

Electronic Notes in Theoretical Computer Science 221 (2008) 115–125

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Towards Computability over Effectively Enumerable Topological Spaces

Margarita Korovina[1](#_bookmark0)

*The University of Manchester, UK and IIS SBRAS, Novosibirsk, Russia*

Oleg Kudinov [2](#_bookmark0)

*Sobolev Institute of Mathematics Novosibirsk, Russia*

**Abstract**

In this paper we study different approaches to computability over *effectively enumerable topological spaces*. We introduce and investigate the notions of computable function, strongly-computable function and weakly- computable function. Under natural assumptions on effectively enumerable topological spaces the notions of computability and weakly-computability coincide.

*Keywords:* Computably enumerable topological space, computability, effective continuity.

# Introduction

In this paper we approach the problem of computability over effectively enumer- able spaces. Since the class of effectively enumerable topological spaces contains effective *ω*-continuous domains, computable metric spaces, and abstract structures with computably enumerable *∃*-theory as proper subclasses, computability over ef- fectively enumerable spaces is crucial problem to investigate. We introduce and study different natural approaches to computability based on well-known enumera- tion operators [[16](#_bookmark17)]. These approaches lead to nonequivalent classes of computable functions over effectively enumerable spaces. The paper is structured as follows. In Section 2 we recall notion and properties of effectively enumerable spaces [[11](#_bookmark12)].

٨ This research was partially supported by CICADA project, RFBR 070100543-a and RFBR-DFG Project GZ: 436 RUS 113/850/01:06-01-04002.

1 Email: [Margarita.Korovina@manchester.ac.uk](mailto:Margarita.Korovina@manchester.ac.uk)

2 Email: [kud@math.nsc.ru](mailto:kud@math.nsc.ru)

1571-0661 © 2008 Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2008.12.011

In Section 3 we propose and study approaches to computability over effectively enumerable spaces.

# Basic notions and Definitions

Let (*X, τ, ν*) be a topological space, where *X* is a non-empty set, *τ∗ ⊆* 2X is a base of the topology *τ* and *ν* : *ω → τ∗* is a numbering. Let *D*k denote the *k*-th finite set with respect to the standard numbering of the finite sets.

**Definition 2.1** A topological space (*X, τ, ν*) is **effectively enumerable** if the following conditions hold.

* 1. There exists a computable function *g* : *ω × ω × ω → ω* such that

*νi ∩ νj* = *νg*(*i, j, n*)*.*

n*∈*ω

* 1. The set *{i|νi /*= *∅}* is computably enumerable.

**Definition 2.2** An effectively enumerable topological space (*X, τ, ν*) is **strongly effectively enumerable** if there exists a computable function *h* : *ω × ω → ω* such that

*X \ cl*(*νi*) = *νh*(*i, j*)*.*

j*∈*ω

Now we show that the topological spaces corresponding to computable metric spaces likewise corresponding to effective *ω*-continuous domains are proper natural subclasses of effectively enumerable topological spaces.

For the definition of computable metric space we refer to [[14](#_bookmark15),[23](#_bookmark24),[2](#_bookmark5)].

**Theorem 2.3** *If M* = (*M, ν,* **B***, d*) *is a computable metric space then* (*M, τ*d*, ν∗*) *is a strongly effectively enumerable topological space.*

**Proof.** Let *M* = (*M, ν,* **B***, d*) be a computable metric space, where **B** *⊆ M* is countable and dense in *M* , *ν* : *ω →* **B** is a numbering, and *d* : *M × M →* R is a distance function computable on (**B***, ν*). We use a computable representation of the rational numbers (Q+*, μ*), the standard pairing function *c* : *ω × ω → ω*, and the inverse function (*l, r*) : *ω → ω × ω*. Let *τ*d be topology induced by *d*, *ν∗* be a numbering of the base of *τ*d such that *ν∗*(*n*) = *B*(*νl*(*n*)*, μr*(*n*)), where *B*(*x, y*) is an open ball with the center *x* and the radius *y*.

It is easy to see that

*ν∗n ∩ ν∗m* = *∪{B*(*x, q*)*|x ∈* **B***,q ∈* Q+*, d*(*νl*(*n*)*, x*)+ *q < d*(*νl*(*n*)*, μr*(*n*)) and

*d*(*νl*(*m*)*, x*)+ *q < d*(*νl*(*m*)*, μr*(*m*))*}*

is an effectively open set. So,

*ν∗n ∩ ν∗m* = *ν∗χ*(*n, m, k*) for a computable function *χ.*

k*∈*ω

Since *ν∗n /*= *∅↔ μr*(*n*) *>* 0, the set *{n|ν∗n /*= *∅}* is effectively open. Finally, since

*M \ cl*(*ν∗n*)*M \ B*¯(*νl*(*n*)*, μr*(*n*)) =

*∪{B*(*x, q*)*|x ∈* **B***,q ∈* Q+*, d*(*νl*(*n*)*, x*) *> q* + *μr*(*n*)*}*

is an effectively open set, we have

*M \ cl*(*ν∗i*) = *ν∗h*(*i, j*) for a computable function *h.*

j*∈*ω

So, (*M, τ*d*, ν∗*) is a strongly effectively enumerable topological space.

The following proposition shows that the condition of computably enumerability for the set *{*(*i, j*) *| α*(*i*)*α*(*j*)*}* considered in [[23](#_bookmark24)] is too restrictive in the case of metric spaces.

**Proposition 2.4** *There exists a computable metric space* (*M, B, d*) *such that the set {*(*i, j*) *| ν∗*(*i*) = *ν∗*(*j*)*} is not c.e.*

**Proof.** In [[9](#_bookmark10),[10](#_bookmark13)] it was constructed some computable closed set *A ⊂* R that its interior is not effectively open. We put *X* = R *\ A* and consider it as a computable metric space since *X* is effectively open, *B* = *X ∩* Q. It is easy to see that

*x ∈ int*(*A*) *↔ ∃a, b ∈ B∃r*1*, r*2 *∈* Q(*B*X(*a, r*1) = *B*X(*b, r*2) *∧ |x − a| < r*1 *∧*

*|x − b| > r*2)*.*

Hence, if the set *{*(*i, j*) *| ν∗*(*i*) = *ν∗*(*j*)*}* is c.e. for this space *X*, *int*(*A*) is effectively open, a contradiction completes the proof.

Now we compare effectively enumerable topological spaces with *ω*-continuous domains (c.f. [[18](#_bookmark19),[1](#_bookmark6),[4](#_bookmark7)]). First we recall well-known properties of *ω*-continuous do- mains.

**Lemma 2.5** *For an ω-continuous domain D* = (*D, {b*i*}*i*∈*ω*, ±*) *the following prop- erties hold.*

1. *If a x then there exists n ∈ ω such that a b*n *x.*
2. (*D, τ, ν*) *is a T*0*-space, where τ is generated by the base τ∗* = *{U*b *} ∪ {∅}*

*n*

=

*and the numbering ν* : *ω → τ∗ is deﬁned as follows: ν*0 = *∅, νk* = *U*b

*{x|b*k*−*1 *x}, k >* 0*.*

*k−*1

**Definition 2.6** An *ω*-continuous domain *D* = (*D, {b*i*}*i*∈*ω*, ±*) is called **weakly effective** if *{< n, m > |b*n *b*m*}* is computably enumerable.

**Theorem 2.7** *Every weakly effective ω-continuous domain is an effectively enu- merable topological space.*

**Proof.** Let *D* = (*D, {b*i*}*i*∈*ω*, ±*) be a weakly effective *ω*-continuous domain. The

topology *τ* is generated by the base *τ∗* = *{U*b *|n ∈ ω} ∪{∅}*, where *U*a = *{x|a x}*,

*n*

and *ν* : *ω → τ∗* is the standard numbering. We show now that

*U*b*n ∩ U*b*m* =

b*s* b*n*,b*m*

*U*b*s .*

If *x ∈ U*b*s* for *b*s *b*n*, b*m then, by definition, *x b*s. So, *x ∈ U*b*n ∩ U*b*m* . Suppose *x ∈ U*b*n ∩ U*b*m* . By definition, *x b*n and *x b*m. So, there exist *s*1 and *s*2 such that *x b*s1 *b*n and *x b*s2 *b*m.

Since *{b*i*|b*i *x}* is directed, there exists *b*s *b*n*, b*m such that *x ∈ U*b*s* . By weak effectiveness, the set *{n|U*b*n /*= *∅}* is computably enumerable.

The following results show that the effectively enumerable spaces enlarge the ef- fective *ω*-continuous domains and the computable metric spaces. We consider struc- tures with topologies induced by *∃*-formulas. Suppose *A* = *⟨A, σ*0*⟩* = *⟨A, σ*P *, /*=*⟩* is an abstract structure, where *A* contains more than one element, *σ*P is a countable set of basic predicates.

The topology *τ A* is formed by the base which is the set of subsets definable by

Σ

existential formulas with positive occurrences of predicates from *σ*0. The following

proposition is straightforward from the definition of effectively enumerable topolog- ical space.

**Theorem 2.8** *[*[*11*](#_bookmark12)*] The topological space* *X, τ A* *is effectively enumerable if and*

Σ

*only if Th∃*(*X*) *is computable enumerable.*

As the example of a structure which is an effectively enumerable space we con- sider the set of continuous functions *C*(R). Let us note that *C*(R) does not belong to the metric spaces and to the *ω*-continuous domains as well.

We consider the structure *C*(R) = (*C*(R)*, P*1*,..., P*12*, /*=), where the predicates

*P*1*,..., P*12 are interpreted for every *f, g ∈ C*(R) as follows.

The first group formalises relations between infimum and sumpemum of two func- tions on [0*,* 1]:

*C*(R) *|*= *P*1(*f, g*) *↔* sup *f|*[0,1] *<* sup *g|*[0,1];

*C*(R) *|*= *P*2(*f, g*) *↔* sup *f|*[0,1] *<* inf *g|*[0,1];

*C*(R) *|*= *P*3(*f, g*) *↔* sup *f|*[0,1] *>* inf *g|*[0,1];

*C*(R) *|*= *P*4(*f, g*) *↔* inf *f|*[0,1] *>* inf *g|*[0,1]*.*

The second group formalises properties of operations on *C*(R).

*C*(R) *|*= *P*5(*f, g, h*) *↔ f* (*x*)+ *g*(*x*) *< h*(*x*); for every *x ∈* [0*,* 1];

*C*(R) *|*= *P*6(*f, g, h*) *↔ f* (*x*) *· g*(*x*) *< h*(*x*) for every *x ∈* [0*,* 1];

*C*(R) *|*= *P*7(*f, g, h*) *↔ f* (*x*)+ *g*(*x*) *> h*(*x*) for every *x ∈* [0*,* 1];

*C*(R) *|*= *P*8(*f, g, h*) *↔ f* (*x*) *· g*(*x*) *> h*(*x*) for every *x ∈* [0*,* 1]*.*

The third group formalises relations between functions *f* and *λx.x*.

*C*(R) *|*= *P*9(*f* ) *↔ f* (*x*) *> x*; for every *x ∈* [0*,* 1];

*C*(R) *|*= *P*10(*f* ) *↔ f* (*x*) *< x* for every *x ∈* [0*,* 1]*.*

The fourth group formalises relations between a function *h* and the composition of

functions *f* and *g*.

*C*(R) *|*= *P*11(*f, g, h*) *↔ f* (*g*(*x*)) *< h*(*x*) for every *x ∈* [0*,* 1];

*C*(R) *|*= *P*12(*f, g, h*) *↔ f* (*g*(*x*)) *> h*(*x*) for every *x ∈* [0*,* 1]*.*

We recall the notion of compact open topology *τ*c*−*o on *C*(*X, Y* ). Let (*X, α*) and (*Y, β*) be topological spaces, *K ⊆ X* be a compact set, and *O⊆ Y* be an open set. Then subbase of the compact open topology is defined by sets of the type

*UK* = *{f ∈ C*(*X, Y* )*|f* (*K*) *⊂ O}.*

*O*

Since, by Weierstrass Theorem [[21](#_bookmark22)], Q[*x*] is dense in *C*(R), the base *τ∗* of the

c*−*o

topology *τ*c*−*o and its numbering are defined as follows:

1. The base *τ∗* is the finite intersections of the following sets

c*−*o

*U* a,b = *{f|p −* 1 *< f|*

1

*< p* + *},* where *b ∈* Q*,p ∈* Q[*x*] and deg(*p*) = *n.*

p,n

*n* [a,b] *n*

1. The numbering *ν* : *ω → τ∗* is standard.

**Proposition 2.9** *On the structure C* = (*C*(R)*, P*1*,..., P*12*, /*=) *the compact open topology τ*c*−*o *coincides with τ C.*

Σ

**Proof.** *⊆*)*.* It is easy to see that, for 1 *≤ i ≤* 12 the sets *{f*¯*|C*(R) *|*= *P*i(*f*¯)*}* and projections of them belong to *τ*c*−*o. By induction, *τ C*(R) *⊆ τ*c *.*

Σ *o*

*⊇*)*.* By definition, it is sufficient to show that the relations *f|*[a,b] *> g|*[a,b] and *f|*[a,b] *< g|*[a,b] are *∃*–definable. Note that *W*a,b = *{χ|χ*(0) *< a* and *χ*(1) *> b} ⊆ C*[0*,* 1] is

*∃*–definable set in the language *{P*i*, /*=*}*i*≤*12. Since,

*f|*[a,b] *< g|*[a,b] *↔ ∃χ ∈ W*a,b*∃h* (*f ◦ χ < h < g ◦ χ*) *,*

the relations *f|*[a,b] *> g|*[a,b] and *f|*[a,b] *< g|*[a,b] are *∃*–definable.

**Theorem 2.10** *The topological space* (*C*(R)*, τ*c*−*o*, ν*) *is effectively enumerable.*

**Proof.** Existence of a computable function *g* : *ω × ω × ω → ω*, such that

*νi ∩ νj* = *νg*(*i, j, n*)*,*

n*∈*ω

follows from the definition of *ν*. By quantifier elimination on R, the set *{i|νi /*= *∅}*

ic computably enumerable. Indeed, by Weierstrass Theorem [[21](#_bookmark22)], existence of *g ∈*

*C*(R) such that *g ∈*

i*∈*I

p*i*,n*i*

*U* a*i*,b*i* is equivalent to existence of *m ∈ ω* and polynomial

*p ∈* Q[*x*] of degree *m* such that *p ∈*

i*∈*I

p*i*,n*i*

*U* a*i*,b*i* . By quantifier elimination on R, we

can effectively check this property.d

We recall the notion of specialisation order on *T*0-spaces.

**Definition 2.11** Let (*X, τ* ) be a *T*0-space. A binary relation *≤* on *X* is called

*specialisation order* if *y ≤ x ↔ y ∈ cl*(*{x}*).

**Remark 2.12** Let us note that every partial continuous function *f* on a *T*0-space is monotone on dom*f* with respect to the specialisation order.

We recall the notion of core-compact topological space.

**Definition 2.13** A topological space (*X, τ* ) is said to be *core-compact* iff the lattice

O(*X*) of the open subsets is continuous.

It is well-known that locally compact spaces and continuous domains are core- compact [[8](#_bookmark11)]. Below we slightly modify the definition of strong inclusion. Let *≤* be the specialisation order. Denote *y*ˇ = *{z ∈ X|y ≤ z}* = k:y*∈*βk *βk*.

**Definition 2.14** Let (*X, τ, ν*) be an effectively enumerable core-compact *T*0-space, where *X* is a non-empty set, *τ∗ ⊆* 2X is a base of the topology *τ* and *α* : *ω → τ∗* is a numbering. Let *E ⊆ ω*2 be a computably enumerable relation. We say that *E* is compact-like strong inclusion (abbreviated as *clsi* ) if the following conditions hold.

(E 1). If *kEm*, then s*∈*D*k αs αm*.

(E 2). *αn* = mE*'*n *αm* for every *n, m ∈ ω* where *E'* = *{< n, m >∈ ω*2*|∃k*(*D*k =

*{n}∧ kEm*)*}*.

(E 3). If j*∈*J *αj* = *x*ˇ *{y ∈ X|x ≤ y}* for *x ∈ αm* and *J ⊆ ω*, then *kEm* for a finite

*D*k *⊆ J* .

(E 4). If *kEn* and for all *j ∈ D*k *l*j*Ej* and *D*s = j*∈*D*k D*l*j* , then *sEn*. (E 5). If *sEn* and *sEm*, then *∃k* (*kE'n ∧ kE'm ∧ sEk*).

The basic examples are Euclidian spaces (Rn*,τ* ), where the topology *τ* is formed by the base which is the set of balls *B*(*p, r*) with *p ∈* Qn and *r ∈* Q+. It is easy to see that s*∈*D*k αs αm* if and only if *cl*( *α*s*∈*D*k s*) *⊆ αm*. Put *kEm* *cl*( s*∈*D*k αs*) *⊆ αm*. By decidability of *Th*(R), the properties (*E*1) *−* (*E*5) hold.

# Computability on Effectively Enumerable Topologi- cal Spaces

Now we introduce notions of computable function over effectively enumerable topo- logical spaces based on the well-known definition of enumeration operator.

**Definition 3.1** [[16](#_bookmark17)] A function Γe : *P*(*ω*) *→ P*(*ω*) is called **enumeration opera- tor** if

Γe(*A*) = *B ↔ B* = *{j|∃i c*(*i, j*) *∈ W*e*, D*i *⊆ A},*

where *W*e is the *e*-th computably enumerable set, and *D*i is the *i*-th finite set.

**Definition 3.2** Let *X* = (*X, τ, α*) be an effectively enumerable topological space and *Y* = (*Y, λ, β*) be an effectively enumerable *T*0-space.

A partial function *F* : *X → Y* is called **computable** if there exists an enumeration operator Γe : *P*(*ω*) *→ P*(*ω*) such that, for every *x ∈ X*,

* 1. If *x ∈ dom*(*F* ) then

Γe(*{i ∈ ω|x ∈ αi}*) = *{j ∈ ω|F* (*x*) *∈ βj}.*

* 1. If *x /∈ dom*(*F* ) then, for all *y ∈ Y*

*{βj|j ∈* Γe(*A*x)*} /*= *{βj|j ∈ B*y*},*

j*∈*ω j*∈*ω

where *A*x = *{i ∈ ω|x ∈ αi}* and *B*y = *{j ∈ ω|y ∈ βj}*.

**Theorem 3.3** *Let X* = (*X, τ, α*) *be an effectively enumerable topological space and Y* = (*Y, λ, β*) *be an effectively enumerable T*0*-space. For a total function F* : *X → Y the following are equivalent.*

1. *F is computable;*
2. *There exists a computable function h* : *ω × ω → ω such that F−*1(*βj*) =

i*∈*ω *αh*(*i, j*)*.*

**Proof.** Let *F* : *X → Y* be computable. By definition, we have Γe (*{i|x ∈ αi}*) =

*{j|F* (*x*) *∈ βj}*. Since *X* is effectively enumerable, there exists a computable function

*H* : *ω × ω → ω* such that

*αi* = *αH*(*k, s*)*.*

i*∈*D*k* s*∈*ω

So,

*x ∈ F−*1(*βj*) *↔ F* (*x*) *∈ βj ↔ ∃k* (*D*k *⊆ {i|x ∈ αi}∧ c*(*k, j*) *∈ W*e) *↔*

*x ∈ αi ↔*  *∃sx ∈ αH*(*k, s*) *↔*

c(k,j)*∈*W*e* c(k,j)*∈*W*e*

*x ∈*

c(k,j)*∈*W*e*,s*∈*ω

*αH*(*k, s*) *↔ x ∈* *αh*(*j, m*)

m*∈*ω

for a computable function *h* : *ω × ω → ω*.

Now suppose *F−*1(*βj*) = i*∈*ω *αh*(*i, j*). Then, there exists *e* such that, for *A*x =

*{x|x ∈ αi}*,

Γe(*A*x) = *{j|∃s h*(*j, s*) *∈ A*x*}* = *{j|x ∈ F−*1(*βj*)*}* = *{j|F* (*x*) *∈ βj}.*

**Proposition 3.4** *Let X* = (*X, τ, α*) *be an effectively enumerable topological space and Y* = (*Y, λ, β*) *be an effectively enumerable T*0*-space.*

1. *If F* : *X → Y is a computable function, then F is continuous at every points of* dom *F.*
2. *A total function F* : *X → Y is computable if and only if F is effectively continuous.*

**Proof.** The first claim is straightforward form Definition [3.2](#_bookmark1). The second claim is based on Theorem [3.3](#_bookmark2).

**Definition 3.5** Let *X* = (*X, τ, α*) be an effectively enumerable topological space and *Y* = (*Y, λ, β*) be an effectively enumerable *T*0-space.

A partial function *F* : *X → Y* is called **strongly computable** if there exists an enumeration operator Γe : *P*(*ω*) *→ P*(*ω*) such that

1. If *x ∈* dom*F* , then Γe(*A*x) = *B*F (x), where *A*x = *{i ∈ ω|x ∈ αi}, B*y = *{j ∈*

*ω|y ∈ βj}*.

1. If *x /∈* dom*F* and Γe(*A*x) = *J* , then *{β*i*|j ∈ J} /⊆ y*ˇ for every *y ∈ Y* .

**Remark 3.6** Let us note that the notion of strongly computability is invariant under computably equivalent numberings of topologies bases.

Now we compare our notion of strongly computability with strongly (*ρ*c *, ρ*c )-

X Y

computability for *F* : *X → Y* , where *X* and *Y* are computable metric spaces,

and *ρ*c *, ρ*c are Cauchy-representations of them. For the definitions of Cauchy-

X Y

representation and strongly (*ρ*c *, ρ*c )-computability we refer to [[23](#_bookmark24)].

X Y

**Theorem 3.7** *Let X* = (*X, λ, B*X *, d*X) *and Y* = (*Y, β, B*Y *, d*Y ) *be computable met- ric spaces and* (*X, τ*X *, α∗*)*,* (*Y, τ*Y *, β∗*) *be corresponding them effectively enumerable topological spaces. For every total function F* : *X → Y , the following are equivalent.*

1. *F is strongly* (*ρ*c *, ρ*c )*-computable;*

X Y

1. *F is strongly computable as a function from one effectively enumerable topo- logical space to another (c.f. Deﬁnition* [*3.5*](#_bookmark3)*).*

**Proof.** It is easy to see that there exists an effective procedure which given a Cauchy-representation *ρ*c (*z*) produces *A*z = *{i|z ∈ α∗i}* as well as there exists an effective procedure which given *A*z produces a Cauchy-representation *ρ*c (*z*) for every *z ∈ X*. By Definition [3.5](#_bookmark3) and the definition of (*ρ*c *, ρ*c )-computability, both

X

X

X Y

computabilities coincide, details are routine.

**Theorem 3.8** *For total functions the notions of computability and strongly com- putability coincide.*

**Remark 3.9** Below in the case of total functions we use notation ”computable” for both computable and strongly computable functions.

Let (N*, τ, ν*)*,* be a *T*0–space, where N is the natural numbers, *τ* is the discrete topology and *ν* is its numbering defined as follows:

*ν* 0 = *∅*; *νn* +1 = *{n}.*

**Proposition 3.10** *For* (N*, τ, ν*)*, the class of partial strongly computable functions coincides with the partial recursive functions.*

**Proof.** Suppose *f* : N *→* N is strongly computable. Since the specialisation order on N coincides with the equality on N, there exists an enumeration operator Γe : *P*(*ω*) *→ P*(*ω*) such that

*n* +1 *∈* Γe(*D*) *↔ ∃x* (*x* +1 *∈ D ∧ f* (*x*) = *n*) *.*

Suppose *D* is finite. Note that if *x /∈* dom*f* , then for all *y ∈ Y* ,

*{βj|j ∈* Γe(*A*x)*} /⊆ {y}.*

j*∈*ω

Hence, *f* (*x*) = *n ↔ ∃D* (*D* is finite *∧ x* = 1 *∈ D ∧ n* +1 *∈* Γe(*D*)), i.e., *f* is a par- tial recursive function.

Suppose *f* is a partial recursive function. Put Γe(*A*) = *{f* (*x*)+ 1*|x* +1 *∈ A}*. It is easy to see that Γe(*A*) is a required enumeration operator.

**Theorem 3.11** *For partial functions, the strongly computable functions is a proper subclass of the computable functions.*

**Proof.** Let us consider *T*0–space (N*, τ, ν*). It is easy to see that a computable function is representable as *h*1*\h*2 for some partial recursive functions *h*1*, h*2 whereas the strongly computable functions coincide with the partial recursive functions.

**Definition 3.12** Let *X* = (*X, τ, α*) be an effectively enumerable topological space and *Y* = (*Y, λ, β*) be an effectively enumerable *T*0-space.

A partial function *F* : *X → Y* is called **weakly computable** if there exists an enumeration operator Γe : *P*(*ω*) *→ P*(*ω*) such that, for every *x ∈ X*,

1. If *x ∈ dom*(*F* ), then

Γe(*A*x) = *J* and *β*j = *F*ˇ(*x*)

j*∈*J

1. If *x /∈ dom*(*F* ), then

Γe(*A*x) = *J* and *β*j */*= *y*ˇ for any *y ∈ Y.*

j*∈*J

**Proposition 3.13** *The computable functions is a proper subclass of weakly com- putable functions.*

Let us consider the real numbers with two topologies *τ*R and *τ*A, where *τ*R is the standard topology and *τ*A is defined as follows. We fix a se t *A* which is open but not effectively open. The topology *τ*A is induced by the base

*τ∗* = *{*(*a, b*)*|a, b ∈* Q*}∪ {*(*a, b*) *∩ A|a, b ∈* Q*}∪ {*(*−∞,* +*∞*)*}.* We take *f* = *id* : (R*, τ*R*, α*) *→* (R*, τ*A*, β*), where *β* is defined as follows.

A

*β*(2*n*) = *αn*; *β*(2*n* + 1) = *A ∩ αn.*

Since preimage of *A* is not effectively open, *f* is not computable whereas *f* is weakly computable. Indeed, it is easy to see that Γe(*Y* ) = 2*Y* = *{*2*m|m ∈ Y }* is a corresponding enumeration operator.

**Theorem 3.14** *Let X* = (*X, τ, α*) *be an effectively enumerable topological space and Y* = (*Y, λ, β*) *be an effectively enumerable core-compact T*0*-space endowed by some clsi-relation E ⊆ ω*2*. A partial function F* : *X → Y is computable if and only if F is weakly computable.*

**Proof.** If *F* is computable it is easy to see that the corresponding operator Γe satisfy the conditions of Definition [3.12](#_bookmark4).

Let *F* be a weakly computable function and Γe be a corresponding enumeration operator. We construct a new enumeration operator Γe*'* as follows.

*m ∈* Γe*'* (*A*) *↔ m ∈* Γe(*A*) *∨ ∃k∃s*[*D*s *⊆ A ∧ D*k *⊆* Γe(*D*s) *∧ kEm*]*.*

By the properties (*E*1) and (*E*3) of the clsi-relation *E* it follows that

*{α*j*|j ∈* Γe*'* (*A*)*}* = *{α*j*|j ∈* Γe(*A*)*}.*

Hence, whereas,

if *x ∈* dom*F,* then *αm ∈ F* (*x*) *↔ m ∈* Γe*'* (*A*x)*,*

if *x /∈* dom*F,* t*hen* *{α*j*|j ∈* Γe*'* (*A*x)*} /*= *z*ˇ for any *z ∈ Y.*

So, *F* is computable.

# Conclusion and Related work

We investigated computability over effectively enumerable topological spaces which contain computable metric spaces and effective *ω*-continuous domains as proper subclasses. It has been shown that computability over effectively enumerable topo- logical spaces corresponds to effective continuity. There has been a considerable interest in computability theory in the question of whether computable maps are continuous with respect to natural topologies. Myhill and Shepherdson [[15](#_bookmark16)] have shown that every computable operator on the set of partial recursive functions is effectively continuous and vice versa. Kreisel, Lacombe and Shoenfield [[13](#_bookmark14)] have proven analogous results for the total recursive functions. These results have been generalised to effectively given Scott domains [[6](#_bookmark8),[17](#_bookmark18),[22](#_bookmark23)], recursive metric spaces [[14](#_bookmark15)], separable countable *T*0-spaces with a witness for noninclusion [[20](#_bookmark21)]. It was shown that in general the correspondence between computability and effective continuity does not hold [[7](#_bookmark9),[13](#_bookmark14),[24](#_bookmark25)]. For historical remarks we refer to [[19](#_bookmark20)].

The main advantages of the class of effectively enumerable topological spaces are the following:

* The class of effectively enumerable topological spaces is not restricted to countable spaces.
* The class of effectively enumerable topological spaces contains computable metric spaces, *ω*-continuous domains.
* Different notions of computability of partial functions is formalised and investi- gated.
  + For total functions, computability is equivalent to effective continuity.

# References

1. Abramsky,S., and A. Jung, *Domain Theory*, in Handbook of Logic in Computer Science, S. Abramsky,

D. Gabbay and T. S. E. Maibaum, eds., (Oxford University Press), 1994, 1–168

1. Vasco Brattka, Vasco, Gero Presser, *Computability on subsets of metric spaces*, Theoretical Compututer Sciene **305(1-3)** (2003), 43–76.
2. Brattka, V., and K. Weihrauch, *Computability on subsets of euclidean space I: Closed and compact sets*, Theoretical Compututer Sciene, **219** (1999), 65–93.
3. Edalat, A., and P. S¨onderhauf, *A Domain-theoretic Approach to Real Number Computation*, Theoretical Computer Science **210** (1998), 73–98.
4. Ershov, Yu. L., “ Numbering Theorey” (in Russian), Nauka, Moscow, 1977.
5. Ershov, Yu. L., *Model* C *of partial continuous functionals*, In Logic colloquium 76, North-Holland, Amsterdam (1977), 455–467.
6. Friedberg, R., *Un contre-exapmle relatif aux fonctionelles recursives*, Comptes Rendus de ’Academie des Science **247** (1958), 852–854.
7. Gierz, G., K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislov, and D.S. Scott, “ Continuous Lattices and Domains”, Cambridge University Press, 2003.
8. Hertling, P., *An effective Riemann Mapping Theorem*, Theoretical Computer Science **219** (1999), 225– 265.
9. Hertling, P., *Is the Mandelbrot set computable?* Math. Log. Quat. **51, No. 1** (2005), 5–18.
10. Korovina, Margarita, and Oleg Kudinov, *Basic Principles of* Σ*-definability and Abstract Computability*, Fachbereich Mathematik **08–01**, 2008.
11. Korovina, Margarita, and Oleg Kudinov, *The Uniformity Principle for* Σ*-definability with Applications to Computable Analysis*, In S.B. Cooper, B. L¨owe, and A. Sorbi, editors, *CiE’07,* Lecture Notes in Computer Science, **4497** (2007) 416–425.
12. Kreisel, G., D. Lacombe, and J. Shoenfield, *Partial recursive functions and effective operators*, Constructivity in mathematics, North-Holland, Amsterdam (1959), 290–297.
13. Moschovakis, Y. N., *Recursive Metric Spaces*, Fund. Math. **55** (1964), 215–238.
14. Myhill, J., and J.C. Shepherdson, *Effective operators on partial recursive functions*, Zeitschrift fur mathematische Logik Grundlagen def Mathematik **1** (1955), pp 310-317.
15. Rogers, H., “ Theory of Recursive Functions and Effective Computability”, McGraw-Hill, New York, 1967.
16. Sciore, E., and A. Tang, *Computability theory in admissible domains*, In Proc. 10th annual ACM symposium on theory of computing, ACM, New York (1978), 95–104.
17. Scott, D., *Outlines of mathematical theory of computation*, In Proc. 4th Annual Princeton Conf. on Information Sciences and Systems, Princeton University Press (1970), 169-176.
18. Spreen, D., *On Effective Topological Spaces*, JSL **63, 1** (1998), 185–221.
19. Spreen, D., and P. Young, *Effective Operators in a Topological Setting*, Lecture Notes in Mathematics

**1104** (1984), 436–451.

1. Weierstrass, K., *Sitzungsber*, Acad. Berlin **1885-S. 633-9**, 789-805.
2. Weihrauch, Klaus, *Berechenbarkeit auf cpo’s* , Schriften zur Angewandten Mathematik und Informatik, Aachen **63** (1980).
3. Weihrauch, Klaus, “ Computable Analysis”, Springer Verlag, Berlin, 2000.
4. Young, P., *An effective operator, continuous but not partial resursive*, Proc. of AMS **19** (1968), 103-108.