x****nk* ) converging within *A*. Equivalence “i)*⇔*ii)” (Heine–Borel) carries over to the effective setting [[21](#_bookmark62),

Lemma 5.2.5] [[3](#_bookmark46), Theorem 4.6]. Regarding sequential compactness iii), a Specker

Sequence (compare the proof of Lemma [2.10](#_bookmark16)c) yields the counter-example of a recur- sive rational sequence in *A* := [0*,* 1] having no recursive *fast* converging subsequence, that is, no computable accumulation point. This leaves the question whether every bounded recursive sequence admits an at least *naively* computable accumulation point. Simply taking the *largest* one (compare the proof of Proposition [3.2](#_bookmark19) in case *d* = 1) does not work in view of [[26](#_bookmark70), Theorem 6.1]. Also effectivizing the Bolzano– Weierstraß selection argument yields only an accumulation point computable rela- tive to *∅''*:

**Observation 3.5** *Let* (*xn*) *⊆* [0*,* 1] *be a bounded sequence. For each m ∈* N *choose k* = *k*(*m*) *∈* N *such that there are inﬁnitely many n with xn ∈ B◦*(*xk,* 2*−m*)*. Bound- edness and pigeonhole principle, inductively for m* = 1*,* 2*,.. ., assert the existence of smaller and smaller (length* 2*−m) sub-intervals each containing inﬁnitely many members of that sequence:*

*∃a, b ∈* Q *∀N ∃n ≥ N* : *xn ∈* (*a, b*) *∧ |b − a|≤* 2*−m .* (2)

*This is a* Σ3*–formula; and thus semi-decidable* relative *to ∅'', see for instance* [[20](#_bookmark65),

Post’s Theorem *§*IV.2.2]*.*

In fact *∅''* is the best possible as we establish, based on Section [3.2](#_bookmark25),

**Theorem 3.6** *There exists a recursive rational sequence* (*xn*) *⊆* [0*,* 1] *containing no* naively *computable accumulation point.*

This answers a recent question in Usenet [[14](#_bookmark58)]. The sequence constructed is rather complicated—and must be so in view of the following counter-part to Fact [2.5](#_bookmark9)a) and Observation [2.4](#_bookmark8):

**Lemma 3.7** *Let* (*xn*) *⊆* [0*,* 1]*d be a computable real sequence and let A denote the set of its accumulation points.*

* + - 1. *Every isolated point x of A is naively computable.*
      2. *If* Card(*A*) *<* c*, then A contains a naively computable point.*

**Proof.** *A* is closed non-empty and thus, if in addition free of isolated points, perfect; so b) follows from a). Let *{x}* = *A ∩* [*u, v*]= *A ∩* (*r, s*) with rational *u < r < s <*

*v*. A subsequence (*xnm* ) contained in (*r, s*) will then necessarily converge to *x*. Naive computability of *x* thus follows from selecting such a subsequence effectively: Iteratively for *m* = 1*,* 2*,...* use dove-tailing to search for (and, as we know it exists, also find) some integer *nm > nm−*1 with “*xnm ∈* (*r, s*)”. The latter property is indeed semi-decidable, for instance by virtue of [[24](#_bookmark68), Lemma 4.1c].

We have been pointed out [[25](#_bookmark69)] that Theorem [3.6](#_bookmark24) admits an easy proof based on a standard diagonalization over an enumeration of all recursive rational sequences. However we prefer an alternative approach because the uniform Proposition [3.9](#_bookmark26) below may be of interest of its own. Indeed, Theorem [3.6](#_bookmark24) follows from applying to Proposition [3.9](#_bookmark26) a relativization of Fact [1.2](#_bookmark3) which is an easy consequence of for example the proof of [[21](#_bookmark62), Theorem 4.2.8], namely

**Scholium 3.8** *For any oracle O, there exists a non-empty closed set A ⊆* [0*,* 1]

*co-r.e.* relative *to O, containing no point Cauchy–computable* relative *to O.*

* 1. *Co-R.E. Closed Sets Relative to ∅'*

[[7](#_bookmark51),

Theorem

9] has given a nice characterization of real numbers Cauchy–

computable *relative* to the Halting oracle. We do similarly for co-r.e. closed real sets:

**Proposition 3.9** *A closed subset A ⊆* R*d is ψd–computable* relative *to ∅' if and only if it is the set of accumulation points of a recursive rational sequence or, equivalently, of an enumerable inﬁnite subset of rationals.*

*>*

This follows (uniformly and for simplicity in case *d* = 1) from Claims a-e) of

**Lemma 3.10**

1. *Let closed A ⊆* R *be co-r.e. relative to ∅'. Then there is a recursive double*

*sequence of open rational intervals B◦*

*m,n*

*recursive) function M* : N *→* N *such that*

= (*um,n, vm,n*) *and a (not necessarily*

*i) ∀N ∈* N *∀m ≥ M* (*N* ) *∀n ≤ N* : *B◦*

*m,n*

*◦*

*M* (*N* )*,n*

= *B*

= *...* =: *B◦*

*(B◦ ,..., B◦*

*∞,n*

*each stabilizes beyond m ≥ M* (*N* )*)*

*ii)*

*m,*1

*A* = R *\*

*m,N*

*◦*

*B*

*.*

*n ∞,n*

1. *From a double sequence B◦ of open rational intervals as in a i+ii), one can*

*m,n*

*effectively obtain a rational sequence* (*ql*) *whose set of accumulation points coincides with A.*

1. *Given a rational sequence* (*ql*)*, a Turing machine can enumerate a subset Q of rational numbers having the same accumulation points. (Recall that a sequence may repeat elements but a set cannot.)*
2. *Given an enumeration of a subset Q of rational numbers, one can effectively*

*generate a double sequence of open rational intervals B◦ satisfying i+ii) above*

*m,n*

*where A denotes the set of accumulation points of Q.*

1. *If a double sequence of open rational intervals B◦ with i) is recursive, then*

*m,n*

*the set A according to ii) is co-r.e. relative to ∅'.*

1. *Let N ∈* N*,* ***u****n,* ***v****n ∈* Q*d, and* ***x*** *∈* R*d with* ***x*** */∈* *N*

(***u****n,* ***v****n*)*. Then, to every*

*d*  *N*

*ϵ >* 0*, there is some* ***q*** *∈* Q

*\*

*n*=1(***u****n,* ***v****n*) *such that* ***x*** *−* ***q*** *≤ ϵ.*

*n*=1

**Proof** omitted.

# Connected Components

Instead of asking whether a set contains a computable point, we now turn to the question whether it has a ‘computable’ connected component. Proofs here are more complicated but the general picture turns out rather similar to Section [2](#_bookmark4):

* If the co-r.e. closed set under consideration contains finitely many components, each one is again co-r.e. (Section [4.1](#_bookmark33)).
* If there are countably many, some is co-r.e. (Section [4.2](#_bookmark36)).
* There exists a compact co-r.e. set of which none of its (uncountably many) connected components is co-r.e. (Observation [4.3](#_bookmark32)).

Recall that for a topological space *X*, the connected component *C*(*X, x*) of *x ∈ X* denotes the union over all connected subsets of *X* containing *x*. It is connected and closed in *X*. *C*(*X, x*) and *C*(*X, y*) either coincide or are disjoint.

**Proposition 4.1** *Fix d ∈* N*.*

* 1. *Every (path* [10](#_bookmark28) *–) connected component of an r.e. open set is r.e. open.*

*More precisely (and more uniformly) the following mapping is well-deﬁned and*

(*θd , ρd, θd* )*–computable:*

*< <*

(*U,* ***x***): ***x*** *∈ U ⊆* R*d open*} *e* (*U,* ***x***) *'→ C*(*U,* ***x***) *⊆* R*d open.*

10 An open subset of Euclidean space is connected if and only if it is path-connected.

* 1. *The following mapping is well-deﬁned and* (*ψd, ρd, ψd*)*–computable:*

*> >*

(*A,* ***x***): ***x*** *∈ A ⊆* [0*,* 1]*d closed*} *e* (*A,* ***x***) *'→ C*(*A,* ***x***) *⊆* [0*,* 1]*d closed.*

**Proof.** First observe that closedness of *C*(*A,* ***x***) in closed *A ⊆* [0*,* 1]*d* means com- pactness in R*d*. Similarly, open *U* is locally (even path-) connected, hence *C*(*U,* ***x***) open in *U* and thus also in R*d*.

1. Let (*B*1*, B*2*,..., Bm,.. .*) denote a sequence of open rational balls exhausting *U* , namely given as a *θd* –name of *U* . Since the non-disjoint union of two connected subsets is connected again,

*<*

***x*** *∈ Bm*1 *∧ Bmi ∩ Bmi*+1 */*= *∅ ∀i < n* (3) implies *Bmn ⊆ C*(*U,* ***x***) for any choice of *n, m*1*,..., mn ∈* N. Conversely,

for instance by [[18](#_bookmark66), Satz 4.14], there exists to every ***y*** *∈ C*(*U,* ***x***) a finite

subsequence *Bmi* (*i* = 1*,..., n*) satisfying ([3](#_bookmark29)) with ***y*** *∈ Bmn* . Condition ([3](#_bookmark29)) being semi-decidable, one can enumerate all such subsequences and use them to exhaust *C*(*U,* ***x***). Nonuniformly, every connected component contains by openness a rational (and thus computable) ‘handle’ ***x***.

1. Recall the notion of a *quasi-*component [[10](#_bookmark54), *§*46.V]

*Q*(*A,* ***x***) := *S, S* := *S ⊆ A* : *S* clopen in *A,* ***x*** *∈ S*} (4)

*S∈S*(*A,****x***)

where “clopen in *A*” means being both closed and open in the relative topology of *A*. That is, *S* is closed in R*d*, and so is *A\ S*! By the *T*4 separation property (normal space), there exit disjoint open sets *U, V ⊆* R*d* such that *S ⊆ U* and *A \ S ⊆ V* . In particular *S* = *A ∩ U* , *U ∩ V* = *∅*, and *A ⊆ U ∪ V* :

*S*(*A,* ***x***) = *A ∩ U* *U, V ⊆* R*d* open*,U ∩ V* = *∅,* ***x*** *∈ U, A ⊆ U ∪ V* } *.* (5)

Both *U* and *V* are unions from from the topological base of open rational balls; w.l.o.g. *ﬁnite* such unions by compactness of *A*: *U* = *B*1 *∪ ... ∪ Bn* = *B*1 *∪ ... ∪ Bn* and *V* = *B' ∪ ... ∪ B'* . Therefore *Q*(*A,* ***x***) coincides with

1 *m*

*A ∩* *B*1 *∪ ... ∪ Bn* *B*1*,..., Bn, B' ,..., B'* open rational balls*,*

1

*m*

*Bi ∩ B'* = *∅,* ***x*** *∈ B*1*, A ⊆ B*1 *∪ ... ∪ B'* } *.* (6)

*j*

*m*

Conditions “*Bi ∩ B'* = *∅*” and “***x*** *∈ B*1” are semi-decidable; and so is “*A ⊆*

*j*

*B*1 *∪ ... ∪ B'* ”, see for example [[24](#_bookmark68),

*m*

*>*

Lemma

* 1. b]. Hence *Q*(*A,* ***x***) is *ψd*–

computable via the intersection ([6](#_bookmark30)) by virtue of the countable variant of [[21](#_bookmark62), Theorem 5.1.13.2], compare [[21](#_bookmark62), Example 5.1.19.1]. Now finally, *Q*(*A,* ***x***)= *C*(*A,* ***x***) since components and quasi-components coincide for compact spaces [[10](#_bookmark54), Theorem *§*47.II.2].

Effective boundedness is essential in Proposition [4.1](#_bookmark27)b): one can easily see that

*A '→ C*(*A,* ***x***) is in general (*ψ*2*, ψ*2)–discontinuous for fixed computable ***x*** *∈ A* when

*> >*

a bound on *A* is unknown. Non-uniformly, we have the following (counter-)

**Example 4.2** The following indicates an unbounded co-r.e. closed set *A ⊆* R2:



Here *ne* denotes the number of steps performed by the Turing machine with Go¨del index *e* before termination (on empty input), *ne* = *∞* if it does not terminate (i.e. *e /∈ H*).

Consider the connected component *C* of *A* with computable handle (0*, —*1): Were it co-r.e., then one could semi-decide “(*e,* 0) */∈ C*” [[24](#_bookmark68), Lemma 4.1c], equivalently: semi-decide “*e /∈ H*”: contradiction.

As opposed to the open case a), a computable ‘handle’ ***x*** for a compact connected component *C*(*A,* ***x***) need not exist; hence the non-uniform variant of b) may fail:

**Observation 4.3** *A co-r.e. closed subset of* [0*,* 1] *obtained from Fact* [*1.2*](#_bookmark3) *has un- countably many connected components, all singletons and none co-r.e.*

Indeed if *A ⊆* [0*,* 1] has positive measure, it must contain uncountably many points

*x*. Each such *x* is a connected component of its own: otherwise *C*(*A, x*) would be a non-empty interval and therefore contain a rational (hence computable) element: contradiction.

Regarding that the counter-example according to Observation [4.3](#_bookmark32) has uncount- ably many connected components, it remains to study—in analogy to Section [2.2](#_bookmark13)— the cases of countably infinitely many (Section [4.2](#_bookmark36)) and of

*4.1 Finitely Many Connected Components*

Does every bounded co-r.e. closed set with *ﬁnitely* many connected components have a co-r.e. closed connected component? Proposition [4.1](#_bookmark27)b) stays inapplicable because there still need not exist a computable handle:

**Example 4.4** Let *A ⊆* [0*,* 1] denote a non-empty co-r.e. closed set without com- putable points (recall Fact [1.2](#_bookmark3)). Then (*A ×* [0*,* 1]) *∪* ([0*,* 1] *× A*) *⊆* [0*,* 1]2 is (even *path-*) *connected* non-empty co-r.e. closed, devoid of computable points.

Nevertheless, Proposition [4.5](#_bookmark35)b+c) exhibits a (partial) analog to Corollary [2.9](#_bookmark15). To this end, observe that a point ***x*** in some set *A ⊆* R*d* is isolated if and only if *{****x****}* is open in *A*.

**Proposition 4.5** *Let ∅ /*= *A ⊆* R*d be closed.*

1. *If A has ﬁnitely many connected components, then each such connected com- ponent is open in A.*
2. *If A is co-r.e. and C*(*A,* ***x***) *a bounded connected component of A open in A, then C*(*A,* ***x***) *is also co-r.e.*
3. *If A is r.e. and C*(*A,* ***x***) *a bounded connected component of A open in A, then*

*C*(*A,* ***x***) *is also r.e.*

**Proof** omitted.

**Corollary 4.6** *If bounded co-r.e. closed A ⊆* R*d has only ﬁnitely many connected components, then each of them is itself co-r.e.*

* 1. *Countably Inﬁnitely Many Connected Components*

By Proposition [4.5](#_bookmark35)a+b), if bounded co-r.e. closed *A ⊆* R*d* has finitely many com- ponents, each one is itself co-r.e. In the case of countably infinitely many connected components, we have seen in Example [4.2](#_bookmark31) a bounded co-r.e. closed set containing a connected component which is not co-r.e.; others of its components on the other hand are co-r.e. In fact it holds the following counterpart to Fact [2.5](#_bookmark9)b):

**Lemma 4.7** *Let ∅ /*= *A ⊆* R*d be compact with no connected component open in A. Then A has as many connected components as cardinality of the continuum.*

Proposition [4.5](#_bookmark35)b) implies

**Corollary 4.8** *Let A ⊆* R*d be compact and co-r.e. with countable many connected components. Then at least one such component is again co-r.e.*

**Proof (of Lemma** [**4.7**](#_bookmark37)**).** By [[10](#_bookmark54), Theorem *§*46.V.3], there exists a continuous function *f* : *A → {*0*,* 1*}ω* such that the point inverses *f−*1(*σ*¯) coincide with the quasi-components of *A*; and these in turn with *A*’s connected components [[10](#_bookmark54),

Theorem

*§*47.II.2]. Since *A* is compact and *f* continuous, *f* [*A*] *⊆ {*0*,* 1*}ω* is

compact, too. Moreover every isolated point *{σ*¯*}* of *f* [*A*] yields *f−*1(*σ*¯) (closed and) open a component in *A*. So if *A* has no open component, *f* [*A*] must be perfect—and thus of continuum cardinality by virtue of Fact [2.5](#_bookmark9)b).

Corollary [4.8](#_bookmark38) and Example [4.2](#_bookmark31) leave open the following

**Question 4.9** *Is there a* bounded *co-r.e. closed set with countably many connected components, one of which is* not *co-r.e.?*

In view of Proposition [4.1](#_bookmark27)b), this component must not contain a computable point.

* 1. *Related Work*

*i*

An anonymous referee has directed our attention to the following interesting result which appeared as [[15](#_bookmark59), Theorem 2.6.1]:

**Fact 4.10** *For any co-r.e. closed X ⊆* [0*,* 1]*d, the following are equivalent:*

1. *X contains a nonempty co-r.e. closed connected component,*
2. *X is the set of ﬁxed points of some computable map g* : [0*,* 1]*d →* [0*,* 1]*d,*
3. *the image f* (*X*) *contains a computable number for any computable f* : *X →* R*.*

# Co-R.E. Closed Sets *with* Computable Points

The co-r.e. closed subsets of R devoid of computable points according to Fact [1.2](#_bookmark3) lack convexity:

**Observation 5.1** *Every non-empty co-r.e. interval I ⊆* R *trivially has a com- putable element:*

*Either I contains an open set (and thus lots of rational elements x ∈ I) or it is a singleton I* = *{x}, hence x computable* [[3](#_bookmark46), Proposition 3.6]*.*

(It is not possible to continuously ‘choose’, even in a multi-valued way, some *x ∈ I*

from a *ψ>*–name of *I*, though. . . ) This generalizes to higher dimensions:

**Theorem 5.2** *Let ∅ /*= *A ⊆* R*d be co-r.e. closed and convex. Then there exists a computable point* ***x*** *∈ A.*

**Proof** omitted.

* 1. *Star-Shaped Sets*

A common weakening of convexity is given in the following

**Definition 5.3** *A set A ⊆* R*d is* star-shaped *if there exists a (so-called* star-*) point* ***s*** *∈ A such that, for every* ***a*** *∈ A, the line segment* [11](#_bookmark40) [***s****,* ***a***] := *{λ****s*** + (1 *— λ*)***a*** :0 *≤ λ ≤* 1*} is contained in A.*

*The* set of star-points *S*(*A*) *is the collection of* all *star-points of A.*

So *A* is convex if and only if *A* = *S*(*A*); *A* is star-shaped if and only if *S*(*A*) */*= *∅*; and star-shape implies (even simply-)connectedness.

Fig. 1. A convex, a star-shaped, a simply-connected, and a connected set.

11 The reader is not in danger of confusing this with the same notion [***s****,* ***a***] standing for the cube Q [*si, ai*]

in Sections [2](#_bookmark4) and [3](#_bookmark18).

**Lemma 5.4** *S*(*A*) *⊆ A is convex. Moreover if A is closed, then so is S*(*A*)*.*

**Proof** omitted.

**Theorem 5.5** *Let ∅ /*= *A ⊆* R2 *be co-r.e. closed and star-shaped. Then A contains a computable point.*

In view of Lemma [5.4](#_bookmark41) this claim would follow from Theorem [5.2](#_bookmark39) if, for every star- shaped co-r.e. closed *A*, its set *S*(*A*) of star-points were co-r.e. again. However we have been shown the latter assertion to fail already for very simple compact subsets in 2D [[16](#_bookmark60)].

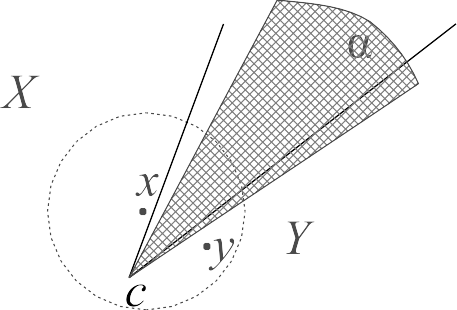


Fig. 2. Illustration to the proof of Theorem [5.5](#_bookmark42) for the case *S*(*A*)= *{****c****}*.

**Proof (of Theorem** [**5.5**](#_bookmark42)**).** If *A* has non-empty interior, it contains a rational (and thus computable) point. Otherwise suppose the convex set *S*(*A*) to have dimension one, i.e. *S*(*A*) = [***x****,* ***y***] with distinct ***x****,* ***y*** *∈ A*. Were *S*(*A*) a *strict* subset of *A*, *A* would contain an entire triangle (compare the proof of Lemma [5.4](#_bookmark41)) contradicting *A◦* = *∅*. Hence *S*(*A*)= *A* is co-r.e. and contains a computable point by Theorem [5.2](#_bookmark39). It remains to treat the case of *S*(*A*)= *{****c****}* Ç *A*, *A* consisting of semi-/rays origi- nating from ***c*** as indicated in Figure [2](#_bookmark43). Consider some rational square *Q* containing ***c*** in its interior but not the entire *A*. If the square’s boundary, intersected with *A*, contains an isolated point, this point will be computable according to [[21](#_bookmark62), The- orem 5.1.13.2] and Section [2.2](#_bookmark13). Otherwise *Q◦ \ A* consists of uncountably many (Observation [2.4](#_bookmark8)) connected components. Let *X* and *Y* denote two *non-*adjacent ones of them, each r.e. open according to Proposition [4.1](#_bookmark27)a). Also let 0 *< α ≤* 180*◦* be some (w.l.o.g. rational and thus computable) lower bound on the angle be- tween *X* and *Y* . Notice that *X* and *Y* ‘almost touch’ (i.e. their respective closures meet) exactly in the sought point ***c***. Moreover for ***x*** *∈ X* and ***y*** *∈ Y* , elementary trigonometry confirms that ***x*** *—* ***c *** 2 *≤ ****x*** *—* ***y *** 2*/*(2 sin *α* ). Based on effective enu- merations of all rational ***x*** *∈ X* and all rational ***y*** *∈ Y* , we thus obtain arbitrary good approximations to ***c***.

2

Regarding further weakenings of the prerequisites of Theorem [5.5](#_bookmark42), we ask

**Question 5.6** *a) For d ∈* N*, does every non-empty star-shaped co-r.e. closed subset of* [0*,* 1]*d contain a computable point?*

*b) Does every (connected and) simply-connected co-r.e. closed non-empty subset of* [0*,* 1]2 *contain a computable point?*

Mere connectedness is not sufficient: recall Example [4.4](#_bookmark34). This immediately extends to a (counter-)example giving a negative answer to Question [5.6](#_bookmark44)b) in 3D:

**Example 5.7** Let *A ⊆* [0*,* 1] denote a non-empty co-r.e. closed set without com- putable points. Then (*A ×* [0*,* 1]2) *∪* ([0*,* 1] *× A ×* [0*,* 1]) *∪* ([0*,* 1]2 *× A*) *⊆* [0*,* 1]3 is simply-connected non-empty co-r.e. closed devoid of computable points.

# References

1. *Beeson M.J.*: “*Foundations of Constructive Mathematics* ”, Springer (1985).
2. *Brattka V.*: “Effective Borel Measurability and Reducibility of Functions”, pp.19–44 in *Mathematical Logic Quarterly* vol.**51:1** (2005).
3. *Brattka V., Klaus, Weihrauch*: “Computability on Subsets of Euclidean Space I: Closed and Compact Subsets”, pp.65-93 in *Theoretical Computer Science* vol.**219** (1999).
4. *Cenzer D., J.B. Remmel* : “Π0

1

Classes in Mathematics”, pp.623–821 in

Yu.L.

Ershov,

S.S. Goncharov, A. Nerode, J.B. Remmel (Eds.) *Handbook of Recursive Mathematics* vol.**2**, Elsevier (1998).

1. *Ge X., A. Nerode*: “On Extreme Points of Convex Compact Turing Located Sets”, pp.114-128 in *Logical Foundations of Computer Science*, Springer LNCS vol.**813** (1994).
2. *Gherardi G.*: “An Analysis of the Lemmas of Urysohn and Urysohn-Tietze According to Effective Borel Measurability”, pp.199–208 in *Proc. 2nd Conference on Computability in Europe* (CiE’06), Springer LNCS vol.**3988**.
3. *Ho C.-K.*: “Relatively recursive reals and real functions”, pp.99–120 in *Theoretical Computer Science*

vol.**210** (1999).

1. *Kechris A.S.*: “*Classical Descriptive Set Theory* ”, Springer (1995).
2. *Kreisel G., D. Lacombe*: “Ensembles r´ecursivement measurables et ensembles r´ecursivement ouverts ou ferm´es”, pp.1106–1109 in *Compt. Rend. Acad. des Sci. Paris* vol.**245** (1957).
3. *Kuratowski K.*: “*Topology Vol.II* ”, Academic Press (1968).
4. *Kushner B.*: “*Lectures on Constructive Mathematical Analysis*”, vol.**60**, American Mathematical Society (1984).
5. *Lacombe D.*: “Les ensembles r´ecursivement ouverts ou ferm´es, et leurs applications `a l’analyse r´ecursive I”, pp.1040–1043 in *Compt. Rend. Acad. des Sci. Paris* vol.**245** (1957).
6. *Lacombe D.*: “Les ensembles r´ecursivement ouverts ou ferm´es, et leurs applications `a l’analyse r´ecursive II”, pp.28–31 in *Compt. Rend. Acad. des Sci. Paris*, vol.**246** (1958).
7. *Lagnese G.*: “Can someone give me an example of... ”, in Usenet [http://cs.nyu.edu/pipermail/fom/](http://cs.nyu.edu/pipermail/fom/2006-February/009835.html) [2006-February/009835.html](http://cs.nyu.edu/pipermail/fom/2006-February/009835.html)
8. *Miller J.S.*: “Pi-0-1 Classes in Computable Analysis and Topology”, PhD thesis, Cornell University, Ithaca, USA (2002).
9. *Miller J.S.*, Personal Communication (June 21, 2007).
10. *Morris M.* (Editor): “*American Heritage Dictionary of the English Language*”, American Heritage Publishing (1969).
11. *Querenburg, B. von*: “*Mengentheoretische Topologie*”, Springer (1979).
12. *Specker E.*: “Der Satz vom Maximum in der rekursiven Analysis”, pp.254–265 in *Constructivity in Mathematics* (A. Heyting Edt.), Studies in Logic and The Foundations of Mathematics, North-Holland (1959).
13. *Soare R.I.*: “*Recursively Enumerable Sets and Degrees*”,
14. *Weihrauch K.*: “*Computable Analysis*”, Springer (2000).
15. *Weihrauch K., X. Zheng*: “Computability on continuous, lower semi-continuous and upper semi- continuous real functions”, pp.109–133 in *Theoretical Computer Science* vol.234 (2000).
16. *Zaslavski˘ı I.D., G.S. Tse˘ıtin* : “On singular coverings and related properties of constructive functions”, pp.458–502 in *Trudy Mat. Inst. Steklov.* vol.**67** (1962);

English transl. in *Amer. Math. Soc. Transl.* (2) **98** (1971).

1. *Ziegler M.*: “Computable operators on regular sets”, pp.392–404 in *Mathematical Logic Quarterly* vol.**50** (2004).
2. *Zheng X.*, Personal Communication (June 21, 2007).
3. *Zheng X., K. Weihrauch*: “The Arithmetical Hierarchy of Real Numbers”, pp.51–65 in *Mathematical Logic Quarterly* vol.**47:1** (2001).