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Some Results on Poset Models Consisting of Compact Saturated Subsets

## Qingyu He[1](#_bookmark0)*,*[3](#_bookmark0)

*School of Mathematics Science Yangzhou University Yangzhou 225002, China*

## Gaolin Li[1](#_bookmark0)*,*[4](#_bookmark0)

*School of Mathematics and Statistics Yancheng Teachers University Yancheng 224002, China*

## Xiaoyong Xi[1](#_bookmark0)*,*[5](#_bookmark0)

*School of Mathematics and Statistics Jiangsu Normal University*

*Xuzhou 221116, China*

## Dongsheng Zhao[2](#_bookmark0)*,*[6](#_bookmark0)

*Mathematics and Mathematics Education National Institute of Education Nanyang Technological University*

*1 Nanyang Walk 637616, Singapore*

**Abstract**

Given a topological space *X*, the set *Q*(*X*) of all nonempty saturated compact subsets of *X* is a poset with respect to the reverse inclusion order. The posets of the form *Q*(*X*) play important roles in several aspects of domain theory. In this paper, we investigate some further properties of such posets, in particular their links to the dcpo models of *T*1 topological spaces.

*Keywords:* Scott topology; maximal point space; dcpo model; compact saturated subset; K-filter defined space; *k*-space

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A dcpo model of a topological space *X* is a dcpo (directed complete poset) *P* such that *X* is homeomorphic to the maximal point space of *P* with the subspace topology of the Scott space of *P* . Although it has been proved that every *T*1 topological space has a dcpo model [[11](#_bookmark15)], one may not be able to construct a simply defined dcpo model for a general *T*1 space, if it is not a complete metrizable space. In [[12](#_bookmark16)], the authors considered the dcpo models of the form *CK*(*X*) consisting of all nonempty compact closed subsets of space *X* with the reverse inclusion order. The key notion employed in [[12](#_bookmark16)] is the CK-filter defined topology, described using the nonempty compact closed subsets. They proved that if a *T*1 space is CK-filter defined, then the dcpo (*CK*(*X*)*, ⊇*) is a bounded complete dcpo model of *X*. In particular, for every Hausdorff *k*-space (namely, compactly generated space), *CK*(*X*) is a bounded complete dcpo model of *X*. In the current paper, we shall make use the set *Q*(*X*) of all nonempty compact saturated subsets of a topological space *X* in the place of *CK*(*X*) to investigate the corresponding problems considered in [[12](#_bookmark16)]. The main results include: (1) For any *T*1 well-filtered K-filter defined space (to be defined in Section 2) *X*, *Q*(*X*) is a dcpo model of *X*; (2) a Hausdorff space is a *k*-space iff it is CK-filter defined; (3) a first countable, coherent *T*1 space is well-filtered if and only if it is sober. The result (2) answers a problem posed in [[12](#_bookmark16)] and establishes a new characterization of Hausdorff *k*-spaces.

# Preliminaries

In this section, we recall some basic notions and results to be used in the sequel, most of them can be found in [[1,](#_bookmark8) [2](#_bookmark9)].

Let *P* be a poset. For *D ⊆P* , we use W*D* (resp., V*D*) to denote the supremum

(resp., infimum) of *D* if it exists.

A subset *A* of a poset *P* is called an *upper* (resp., a *lower*) set if *A* = *↑A* = *{x ∈ P | a ≤ x* for some *a ∈ A}* (resp., *A* = *↓A* = *{x ∈ P | x ≤ a* for some *a ∈ A}*). A subset *D* of *P* is *directed* if it is nonempty and every finite subset of *D* has an upper bound in *D*. A poset is called a *directed complete poset* (dcpo, for short) if every directed subset in it has a supremum. A poset is called *bounded complete* if every subset that is bounded above has a supremum. In particular, a bounded complete poset has a bottom element, the supremum of empty set.

For two elements *x* and *y* in a poset *P* , We say that *x* is *way below y*, written as *x y* if for any directed set *D ⊆ P* with existing W *D* and W *D ≥ y*, there

is some *d ∈ D* such that *x ≤ d*. An element *x ∈ P* is called *compact* if *x x*.

The set of all compact elements of *P* will be denoted by *K*(*P* ). A poset *L* is said to be *continuous* (resp., *algebraic*) if every element is the directed supremum of

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2 Supported by the NIE AcRF Project, Singapore (RI 3/16 ZDS)

3 Email: [smileheqingyu@163.com](mailto:smileheqingyu@163.com)

4 Email: [ligaolin1981@126.com](mailto:ligaolin1981@126.com)

5 Email: [littlebrook@jsnu.edu.cn](mailto:littlebrook@jsnu.edu.cn)

6 Email: [dongsheng.zhao@nie.edu.sg](mailto:%20dongsheng.zhao@nie.edu.sg)

(resp., compact) elements that are way below it. A continuous dcpo is often called a *domain*.

A subset *U ⊆P* of poset *P* is *Scott open* iff (i) *U* = *↑U* , and (ii) for any directed subset *D ⊆P* , W *D ∈ U* implies *D ∩ U /*= *∅* whenever W *D* exists. The collection of

all Scott open sets of *P* forms a topology, called the *Scott topology* of *P* and denoted

by *σ*(*P* ). The space Σ*P* = (*P, σ*(*P* )) is called the Scott space of *P* .

For a topological space *X*, a nonempty subset *A ⊆ X* is said to be *irreducible* if for any finite family *{Ci}i∈F* of closed sets, whenever *A ⊆ ∪i∈F Ci*, then *A ⊆ Ci* for some *i ∈ F* . A topological space *X* is called *sober* if for every irreducible closed set *C*, there exists a unique *x ∈ X* such that cl(*{x}*)= *C*, where cl(*{x}*) means the closure of *{x}*. The *specialization order* on a *T*0 space *X* is defined by that *x ≤ y* iff *x ∈* cl(*{y}*). Alternatively *x ≤ y* iff every open set that contains *x* must also contain *y*. Thus open sets are upper sets and closed sets are lower sets. A subset of *X* is called *saturated* if it is an intersection of open subsets, equivalently if it is an upper set with respect to the specialization order. Notice that for a poset *L*, a subset *A* is saturated in (*L, σ*(*L*)) iff *A* is an upper set. A topological space *X* is called *well-ﬁltered* if for each filter basis *C* of compact saturated sets and each open set *U* with *C ⊆ U* , there is a *K ∈ C* with *K ⊆ U* . It is known by [[1](#_bookmark8), Theorem II-1.21] that if *X* is sober, then *X* is well-filtered. A topological space is said to be *coherent* if the intersection of any two compact saturated sets is again compact.

A dcpo *L* is said to be well-filtered (resp., compact, sober, coherent) if Σ*P* is a well-filtered (resp., compact, sober, coherent) space.

**Proposition 1.1** ( [[1](#_bookmark8), Proposition I-1.24.2.]) *Let X be a topological space. If X is well-ﬁltered, then K* = *C is a nonempty compact saturated set for each ﬁlter base C of nonempty compact saturated sets C.*

# Upper Vietoris topologies and Scott topologies on

*Q*(*X*)

In this section, instead of nonempty compact closed subsets of a *T*1 space used by Zhao and Xi in [[12](#_bookmark16)], we use nonempty compact saturated subsets of a topological space *X* to form a dcpo model of *X*.

For any *T*0 well-filtered space (*X, τ* ), let *Q*(*X, τ* ) (*Q*(*X*) for short) be the set of all nonempty compact saturated subsets of *X*. The poset (*Q*(*X*)*, ⊇*) is directed

complete: for any directed subset *D ⊆ Q*(*X*), *D* = W *D*.

*Q*(*X*)

**Definition 2.1** An upper set *U* of a topological space (*X, τ* ) is called *K*-open if, for any filtered family *{Ki}i∈I ⊆ Q*(*X*) with *i∈I Ki* =*↑ x ⊆ U* for some *x ∈ U* , then *Ki ⊆ U* for some *i ∈ I*.

**Definition 2.2** An upper set *U* of a topological space (*X, τ* ) is called *K∗*-open if, for any filtered family *{Ki}i∈I ⊆ Q*(*X*) with *i∈I Ki ⊆ U* , then *Ki ⊆ U* for some *i ∈ I*.

Let *τK* be the set of all K-open sets of *X*. Obviously, *∅* and *X* are K-open. It

is easy to verify that *τK* is indeed a topology on *X*. We call *τK* the *K-generated topology*.

Clearly, the intersection of two K*∗*-open sets is K*∗*-open. In general, the union of two K*∗*-open sets may not be K*∗*-open. So, all K*∗*-open sets may not form a topology. The topology generated by K*∗*-open sets and *∅* as a base is denoted by *τK∗* , called the *K∗-generated topology*.

It is easy to see that every K*∗*-open set is K-open. For a well-filtered space, every open set of *X* is K*∗*-open. Thus, we have the following result.

**Proposition 2.3** *Let* (*X, τ* ) *be a well-ﬁltered space. Then we have τ ⊆ τK∗ ⊆ τK.*

**Definition 2.4** Let (*X, τ* ) be a *T*0 space. Then

* 1. (*X, τ* ) is called K-filter defined if *τK* = *τ* .
  2. (*X, τ* ) is called K*∗*-filter defined if *τK∗* = *τ* .

**Remark 2.5** 1) For every well-filtered dcpo *L*, the space Σ*L* = (*L, σ*(*L*)) is K-filter defined.

2) The well-known non-sober dcpo constructed by Johnstone [[3](#_bookmark10)] is K-filter de- fined but not well-filtered.

Recall that in [[7](#_bookmark13)] the upper Vietoris topology has a basis of open sets of the form *2U* = *{K ∈ Q*(*X*) *| K ⊆U}* where *U* ranges over the open subsets of *X*. The specialization order for the upper Vietoris topology on *Q*(*X*) agrees with the Smyth preorder *A ≤ B*, i.e., *B ⊆A*.

Note that, the specialization order on *T*1 space (*X, τ* ) reduces to the discrete order. So, for a *T*1 space *X*, compact saturated subsets of *X* are the same as compact subsets of *X*. Let *ηX* : *X → Q*(*X*) be the mapping given by *ηX* (*x*)= *{x}* for all *x*.

**Proposition 2.6** *Let X be a T*1 *well-ﬁltered space and Q*(*X*) *endowed with the upper Vietoris topology. Then ηX* : *X → Max*(*Q*(*X*)) *is a homeomorphism.*

Denote the upper Vietoris topology (resp., the Scott topology) on *Q*(*X*) by *UV* (*Q*(*X*)) (resp., *σ*(*Q*(*X*))). Clearly, for a well-filtered space *X*, *UV* (*Q*(*X*)) *⊆ σ*(*Q*(*X*)).

**Proposition 2.7** *If* (*X, τ* ) *is T*1*, well-ﬁltered and K-ﬁlter deﬁned, then*

*UV* (*Q*(*X*)) *|Max*(*Q*(*X*))= *σ*(*Q*(*X*)) *|Max*(*Q*(*X*)) *.*

**Proof.** It suffices to show that *UV* (*Q*(*X*)) *|Max*(*Q*(*X*))*⊇ σ*(*Q*(*X*)) *|Max*(*Q*(*X*)). For any *U ∈ σ*(*Q*(*X*)), let *U* = *{x | {x} ∈ U ∩ Max*(*Q*(*X*))*}*. It is easy to see that *U ∩ Max*(*Q*(*X*)) = *2U ∩ Max*(*Q*(*X*)). Next we show that *U ∈ τ* . Since *U ∈*

*σ*(*Q*(*X*)), for any *x ∈ U* and any filtered family *{Ki}i∈I ⊆ Q*(*X*), if W *Ki* =

*i∈I*

*i∈I Ki* = *{x} ∈ U* , then there exists some *i ∈ I* such that *Ki ∈ U* , which implies that *Ki ⊆ U* . Thus, *U ∈ τK*. Since (*X, τ* ) is K-filter defined, we have *U ∈ τK* = *τ* .*2*

**Theorem 2.8** *Let X be a T*1 *well-ﬁltered K-ﬁlter deﬁned space. Then Q*(*X*) *is a dcpo model of X.*

**Proof.** It follows from Propositions [2.6](#_bookmark1) and [2.7](#_bookmark2). *2*

**Proposition 2.9** *Let* (*X, τ* ) *be a T*1 *and well-ﬁltered space. If*

*UV* (*Q*(*X*)) *|Max*(*Q*(*X*))= *σ*(*Q*(*X*)) *|Max*(*Q*(*X*))*, then* (*X, τ* ) *is K∗-ﬁlter deﬁned.*

**Proof.** Let *U ∈ τK∗* . We show that *U ∈ τ* . Let *U* = *{K ∈ Q*(*X*) *| K ⊆ U}*. Then *{x} ∈ U* holds for any *x ∈ U* . Since *U ∈ τK∗* , *U ∈ σ*(*Q*(*X*)). As *UV* (*Q*(*X*)) *|Max*(*Q*(*X*))= *σ*(*Q*(*X*)) *|Max*(*Q*(*X*)), there exists *V ∈ τ* such that

*{x} ∈ 2V ∩ Max*(*Q*(*X*)) *⊆ U ∩ Max*(*Q*(*X*)). So, we have *x ∈ V ⊆ U* , and thus *U ∈ τ* . *2*

**Example 2.10** Let *X* = R be the set of all real numbers and *τ* the topology on *X*, where *U ∈ τ* if and only if *U* = *V − A* for some Euclidean open set *V* and a countable set *A*. Then, *Q*(*X*) is the family of all nonempty finite subsets of R. Thus, every subset is K-open, and thus *τK* is a discrete topology. So, (*X, τK*) is not K-filter defined. Also *σ*(*Q*(*X*)) *|Max*(*Q*(*X*)) is discrete, which reveals that *Q*(*X*) can not be a dcpo model of *X*.

**Proposition 2.11** *Every T*1 *ﬁrst countable and well-ﬁltered space is K-ﬁlter de- ﬁned.*

**Proof.** Let (*X, τ* ) bea *T*1 first countable and well-filtered space. It suffices to show *τK ⊆ τ* . Let *U ∈ τK*. For any *x ∈ U* , we need to show that there exists a *V ∈ τ* such that *x ∈ V ⊆ U* . Let *{Vi | i ∈* N*}* be a neighborhood base of *x* with *Vi*+1 *⊆ Vi*. Suppose that *x ∈ Vi* ¢ *U* for all *i ∈* N. Take *xi ∈ Vi \ U* (*∀i ∈* N). Then *{xi | i ∈* N*}* converges to *x*. Set *Ai* = *{xj | j ∈* N*,j ≥ i}∪{x}*. Then *Ai* is compact for all *i ∈* N.

Since (*X, τ* ) is well-filtered, *∞ Ai* = *{x} ⊆ U* . However, *Ai* ¢ *U* for all *i ∈* N,

*i*=0

which contradicts *U ∈ τK*. *2*

By Theorem [2.8](#_bookmark3) and Proposition [2.11](#_bookmark4), we have the following result.

**Corollary 2.12** *Let X be a T*1 *ﬁrst countable and well-ﬁltered space. Then Q*(*X*)

*is a dcpo model of X.*

# Compact subsets in K*∗*-generated topological spaces

For a well-filtered space *X*, the K*∗*-generated topology may contain more open subsets than the original topology. In this section, we show that the compact subsets with respect to these two topologies are the same.

**Lemma 3.1** *Let X be a T*1 *well-ﬁltered space. For any K ∈ Q*(*X, τ* ) *and any closed set C in* (*X, τK∗* )*, we have C ∩ K ∈ Q*(*X, τ* )*.*

**Proof.** It suffices to verify the compactness of *C ∩ K* for each *K ∈ Q*(*X, τ* ) and any closed set *C* in (*X, τK∗* ). Suppose that *{Ui}i∈I* is a directed family of open sets in (*X, τ* ) such that *i∈I Ui ⊇ C ∩ K* and *Ui ∩* (*C ∩ K*) */*= *∅* for any *i ∈ I*, while *Ui /⊇ C ∩ K* for any *i ∈ I*. Then for any *i ∈ I*, (*X − Ui*) *∩* (*C ∩ K*) */*= *∅*. Set *Di* = *K ∩* (*X − Ui*)(*∀i ∈ I*). Then each *Di* is a compact saturated subset of *K* and

*i∈I Di ⊆ K − C*. Since *C* is closed in (*X, τK∗* ), there exists some *i*0 *∈ I* such that

*Di*0 = *K ∩* (*X − Ui*0 ) *⊆ K − C*, which contradicts (*X − Ui*0 ) *∩* (*C ∩ K*) */*= *∅*. *2*

Let *Q*(*X, τK∗* ) be the set of all compact saturated sets of (*X, τK∗* ).

**Theorem 3.2** *Let X be a T*1 *well-ﬁltered space. Then*

*Q*(*X, τ* )= *Q*(*X, τK∗* )*.*

**Proof.** It is easy to check that *Q*(*X, τK∗* ) *⊆ Q*(*X, τ* ). Conversely, we show that *Q*(*X, τ* ) *⊆ Q*(*X, τK∗* ). Let *K ∈ Q*(*X, τ* ) and *{Ci | i ∈ I}* be a family of closed subsets in (*X, τK∗* ) with *{K ∩ Ci | i ∈ I}* satisfying finite intersection property.

It suffices to show the compactness of *K* in *τK∗* . Next, we need to check that

*i∈I* (*K ∩ Ci*) */*= *∅*.

By Lemma [3.1](#_bookmark5), we have *K ∩ Ci ∈ Q*(*X, τ* ) for all *i ∈ I*. Suppose that *i∈I* (*K ∩ Ci*)= *∅*. Since (*X, τ* ) is well-filtered, there exists some *i*0 *∈ I* such that *K ∩Ci*0 = *∅*, which contradicts the finite intersection property. Thus, *K ∈ Q*(*X, τK∗* ), which implies that *Q*(*X, τ* ) *⊆ Q*(*X, τK∗* ). *2*

**Corollary 3.3** *Let X be a T*1 *well-ﬁltered space. Then* (*X, τK∗* ) *is K∗-ﬁlter deﬁned.*

# CK-filter defined topologies and *k*-spaces

In this section, we give a positive answer to a question raised by Zhao and Xi in [[12](#_bookmark16)].

For any *T*0 space (*X, τ* ), let *CK*(*X*) be the set of all nonempty closed compact subsets of *X*. The poset (*CK*(*X*)*, ⊇*) is directed complete: for any directed subset

*D ⊆ CK*(*X*), *D* = W *D*.

*CK*(*X*)

**Definition 4.1** ( [[12](#_bookmark16)]) A subset *U* of a topological space (*X, τ* ) is called *CK*-open if, for any filtered family *F ⊆ CK*(*X*) with *|* *F |*= 1, that is, *F* is a singleton, and *F ⊆ U* , then *F ⊆ U* for some *F ∈ F*. The topology consisting of all CK-open subsets is called the CK-generated topology.

**Definition 4.2** ( [[12](#_bookmark16)]) A subset *U* of a topological space (*X, τ* ) is called *CK∗*-open if, for any filtered family *F ⊆ CK*(*X*) with *F ⊆ U* , then *F ⊆ U* for some *F ∈ F*. The topology generated by all CK*∗*-open subsets as a basis is called the CK*∗*-generated topology.

Obviously, every open set of *X* is CK*∗*-open, and every CK*∗*-open set is CK- open. It is shown in [[12](#_bookmark16), Lemma 2.9] that a subset of a Hausdorff space is CK-open if and only if it is CK*∗*-open. The next example shows that a CK-open set of a *T*1 space need not to be CK*∗*-open.

**Example 4.3** Let (N*, τcof* ) be the set N of all positive integers equipped with the co-finite topology *τcof* . Then (N*, τcof* ) is not well-filtered. However, for any *x ∈* N, any open neighborhood *U* of *x* and filtered family of compact subsets *{Ki}i∈I* ,

*i∈I*

*Ki* = *{x}⊆ U* implies that there exists *i*0 *∈ I* such that *Ki*0

*⊆ U* .

**Definition 4.4** ( [[12](#_bookmark16)]) i) A topological space (*X, τ* ) is called *CK*-filter defined if

*τCK* = *τ* .

ii) A topological space (*X, τ* ) is called *CK∗*-filter defined if *τCK∗* = *τ* .

**Definition 4.5** ( [[12](#_bookmark16)]) A space *X* is a *k*-space (or, *compactly generated space*) if a subset *U* of *X* is open if and only if for any compact set *K*, *U ∩ K* is open in the subspace *K*. Equivalently, a subset *B* is closed if and only if for any compact set *K*, *B ∩ K* is closed in the subspace *K*.

**Lemma 4.6** ( [[12](#_bookmark16), Theorem 2.4]) *Every Hausdorff k-space is CK-ﬁlter deﬁned.*

Xi and Zhao in [[12](#_bookmark16)] asked if Hausdorff CK-filter defined spaces are *k*-spaces. We now give a positive answer for more general case as follows.

**Theorem 4.7** *Every CK-ﬁlter deﬁned space is a k-space.*

**Proof.** Let (*X, τ* ) be CK-filter defined. Assume *U ⊆ X* such that *U ∩ K* is open in

*K* for any compact set *K ⊆ X*. Let *{Ki}i∈I* be a filtered family of compact closed

sets such that *Ki* = *{x} ⊆ U* . Then there is an *i*0 *∈ I* such that *Ki ⊆ Ki* for

*i∈I*

0

*i ≥ i*0. So, we assume that every *Ki* is contained in *Ki*0 . Then every *Ki* is a closed

subset of the compact space *Ki* . Suppose that *Ki − U* = *Ki ∩ UC /*= *∅* for all *i ∈ I*, then, as *U ∩ Ki* is open in *Ki*, *Ki − U* = *Ki ∩ UC* is closed in *Ki*. Hence, every

0

*Ki − U* is closed in *Ki*0 and *{Ki | i ∈ I}* satisfies the finite intersection property.

Now

*i∈I*

(*Ki − U* )=

*i∈I*

*Ki − U* = *∅*, which contradicts that *Ki*0

is compact. So,

there exists an *ij* such that *Ki∗ ⊆ U* . Since (*X, τ* ) is CK-filter defined, *U ∈ τCK* = *τ* ,

and thus *U* is open in *X*. Therefore, *X* is a *k*-space. *2*

By Theorem [4.7](#_bookmark7) and Lemma [4.6](#_bookmark6), we have the following result which gives a new characterization for Hausdorff *k*-spaces.

**Corollary 4.8** *A Hausdorff space is a k-space iff it is CK-ﬁlter deﬁned.*

# First countable spaces with bounded complete dcpo models are sober

It is well known that every sober space is well-filtered. But there exists a well- filtered *T*1 space which is non-sober. There are also dcpos whose Scott spaces are well-filtered but non-sober (see [[6](#_bookmark11)], [[4](#_bookmark12), Example 2.6.1], [[13](#_bookmark17)]). There are even coherent well-filtered spaces which are non-sober.

We now show that if we add the first countability, then coherence and well- filteredness imply sobriety. Using this result we deduce that if a first countable *T*1 space has a bounded complete dcpo model, then it is sober.

**Lemma 5.1** *Let X be a coherent and ﬁrst countable space. Then X is well ﬁltered if and only if it is sober.*

**Proof.** We only need to show that *X* is sober if it is well-filtered. Suppose that

*X* is well-filtered. If *X* is not sober, there is an irreducible closed subset *X*˜ which

is not the closure of any single point. We consider *Max*(*X*˜), the set of maximal points of *X*˜ which is also irreducible, although it may not be closed.

Then for any *x, y ∈ Max*(*X*˜), there exist two countable neighborhood bases

*{V i | i ∈* N*}* with *V i*+1 *⊆ V i* and *{V j | j ∈* N*}* with *V i*+1 *⊆ V i*. Since *Max*(*X*˜)

*x*

*x*

*x*

*y*

*y*

*y*

is irreducible, *V i ∩ V j ∩ Max*(*X*˜) */*= *∅* for any *i, j ∈* N. For any *i ∈* N, we take

*x*

*y*

*a ∈ V i ∩V j ∩Max*(*X*˜). By the choice of each *a* , each neighbourhood of *x* contains

*i*

*x*

*y*

*i*

all *ai*, except finite elements. Thus, *{ai | i ∈* N*} ∪ {x}* is compact. Similarly,

*{ai | i ∈* N*}∪ {y}* is also compact. By the coherence of *X*, we have *{ai | i ∈* N*}* = (*{ai | i ∈* N*}∪ {x}*) *∩* (*{ai | i ∈* N*}∪ {y}*) is compact. Let

*A*0 = *{ai | i ≥* 0*,i ∈* N*}, A*1 = *{ai | i ≥* 1*,i ∈* N*},*

*A*2 = *{ai | i ≥* 2*,i ∈* N*}, ··· , Ak* = *{ai | i ≥ k, i ∈* N*}, ··· .*

Note that *Ak* = (*{ai | i ≥ k, i ∈* N*}∪ {x}*) *∩* (*{ai | i ≥ k, i ∈* N*}∪ {y}*), we have

that *Ak* is compact by the coherence, while *∞ Ak* = *∅*, which contradicts the

*k*=0

well-filteredness. *2*

**Remark 5.2** (1) The set R of all real numbers equipped with the co-countable topology is coherent (because only finite subsets are compact) and well-filtered, but it is non-sober.

(2) The set N of all positive integers equipped with the co-finite topology is first countable and coherent (in this case every subset is compact), but it is non-sober.

For any bounded complete dcpo *P* , the Scott space Σ*P* is coherent [[4](#_bookmark12), Corollary 4.1.8] and well-filtered [[8](#_bookmark14), Corollary 3.2]. It thus follows that the maximal point space Max(*P* ) is also coherent and well-filtered. So the above theorem implies the following result.

**Corollary 5.3** *If a ﬁrst countable space has a bounded complete dcpo model, then it is sober.*

In [[12](#_bookmark16), Corollary 2.14], it was proved that every Hausdorff *k*-space (particularly, every first countable Hausdorff space) has a bounded complete dcpo model. The above result indicates that the sobriety is a necessary condition for a first countable *T*1 space to have a bounded complete dcpo model.

However, we do not know the answer to the following problem.

**Problem 1.** If a *k*-space has a bounded complete dcpo model, must it be sober?

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