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The Duality Theory of General *Z*-continuous Posets

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**Abstract**

In this paper, we research further into *Z*-predistributive and *Z*-precontinuous posets introduced by Ern´e. We focus on duality theorems based on the application of Galois connections whenever *Z* is a closed subset selection. For example, there is a duality between the categories *Z*-**PD***G* and *Z*-**PD***D* of all *Z*-

predistributive posets with weakly *ZΔ*-continuous maps which have a lower adjoint, and maps preserve *Z*-below relation that have an upper adjoint, respectively, as morphisms. We introduce the concept of *Z*0-approximating auxiliary relation, and have made a slight improvement on *Z*-precontinuity, so that there is a generalization of the classical equivalence between domains and auxiliary relations.

*Keywords:* poset, *Z*-predistributive, *Z*-precontinuous, *Z*-closed, Galois connection, auxiliary relation.

# Introduction

The “way-below” relation is an essential ingredient in continuous posets and do- mains [[1,](#_bookmark9)[10,](#_bookmark18)[11](#_bookmark19)] and plays a central role in the applications of computer sciences. Continuous poset is based on the axiom of approximation, where the classical “way- below” relation is associated with all directed subsets which have supremum, but not for arbitrary subsets. In [[16](#_bookmark23)], Wright, J. B., Wagner, E. G. and Thatcher, J. W. introduced the concept of subset systems *Z*, got rid of the restriction to directed subsets and replaced by “*Z*-sets”. After this, a theory of *Z*-continuous posets was developed by Bandelt and Ern´e [[3,](#_bookmark11)[4](#_bookmark12)], Novak [[13](#_bookmark21)], Venugopalan [[15](#_bookmark24)]. The theory has been pursued by others, such as [[2,](#_bookmark10)[8,](#_bookmark16)[9,](#_bookmark17)[19](#_bookmark25)] and so on. *Z*-continuity inherits the basic idea to use variants of the “way-below” relation, associated with *Z*-sets which have a least upper bound.

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In [[8]](#_bookmark16), Ern´e used *Z* to denote a subset selection, which assigns every poset *P* to a certain collection *ZP* of subsets and is more extensive than subset system. Some types of posets with “*Z*-approximation” from below were put forward, for instance, *Z*-predistributive and *Z*-precontinuous posets. The “*Z*-approximation” involves the cut operator *Δ* (others may use *δ*) of subsets instead of the existence of supremum. Ern´e characterized these posets by certain homomorphism properties and adjunctions. In recent years, there are other results about *Z*-precontinuity, see [[12,](#_bookmark20)[14,](#_bookmark22)[17,](#_bookmark26)[18](#_bookmark27)], but none discussed the dual category on these posets. The purpose of our paper is to discuss that.

In Section 3, we give some statements of *Z*-predistributive and *Z*-precontinuous posets. Galois connections play an important role in the framework of category theory. Let *Z* be a closed subset selection. A duality is built up between categories *Z*-**PD***t* and *Z*-**PD***D* of all *Z*-predistributive posets with *ZΔ*-morphisms and *ZΔ*- comorphisms, as morphisms respectively, in particular the full subcategories *Z*-**PC***t* and *Z*-**PC***D* of all *Z*-precontinuous posets. We also show that the image of a *Z*- precontinuous poset under a *ZΔ*-morphism is *Z*-precontinuous. We characterize *Z*-precontinuity with appropriate auxiliary relations in Section 4.

# Preliminaries

Let us recall some basic definitions. For each poset *P* and *A ⊆ P* , we denote

*↓A* := *{y ∈ P |* (*∃ x ∈ A*) *y ≤ x}* and *↓x* := *↓{x}*, *↓A* is said to be the *lower set generated by A*. *Au* := *{x ∈ P |* (*∀y ∈ A*) *y ≤ x}* is called the *upper bound set* of *A*. The least element of *Au* if it exists is called the *supremum* of *A* and is denoted by *A*; *Al* := *{x ∈ P |* (*∀y ∈ A*) *y ≥ x}* is the *lower bound set* of *A*. The *cut operator Δ* is written by *ΔA* := *Aul*. A *subset selection Z* denotes a function which assigns

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to each poset *P* a set *ZP* of subsets of *P* , and *Z* is called a *subset system* if

* 1. there exists a poset *P* such that *ZP* contains some nonempty set;
  2. if *f* : *P → Q* is a monotone map from *P* into a poset *Q*, then *f* (*Z*) *∈ ZQ* for all *Z ∈ ZP* .

By (ii), for any subset *B ⊆ P* , *Z ∈ Z*(*B*) implies *Z ∈ Z*(*P* ). The frequently used examples of subset selections are:

* + - *A* where *AP* is the collection of all lower sets;
    - *B* where *BP* is the collection of all nonempty upper bounded subsets;
    - *C* where *CP* is the collection of all nonempty chains;
    - *D* where *DP* is the collection of all directed subsets;
    - *E* where *EP* is the collection of all one-element subsets;
    - *F* where *FP* is the collection of all finite subsets;
    - *P* where *PP* is the collection of all subsets.

Among these, all except *A* are subset systems. But note that *↓f* (*Z*) *∈ AQ* for all

*Z ∈ AP* .

For any subset selection *Z*, we denote by *ZΛP* = *{↓Z* : *Z ∈ ZP EP}*, the collection of all *Z-ideals*. However, for *A* and subset system *Z*, *ZΛP* is just the set

*{↓Z* : *Z ∈ ZP}*. A subset selection *Z* such that *Y ∈ Z*(*ZΛP* ) implies *Y ∈ ZΛP*

for all posets *P* is called *union-complete*. We denote

*ZΔP* := *{Y ∈ AP | Z ∈ ZP* and *Z ⊆ Y* implies *ΔZ ⊆ Y }*,

this is called *Δ*-*ideal completion* of poset *P* . Obviously, *↓x ∈ ZΔP* for *x ∈ P* and *ΔX ∈ ZΔP* for *X ⊆ P* . The closure *X—* of any subset *X* is defined by *X—* := *{Y ∈ ZΔP | X ⊆ Y }*. Let *σ2* (*P* )= *{P \Y | Y ∈ ZΔP}*, which generalizes the classical Scott topology. It is easy to show that *U ∈ σ2* (*P* ) iff *U* = *↑U* and for all *Z ∈ ZΛP* , *ΔZ ∩ U /*= *∅* implies *Z ∩ U /*= *∅*.

The function *f* is called *weakly ZΔ-continuous* if *f—*1(*↓x*) *∈ ZΔP* for all *x ∈ Q*; *f* is said to be *Z-closed* if for every *Z ∈ ZΛP* implies *↓f* (*Z*) *∈ ZΛQ*. The subset selection *Z* is called *closed* if every monotone map is *Z*-closed. Some tedious manipulation yields that all the subset systems are closed, including subset selection *A*.

# Duality of *Z*-predistributive posets

In this paper, unless otherwise stated, *Z* denotes a subset selection. We will con- sider variants of *Z*-continuity: *Z*-predistributive and *Z*-precontinuous posets, some properties will be given. In order to make connections between categories, we need to define suitable morphisms, that is, *ZΔ*-morphisms and *ZΔ*-comorphisms which involve the Galois connections for any closed subset selection.

Now, we firstly recall the “way-below” relation on posets with respect to *Z*-sets. The *Z-below ideal* of an element *x* in a poset *P* is the set ***↓****2x* = *{Z ∈ ZΛP | x ∈ ΔZ}*. For *x, y ∈ P* , we write *y 2 x* if *Z ∈ ZΛP* and *x ∈ ΔZ* imply *y ∈ Z*, the relation *2* is called *Z-below relation*. Denote the set *{v ∈ P | x 2 v}* by

*↑*

*2x*, and for *A ⊆ P* , ***↓****2*

*↑*

*↑*

*A* = *{u ∈ P |* (*∃ y ∈ A*) *u 2*

*y}*, *↑2*

*A* = *{v ∈ P |* (*∃ y ∈*

1. *y 2 v}*. The properties of the relation *2* are as follows.

**Proposition 3.1** *For any poset P, the following statements hold for x, y, u, v ∈ P:*

* 1. *x 2 y implies x ≤ y;*
  2. *u ≤ x 2 y ≤ v implies u 2 v;*
  3. 0 *2 x whenever P has bottom element* 0 *and x /*= 0*.*

**Remark 3.2** The empty set may confuse us. If *∅∈ ZΛP* whenever *P* has bottom element 0, then 0 *∈ Δ∅*. Thus 0 is impossible to have *Z*-below relationship with itself. Otherwise, 0 *2 x* for all *x ∈ P* .

**Proposition 3.3** *Let P be a poset. Suppose that there exists a Z-set Z ⊆ ↓2x with*

*↓*

*x* = W *Z. Then* ***↓****2x ∈ ZΛP and x* = W *↓2x.*

*↓*

**Definition 3.4** [[8](#_bookmark16)] Let *P* be a poset.

* + 1. *P* is called *Z-predistributive* if *x* = W *↓2x* for each *x ∈ P* ;

*↓*

* + 1. *P* is called *Z-precontinuous* if it is *Z*-predistributive and ***↓****2x ∈ ZΛP* for each

*x ∈ P* .

Actually, *Z*-predistributive posets were called completely *Z*-distributive in [[7](#_bookmark15)]. If *Z* is a subset system, then *Z*-precontinuous is the *Zδ*-continuous poset essentially (see [[18](#_bookmark27)]); for a subset system which requires the existence of a non-singleton *Z*-set, *Z*-predistributive is the weak *sZ*-continuous, and *Z*-precontinuous is *sZ*-continuous in the sense of [[14](#_bookmark22)]. *A*-precontinuous is just the completely precontinuous in [[17](#_bookmark26)], *D*-precontinuous is *s*2-continuous in [[5](#_bookmark13)].

In a continuous poset, the classical “way-below” relation satisfies the *interpo- lation property*, that is, *x y* implies *x u y*. For any union-complete, lower fine subset system *Z*, the interpolation property holds in *Z*-precontinuous poset in the sense of [[18](#_bookmark27)]. We have a similar property for subset selections.

**Proposition 3.5** *Let Z be a union-complete and closed subset selection. Then the*

*Z-below relation of a Z-precontinuous poset P satisﬁes the interpolation property.*

**Proof.** Take *x 2 y* in *P* . Since *P* is a *Z*-precontinuous poset, we have that *y* =

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***↓****2y* and ***↓****2y ∈ ZΛP* . Note that *Δ*(*↓2y*)= *Δ*( *a Z y* ***↓****2a*) and the function *c '→*

*↓*

*↓2c* : *P → ZΛP* is monotone. Then *↓{↓2a* : *a 2 y}* is a *Z*-ideal of *ZΛP* by *Z* being

*↓ ↓*

closed. It suffices that there exists *U ∈ Z*(*ZΛP* ) such that *↓{↓2a* : *a 2 y}* = *↓U*

*↓*

in *ZΛP* . Thus *U ∈ ZΛP* by union-completeness and *Δ*( *a Z y* ***↓****2a*)= *Δ*( *U* ). Hence, *x ∈* ***↓****2a* for some *a 2 y*. *2*

Next, we use the relationship between elements to describe the variants of con- tinuity.

**Theorem 3.6** *Let P be a poset. Then the following conditions are equivalent:*

1. *P is Z-predistributive;*
2. *A ⊆ Δ*(*↓2A*) *for all A ∈ ZΛP;*

*↓*

1. *there is a u ∈ P such that u 2 x with u* ¢ *y whenever x* ¢ *y for x, y ∈ P;*

1. *P \ ↓y* = *{****↑****2 x | x ∈ P \ ↓y} for each y ∈ P;*
2. *P \ ΔA* = ***↑****2* (*P \ A*) *for each A ∈ ZΛP.*

**Proof.** (1) *⇔* (2) *⇔* (3) and (5) *⇒* (4) *⇒* (3), are straightforward.

*2*

(3) *⇒* (5): For every *y ∈* ***↑*** *x* where *x ∈ P \ A*. Assume that *y ∈ ΔA*. Then

*x ∈ A* by the *Z*-below relation, which is contradictory. On the other hand, *P \ΔA ⊆*

***↑****2* (*P \ A*) if the upper bound set of *A* is empty; otherwise, for any *y ∈ P \ ΔA*, then we have *t ∈ Au* such that *y* A *t*. According to (3), there exists *x ∈ ↓2y* such that *x* A *t*, that is, *x* belongs to *P \ ↓t* which contains in the complement of *A*. It follows that *y* is a member of *↑2* (*P \ A*). *2*

*↓*

*↑*

Galois connection is one of the most efficient tools in dealing with complete lattices. Moreover, Galois connections can be used as morphisms to define the

functors between categories. It is a natural point in our study to use them to discuss duality theorems on the variants of *Z*-continuity.

**Definition 3.7** [[10](#_bookmark18)] Let *P* and *Q* be two posets. We say that a pair (*g, d*) of functions *g* : *P → Q* and *d* : *Q → P* is a *Galois connection* or an *adjunction* between *P* and *Q* provided that

1. both *g* and *d* are monotone, and
2. the relations *g*(*s*) *≥ t* and *s ≥ d*(*t*) are equivalent for all pairs of elements (*s, t*) *∈ P × Q*.

In an adjunction (*g, d*), the function *g* is called the *upper adjoint* and *d* the *lower adjoint*.

**Lemma 3.8** *Let* (*g, d*) *be a Galois connection between posets P and Q. Then*

*d*(*ΔA*) *⊆ Δd*(*A*) *for any subset A of Q.*

**Proof.** Let *A* be a subset of poset *Q*. It is formed in case *ΔA* is an empty subset. For each *x ∈ ΔA*, if *d*(*A*)*u* = *∅*, then *Δd*(*A*) = *P* ; otherwise, note that *v ∈ d*(*A*)*u* is equivalent to *g*(*v*) *∈ Au*, then *x ≤ g*(*v*), that is, *d*(*x*) *≤ v*. Therefore *d*(*ΔA*) *⊆ Δd*(*A*) holds in all situations. *2*

The following proposition is needed for later proof, slightly different from in [[12,](#_bookmark20) Proposition 2] and [[6](#_bookmark14), Proposition 1.8].

**Proposition 3.9** *Let f be a map between posets P and Q. Consider the following conditions:*

1. *f is a weakly ZΔ-continuous map;*
2. *for every Z ∈ ZP, f* (*ΔZ*) *⊆ Δf* (*Z*)*.*

*Then (1) implies (2) for any subset selection Z; if Z is a subset system, then (1)*

*e (2). If f is monotone, then conditions are equivalent for all subset selections.*

**Remark 3.10** For an arbitrary subset selection *Z*, the condition of monotonicity of *f* is essential when (2) implies (1) in the above proposition. See the following example as *Z* = *A*. Let P be the set *{a, b, T}* with *a, b ≤T* and Q be the chain **2**. Consider the function *f* which sends *a, b* to1 and *T* to 0, simple verification shows that *f* (*ΔZ*) *⊆ Δf* (*Z*) for every lower set *Z* of *P* . However, a weakly *AΔ*-continuous map is monotone but *f* is not.

From Lemma [3.8](#_bookmark3) and Proposition [3.9](#_bookmark4), a lower adjoint *d* of map *g* between posets is always weakly *ZΔ*-continuous.

There is a well-known duality on posets. The categories **POSET***t* and **POSET***D* have the class of all posets with the order preserving maps *g* which have a lower adjoint *d* and the order preserving maps *d* having an upper adjoint *g* as morphisms, respectively. We know that the categories **POSET***t* and **POSET***D* are dual via functors *D* and *G*, where for any poset *P* we write simply *D*(*P* )= *P* and *G*(*P* ) = *P* ; for every morphism *g* : *P → Q* of **POSET***t*, *D*(*g*) : *Q → P* is the lower adjoint of *g*; for each morphism *d* : *Q → P* of **POSET***D*, *G*(*d*): *P → Q*

is the upper adjoint of *d*. Then our following task is to investigate other duality theories in the context of subset selections. First, we see how the functors *D* and *G* translate certain preservation properties of morphisms.

**Proposition 3.11** *Let Z be a closed subset selection. If* (*g, d*) *is a Galois connec- tion between posets P and Q. Then the following statements are equivalent:*

1. *g is weakly ZΔ-continuous;*
2. *if U ∈ σ2* (*Q*)*, then †d*(*U* ) *∈ σ2* (*P* )*.*

*These conditions imply*

1. *d preserves Z-below relation 2, that is, x 2 y in Q implies d*(*x*) *2 d*(*y*)

*in P.*

*and if Q is Z-predistributive, we have all three conditions are equivalent.*

**Proof.** (1) *⇒* (2): Let *U* be an element of *σ2* (*Q*). We take a *A ∈ ZΛP* with *ΔA ∩ †d*(*U* ) */*= *∅* and should show that *A ∩ †d*(*U* ) */*= *∅*. Then there exists *u ∈ U* such that *d*(*u*) *∈ ΔA* by *ΔA ∩ †d*(*U* ) */*= *∅*. Without loss of generality, let *A* = *↓Z* with *Z ∈ ZP* . It is easy to see that *u ∈ Δg*(*Z*)= *Δ*(*↓g*(*A*)). Since *Z* is closed and *U ∩Δ*(*↓g*(*A*)) */*= *∅*, there exists *t ∈ A* such that *g*(*t*) *∈ U* . We obtain that *t ∈ †d*(*U* ), so *†d*(*U* ) *∈ σ2* (*P* ).

(2) *⇒* (1): Assume that *v ∈ ΔZ* such that *g*(*v*) *∈ g*(*ΔZ*)*\Δg*(*Z*) for *Z ∈ ZP* . This means that we have an element *x ∈ g*(*Z*)*u* such that *g*(*v*) A *x*. Let *U* = *Q \ ↓x*. Then we have *U ∈ σ2* (*Q*) and *g*(*v*) *∈ U* . By hypothesis (2) we know that *†d*(*U* ) *∈ σ2* (*P* ). So, *v ∈ †d*(*U* ), we have a *t ∈ Z* with *t ∈ †d*(*U* ), that is, *d*(*u*) *≤ t* for some *u ∈ U* . It follows that *u ≤ g*(*t*) *≤ x*, a contradiction.

(1) *⇒* (3): Suppose that *x 2 y* in *Q* and *A ∈ ZΛP* with *d*(*y*) *∈ ΔA*. It suffices to show that if *A* = *↓Z* for some *Z ∈ ZP* , then *d*(*x*) *∈ A*. By Proposition [3.9](#_bookmark4), *g*(*d*(*y*)) *∈ Δg*(*Z*) = *Δ*(*↓g*(*Z*)). Recall that *↓g*(*Z*) = *↓g*(*↓Z*) since *g* is monotone, thus *y ∈ Δ*(*↓g*(*A*)) due to *y ≤ g*(*d*(*y*)). Because *Z* is a closed subset selection,

*↓g*(*A*) is a *Z*-ideal, we have *x ∈ ↓g*(*A*). Hence *d*(*x*) *∈ d*(*↓g*(*A*)) *⊆ A*. Therefore,

*d*(*x*) *2 d*(*y*) in *P* .

It remains (3) *⇒* (1). Suppose that *Q* is a *Z*-predistributive poset. We claim *g*(*ΔZ*) *⊆ Δg*(*Z*) for each *Z ∈ ZP* . Assume there is an element *x ∈ g*(*ΔZ*)*\Δg*(*Z*). Then we have an upper bound *y* of set *g*(*Z*) such that *x* ¢ *y* in *Q*. Thus there exists *u* such that *u ∈* ***↓****2x* but *u* ¢ *y* in *Q* by Theorem [3.6.](#_bookmark2) It follows that *d*(*u*) *2 d*(*x*) by hypothesis (3). Naturally *d*(*u*) *∈ ↓Z* since *d*(*x*) *∈ d*(*g*(*ΔZ*)) *⊆ Δ*(*↓Z*), as a result, *u ∈ g*(*↓Z*) *⊆ ↓g*(*Z*). Hence *u ≤ y* and this is the desired contradiction. *2*

For simplicity of presentation, we assume that subset selection *Z* is closed throughout the rest of this section.

**Definition 3.12** Let *S*, *T* be two posets.

* 1. A map *g* : *S → T* is said to be a *ZΔ-morphism* if *g* is weakly *ZΔ*-continuous and has a lower adjoint.
  2. A map *d* : *T → S* is called a *ZΔ-comorphism* if *d* preserves *Z*-below relation

and has an upper adjoint.

* 1. A map *f* : *S → T* is called a *quasiopen* if *U ∈ σ2* (*S*) implies *†f* (*U* ) *∈ σ2* (*T* ).

**Corollary 3.13** *Let g* : *P → Q be a map between posets which has a lower adjoint*

*d. If g is a ZΔ-morphism, then d is a ZΔ-comorphism. If Q is Z-predistributive and d preserves Z-below relation, then g is a ZΔ-morphism.*

We introduce the subcategories of **POSET***t* and **POSET***D*, in order to refor- mulate Proposition [3.11](#_bookmark5) in terms of duality.

**Definition 3.14** We define the following categories.

1. *Z*-**POS***t* is the category of posets with *ZΔ*-morphisms.
2. *Z*-**POS***D* is the category of posets and maps *d* as morphisms which are qua- siopen and have an upper adjoint.
3. *Z*-**PD***t* and *Z*-**PD***D* have the same objects of all *Z*-predistributive posets; the morphisms of *Z*-**PD***t* are *ZΔ*-morphisms and the morphisms of *Z*-**PD***D* are *ZΔ*-comorphisms.
4. *Z*-**PC***t* is the full subcategory of *Z*-**PD***t* consisting of all *Z*-precontinuous posets.
5. *Z*-**PC***D* is the full subcategory of *Z*-**PD***D* consisting of all *Z*-precontinuous posets.

**Theorem 3.15** *The following categories are dual under the functors D and G given through the Galois connection of functions:*

* 1. *Z-***POS***t and Z-***POS***D;*
  2. *Z-***PD***t and Z-***PD***D;*
  3. *Z-***PC***t and Z-***PC***D.*

We explore the constructions of new *Z*-precontinuous posets. The following example gives us some constructions of posets, and the Table 1 will show whether the *Z*-precontinuity is preserved under these structures for a frequent subset selection *Z*.

**Example 3.16** Let *P* and *Q* be two posets. We have the following five kinds of “disjoint” sums: (1) (*Disjoint sum*) *P H Q*, the disjoint union of *P* and *Q* (with the obvious partial ordering: elements *x ∈ P* and *y ∈ Q* are incomparable); (2) (*Coalesced sum*) *P ⊕Q*, the disjoint sum *P HQ* with the bottom elements identified, if they have them; (3) (*Separated sum*) *P* + *Q* = (*P H Q*)0, that is, the disjoint sum with a new bottom element adjoined; (4) *P* +1 *Q* = (*P ⊕ Q*)1, the coalesced sum with a new top element adjoined; (5) *P* +2 *Q*, the coalesced sum with the top elements identified if they have them.

Suppose that *P* and *Q* are *Z*-precontinuous posets where *Z* is the subset selection *B*, *C* or *D* respectively. Then it is easy to check that the sum corresponding to the first three is still *Z*-precontinuous. Pick *P* is the chain **3** and *Q* is a 4-element lattice. It is clear that *A*, *F* and *P*-precontinuity may be destroyed under all these

constructions. If *P* and *Q* are poset N, then *P* +1 *Q* and *P* +2 *Q* are not *B*, *C* or

*D*-precontinuous posets. As shown in Table 1.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| sums  *Z* | (1) | (2) | (3) | (4) | (5) |
| *A* | no | no | no | no | no |
| *B* | yes | yes | yes | no | no |
| *C* | yes | yes | yes | no | no |
| *D* | yes | yes | yes | no | no |
| *F* | no | no | no | no | no |
| *P* | no | no | no | no | no |

Table 1

The *2*-precontinuity of the sums

**Lemma 3.17** *Let* (*g, d*) *be an adjunction between posets P and Q. If g is surjective, then g*ˆ : *ZΛP → ZΛQ deﬁned by g*ˆ(*A*)= *↓g*(*A*) *is surjective. Moreover,* (*g*ˆ*, d*ˆ) *is an adjunction between ZΛP and ZΛQ, where d*ˆ *is similarly deﬁned.*

**Proof.** *g*ˆ is well-defined by *Z* being closed. For all subsets *A* of *P* , *g*(*↓A*) *⊆ ↓g*(*A*) by

the monotonicity of *g*. Since *x ∈ ↓g*(*A*) implies *d*(*x*) *∈ ↓A*, we have *g*(*↓A*)= *↓g*(*A*). Recall that *gd* = id*Q* (see [[10](#_bookmark18), Proposition O-3.7]). Then *g*ˆ(*↓d*(*B*)) = *↓g*(*d*(*B*)) = *B*

for *B ∈ ZΛQ*. As a result, *g*ˆ maps *Z*-ideals of poset *P* onto *Z*-ideals of *Q*. It is easy to show that (*g*ˆ*, d*ˆ) is an adjunction. *2*

**Theorem 3.18** *Let g* : *P → Q be a ZΔ-morphism from Z-precontinuous poset P onto poset Q. Then Q is Z-precontinuous. In particular, the image of a Z- precontinuous poset under a ZΔ-morphism is Z-precontinuous.*

**Proof.** Let *d* be the lower adjoint of *g*. First, we claim that *u 2 d*(*x*) in *P* implies *g*(*u*) *2 x* in *Q*. If *x ∈ Δ*(*↓B*)= *ΔB* for *B ∈ ZQ*, then *d*(*x*) *∈ d*(*ΔB*) *⊆ Δd*(*B*) by Lemma [3.8](#_bookmark3). Thus *u ∈ ↓d*(*B*), this means that *g*(*u*) *∈ g*(*↓d*(*B*)) = *↓B*. We obtain *g*(***↓****2d*(*x*)) *⊆* ***↓****2x*, furthermore, *g*(*↓2d*(*x*)) = ***↓****2x* in *Q* by Proposition [3.11](#_bookmark5). Since

*↓*

*P* is *Z*-precontinuous, *↓2d*(*x*) is a *Z*-ideal of *P* . It follows that *↓2x ∈ ZΛQ* due to

*↓ ↓*

Lemma [3.17](#_bookmark6). Hence, we have

*x* = *g*(*d*(*x*)) *∈ g*(*Δ*[*↓2d*(*x*)]) *⊆ Δg*(*↓2d*(*x*)) = *Δ*(*↓2x*)*,*

*↓ ↓ ↓*

i.e., *x* = W ***↓****2x*. So far, we reach the conclusion that *Q* is a *Z*-precontinuous poset.*2*

# *Z*0-approximating auxiliary relation

In this section, we take a closer look at the *Z*-below relation and auxiliary relations. We use appropriate auxiliary relations to characterize the improved *Z*-precontinuity.

**Definition 4.1** [[10](#_bookmark18)] We say that a binary relation *≺* on a poset *P* is an auxiliary relation, or an auxiliary order, if it satisfies the following conditions for all *u, x, y, v*:

* 1. *x ≺ y* implies *x ≤ y*;
  2. *u ≤ x ≺ y ≤ v* implies *u ≺ v*;
  3. if a bottom element 0 exists, then 0 *≺ x*.

The set of all auxiliary relations on *P* is denoted by Aux(*P* ).

Based on Remark [3.2,](#_bookmark1) for any subset selection *Z*, we denote the *truncated selection Z*0 by *Z*0*P* = *ZP\{∅}*; *Z*0*-ideals* by *Z*0*ΛP* = *{↓Z* : *Z ∈ Z*0*P EP}*. For *x, y ∈ P* , we write *x 2*0 *y* if *Z ∈ Z*0*ΛP* and *y ∈ ΔZ* imply *x ∈ Z*, the re- lation *2*0 is called *Z*0*-below relation* of poset *P* . Similarly, we may define *Z*0*- predistributive* and *Z*0*-precontinuous* posets. *2* and *2*0 are equal whenever

*∅ ∈/ ZP* . Obviously, the *Z*0-below relation is an auxiliary relation.

We may introduce some definitions that help us to characterize *Z*0-precontinuous posets with auxiliary relations.

**Definition 4.2** An auxiliary relation *≺* on a poset *P* is said to be *Z*0*-approximating* iff the set *↓≺x* = *{y ∈ P* : *y ≺ x}* is a *Z*0-ideal and *x* = *↓≺x* for all *x ∈ P* . The set of all *Z*0-approximating auxiliary relations is written App*2*0 (*P* ).

W

**Definition 4.3** A poset *P* is called *Z*0*-meet-precontinuous* if *↓x ∩ Y — ⊆* (*↓x ∩ Y* )*—* for all *x ∈ P* and *Y ∈ Z*0*ΛP* .

As what we have anticipated, every *Z*0-precontinuous poset is *Z*0-meet- precontinuous (see [[12,](#_bookmark20)[14](#_bookmark22)]). Let Low(*P* ) denote the set of all lower sets of *P* . We know the assignment

*≺ '→ s≺* = (*x '→ {y* : *y ≺ x}*)

is an isomorphism from Aux(*P* ) onto monotone functions *s* : *P →* Low*P* , whose inverse associates to each monotone function *s* the relation *≺s* given by *x ≺s y* iff *x ∈ s*(*y*) in [[10](#_bookmark18)]. If *P* is a semilattice, then we consider for each *Z ∈ Z*0*ΛP* the monotone function *mZ* : *P →* Low*P* given by

*m* (*x*)= *↓x ∩ Z* = *x ∧ Z,* if *x ∈ ΔZ,*

*Z*

*↓x,* otherwise*.*

Let *P* be a semilattice. The unary meet operation *∧x* : *P → P* : *y '→ x ∧ y*

is monotone for *x, y ∈ P* , we say that *∧x* is *Z*0*-closed* if for all *Z ∈ Z*0*ΛP* implies

*∧x*(*Z*) *∈ Z*0*ΛP* . The truncated selection *Z*0 is called *∧-closed* if each unary meet operation on all semilattices is *Z*0-closed. There is no doubt that each closed subset selection *Z*0 is *∧*-closed. Let *Z*0 be the all nonempty Frink ideals. Then *Z*0 is a

*∧*-closed subset selection but not closed.

**Proposition 4.4** *For any ∧-closed subset selection Z*0*, a semilattice P is Z*0*-meet- precontinuous iff the unary meet operations ∧x* : *P → P* : *y '→ x ∧ y are weakly Z*0*Δ-continuous.*

**Proof.** For all *Z ∈ Z*0*P* , *ΔZ* is equal to *Z—* by [[12](#_bookmark20), Lemma 1]. We have *∧x*(*ΔZ*)= *x∧Z—* = *↓x∩*(*↓Z*)*—* for a semilattice. Let *P* be a *Z*0-meet-precontinuous. Then *↓x∩* (*↓Z*)*— ⊆* (*↓x ∩ ↓Z*)*— ⊆ Δ*(*x ∧ Z*). Thus *∧x* is weakly *Z*0*Δ*-continuous. Conversely, we just need to prove that *↓x ∩* (*↓Z*)*— ⊆* (*↓x ∩ ↓Z*)*—* for all *x ∈ P* and *Z ∈ Z*0*P* . Indeed,

*↓x ∩* (*↓Z*)*—* = *x ∧* (*ΔZ*) *⊆ Δ*(*x ∧ Z*)= [*↓*(*x ∧ Z*)]*—* = (*↓x ∩ ↓Z*)*—,*

the second inequality holds as the map *∧x* is weakly *Z*0*Δ*-continuous, third equation because *Z*0 is *∧*-closed. This completes the proof. *2*

**Lemma 4.5** *Let Z be a subset selection which truncated selection Z*0 *is ∧-closed.*

*Then for a Z*0*-meet-precontinuous semilattice P, all relations ≺mZ*

*are Z*0*-approximating.*

*for Z ∈ Z*0*ΛP*

**Proof.** Let *x ∈ P* . If *x ∈ ΔZ*, then *{y ∈ P* : *y ≺mZ x}* = *x ∧ Z*. *x ∧ Z* is a *Z*0-ideal since *Z*0 is *∧*-closed. It follows that *Δ*(*x ∧ Z*) = (*x ∧ Z*)*—*. Thus *Δ*(*x ∧ Z*)= *↓x ∩ Z—* = *↓x ∩ ΔZ* = *↓x* by *Z*0-meet-precontinuity. If *x ∈/ ΔZ*, then

W

*{y ∈ P* : *y ≺mZ x}* = *↓x*. We have *x* = *{y ∈ P* : *y ≺mZ x}* in all cases. *2*

**Proposition 4.6** *Let P be a poset and Z*0 *a ∧-closed subset selection. Then the Z*0*-below relation 2*0 *is contained in all Z*0*-approximating auxiliary relations, and is equal to their intersection if P is a Z*0*-meet-precontinuous semilattice.*

**Proof.** It is straightforward that *2*0 is contained in all *Z*0-approximating auxil- iary relations. If *P* is a *Z*0-meet-precontinuous semilattice, then by Lemma [4.5](#_bookmark7) we obtain

*{s≺*(*x*) *|≺∈* App*2* (*P* )*}⊆* *{mZ*(*x*) *| Z ∈ Z*0*ΛP}*

0

= (*↓x ∩ Z*) *∩ ↓x*

*x∈ΔZ x∈/ΔZ*

= *↓x ∩ Z*

*x∈ΔZ*

= ***↓****2*0 *x,*

thus *2*0 includes the intersection of all *Z*0-approximating auxiliary relations. *2*

For a poset, *2*0 is itself not a *Z*0-approximating relation necessarily. But we may now derive the following theorem.

**Theorem 4.7** *Let P be a poset and Z*0 *a ∧-closed subset selection. Consider the following conditions:*

1. *P is Z*0*-precontinuous;*
2. *2*0 *is the smallest Z*0*-approximating auxiliary relation on P;*
3. *there is a smallest Z*0*-approximating auxiliary relation on P.*

*Then (1) e (2) ⇒ (3), and if P is a Z*0*-meet-precontinuous semilattice, then con- ditions are equivalent.*

**Proof.** (1) *e* (2): Observe that *P* is a *Z*0-precontinuous poset iff *2*0 is a *Z*0-approximating auxiliary relation. Then the equivalence follows from Proposi- tion [4.6](#_bookmark8).

(2) *⇒* (3) is clear.

Let *P* be a *Z*0-meet-precontinuous semilattice. Then *Z*0-below relation is equal to the intersection of all *Z*0-approximating auxiliary relations by Proposition [4.6](#_bookmark8). Thus, *2*0 has to be the smallest *Z*0-approximating auxiliary relation. Therefore we have (3) *⇒* (1). *2*

# Conclusion

The collection of all directed subsets is a crucial subset selection in domain theory. The present paper has further exhibited some results of general *Z*-continuity which enriches the *Z*-theory, a generalization of domain theory. In closed subset selec- tions, we investigated the duality theory for *Z*-predistributive (*Z*-precontinuous) posets. Finally, we used *Z*0-approximating auxiliary relations to characterize *Z*0- precontinuous posets. Naturally, it may take into consideration whether these re- sults are suitable for more subset selections.

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