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**Full Length Article**

**Time-fractional effect on pressure waves propagating through a fluid filled circular long elastic tube**



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The pressure waves propagating through an incompressible inviscid fluid that moves in a circular cylindrical long elastic tube are considered. The reductive perturbation method is used to derive the KdV equation from the hydrodynamic equations of the system. The Euler– Lagrange variational technique described by Agrawal has been applied to formulate the time- fractional KdV equation. The derived time-fractional KdV equation is solved by employing the variational-iteration method represented by He. The effects of the tube and fluid pa- rameters and the time fractional order on the propagation of pressure waves are investigated.

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# Introduction

The propagation of pressure waves in fluid that moves in large vessels was studied by many authors, e.g. References [1–16](#_bookmark22). Many of the works studied the small amplitude wave propagation in elastic tubes, ignored the nonlinear effects and focused on the dispersive characteristic [[3–5]](#_bookmark10). When the nonlinear character

appears, either finite amplitude or small-but-finite ampli- tude wave is considered, depending on the nonlinearity order. The propagation of finite waves through fluid filled elastic or viscoelastic tubes was studied [[6–9]](#_bookmark11). Also, the small-but- finite amplitude waves propagating in distensible tubes were investigated [[10–16]](#_bookmark12), where the Korteweg–de Vries (KdV) equa- tion appears due to the balancing between the nonlinearity and the dispersion effects.

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The KdV equation as evolution and interaction model of nonlinear waves is employed to represent a wide range of physi- cal phenomena. The KdV equation was first formulated as an evolution equation governing one-dimensional, small ampli- tude, long surface gravity waves propagating in a shallow water channel [[17]](#_bookmark17). Afterward, the KdV equation has appeared in a number of other physical problems such as ion-acoustic waves, collision-free hydro-magnetic waves, plasma physics, strati- fied internal waves, lattice dynamics, etc. [[18]](#_bookmark18). By means of the KdV model, some theoretical physics phenomena in quantum mechanics domain and continuum mechanics for shock wave formation are explained. The KdV model is also applied in many scientific fields like fluid dynamics, aerodynamics, solitons, tur- bulence, boundary layer behavior and mass transport.

Most of the forces in nature are non-conservative: dissipa- tive and/or dispersive forces. The classical mechanics treated conservative forces using integer differential equations, while the non-conservative forces can be described in terms of the non-integer differential equations. Non-integer differentia- tion and integration is called Fractional Calculus, which is a field of mathematics study that generalizes the traditional defi- nitions of calculus integral and derivative operators. During the last decades, Fractional Calculus has acquired importance due to its applications in various fields of science and engineer- ing, including electrical networks, signal processing, optics, fluid flow, viscoelasticity, rheology, probability and statistics, dy- namical process in self-similar and porous structures, diffusive transport, control theory of dynamical systems, electrochem- istry of corrosion, and so on [[19–35]](#_bookmark19).

differential equations and to an efficient numerical solution for nonlinear and fractional differential equations. The series solution begins with a trial function that can be used as the solution of the linear term of the differential equation.

Our main motive here is to study the time fractional pa- rameter effects on the propagation of solitary pressure waves through a fluid filled elastic tube. Therefore, beside what is con- sidered in Reference [16](#_bookmark16), the derived nonlinear KdV equation is transformed using variational technique described by Agrawal [[24–26]](#_bookmark23) into the time fractional KdV (TFKdV) equation [[45]](#_bookmark28). The TFKdV equation is solved by applying VIM [[46–48]](#_bookmark29) developed by He and the effect of the fractional order is studied.

This paper is organized as follows: The basic set of tube and fluid equations, which governs our system, is presented in section 2. In section 3, the KdV equation is derived by apply- ing the reductive perturbation method [[51]](#_bookmark31). Section 4 is devoted to derive and solve the time fractional KdV (TFKdV) equation using variational methods. Finally, some results and discus- sion are presented in section 5.

# Basic equations of motion of the tube and fluid

A circular cylindrical long tube of un-deformed radius *R*0 , sub- jected to a uniform initial inner pressure *P*0 is considered. A position vector *r* of a general point on the tube is described as [[11,12]](#_bookmark13):

Riewe [[19,20]](#_bookmark19) used fractional derivatives [[21–23]](#_bookmark20) in the action to have non-conservative Euler–Lagrange equations. In terms of Riemann–Liouville fractional derivatives, Agrawal [[24–26]](#_bookmark23) used variational technique to formulate fractional equations of

*r*  [*r*0  *u*(*z*, *t*)]*e*ˆ *r*  *ze*ˆ*z*

*z*  *zZ*

(1a)

(1b)

motion. These Euler–Lagrange equations are employed to in- vestigate different real problems [[27–35]](#_bookmark24).

where *r*0 is the radius of the tube after a finite static defor- mation, *u*(*z*, *t*) is a finite dynamic time dependent deformation

The fractional differential equations are solved by apply-

in the tube radius,

*e*ˆ *r* and

*e*ˆ*z* are the radial and axial unit

ing several methods such as: Fourier transformation method, Laplace transformation method, operational method, and the iteration method [[21–23]](#_bookmark20). However, most of these methods are suitable only for special types of fractional differential equa- tions, called linear with constant coefficients. Recently, there are some works dealing with the solutions of nonlinear frac- tional differential equations using techniques of nonlinear analysis such as: Adomian decomposition method (ADM)

[[36–42]](#_bookmark25), variational-iteration method (VIM) [[43–48]](#_bookmark27), homotopy

vectors, respectively in the cylindrical polar coordinates. *Z* is the axial coordinate before the deformation, *z* is the axial co- ordinate after the static deformation and *z* is the axial stretch ratio. The equation of motion of a small element of the tube’s wall in the radial direction is given by [[11,12]](#_bookmark13)

*R*0**0 2*u*(*z*, *t*)    *R*0** (*z*, *t*) *u*(*z*, *t*)



2 

*z* *t* *z*  **1 *z* 



perturbation method (HPM) [[49,50]](#_bookmark30) and others. Adomian de- composition method [[36–38]](#_bookmark25) succeeded to solve accurately

  ** (*z*, *t*)   *R*0**2 *P*\* *z*, *t*

*z* **2 *H*

(2)

different types of fractional nonlinear differential equations by applying the Adomian polynomials. This method is applied to study many problems arising from applied sciences and en- gineering [[39–42]](#_bookmark26). The variational-iteration method [[43–46]](#_bookmark27) was successfully employed to solve many types of linear, nonlin- ear and fractional differential equations that describe scientific and engineering problems [[46–48]](#_bookmark29). As advantages over ADM, the VIM solves differential equations without using Adomian polynomials and has no linearization or perturbation for solving the nonlinear and fractional problems. The VIM principles for solving the differential equations are given in many papers,

e.g. References [46–48](#_bookmark29). The VIM solution is provided as a con-

where **0 and ** are the mass density and the shear modulus of the tube material, respectively. (*z*, *t*) is the strain energy density function, *P*\*(*z*, *t*) is the fluid pressure at the final inner deformed tube radius *rf* and *H* is the initial un-deformed tube thickness. *r* and *z* are the radial and axial cylindrical coordi- nates, respectively after both static and dynamic deformations and *t* is the time parameter. **1 and **2 are the axial and radial direction stretches, respectively and are represented by [[11,12]](#_bookmark13)

**1  *z*  (3a)

vergent series, which may lead to exact solution for linear

**2  *r*  *u*(*z*, *t*)*R*0

(3b)

  {1  [*u*(*z*, *t*)*z*]2 }12

(3c)

Here *V*(*r*, *z*, *t*) and *W*(*r*, *z*, *t*) are the radial and axial fluid ve- locity components, respectively, *P*(*r*, *z*, *t*) is the fluid pressure

where *r*  *r*0  *R*0 is the radial stretch ratio after finite static deformation.

Equation [(2)](#_bookmark3) has been derived by applying Newton’s second law of motion for a small element of the tube wall material [[11,12]](#_bookmark13). The term in the left hand side is Newton’s second law

force of the wall element. In the right hand side, the first term

and ** *f* is the fluid mass density.

The strain energy density (*z*, *t*) is generally a function of

**1 and **2. Expanding (*z*, *t*) into power series at its equilib- rium, (4) can take the form [[16]](#_bookmark16)

*P*\*(*z*, *t*)  *b*1*H u*  **0*H* 2*u*  *a H* 2*u*  *b*2*H u*2   **0*H u* 2*u*

0

*r z*

0

*r*

*z*

0

is the force due to axial tension of the tube wall, the second

*R*2 ** ** *t*2 1

*z*2 *R*3

**2** *R*

*t*2

term is the force due to the tension in the radial direction and

*R*

 *a*2*H*  *u* 2   *a*1*H*  2*a*2*H*  *u* 2*u*  *H*

the third term is the force due to the inner fluid pressure on the tube wall element.

*R*0  *z* 

 *rR*0

*R*0 

*z*2

*P*0 (7)

0

The value of the fluid pressure at the final radius of the de- formed tube ( *rf* ) can be obtained from (2) as

where the coefficients *a*1, *a*2 , *b*1 and *b*2 are defined by

*P*\*(*z*, *t*) 

**0*H* 2*u*(*z*, *t*) 

*H* (*z*, *t*)

*a*1  1  ,

*rz*  *u*  0

*a*2 

*u*  0

*R*0 2 

*rz* *u*

(8a)

**1**2 *t*2 **1**2*R*0 **2

*R*3

0 3 

2*rz* *u*3

*H*   1 (*z*, *t*) *u*(*z*, *t*)

*b*   *R*0  *R*

2   1 

, *b* 

 *b*1

 *z*  

(4)

1 ** **  0 *u*2 ** *u* 

2

** (8b)

**1**2 *z*   **1 *z* 

*r z r u*  0

*u*  0 *r*

On the other hand, the fluid that filled the tube is consid- ered to be an incompressible inviscid fluid. The ratio of the viscous term to the nonlinear term is assumed to be very small. Therefore, the viscous effect in comparison to the nonlinear

Equations [(5)–(7)](#_bookmark4) give sufficient relations to determine the unknowns *V*(*r*, *z*, *t*), *W*(*r*, *z*, *t*), *u*(*r*, *z*, *t*), and *P*(*r*, *z*, *t*).

effect will be neglected. Based on these assumptions, the in-

compressible inviscid fluid filled the tube has equations of axially symmetrical motion in the cylindrical polar coordi- nates represented by [[12,13]](#_bookmark14)

*V*(*r*, *z*, *t*)  *V*(*r*, *z*, *t*)  *W*(*r*, *z*, *t*)  0 (5a)

# The Evolution equation

The following stretched co-ordinates can be introduced by ap- plying the reductive perturbation theory [[51]](#_bookmark31) as:

*r r* *z*

**  **12(*z*  *gt*), **  **32*t* (10)

*V*(*r*, *z*, *t*)  *V*(*r*, *z*, *t*) *V*(*r*, *z*, *t*)

*t* *r*

 *W*(*r*, *z*, *t*) *V*(*r*, *z*, *t*)  1 *P*(*r*, *z*, *t*)  0 (5b)

where ** is a smallness parameter and *g* is the long wave ap- proximation’s phase velocity. The physical quantities represented in (5)–(8) are expanded as power series in ** as:

*z * *f* *r*

*V*(*r*, **, ** )  **32*V*1(*r*, **, ** )  **52*V*2(*r*, **, ** )  **72*V*3(*r*, **, ** )  … (11a)

*W*(*r*, *z*, *t*)  *V*(*r*, *z*, *t*) *W*(*r*, *z*, *t*)

*t* *r*

*W*(*r*, **, ** )  *W*1(*r*, **, ** )  **2*W*2(*r*, **, ** )  **3*W*3(*r*, **, ** )  … (11b)

 *W*(*r*, *z*, *t*) *W*(*r*, *z*, *t*)  1 *P*(*r*, *z*, *t*)  0 (5c)

*z * *f* *z*

The first equation [(5a)](#_bookmark4) is the incompressibility condition (mass conservation) of the fluid, while [(5b)](#_bookmark5) and [(5c)](#_bookmark6) are con- servative equations of fluid momentum in radial and axial directions, respectively.

These field equations [(5)](#_bookmark4) satisfy boundary conditions at the wall as follows:

The pressure at the final deformed radius of the tube is given

*P*(*r*, **, ** )  *P*1(*r*, **, ** )  **2*P*2(*r*, **, ** )  **3*P*3(*r*, **, ** )  … (11c)

*u*(**, ** )  *u*1(**, ** )  **2*u*2(**, ** )  **3*u*3(**, ** )  … (11d)

Substituting (10) and (11) into (5) and (6), applying (7) and equating the coefficients of different powers of ** , the follow- ing results are obtained:

The coefficient of the first-order of ** is

as

1 *P*1(*r*, **, ** )  0

(12a)

*P*(*rf* , *z*, *t*)  *P*\*(*z*, *t*)

(6a)

** *f* *r*

while the fluid velocity at the wall is assumed to equal the radial velocity of the wall itself (no-slip condition) [[13]](#_bookmark15), so

0 1

with the boundary condition

*V*(*rf* , *z*, *t*)  *dr*  *e*ˆ *r*

 *u*(*z*, *t*)  *u*(*z*, *t*) *W*(*rf* , *z*, *t*) (6b)

*P*1(*r*, **, ** )

*r*  *rf*

 (*b*1*HR*2 )*u* (**, ** )

(12b)

*dt r**rf* *t* *z*

Meanwhile, the coefficients of **32are given as

1 *P*1(*r*, **, ** )  *g* *W*1(*r*, **, ** )  0

** *f* ** **

(13a)

# Time-fractional KdV equation and its solution

*V*1(*r*, **, ** )  *V*1(*r*, **, ** )  *W*1(*r*, **, ** )  0

*r r* **

(13b)

The TFKdV equation in (1 + 1) dimension can be formulated fol- lowing El-Wakil et al. [[45]](#_bookmark28) to have the form [see Appendix A]:

with boundary condition

*RD* *Y*(**, ** )  *AY*(**, ** )  *Y*(**, ** )  *B* 3 *Y*(**, ** )  0,

*V*1(*r*, **, ** ) *r*  *rf*  *g*[*u*1(**, ** )**]

(13c)

0 **

0  **  1,

**

** [0, *T*0 ]

**3

(18)

where

where the Riesz fractional operator *RDU*(**, ** ) is repre-

*g*2  *b*1*Hrf*

(2** *R*2 ) (13d)

0 **

sented by [[21–23]](#_bookmark20)

0 0

*R * 1 1 *d*  ** **

*T*0 ** 

The coefficient of **2 gives the following equation

0*D* *f* (**, ** )  2 (1  ** ) *d* 0 *dt*(**  *t*)

*f* (**, *t*)  ** *dt*(*t*  ** ) *f* (**, *t*)

   1 1 *d*  *T*0 *dt *  *t* ** *f* (**, *t*) (19)

*P*2  *g* *f*

*r*

*V*1  0

**

(14a)

2 (1  ** ) *d* 0

under the boundary condition

The TFKdV equation represented by (18) can be solved using the VIM [[43–46]](#_bookmark27) in terms of the following correction func- tional [see Appendix B]

*P*   **0*Hg*2  *a H* 2*u*1  *b*2*H u*2  *b*1*H u*

2 *r*  *rf*

*R*

*R*

 *rz*

1  **2

3 1 2 2

0 0

(14b)

*Yn*1(**, ** )  *Yn*(**, ** )

0

** 

The coefficients of **52are given in the forms

 ** *d*   

*Y* (**, ** )  *RI*1** *Y* (**, ** ) [** (** 2)(**  1)]

*n*

0 ** 

*n*

**   0

 *RD*1** *AY* (**, ** )  *Y* (**, ** )  *B* 3 *Y* (**, ** ),

*n*  0

*V*2  *V*2  *W*2  0

0 **   *n*

** *n*

**3 *n*



*r r*

** (15a)



(20)

*W* *W*

*W*  1 *P* *W*

where the Riesz integral operator *RI* *Y*(**, ** ) is represented

1  *V*1 1  *W*1 1 

2  *g*

2  0

(15b) *a *

** *r*

** ** *f* ** **

by [[21–23]](#_bookmark20)

with the boundary condition

*RI* *f* (**, ** )  1 1

*b dt *  *t *1 *f* (**, *t*)

*V*2  *u*1  *W*1 *u*1  *g* *u*2

*a *

(15c)

2 (** ) *a*

(21)

*r*  *rf* **

** **

If the parameter ** represents the time-variable in (18), the right Riemann–Liouville fractional derivative in Riesz frac-

Eliminating the second order perturbation quantities *P*2, *V*2 , *W*2 and *u*2 in (14) and (15) employing (12) and (13) to define the first order perturbation quantities *V*1 , *W*1 and *u*1 lead to KdV equation for the first-order perturbation pres- sure *P*1 in the form:

 *Y*(**, ** )  *AY*(**, ** )  *Y*(**, ** )  *B* 3 *Y*(**, ** )  0

tional operator can be interpreted as future states of the system. Therefore, this term may be neglected in calculations, when the present state of the system does not depend on the results of the future development [[23]](#_bookmark21).

The initial value of the state variable can be taken to rep- resent the zero-order correction of the solution. This initial value

is taken as the solution of the ordinary KdV equation at time

** **

**3

(16)

equal to zero, in this case as:

where *Y*(**, ** ) represents the first-order perturbation pressure

*P*1 . The nonlinear coefficient *A* and dispersion coefficient *B*

*Y*0(**, ** )  *Y*(**, 0)  *A*0 sech2(*c*)

(22a)

are defined, respectively by

where *A*0  3*vA*, *c* 

*v* (4*B*) .

*A*  *gR*0(2*R*0*b*1  *b*2*rf* )(*b*2*Hr* )

1 *f*

(17a)

Substituting the zero-order solution (22a) into (20) leads to the first order approximation as

*B*  (*g*2)[**0*Hrf*

(2** *f rz* )  *a R*2 *b*1  *r*2 8]

(17b)

*Y*1(**, ** )  *A*0 sech2(*c*)  2*A*0*c* sinh(*c*)sech3(*c*)

1 0 *f*

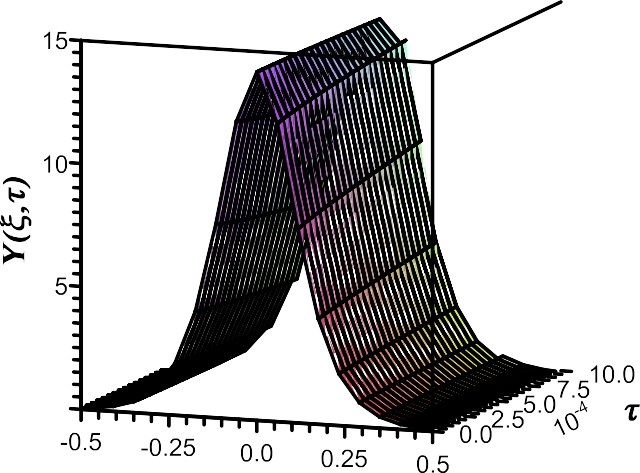
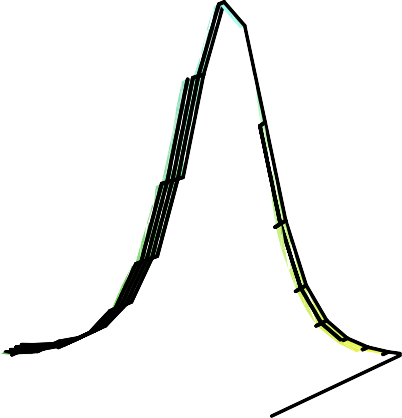
 [4*c*2*B*  (*A*0*A*  12*c*2*B*)sec*h*2(*c*)]** (**  1)

(22b)

To study the effect of time fractional derivative on the pres- sure pulses propagation in inviscid non-Newtonian incompressible fluid, the KdV equation [(16)](#_bookmark7) must be represented in terms of time-fractional form as in the following section.

Substituting this equation into (20) and using the Math- ematical package like Maple or Mathematica lead to the second order approximation in the form

*Y*2(**, ** )  *A*0 sech2(*c*)  2*A*0*c* sinh(*c*)sech(*c*)3



 [4*c*2*B*  (*A*0*A*  12*c*2*B*)sech(*c*)2 ]

 2*A*0*c*2 sech(*c*)2[32*c*4*B*2

 16*c*2*B*(5*A*0*A*  63*c*2*B*)sech(*c*)2

**

(**  1)

 2(3*A*2*A*2  176*A c*2*AB*  1680*c*4*B*2 )sech(*c*)4

0 0

 7(*A*2*A*2  42*A c*2*AB*  360*c*4*B*2 )sech(*c*)6 ]

** 2**

0 0 (2**  1)

 4*A*2*c*3 sinh(*c*)sech(*c*)5[32*c*4*B*2

0

 24*c*2*B*(*A*0*A*  14*c*2*B*)sech(*c*)2

 4(*A*2*A*2  32*A c*2*AB*  240*c*4*B*2 )sech(*c*)4

0 0

 5(*A*2*A*2  24*A c*2*AB*  144*c*4*B*2 )sech(*c*)6 ]

0 0

(2**  1) ** 3**

[(**  1)]2 (3**  1)

(22c)

The higher order approximations can be calculated using a symbolic mathematical package to the appropriate order where the infinite approximation leads to exact solution.

# Results and discussion



### Fig. 1 – The pressure wave with position and time for

***R*0**  **0.38 *cm* , *H***  **0.02 *cm* ,** **  **0.4 , *rf***  **0.75 *cm*,**

****0**  **1.03 *gmcm*3 ,** *****f***  **1.05 *gmcm*3 , *a*1**  **78.692,**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6, *v***  **8 *cms* and**

**  **0.8 .**

Blood is known to be an incompressible non-Newtonian fluid. The order of hematocrit ratio (red cell concentration) and the

deformability of red blood cells are the main factors that make blood behave like a Newtonian fluid. Experimental observa- tions show that blood behaves as a Newtonian fluid when the shear rate is high. Blood viscosity can be considered very small to be neglected with respect to its nonlinear term. Due to these observations, it can be assumed that blood can be treated as incompressible inviscid fluid.

The reductive perturbation method [[51]](#_bookmark31) is applied to drive the KdV equation for a system of fluid filled elastic tube. A varia- tional method suggested by Agrawal [[24–26]](#_bookmark23) is used to derive the TFKdV equation. The VIM [[43–48]](#_bookmark27) developed by He [[46]](#_bookmark29) is employed to solve the derived equation by applying the Riemann–Liouville definition for the fractional derivative.

The numerical results are made physically relevant by ap- plying parameters close to experimental data in dogs [[52,53]](#_bookmark32). The average values of the parameters are founded experimen- tally as *R*0  0.38 *cm*, *H*  0.02 *cm* , **  0.4, *rf*  0.75 *cm*,

**0  1.03 *gmcm*3 , ** *f*  1.05 *gmcm*3 , *a*1  78.692, *b*1  296.105,

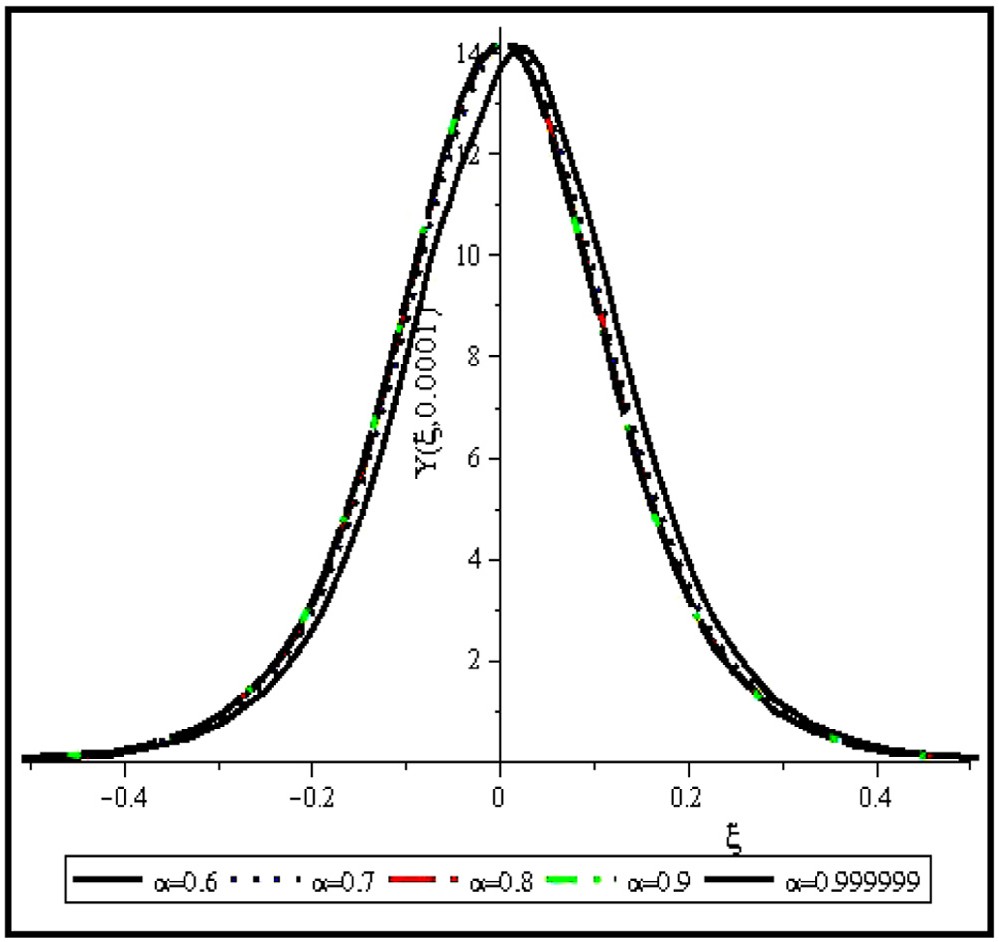
*b*2  991.496 , *z*  *r*  1.6 and *v*  8 *cms*. These experimental data are employed to study the pressure waves in dogs’ blood and the wave forms are represented in different figures.

The pressure wave against space and time at fractional pa- rameter **  0.8 has soliton shape as presented in [Fig. 1](#_bookmark8), which means that the pressure wave has constant shape through the artery. Effect of the variation of ** on the pressure wave is

studied in [Fig. 2](#_bookmark8). This figure shows that the fractional order of differentiation has a small effect only on the position of the wave peak. The initial radius of the blood-vessel *R*0 effect on the blood pressure wave is represented in [Fig. 3](#_bookmark9), which shows that the increase of *R*0 decreases both the amplitude and width of the pressure wave. In [Fig. 4](#_bookmark9), the increase of the thickness of the blood-vessel wall *H* increases both of the amplitude and

width of the blood pressure wave. The shear modulus of the blood-vessel wall material ** effect on the blood pressure waves

is given in [Fig. 5](#_bookmark9). It is shown that the increase of ** increases both the amplitude and width of the pressure wave. [Figure 6](#_bookmark9) represents the effect of the final radius of the blood-vessel *rf* on the pressure waves. The representation shows that the in- crease of *rf* decreases the amplitude while it increases the width of the pressure wave.



**Fig. 2 – The pressure wave with position at different values of** ** **for *R*0**  **0.38 *cm*, *H***  **0.02 *cm* ,** **  **0.4 , *rf***  **0.75 *cm*,**

****0**  **1.03 *gmcm*3 ,** *****f***  **1.05 *gmcm*3 , *a*1**  **78.692,**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6 and *v***  **8 *cms*.**



### Fig. 3 – The pressure wave with position at different values of *R*0 for *H*  0.02 *cm* , **  0.4 , *rf*  0.75 *cm*,



****0**  **1.03 *gmcm*3 ,** *****f***  **1.05 *gmcm*3 , *a*1**  **78.692,**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6, *v***  **8 *cms* and**

**  **0.8 .**



### Fig. 5 – The pressure wave with position at different values of ** for *R*0  0.38 *cm*, *H*  0.02 *cm* , *rf*  0.75 *cm*,

****0**  **1.03 *gm cm*3 ,** *****f***  **1.05 *gm cm*3 , *a*1**  **78.692,**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6, *v***  **8 *cms* and**

**  **0.8 .**

### Fig. 4 – The pressure wave with position at different values of *H* for *R*0  0.38 *cm*, **  0.4 , *rf*  0.75 *cm*,



****0**  **1.03 *gmcm*3 ,** **  **1.05 *gmcm*3 , *a***  **78.692,**

### Fig. 6 – The pressure wave with position at different values of *rf* for *R*0  0.38 *cm*, *H*  0.02 *cm* , **  0.4 ,

***f* 1** **

 **1.03 *gm cm*3 ,** **

 **1.05 *gm cm*3 , *a***

 **78.692,**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6, *v***  **8 *cm s* and 0 *f* 1**

**  **0.8 .**

***b*1**  **296.105 , *b*2**  **991.496,** *****z***  *****r***  **1.6, *v***  **8 *cms* and**

**  **0.8 .**

The above calculations show that the blood-vessel charac- teristics modulate the shape of the pressure waves in the blood for both the amplitude and width while the fractional order of the evolution equation that describes the pressure wave propagation has a small effect on these waves.

# Acknowledgements

All of the authors would like to pass their great thanks to their Professor S. A. El-Wakil, who is celebrating his Diamond Birthday,

for his continuous encouragement, suggestions, and review of the work.

The variation of the functional with respect to *V*(**, ** ) leads

to

** *J*(*V*) 

*d* *d*  *F*  ** *DV*   *F*  *V*

# Appendix A

*R* *T*

  *DV*  **  *V*  **

 0 ** **

0

## *Time-fractional KdV-equation formulation*

  *F*  *V*   *F*  *V* 

(A8)

The TFKdV equation in (1 + 1) dimension can be formulated



 *V* 

 *V*  

as follows [[45]](#_bookmark28):

Using a potential function *V*(**, ** ) where *Y*(**, ** )  *V*(**, ** )

gives the potential equation of the regular KdV equation [(16)](#_bookmark7)

Integrating the right-hand side of this equation by parts leads

to

in the form

** *J*(*V*) 

*d* *d*  *D*  *F*     *F*    *F*   2  *F*  *V*

*R* *T*

 ** *T*0   *D* *V* 

**  *V* 

 *V* 

**2  *V*  

*V* (**, ** )  *AV*(**, ** )*V*(**, ** )  *BV*(**, ** )  0 (A1)

 0 ** **

** 

(A9)

The functional of this equation can be represented by

This is done by taking the fractional integrating by parts formula [[21–23]](#_bookmark20)

*J*(*V*)  *R d**T dV*(**, ** )[*c*1*V* (**, ** )

*b dtf* (*t*) *Dg*(*t*) 

*b dtg*(*t*) *D* *f* (*t*)

(A10)

 *c AV* (**, ** )*V* (**, ** )  *c BV*

2

** **

3

**

(**, ** )]

(A2) *a a t*

*a t b*

where *c* , *c* , and *c*

are unknown constants to be deter-

where the right Riemann–Liouville fractional derivative

1 2 3

*D* *f* (*x*, *t*) is defined by [[21–23]](#_bookmark20)

mined. Integrating this equation by parts where *V* *R*

gives

 *V* *T*  0

*t b*

*D* *f* (*x*, *t*) 

(1)*n*

*dn*  *b d*(*t*  ** )*n*** 1 *f* (*x*, ** ),

*t b* (*n*  ** ) *dtn* *t* 

*J*(*V*)  *R d**T d* {[*c*1*V* (**, ** )*V*(**, ** )

 1 *c*2*A*[*V*(**, ** )]3  *c*3*B*[*V*(**, ** )]2 ]}

(A3)

*t* [*a*, *b*], *n*  1  **  *n*

(A11)

2

Taking the variation for the result with respect to *V*(**, ** )

leads to *c*  1 , *c*  1 , *c*  1 .

Optimizing this variation of the functional *J*(*V*), i.e.,

** *J*(*V*)  0 , gives the Euler–Lagrange equation for the time- fractional of the potential equation in the form

1 2 2 3 3 2

So, the functional gives the Lagrangian of the potential equa-

tion as

*D*  *F*     *F*    *F*   2  *F*   0

** *T*0   *D* *V* 

**  *V* 

 *V* 

**2  *V* 

(A12)

0 ** ** **

*L*(*V*, *V*, *V* )   1 *V* (**, ** )*V*(**, ** )  1 *c*2*A*[*V*(**, ** )]3  1 *B*[*V*(**, ** )]2

2 6 2

(A4)

Substituting from the Lagrangian equation into this Euler– Lagrange formula gives

Similar to this form, the Lagrangian of the potential equa-

 1[** *D* *V* (**, ** )]  1 [ *DV* (**, ** )]  *AV* (**, ** )*V* (**, ** )  *BV*

(**, ** )  0

tion in the time-fractional domain can be written as

*F*(0 *DV*, *V*, *V* )   1 [ *DV*(**, ** )]*V* (**, ** )

2 *T*0 **

2 0 **

** ** ** **

(A13)

** ** 2 0 ** **

 1 *A*[*V* (**, ** )]3  1 *B*[*V* (**, ** )]2 (A5)

Substituting for the potential function *V*(**, ** )  *Y*(**, ** ) gives the time-fractional KdV equation for the state function *Y*(**, ** )

6 ** 2 **

in the form

where the left Riemann–Liouville fractional derivative

 1 [** *D* *Y*(**, ** )]  1 [ *D* *Y*(**, ** )]  *AY*(**, ** )*Y* (**,** )  *BY*

(**, ** )  0

0*D* *f* (*x*, *t*) is represented by [[21–23]](#_bookmark20)

*t*

2 *T*0

2 0 **

** **

(A14)

*D* *f* (*x*, *t*)  1 *dn*  *t d*(**  *t*)*n*** 1 *f* (*x*, ** ),

*a t* (*n*  ** ) *dtn* *a* 

** **

*t* [*a*, *b*], *n*  1  **  *n* (A6)

where the fractional derivatives 0*D* *Y*(**, ** ) and ** *DT*0 *Y*(**, ** ) are the left and right Riemann–Liouville fractional derivatives respectively.

The functional of the time-fractional potential equation can be represented in the form

*R* *T *

*J*(*V*) 

*dx dtF*(0 *DV*, *V*, *V*, *V* )

(A7)

# Appendix B

## *Time-fractional KdV equation solution*

where the time-fractional Lagrangian *F*(0*DV*, *V*, *V* , *V*

) is

The TFKdV equation represented by (18) can be solved using

defined by (A5).

** ** **

the VIM [[43–48]](#_bookmark27) as follows:

Affecting from left by the fractional operator *RD*1** on (18)

1. [Fatima A, Taj Z. Behavior of viscoelastic fluid in presence of](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0020)

leads to

 *Y*(**, ** )  *RD* 1*Y*(**, ** )

0 **

**

**  2

**  0 (**  1)

0 **

[diffusion of chemically reactive species. Int J Sci Eng Res](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0020) [2014;5(8):487–92.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0020)

1. [Rachev AJ. Effects of transmural pressure and muscular](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0025) [activity on pulse waves in arteries. J Biomech Eng ASME](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0025) [1980;102:119–23.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0025)

 *RD*1** *AY*(**, ** )  *Y*(**, ** )  *B* 3 *Y*(**, ** ),

1. [Demiray H. Wave propagation through a viscous fluid](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0030)

0 **  **

**3 

[contained in a pre-stressed thin elastic tube. Int J Eng Sci](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0030)

0  **  1,

** [0, *T*0 ]

(B1)

[1992;30:1607–20.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0030)

1. [Choy YY, Tay KG, Ong CT. NLS equation with variable](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0035)

Taking into account the following fractional derivative prop- erty [[21–23]](#_bookmark20)

*RD*[*RD* *f* (*t*)]  *RD* ** *f* (*t*)  *k RD*  *j f* (*t*) (*t*  *a*)**  *j* ,

[coefficient in a stenosed elastic tube filled with an averaged](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0035) [inviscid fluid. World Appl Sci J 2012;16(4):622–31.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0035)

1. [Ando K, Sanada T, Inaba K, Damazo JS, Shepherd JE,](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0040) [Colonius T, et al. Shock propagation through a bubbly liquid](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0040)

[in a deformable tube. J Fluid Mech 2011;671:339–63.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0040)

1. *b a*
2. *a b*

 *a b j* 1

(1  **  *j*)

*t*  *a*

1. [Ikenaga Y, Nishi S, Komagata Y, Saito M, Lagree P-Y, Asada](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0045) [T, et al. Experimental study on the pressure and pulse wave](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0045)

*k*  1  **  *k* (B2)

The Riesz fractional derivative *RD* 1 is considered as Riesz

0

**

[propagation in viscoelastic vessel tubes – effects of liquid](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0045) [viscosity and tube stiffness. IEEE Trans Ultrason Ferroelectr](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0045) [Freq Control 2013;60(11):2381–8.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0045)

fractional integral *RI*1** that is defined by [[21–23]](#_bookmark20)

0 **

1. [Choy YY, Tay KG, Ong CT. Modulation of nonlinear waves in](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0050) [an inviscid fluid (blood) contained in a stenosed artery. Appl](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0050)

*RI* *f* (*t*)  1 [ *I* *f* (*t*)  *I* *f* (*t*)]  1 1

*b d* *t*  ** ** 1 *f* (** ),

**  0

[Math Sci 2013;7(101):5003–12.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0050)

0 ** 2 0 ** ** *b*

2 (** ) *a*

(B3)

1. [Abdou MA, Hendi A, Alanzi HK. New exact solutions of KdV](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0055) [equation in an elastic tube filled with a variable viscosity](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0055) [fluid. Stud Nonlinear Sci 2012;3(2):62–8.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0055)

The iterative correction functional of (B1) is given as

*Yn*1(**, ** )  *Yn*(**, ** )

1. [Gaik TK, Demiray H. Forced Korteweg-de Vries-Burgers](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0060) [equation in an elastic tube filled with a variable viscosity](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0060) [fluid. Chaos Soliton Fract 2008;38(4):1134–45.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0060)

  

**

*R* 1**

** **  2

1. [Demiray H. Nonlinear wave modulation in a fluid-filled](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0065)

0 *d* **(** ) **  *Yn*(**, ** )  0*I* *Yn*(**, ** ) **   0 (**  1)



 *RD*1** *AY*˜ (**, ** )  *Y*˜ (**, ** )  *B* 3 *Y*˜ (**, ** )

[elastic tube with stenosis. Z Naturforsch A 2008;](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0065) [63a:24–34.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0065)

1. [Cascaval RC. Variable coefficient KdV equations and waves](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0070)

0 **   *n*



** *n*

**3 *n*



(B4)

[in elastic tubes. In: Goldstein GR, Nagel R, Romanelli S,](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0070) [editors. Evolution equations. New York, USA: Marcel Dekker,](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0070)

where the function *Yn*(**, ** ) is considered as a restricted variation function, i.e., ** *Yn*(**, ** )  0 . The extreme of the variation of (B4) using the restricted variation function leads to

** ** *Y* (**, ** )  ** *Y* (**, ** )  *d* **(** )** *Y* (**, ** )

[Inc.; 2003.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0070)

1. [Demiray H. Weakly nonlinear waves in a fluid with variable](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0075) [viscosity contained in a pre-stressed thin elastic tube. Chaos](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0075) [Soliton Fract 2008;36(1):196–202.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0075)
2. [Demiray H. Variable coefficient modified KdV equation in](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0080) [fluid-filled elastic tubes with stenosis. Chaos soliton Fract](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0080)

[2009;42(2):358–64.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0080)

*n*  1

*n* 0

**  *n*

1. [Elgarayhi A, El-Shewy EK, Mahmoud AA, Elhakem AA.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0085)

 ** *Y* (**, ** )  **(** )** *Y* (**, ** )  *d*   **(** )** *Y* (**, ** )  0

**

[Propagation of nonlinear pressure waves in blood. ISRN](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0085)

*n n* 0

**  *n*

(B5)

[Comput Biol 2013;2013:436267.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0085)

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This relation leads to the stationary conditions 1  **(** )  0

and  **(** )  0, which leads to the Lagrangian multiplier as

** 

**(** )  1 . Therefore, the iterative correction functional has the form

[of long stationary waves. Phil Mag Ser 5 1892;39(240):422–43.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0090)

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*Yn*1(**, ** )  *Yn*(**, ** ) 

*d*   

0 

**

** 



*Yn*(**,** )  *RI*1** *Yn*(**, ** )

** **  2

**   0 (**  1)

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 *RD*1** *AY* (**, ** )  *Y* (**, ** )  *B* 3 *Y* (**, ** )

0 ** 

[Sons Inc.; 1993.](http://refhub.elsevier.com/S2314-808X(15)00062-7/sr0110)

0 **   *n*



** *n*

**3 *n*



(B6)

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