Available online at [www.sciencedirect.com](http://www.sciencedirect.com/)

[Electronic Notes in Theoretical Computer Science 346 (2019) 125–133](https://doi.org/10.1016/j.entcs.2019.08.012)

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

The Chromatic Index of Proper Circular-arc Graphs of Odd Maximum Degree which are Chordal

João Pedro W. Bernardi[a](#_bookmark0)*,*[1](#_bookmark2)*,*[3](#_bookmark2) Murilo V. G. da Silva[a](#_bookmark0)*,*[1](#_bookmark2)*,*[4](#_bookmark2) André Luiz P. Guedes[a](#_bookmark0)*,*[1](#_bookmark2)*,*[5](#_bookmark2) Leandro M. Zatesko[a](#_bookmark0)*,*[b](#_bookmark1)*,*[1](#_bookmark2)*,*[2](#_bookmark2)*,*[6](#_bookmark2)

a *Department of Informatics, Federal University of Paraná, Curitiba, Brazil*

b *Federal University of Fronteira Sul, Chapecó, Brazil*

Abstract

The complexity of the edge-coloring problem when restricted to chordal graphs, listed in the famous D. John- son’s NP-completeness column of 1985, is still undetermined. A conjecture of Figueiredo, Meidanis, and Mello, open since the late 1990s, states that all chordal graphs of odd maximum degree Δ have chromatic index equal to Δ. This conjecture has already been proved for proper interval graphs (a subclass of proper circular-arc *∩* chordal graphs) of odd Δ by a technique called pullback. Using a new technique called multi-pullback, we show that this conjecture holds for all proper circular-arc *∩* chordal graphs of odd Δ.

We also believe that this technique can be used for further results on edge-coloring other graph classes.

*Keywords:* Pullback, circular-arc, chromatic index, edge-coloring, chordal

# Introduction

Circular-arc graphs are the intersection graphs of a finite set of arcs on a circle. If no arc properly contains another, the graph is said to be a *proper circular-arc graph*. If all the arcs have the same length, the graph is said to be a *unit circular-arc graph*. Although the class of the circular-arc graphs is well studied, very little is known about deciding the chromatic index of these graphs, except for the subclass consisting of the *n*-vertex proper circular-arc graphs of odd maximum degree Δ

1 Partially supported by CNPq (Proc. 428941/2016-8 and a Master’s grant).

2 Partially supported by UFFS (Proc. 23205.001243/2016-30).

3 [winckler@ufpr.br](mailto:winckler@ufpr.br)

4 [murilo@inf.ufpr.br](mailto:murilo@inf.ufpr.br)

5 [andre@inf.ufpr.br](mailto:andre@inf.ufpr.br)

6 [leandro.zatesko@uffs.edu.br](mailto:leandro.zatesko@uffs.edu.br)

<https://doi.org/10.1016/j.entcs.2019.08.012>

1571-0661/© 2019 The Author(s). Published by Elsevier B.V.

This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

which have *n /≡*1*,* Δ (mod (Δ + 1)) and a maximal clique of size two, or which have

*n ≡* 0 (mod (Δ + 1)) [[1](#_bookmark17)].

Circular-arc graphs form a superclass of interval graphs. An important difference between these two classes is that interval graphs have a linear number of maximal cliques (in the number of vertices), while circular-arc graphs may have an exponential number of maximal cliques. This may suggest why some problems are more difficult for circular-arc graphs than for interval graphs. For instance, vertex-coloring is polynomial for interval graphs, but NP-hard for circular-arc graphs [[4](#_bookmark20)].

The NP-hard [[5](#_bookmark21)] edge-coloring problem is the problem of determining the min- imum amount of colors needed to color the edges of a graph such that no two adjacent edges receive the same color. This amount is called the chromatic index of *G*, denoted *χj*(*G*). By definition, *χj*(*G*) *≥* Δ(*G*) for any graph *G*. The cele- brated Vizing’s Theorem brings that *χj*(*G*) *≤* Δ(*G*)+1 [[10](#_bookmark26)]. Therefore, graphs which satisfy *χj*(*G*)= Δ(*G*) are referred to as *Class 1* graphs, and those satisfying *χj*(*G*)= Δ(*G*)+1 are referred to as *Class 2*. For instance, a complete graph *Kn* is *Class 1* if *n* is even, and *Class 2* otherwise.

We solve the edge-coloring problem in the class of *proper circular-arc ∩ chordal* (PCAC) graphs of odd maximum degree, that is, we prove that all these graphs are *Class 1* and our proof yields a polynomial-time exact edge-coloring algorithm for these graphs. It is important to remark that even for proper interval graphs (often referred to as *indiﬀerence* graphs in the literature), which form an important subclass of PCAC graphs, the problem is solved only for graphs with odd maximum degree Δ, by a technique called *pullback* [[2](#_bookmark18)]. Later, this technique was also used to solve the edge-coloring problem for all dually chordal graphs (which form a superclass of interval graphs) of odd Δ [[3](#_bookmark19)]. The complexity of determining the chromatic index of chordal graphs is one of the problems in the famous D. Johnson’s NP*-completeness*

*column* [[6](#_bookmark22)] which are still open, even restricted to graphs of odd Δ.

To solve the problem for the PCAC graphs of odd maximum degree, we design a new technique called *multi-pullback*, which we suspect that can be used for other graph classes.

This paper is organized as follows: the remaining of this section is dedicated to some preliminary definitions; in Section [2](#_bookmark3) we discuss the pullback functions intro- duced in [[2](#_bookmark18)] and present our multi-pullback functions; then, in Section [3](#_bookmark8) we present our results on PCAC graphs using the multi-pullback functions introduced in Sec- tion [2](#_bookmark3).

**Preliminary definitions**

In this paper, graph-theoretical definitions follow their usual meanings in the literature. In particular, *G* = (*V* (*G*)*, E*(*G*)) is a graph, *V* (*G*) is the set of vertices of *G* and *E*(*G*) is the set of edges of *G*. An edge *uv* is said to be *incident* to the vertices *u* and *v*, and the vertices *u* and *v* are said to be *neighbors*. The *degree* of a vertex *u*, denoted *dG*(*u*), is the number of edges that are incident to the vertex *u*. The *maximum degree* of *G* is Δ(*G*) := max*{dG*(*u*): *u ∈ V* (*G*)*}*. The *open neighborhood* of *u* is the set *NG*(*u*) := *{v* : *uv ∈ E*(*G*)*}*. The *closed neighborhood* of *u* is the set

*NG*[*u*] := *NG*(*u*) *∪ {u}*.

If *NG*[*u*] = *V* (*G*), then the vertex *u* is said to be *universal* in *G*. We say that a graph *H* is a *subgraph* of *G* if *V* (*H*) *⊂ V* (*G*) and *E*(*H*) *⊂ E*(*G*). Let *U ⊂ V* (*G*). The subgraph of *G induced* by *U* is defined by *G*[*U* ] := (*U, {uv ∈ E*(*G*): *u, v ∈ U}*). Let *F ⊂ E*(*G*). The subgraph of *G induced* by *F* is defined by *G*[*F* ] := (*{u* : *uv ∈ F* for some *v ∈ V* (*G*)*, F* ). The *core* of a graph is the subgraph induced by its vertices of maximum degree. The *semi-core* of a graph is the subgraph induced by the vertices of maximum degree and their neighbors.

A *k-edge-coloring* of *G* is a proper edge-coloring of *G* with *k* colors, that is, an assignment of colors to the edges of a graph in such a way that no two adjacent edges receive the same color and that at most *k* colors are used. A set *U ⊂ V* (*G*) is said to be a *clique* if it induces a complete graph in *G*. A clique is said to be *maximal* if it is not properly contained in any other clique. A *simplicial vertex* in *G* is a vertex that belongs to only one maximal clique of *G*.

# Pullback and multi-pullback functions

A function *f* : *V* (*G*) *→ V* (*Gj*) is said to be a *pullback* if it is a homomorphism (i.e. for all *uv ∈ E*(*G*) we have *f* (*u*)*f* (*v*) *∈ E*(*Gj*)), and if *f* is injective when restricted to *NG*[*u*] for all *u ∈ V* (*G*).

**Lemma 2.1 ([**[**2**](#_bookmark18)**,**[**3**](#_bookmark19)**])** *If f is a pullback from G to Gj and λj is an edge-coloring of Gj, then the function λ*(*uv*) := *λj*(*f* (*u*)*f* (*y*)) *is an edge-coloring of G.*

*b*

*b*

*G c c*

*a b a b a*

*Gj* 0 *c* 2

*a a*

0 1 2 Δ 0 1

=3 1 *c*

Δ=3

Figure 1. Example of a pullback from an indifference graph *G* to *G′* := *K*4

**Definition 2.2** Let *G* = (*V, E*) be a graph with *E /*=*∅* and let *{E*1*,..., Et}* be a partition of *E*. A *multi-pullback F* from *G* to a collection of *t* graphs *{Gj ,..., Gj }*

1 *t*

is a collection of *t* functions *{f*1*,..., ft}* such that:

* 1. *fi* is a pullback from *G*[*Ei*] to *Gj* ;

*i*

* 1. there is some positive integer *k* and some collection of *k*-edge-colorings *λj ,..., λj*

1 *t*

of *Gj ,..., Gj* , respectively, such that the edge-colorings obtained from *λj ,..., λj*

1 *t* 1 *t*

and the pullbacks *f*1*,..., ft* do not create any color conflict on the edges of *G*, that is, the function defined by

*λ*(*uv*) := *λj* (*fi*(*u*)*fi*(*v*)) , being *Ei the* set of the partition to which *uv* belongs, is a proper *k*-edge-coloring of *G*.

*i*

Observe the necessity of including ([ii](#_bookmark5)) in Definition [2.2](#_bookmark4), otherwise the pullbacks *f*1*,..., ft* could define *non-compatible* edge-colorings (that is, color conflicts could be created when assembling all edge-colorings in order to construct the edge-coloring of *G*). Also in the definition, observe that disjointness is assumed only among the sets of the partition *{E*1*,..., Et}*, but not among the domains of the functions in *F* , which are sets of vertices, not edges. This means that a single vertex *u* can be

mapped to a vertex *v* of *Gj* by a pullback *fi* and to a different vertex *w* of *Gj* by

*i*

*j*

a pullback *fj*, depending on which *role* we want *u* to assume in order to color each

edge incident to *u*.

Figure [2](#_bookmark7) shows an example of a collection of functions *{f*1*, f*2*, f*3*}* which can be verified to be a multi-pullback from a PCAC graph *G* with Δ=5 to the *K*6, under

the 5-edge-colorings *λj*

1

2

3

= *λj*

= *λj*

=: *λj* of the *K*6 defined by

*λj*(*uv*)= (*u* + *v*) mod Δ , if neither *u* nor *v* is Δ; (2*v*) mod Δ , if *u* = Δ.

(

(1)

*G* 1 = *Gj*

*G*

2

3

*j*

= *Gj*

= *K*6

0 3

1

2

Δ

4

*G*[*E*1]

1 Δ 2

*G*[*E*2]

*G*[*E*3]

3

0

2*∗*

0

1*∗* 4

4

Figure 2. Example of a multi-pullback from *G* to the *K*6 under the 5-edge-coloring defined in ([1](#_bookmark6)). Observe that the vertex marked with an asterisk is mapped to two distinct vertices by *f*2 and by *f*3, with no color conflict being created.

# The result

Proper circular-arc graphs have the consecutive 1’s property [[9](#_bookmark25)], i.e. there is a circular order for the vertices in such a way that for every edge *−u→v* under the clockwise

orientation of the edges along this order, all the vertices clockwise between *u* and *v* induce a complete graph in the original undirected graph. This order is called a *proper circular-arc order*.

**Lemma 3.1** *Let G be a PCAC graph of odd maximum degree. If G has a universal vertex, or if the semi-core of G is an indiﬀerence graph, then G is* Class 1*.*

**Proof** Observe first that if *G* has a universal vertex, then *G* is a subgraph of *K*Δ(*G*)+1 and hence *Class 1*. On the other hand, if the semi-core of *G* is an indiffer- ence graph, then *G* is also *Class 1* because the chromatic index of a graph is equal to the chromatic index of its semi-core [[7](#_bookmark23)], and because all indifference graphs of odd maximum degree are *Class 1* [[2](#_bookmark18)]. *2*

Lemma [3.2](#_bookmark10) below provides a full characterization of the structure of proper circular-arc *∩* chordal graphs which do not satisfy Lemma [3.1](#_bookmark9).

**Lemma 3.2** *If G is a PCAC graph of odd maximum degree with no universal vertex such that the semi-core S of G is not an indiﬀerence graph, then S* = *G and there is a* 6*-partition {YA, YAB, YB, YBC, YC, YAC} of V* (*G*) *which splits any proper circular- arc order σ of G into six contiguous subsequences of σ in a manner that, being the cardinality of each set in the partition denoted by lowercase y with the corresponding subscript:*

* 1. *the graph G has exactly four maximal cliques, which can be given by XA* :=

*{YAB ∪ YA ∪ YAC}, XB* := *{YBC ∪ YB ∪ YAB}, XC* := *{YAC ∪ YC ∪ YBC},*

*and Z* = *{YAB ∪ YAC ∪ YBC},wherein XA is assumed without loss of generality to be of maximum size among the cliques XA, XB, and XC, which are the cliques which appear contiguously in σ (that is, all the vertices in each of these cliques appear consecutively in σ);*

* 1. *all the vertices in YAB and in YAC have degree* Δ(*G*) *in G;*
  2. Δ(*G*)= *yA* + *yB* + *yAB* + *yBC* + *yAC −* 1= *yA* + *yC* + *yAB* + *yBC* + *yAC −* 1*;*
  3. *yA ≥ yB* = *yC;*

**Proof** Let *σ* be a proper circular-arc order of *G* and let (*X*0*, X*1*,* *, Xt−*1) be the

maximal cliques that appear contiguously in *σ*. We must have *t ≥* 3, otherwise it can be straightforwardly shown that *G* is an indifference graph.

We claim that there is no *Xi* such that *Xi ⊂ X*(*i−*1) mod *t ∪ X*(*i*+1) mod *t*. If this claim holds, an induced cycle of size *t* is easily obtained by choosing one vertex from each *Xi ∩ X*(*i*+1) mod *t*. Because *G* is chordal and it is not an indifference graph, we have *t* = 3. These three maximal cliques of *G* that appear contiguously in *σ* are *XA*, *XB*, and *XC*, respectively.

Since *G* is not an indifference graph, we have that the intersection of two con- secutive cliques in *σ* is not empty (otherwise in any circular-arc model of *G* there would be a point on the circumference which would be uncovered by any arc). We define the sets *YAB* := *XA ∩ XB*, *YBC* := *XB ∩ XC*, and *YAC* := *XA ∩ XC*, and also *YA* := *XA \* (*YAB ∪ YAC*), *YB* := *XB \* (*YAB ∪ YBC*), and *YC* := *XC \* (*YAC ∪ YBC*).

As all the vertices in *YAB ∪ YBC ∪ YAC* are neighbors of each other, there is a fourth maximal clique *Z* := *YAB ∪ YBC ∪ YAC* that does not appear contiguously in *σ*.

Up to this point, we have proven that if the claim holds then there are at least three maximal cliques (*XA*, *XB*, and *XC*) which appear contiguously in *σ*, as well as the fourth clique *Z*. We have also proven that the sets *YAB*, *YBC*, and *YAC* are not empty. We can further demonstrate that the sets *YA*, *YB*, and *YC* are non-empty, which is equivalent to prove that each of the cliques *XA*, *XB*, and *XC* has a simplicial

vertex. If *YA* = *∅*, then every vertex of *YBC* is universal (see Figure [3](#_bookmark14)), contradicting the hypothesis. The non-emptiness of *YB* and *YC* follows analogously.

*XA*

*YA*

*YAB*

*YAC*

*YB*

*YC*

*XB*

*Y*

*BC*

*XC*

Figure 3. Structure of a PCAC graph according to Lemma [3.2.](#_bookmark10)

Now we shall prove the claim and that *XA*, *XB*, and *XC* are the only maximal cliques contiguously in *σ*. Assume for the sake of contradiction that there is a fourth maximal clique *XD* contiguously in *σ*. Since the intersections *YAB*, *YBC*, and *YAC* are all non-empty, the clique *XD* must be contained in the union of two cliques from

*{XA, XB, XC}*. Without loss of generality, *XD ⊂ XA ∪XB*. By the same arguments presented above, the four sets (*XD ∩ XA*) *\* (*XB ∪ XC*), (*XD ∩ XB*) *\* (*XA ∪ XC*), (*XB ∩ XC*) *\ XD*, and (*XC ∩ XA*) *\ XD* are all non-empty. Ergo, by choosing

four vertices, one from each of these sets, we obtain an induced cycle of size four, contradicting the fact that *G* is chordal. Hence, we have proven that *XD* cannot exist and also that the claim holds. Furthermore, since all the vertices in *YA*, *YB*, and *YC* are simplicial, the only maximal clique which can be formed not contiguously in *σ* is the clique *Z* (recall Figure [3](#_bookmark14)).

Assuming without loss of generality that *XA* is of maximum size among *XA*, *XB*, and *XC*, it remains to demonstrate ([ii](#_bookmark11))–([iv](#_bookmark13)). Clearly, the vertices of maximum degree in *G* are in *YAB ∪ YBC ∪ YAC*. We shall demonstrate that either *YAB* and *YAC*, or all the sets from *{YAB, YAC, YBC}* have vertices of maximum degree (this

proves ([ii](#_bookmark11))). If only one set *I* from *{YAB, YAC, YBC}* has vertices of maximum de- gree in *G*, then surely *I /*=*YBC*, because of the assumption on the cardinality of *XA*. If *I* = *YAB*, then the semi-core of *G* is an indifference graph, because the order *YB, YBC, YAB, YAC, YA* is an indifference order [7.](#_bookmark15) The case *I* = *YAC* follows

analogously. Remark that this also proves that the semi-core of *G* equals *G*.

Notice that vertices which belong to the same set from *{YA, YAB, YB, YBC, YC, YAC}* have the same closed neighborhood and hence the same degree. Let *u* be a vertex in *YAB*, *v* a vertex in *YAC*, and *w* a vertex in *YBC*. We know that Δ(*G*)= *dG*(*u*)= *dG*(*v*) *≥ dG*(*w*) and also that:

*dG*(*u*)= *yBC* + *yB* + *yAB* + *yA* + *yAC −* 1 ; *dG*(*v*)= *yAB* + *yA* + *yAC* + *yC* + *yBC −* 1 ; *dG*(*w*)= *yAB* + *yB* + *yBC* + *yC* + *yAC −* 1 .

7 An indifference order of an indifference graph is a linear order of the vertices so that vertices belonging to the same maximal clique appear consecutively in this order [[8]](#_bookmark24).

From these equations, we have ([iii](#_bookmark12)) and also that *yB* = *yC* and *yA ≥ yB*, completing the proof of ([iv](#_bookmark13)). *2*

**Theorem 3.3** *Every proper circular-arc ∩ chordal graph with odd maximum degree is Class 1.*

**Proof** In view of Lemma [3.1](#_bookmark9), let *G* be a PCAC graph of odd maximum degree with no universal vertex such that the semi-core of *G* is not an indifference graph. Let also *{YA, YAB, YB, YBC, YC, YAC}* be a partition of *V* (*G*) as in Lemma [3.2](#_bookmark10) (recall Figure [3](#_bookmark14)). Let *{E*1*, E*2*, E*3*, E*4*}* be the partition of *E*(*G*) defined by:

*E*1 := *E*(*G*[*YA ∪ YAB ∪ YB ∪ YBC ∪ YAC*]) ;

*E*2 := *{uv* : *u ∈ YAC* and *v ∈ C}* ; *E*3 := *{uv* : *u ∈ YBC* and *v ∈ C}* ; *E*4 := *E*(*G*[*C*]) .

Let *V* (*K*Δ(*G*)+1)= *{*0*,...,* Δ(*G*)*}* and *V* (*KyC* )= *{*0*,.* *,c −* 1*}*. We shall construct

a multi-pullback *{f*1*, f*2*, f*3*, f*4*}* with *fi* : *Vi → Gj* , for all *i ∈ {*1*,.* *,* 4*}*, being

*i*

*V*1 := *YA ∪ YAB ∪ YB ∪ YBC ∪ YAC* ,

*V*2 := *YAC ∪ YC* , *V*3 := *YBC ∪ YC* , *V*4 := *YC* ,

and being *Gj*

1

:= *Gj*

:= *Gj*

:= *K*Δ(*G*)+1 and *Gj*

= *KyC*

, under the edge-colorings

*λj , λj , λj , λj* defined by *λj* := *λj* := *λj*

2

3

4

:= *λj*, wherein *λj* is the Δ(*G*)-edge-coloring

1 2 3 4 1 2 3

of *K*Δ(*G*)+1 defined in ([1](#_bookmark6)), and *λj* is the Δ(*G*)-edge-coloring of *Ky*

4

*C*

defined by

*fj* (*uv*)= (2*yAC* + *yA* + *yC* + *yBC* + *u* + *v*) mod Δ(*G*) .

4

Remark that *λj* is an optimal edge-coloring of *K*Δ(*G*)+1 (which is *Class 1* since Δ(*G*)

is odd) and that *λ*4 is surely not optimal, since *c <* Δ(*G*) *−* 2.

Remark by Lemma [3.2](#_bookmark10) that *|V*1*|* = Δ(*G*)+ 1. In order to define *f*1, take any bijective labeling function satisfying:

*f*1(*YAC*)= *{*0*,..., yAC −* 1*}* ;

*f*1(*YA*)= *{yAC,..., yAC* + *yA −* 1*}* ;

*f*1(*YB*)= *{yAC* + *yA,..., yAC* + *yA* + *yB −* 1*}* ;

*f*1(*YBC*)= *{yAC* + *yA* + *yB,..., yAC* + *yA* + *yB* + *yBC −* 1*}* ;

*f*1(*YAB*)= *{yAC* + *yA* + *yB* + *yBC,..., yAC* + *yA* + *yB* + *yBC* + *yAB −* 1*}* .

Here, we use *f*1(*Z*) to denote *z∈Z{f*1(*z*)*}*. Notice that we have used Δ(*G*)+1 distinct labels, from 0 to Δ(*G*), and it is easy to realize that this labeling is a *pullback* from *G*[*E*1] to the *Gj* .

1

It remains to color the edges incident to the vertices of *YC*, that is, it remains to define *f*2*,..., f*4. Remark that *G*[*E*2 *∪ E*3] is a bipartite graph, with parts *YC* and

*YBC ∪ YAC*, and *G*[*E*4] is a complete graph. Figure [4](#_bookmark16) represents the sets *YBC*, *YC*, and *YAC*.

AC

C

BC

Figure 4. The sets *YBC* , *YC* , and *YAC*

Recall that *G*[*E*2] is the bipartite graph induced by the edges between *YAC* and *YC*, and notice that the edges incident to vertices in *YAC* are not incident to vertices in *YB*. This is why we can define *f*2 by assigning to the vertices of *YAC* the same labels which they have been assigned by *f*1, and to the vertices of *YC* the same labels assigned to the vertices of *YB* by *f*1, in the manner that we clarify in the sequel. As *yB* = *yC*, there will be enough labels for all the vertices of *YC*.

Analogously, the graph *G*[*E*3] is the bipartite graph induced by the edges between *YBC* and *YC*. Notice that vertices in *YBC* are not neighbors of vertices in *YA*, therefore, the labels assigned by *f*1 to the vertices in *YA* can be reused by *f*3 to the vertices in *YC* (in the manner that we clarify in the sequel), if the vertices in *YBC* are assigned by *f*3 the same labels which they have been assigned by *f*1. Recall that *yA ≥ yC*, so there will be enough labels.

To complete the proof, it remains only to define which are the three labels as- signed to each vertex in *YC* by *f*2, *f*3, and *f*4, and to show that the edge-coloring obtained through these pullbacks do not create color conflicts in *G*. Let *YC* =

*{u*0*,..., uyC−*1*}*. We define for each *ui ∈ YC* the triplet (*f*2(*ui*)*, f*3(*ui*)*, f*4(*ui*)) := (*yAC* + *yA* + *i, yAC* + *i, i*). Let *λ* be the Δ(*G*)-edge-coloring of *G* as in Definition [2.2](#_bookmark4).

We show that *λ* is a proper edge-coloring, for which it suffices to show that all the colors of the edges incident to the same vertex *ui* in *YC* are different.

The colors of the edges incident to *ui* can be verified to be as follows (all the colors listed below are modΔ(*G*), but this information is omitted for a clear description):

* the colors of the edges of *G*[*E*2] that are incident to *ui* are the *yAC* colors from the set

*{yAC* + *yA* + *i,...,* 2*yAC* + *yA* + *i −* 1*}* ;

* the colors of the edges of *G*[*E*3] that are incident to *ui* are the *yBC* colors from the set

*{*2*yAC* + *yA* + *yB* + *i,...,* 2*yAC* + *yA* + *yB* + *yBC* + *i −* 1*}* ;

* the colors of the edges of *G*[*E*4] that are incident to *ui* are the *yC −* 1 colors from the set

*{*2*yAC*+*yA*+*yB*+*yBC*+*i,...,* 2*yAC*+*yA*+2*yB*+*yBC*+*i}\{*2*yAC*+*yB*+*yB*+*yBC*+2*i}* ;

Notice that, at the edges incident to *ui*, the *yB* colors between (2*yAC* + *yA* + *i*) mod Δ(*G*) and (2*yAC* + *yA* + *yB* + *i −* 1) mod Δ(*G*) are not used, as well as the color

(2*yAC* + *yA* + *yB* + *yBC* + 2*i*) mod Δ(*G*). As *yAC* + *yB* + *yBC* + *yC ≤* Δ(*G*) =

*yAC* + *yBC* + *yAB* + *yA* + *yC −* 1, there is no color conflict at *ui*.

Since we have shown that there is no color conflict at any vertex *ui ∈ YC*, we conclude that *G* is *Class 1*. *2*

# References

1. Bernardi, J. P. W., S. M. Almeida and L. M. Zatesko, *On total and edge-colouring of proper circular-arc*

*graphs*, in: *Proc. 38th Congress of the Brazilian Computer Society (CSBC ’18/III ETC)*, Natal, 2018,

pp. 73–76.

URL <http://natal.uern.br/eventos/csbc2018/wp-content/uploads/2018/08/Anais-ETC-2018.pdf>

1. Figueiredo, C. M. H., J. Meidanis and C. P. Mello, *On edge-colouring indiflerence graphs*, Theor. Comput. Sci. **181** (1997), pp. 91–106.
2. Figueiredo, C. M. H., J. Meidanis and C. P. Mello, *Total-chromatic number and chromatic index of dually chordal graphs*, Inf. Process. Lett. **70** (1999), pp. 147–152.
3. Garey, M. R., D. S. Johnson, G. L. Miller and C. H. Papadimitriou, *The complexity of coloring circular arcs and chords*, SIAM J. Algebraic Discrete Methods **1** (1980), pp. 216–227.
4. Holyer, I., *The* NP*-completeness of edge-colouring*, SIAM J. Comput. **10** (1981), pp. 718–720.
5. Johnson, D. S., *The* NP*-completeness column: an ongoing guide*, J. Algorithms **6** (1985), pp. 434–451.
6. Machado, R. C. S. and C. M. H. Figueiredo, *Decompositions for edge-coloring join graphs and cobipartite graphs*, Discrete Appl. Math. **158** (2010), pp. 1336–1342.
7. Roberts, F. S., *Indiflerence graphs*, in: *Proc. 2nd Ann Arbor Graph Theory Conference*, Ann Arbor, USA, 1969, pp. 139–146.
8. Tucker, A., *Matrix characterizations of circular-arc graphs.*, Pacific J. Math. **39** (1971), pp. 535–545.

URL https://projecteuclid.org:443/euclid.pjm/1102969574

1. Vizing, V. G., *On an estimate of the chromatic class of a p-graph (in Russian)*, Diskret. Analiz. **3**

(1964), pp. 25–30.