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Uniform Domains and Uniform Spaces (Abstract)

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Abstract

In a previous work, the current author showed that every compact metric space X can be represented in an omega-algebraic domain D so that X is the retract of the set L(D) of limit (i.e. non-finite) elements of D. This means that every infinite strictly increasing sequence in the set K(D) of finite elements of D can be considered as identifying one point of X, and thus this domain structure can be used to define computation over the space X. In this article, we show a condition on an omega-algebraic domain D which ensures that L(D) has a (separable complete) metric space as its retract. We introduce the notion of a uniform domain, and explain that it corresponds to a uniform space with countable weight. Here, we use the word domain for an omega-algebraic dcpo.

*Keywords:* Uniform space, domain representation, stratified domain

In [[3](#_bookmark4)], the current author showed that every compact metric space *X* can be represented in an *ω*-algebraic domain *D* so that *X* is the retract of the set *L*(*D*) of limit (i.e. non-finite) elements of *D*. This means that every infinite strictly increasing sequence in the set *K*(*D*) of finite elements of *D* can be considered as identifying one point of *X*, and thus this domain structure can be used to define computation over the space *X*. In this article, we show a condition on an *ω*-algebraic domain *D* which ensures that *L*(*D*) has a (separable complete) metric space as its retract. Following [[2](#_bookmark2)], we introduce the notion of a *uniform domain*, and explain that it corresponds to a uniform space with countable weight [[1](#_bookmark3)]. Here, we use the word *domain* for an *ω*-algebraic dcpo.

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Definition 0.1 When *P* is a poset, we define the *level* of *d* ∈ *P* as the max- imal length of a chain ⊥*P* = *a*0 Ç *a*1 Ç *...* Ç *an* = *d*, when it exists, and we write *Kn*(*P* ) for the set of level-*n* elements of *P* . A poset *P* is *stratiﬁed* if each *e* ∈ *P* has a level, that is, if we have *P* = *K*0(*P* ) ∪ *K*1(*P* ) ∪ A domain *D*

is *stratiﬁed* if *K*(*D*) is a stratified poset, and *D* is *evenly stratiﬁed* if all the paths to *d* have the same length for every *d* ∈ *K*(*D*). We call *Kn*(*D*) ∩ *Kx* the set of *level-n approximations* of *x*.

Definition 0.2 Let *D* be a stratified domain and *d* ∈ *Km*(*D*). We denote by *d*∗ ⊂ *Km*(*D*) the set of elements of *Km*(*D*) which are compatible with *d*. If, for each *d* ∈ *Km*(*D*), there exists a lower bound of *d*∗ in *Kn*(*D*), we define that *n <*∗ *m*. When *D* be a stratified domain and for each *n* ∈ N, there is a *m* ∈ N such that *n <*∗ *m*, we say that *D* is a *uniform domain*.

Here, *a* is compatible with *b* means that *a* and *b* have an upper bound in

*D* (which also implies that *a* and *b* have an upper bound in *K*(*D*)).

Definition 0.3 Let *P* be a poset.

1. *x* ∈ *P* is a *minimal element* if *y* ≤ *x* implies *y* = *x* for all *y* ∈ *P* . We write *MP* for the set of all minimal elements of *P* .
2. We say that *P* has *enough minimal elements* if, for all *y* ∈ *P* , there exists

*x* ∈ *MP* such that *x* ≤ *y*.

Theorem 0.4 *Let D be a uniform domain.*

1. *L*(*D*) *has enough minimal elements.*
2. *ML*(*D*) *is a retract of L*(*D*)*.*
3. *ML*(*D*) *is a Hausdorff space.*

Note that many of the domains studied in computer science, for example,

*Pω* = {*a* | *a* ⊆ *N* } and Plotkin’s *T ω* do not have enough minimal elements. The proof for this theorem is analogous to that of the existence of a minimal Cauchy filter in a uniform space. When *d* ∈ *K*(*D*), we define *d*ˆ as the subset

↑*d* ∩ *ML*(*D*) of *ML*(*D*).

Theorem 0.5 *When D is a uniform domain, D induces a complete unifor- mity µ of countable weight on ML*(*D*)*, deﬁned through the base consisting of the coverings* B = {V0*,* V1*,.* } *deﬁned as* V*i* = {*d*ˆ | *d* ∈ *Ki*(*D*)}*.*

Since the weight of the uniformity constructed in Theorem [0.5](#_bookmark1) is countable, we have the following.

Corollary 0.6 *When D is a uniform domain, ML*(*D*) *is metrizable.*

On the other hand, we have the following.

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Theorem 0.7 *Let* (*X, µ*) *be a complete uniform space with a countable weight,* *and* U0 > U1 > *... be a sequence of open coverings which forms a base of µ.* *From this sequence, we can form a evenly-stratiﬁed uniform domain D such that X is homeomorphic to ML*(*D*)*.*

Thus, a uniform domain can be considered as a uniform space with a selection of a base.

# References

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