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T-homotopy and Refinement of Observation (I): Introduction

Philippe Gaucher

*Preuves Programmes et Syst`emes Universit´e Paris 7–Denis Diderot Case 7014*

*2 Place Jussieu*

*75251 PARIS Cedex 05*

*France*

*Email:* [*gaucher@pps.jussieu.fr*](mailto:gaucher@pps.jussieu.fr)

[*http://www.pps.jussieu.fr/∼gaucher/*](http://www.pps.jussieu.fr/~gaucher/)

**Abstract**

This paper is the extended introduction of a series of papers about modelling T-homotopy by refinement of observation. The notion of T-homotopy equivalence is discussed. A new one is proposed and its behaviour with respect to other construction in dihomotopy theory is explained.

*Keywords:* Concurrency, homotopy, directed homotopy, model category, refinement of observation, poset, cofibration

# About deformations of HDA

The main feature of the two algebraic topological models of *higher dimensional automata* (or HDA) introduced in [[8](#_bookmark13)] and in [[4](#_bookmark6)] is to provide a framework for mod- elling continuous deformations of HDA corresponding to subdivision or refinement of observation. Globular complexes and flows are specially designed to modelling the *weak S-homotopy equivalences* (the spatial deformations) and the *T-homotopy equivalences* (the temporal deformations). The first descriptions of spatial defor- mation and of temporal deformation dates back from the informal and conjectural paper [[3](#_bookmark7)].

Let us now explain a little bit what the spatial and temporal deformations consist of before presenting the results. The computer-scientific and geometric explanations of [[8](#_bookmark13)] must of course be preferred for a deeper understanding.

In dihomotopy theory, processes running concurrently cannot be distinguished by any observation. For instance in Figure [1](#_bookmark0), each axis of coordinates represents one process and the two processes are running concurrently. The corresponding

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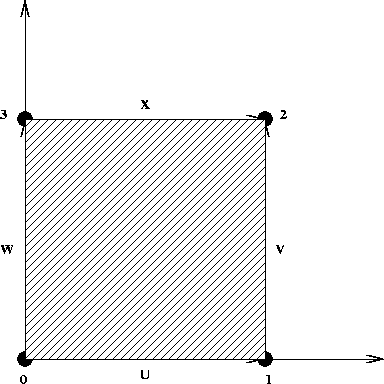


Fig. 1. Two concurrent processes

geometric shape is a full 2-cube. This example corresponds to the flow ***C***2 defined as follows:

* Let us introduce the flow *∂****C***2 defined by (*∂****C***2)0 = *{*0*,* 1*,* 2*,* 3*}*, P0*,*1*∂****C***2 = *{U}*, P1*,*2*∂****C***2 = *{V }*, P0*,*3*∂****C***2 = *{W}*, P3*,*2*∂****C***2 = *{X}*. The flow *∂****C***2 corresponds to an empty square, where the execution paths *U ∗ V* and *W ∗ X* are *not* running concurrently.
* Then consider the pushout diagram

Glob(**S**0) *q*  *∂****C***

2

J J

Glob(**D**1) ***C*** 2

with *q*(**S**0) = *{U ∗ V, W ∗ X}* (the globe functor Glob(*−*) is defined below). The presence of Glob(**D**1) creates a *S-homotopy* between the execution paths *U ∗ V* and *W ∗ X*, modelling this way the concurrency.

It does not matter for P0*,*2***C***2 to be homeomorphic to **D**1 or only homotopy equiv- alent to **D**1, or even only weakly homotopy equivalent to **D**1. The only thing that matters is that the topological space P0*,*2***C***2 be weakly contractible. Indeed, a hole like in the flow *∂****C***2 (the space P0*,*2*∂****C***2 is the discrete space *{U ∗ V, W ∗ X}* ) means that the execution paths *U ∗V* and *W ∗X* are not running concurrently, and therefore that they are distinguishable by observation. This kind of identification is well taken into account by the notion of weak S-homotopy equivalence. This notion is introduced in [[8](#_bookmark13)] in the framework of globular complexes, in [[4](#_bookmark6)] in the framework of flows and it is proved that these two notions are equivalent in [[5](#_bookmark10)].

In dihomotopy theory, it is also required to obtain descriptions of HDA which are invariant by refinement of observation. The simplest example of refinement of observation is represented in Figure [2](#_bookmark1), in which the directed segment *U* is divided in two directed segments *U'* and *U''*. This kind of identification is well taken into account by the notion of T-homotopy equivalence. This notion is introduced in [[8](#_bookmark13)]

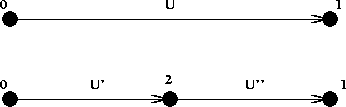


Fig. 2. The most simple example of T-homotopy equivalence

in the framework of globular complexes, and in [[5](#_bookmark10)] in the framework of flows. The latter paper also proves that the two notions are equivalent. In the case of Figure [2](#_bookmark1), the T-homotopy equivalence is the unique morphism of flows sending *U* to *U' ∗ U''*. Each weak S-homotopy equivalence as well as each T-homotopy equivalence

preserves as above the initial states and the final states of a flow. More generally, any good notion of dihomotopy equivalence must preserve the *branching and merging homology theories* introduced in [[7](#_bookmark12)]. This paradigm dates from the beginning of

dihomotopy theory: a dihomotopy equivalence must not change the topological configuration of branching and merging areas of execution paths [[9](#_bookmark14)]. It is also clear that any good notion of dihomotopy equivalence must preserve the *underlying homotopy type*, that is the topological space, defined only up to weak homotopy equivalence, obtained after removing the time flow. In the case of Figure [1](#_bookmark0) and Figure [2](#_bookmark1), this underlying homotopy type is the one of the point.

# Prerequisites and notations

The initial object (resp. the terminal object) of a category *C*, if it exists, is denoted by ∅ (resp. **1**).

Let *C* be a cocomplete category. If *I* is a set of morphisms of *C*, then the class of morphisms of *C* that satisfy the RLP (*right lifting property* ) with respect to any morphism of *I* is denoted by **inj**(*I*) and the class of morphisms of *C* that are transfinite compositions of pushouts of elements of *I* is denoted by **cell**(*I*). Denote by **cof** (*I*) the class of morphisms of *C* that satisfy the LLP (*left lifting property* ) with respect to any morphism of **inj**(*I*). It is a purely categorical fact that **cell**(*I*) *⊂* **cof** (*I*). Moreover, any morphism of **cof** (*I*) is a retract of a morphism of **cell**(*I*). An element of **cell**(*I*) is called a *relative I-cell complex*. If *X* is an object of *C*, and if the canonical morphism ∅ *−→ X* is a relative *I*-cell complex, one says that *X* is a *I-cell complex*.

Let *C* be a cocomplete category with a distinguished set of morphisms *I*. Then let **cell**(*C,I*) be the full subcategory of *C* consisting of the objects *X* of *C* such that the canonical morphism ∅ *−→ X* is an object of **cell**(*I*). In other terms, **cell**(*C,I*) = (∅ *↓C*) *∩* **cell**(*I*).

Possible references for *model categories* are [[11](#_bookmark16)], [[10](#_bookmark15)] and [[2](#_bookmark8)]. The original ref- erence is [[14](#_bookmark18)] but Quillen’s axiomatization is not used in this paper. The axiom- atization from Hovey’s book is preferred. If *M* is a cofibrantly generated model category with set of generating cofibrations *I*, let **cell**(*M*) := **cell**(*M,I*). A cofi- brantly generated model structure *M* comes with a *coﬁbrant replacement functor Q* : *M −→* **cell**(*M*).

A *partially ordered set* (*P,* ≤) (or *poset* ) is a set equipped with a reflexive an- tisymmetric and transitive binary relation ≤. A poset is *locally ﬁnite* if for any (*x, y*) *∈ P × P* , the set [*x, y*] = *{z ∈ P, x* ≤ *z* ≤ *y}* is finite. A poset (*P,* ≤) is *bounded* if there exist ^0 *∈ P* and ^1 *∈ P* such that *P ⊂* [^0*,* ^1] and such that ^0 */*= ^1. Let ^0 = min *P* (the bottom element) and ^1 = max *P* (the top element).

The category **Top** of *compactly generated topological spaces* (i.e. of weak Haus- dorff *k*-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [[1](#_bookmark9),[13](#_bookmark19)], the appendix of [[12](#_bookmark17)] and also the preliminaries of [[4](#_bookmark6)]). For the sequel, any topological space will be supposed to be compactly generated. A *compact space* is always Hausdorff.

The time flow of a higher dimensional automaton is encoded in an object called a *flow* [[4](#_bookmark6)]. A flow *X* consists of a set *X*0 called the 0*-skeleton* and whose elements correspond to the *states* (or *constant execution paths*) of the higher dimensional automaton. For each pair of states (*α, β*) *∈ X*0 *× X*0, there is a topological space P*α,βX* whose elements correspond to the *(nonconstant) execution paths* of the higher dimensional automaton *beginning at α* and *ending at β*. If *x ∈* P*α,βX* , let *α* = *s*(*x*) and *β* = *t*(*x*). For each triple (*α, β, γ*) *∈ X*0*×X*0*×X*0, there exists a continuous map

*∗* : P*α,βX ×* P*β,γX −→* P*α,γX* called the *composition law* which is supposed to be associative in an obvious sense. The topological space P*X* = .(*α,β*)*∈X*0*×X*0 P*α,βX* is called the *path space* of *X*. The category of flows is denoted by **Flow**. A point *α* of

*X*0 such that there are no non-constant execution paths ending to *α* (resp. starting

from *α*) is called an *initial state* (resp. a *ﬁnal state*). A morphism of flows *f* from *X* to *Y* consists of a set map *f* 0 : *X*0 *−→ Y* 0 and a continuous map P*f* : P*X −→* P*Y* preserving the structure. A flow is therefore “almost” a small category enriched in **Top**.

The category **Flow** is equipped with the unique model structure such that [[4](#_bookmark6)]:

* The weak equivalences are the *weak S-homotopy equivalences*, i.e. the morphisms of flows *f* : *X −→ Y* such that *f* 0 : *X*0 *−→ Y* 0 is a bijection and such that P*f* : P*X −→* P*Y* is a weak homotopy equivalence.
* The fibrations are the morphisms of flows *f* : *X −→ Y* such that P*f* : P*X −→* P*Y*

is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set *Igl* = *Igl ∪ {R, C}* with

+

*Igl* = *{*Glob(**S***n−*1) *⊂* Glob(**D***n*)*,n* ≥ 0*}*

where **D***n* is the *n*-dimensional disk, where **S***n−*1 is the (*n −* 1)-dimensional sphere, where *R* and *C* are the set maps *R* : *{*0*,* 1*} −→ {*0*}* and *C* : ∅ *−→ {*0*}* and where for any topological space *Z*, the flow Glob(*Z*) is the flow defined by Glob(*Z*)0 = *{*^0*,* ^1*}*, PGlob(*Z*) = *Z*, *s* = ^0 and *t* = ^1, and a trivial composition law. The set of generating trivial cofibrations is

*Jgl* = *{*Glob(**D***n × {*0*}*) *⊂* Glob(**D***n ×* [0*,* 1])*,n* ≥ 0*}.*

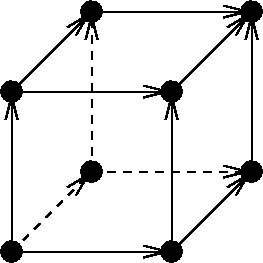


Fig. 3. The full 3-cube

# Why adding new T-homotopy equivalences ?

It turns out that the T-homotopy equivalences, as defined in [[5](#_bookmark10)], are the deforma- tions which locally act like in Figure [2](#_bookmark1) [1](#_bookmark4) . So it becomes impossible with this old definition to identify the directed segment of Figure [2](#_bookmark1) with the full 3-cube of Fig- ure [3](#_bookmark2) by a zig-zag sequence of weak S-homotopy and of T-homotopy equivalences preserving the initial state and the final state of the 3-cube since any point of the 3-cube is related to three distinct edges (cf. Theorem [3.4](#_bookmark3)). This contradicts the fact that concurrent execution paths cannot be distinguished by observation. More precisely, one has:

**Proposition 3.1** *Let X and Y be two flows. There exists a unique structure of flows X ⊗ Y on the set X × Y such that*

* 1. (*X ⊗ Y* )0 = *X*0 *× Y* 0
  2. P(*X ⊗ Y* ) = (P*X ×* P*Y* ) *∪* (*X*0 *×* P*Y* ) *∪* (P*X × Y* 0)
  3. *s*(*x, y*) = (*s*(*x*)*, s*(*y*))*, t*(*x, y*) = (*t*(*x*)*, t*(*y*))*,* (*x, y*) *∗* (*z, t*) = (*x ∗ z, y ∗ t*)*.*

**Definition 3.2** *The* directed segment ***I*** *is the flow* Glob(*Z*) *with Z* = *{u}.*

**Definition 3.3** *Let n* ≥ 1*. The* full *n*-cube ***C****n is by deﬁnition the flow Q*(***I****⊗n*)*, where Q is the coﬁbrant replacement functor.*

Notice that for *n* ≥ 2, the flow ***I****⊗n* is not cofibrant. Indeed, the composition law contains relations. For instance, with *n* = 2, one has (^0*, u*) *∗* (*u,* ^1) = (*u,* ^0) *∗* (^1*, u*)

**Theorem 3.4** *Let n* ≥ 3*. There does not exist any zig-zag sequence*

***C****n* = *X*0

*f*0 *X* ¸*f*,1

*X*  *f*2 *. .* *.* ¸*f*,2*n−*1*X* = ***I***

*where each Xi is an object of* **cell**(**Flow**) *and where each morphism fi is either a S-homotopy equivalence* [2](#_bookmark5) *or a T-homotopy equivalence.*

1

2

2*n*

We must suppose in the statement of Theorem [3.4](#_bookmark3) that each flow *Xi* belongs to

**cell**(**Flow**) because T-homotopy is only defined between this kind of flow.

1 This fact was of course not known when [[8](#_bookmark13)] was being written down. The definition of T-homotopy equivalence presented in that paper was based on the notion of homeomorphism and it sounded so natural...

2 Recall that a morphism between two objects of **cell**(**Flow**) is a weak S-homotopy equivalence if and only if it is a S-homotopy equivalence.

*A*,≤ *B* *,*

≤ *cccc*

*cccc*

^0 *,,*

*,,*

*,*≤

*,,,,*≤

*,,,*zz

*,,* ^1 ¸

*,,*

≤*,,,,*

*,,,*zz*,,,,,*

*,*

*C*

Fig. 4. Example of finite bounded poset

# Full directed ball

We need to generalize the notion of subdivision of the directed segment ***I***.

**Definition 4.1** *A flow X is* loopless *if for every α ∈ X*0*, the space* P*α,αX is empty.*

A flow *X* is loopless if and only if the transitive closure of the set *{*(*α, β*) *∈*

*X*0 *× X*0 such that P*α,βX /*= ∅*}* induces a partial ordering on *X*0.

**Definition 4.2** *A* full directed ball *is a flow* ***D*** *such that:*

* *the* 0*-skeleton* ***D***0 *is ﬁnite*
* ***D*** *has exactly one initial state* ^0 *and one ﬁnal state* ^1 *with* ^0 */*= ^1
* *each state α of* ***D***0 *is between* ^0 *and* ^1*, that is there exists an execution path from*

^0 *to α, and another execution path from α to* ^1

* ***D*** *is loopless*
* *for any* (*α, β*) *∈* ***D***0 *×* ***D***0*, the topological space* P*α,β****D*** *is empty or weakly con- tractible.*

Let ***D*** be a full directed ball. Then the set ***D***0 can be viewed as a finite bounded poset. Conversely, if *P* is a finite bounded poset, let us consider the *flow F* (*P* ) *associated to P* : it is of course defined as the unique flow (up to isomorphism) *F* (*P* ) such that *F* (*P* )0 = *P* and P*α,βF* (*P* ) = *{u}* if *α < β* and P*α,βF* (*P* ) = ∅ otherwise. Then *F* (*P* ) is a full directed ball and for any full directed ball ***D***, the two flows ***D*** and *F* (***D***0) are weakly S-homotopy equivalent.

Let ***E*** be another full directed ball. Let *f* : ***D*** *—→* ***E*** be a morphism of flows preserving the initial and final states. Then *f* induces a morphism of posets from ***D***0 to ***E***0 such that *f* (min ***D***0) = min ***E***0 and *f* (max ***D***0) = max ***E***0. Hence the following definition:

**Definition 4.3** *Let 7 be the class of morphisms of posets f* : *P*1 *—→ P*2 *such that:*

1. *The posets P*1 *and P*2 *are ﬁnite and bounded.*
2. *The morphism of posets f* : *P*1 *—→ P*2 *is one-to-one; in particular, if x and y*

*are two elements of P*1 *with x < y, then f* (*x*) *< f* (*y*)*.*

1. *One has f* (min *P*1) = min *P*2 *and f* (max *P*1) = max *P*2*.*

*Then a* generalized T-homotopy equivalence *is a morphism of* **cof** (*{Q*(*F* (*f* ))*,f ∈ 7 }*) *where Q is the coﬁbrant replacement functor of* **Flow***.*

In a HDA, a *n*-transition, that is the concurrent execution of *n* processes, is represented by the full *n*-cube ***C****n*. The corresponding poset is the product poset

*{*^0 *<* ^1*}n*. In particular, the poset corresponding to the full directed ball of Figure [3](#_bookmark2)

is *{*^0 *<* ^1*}*3 = *{*^0 *<* ^1*}× {*^0 *<* ^1*}× {*^0 *<* ^1*}*.

The poset corresponding to Figure [1](#_bookmark0) is the poset *{*^0 *<* ^1*}*2 = *{*^0 *<* ^1*}× {*^0 *<* ^1*}*. If for instance *U* is subdivided in two processes as in Figure [2](#_bookmark1), the poset of the full directed ball of Figure [1](#_bookmark0) becomes equal to *{*^0 *<* 2 *<* ^1*}× {*^0 *<* ^1*}*.

One has the isomorphism of flows ***I****⊗n ∼*= *F* (*{*^0 *<* ^1*}n*) for every *n* ≥ 1. The flow

***C****n* (*n* ≥ 1) is identified to ***I*** by the zig-zag sequence of S-homotopy and generalized T-homotopy equivalences

***I*** ¸, *Q*(***I***)

where *gn* : *{*^0 *<* ^1*} —→ {*^0 *<* ^1*}n ∈7* .

*Q*(*F* (*gn*)) *Q*( ***I****⊗n*)*,*

# Is this new definition well-behaved ?

First of all, we must verify that each old T-homotopy equivalence as defined in [[5](#_bookmark10)] will be a particular case of this new definition. And indeed, one has:

**Theorem 5.1** *Let X and Y be two objects of* **cell**(**Flow**)*. Let f* : *X —→ Y be a T-homotopy equivalence as deﬁned in [*[*5*](#_bookmark10)*]. Then f can be written as a composite X —→ Z —→ Y where g* : *X —→ Z is a generalized T-homotopy equivalence and where h* : *Z —→ Y is a weak S-homotopy equivalence.*

The two other tests consist of verifying that the branching and merging homology theories [[7](#_bookmark12)], as well as the underlying homotopy type functor [[5](#_bookmark10)] are preserved with this new definition of T-homotopy equivalence. And indeed, one has:

**Theorem 5.2** *Let f* : *X —→ Y be a generalized T-homotopy equivalence. Then* *for any n* ≥ 0*, the morphisms of abelian groups H−*(*f* ) : *H−*(*X*) *—→ H−*(*Y* ) *and*

*n n n*

*H*+(*f* ) : *H*+(*X*) *—→ H*+(*Y* ) *are isomorphisms of groups where H− (resp.H*+*)*

*n n n n n*

*is the n-th branching (resp. merging) homology group. And the continuous map*

*|f|* : *|X| —→ |Y | is a weak homotopy equivalence where |X| denotes the underlying homotopy type of the flow X.*

# Conclusion

This new definition of T-homotopy equivalence seems to be well-behaved. It will hopefully have a longer lifetime than other ones that the author proposed in the past. It is already known after [[6](#_bookmark11)] that it is impossible to construct a model structure on **Flow** such that the weak equivalences are exactly the weak S-homotopy equivalences and the generalized T-homotopy equivalences. So new models of dihomotopy will be probably necessary to understand the T-homotopy equivalences.

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