1. Consider the double integral

Total: 27 points.

$$\mathbf{I} = \iint_R xy^3 dA$$

where R is the region in the xy-plane bounded by the four curves $xy = 1$, $xy = 3$, $y = x$, and $y = 3x$.

(a) [4 pts] Define $\mathbf{T}$ to be the transformation

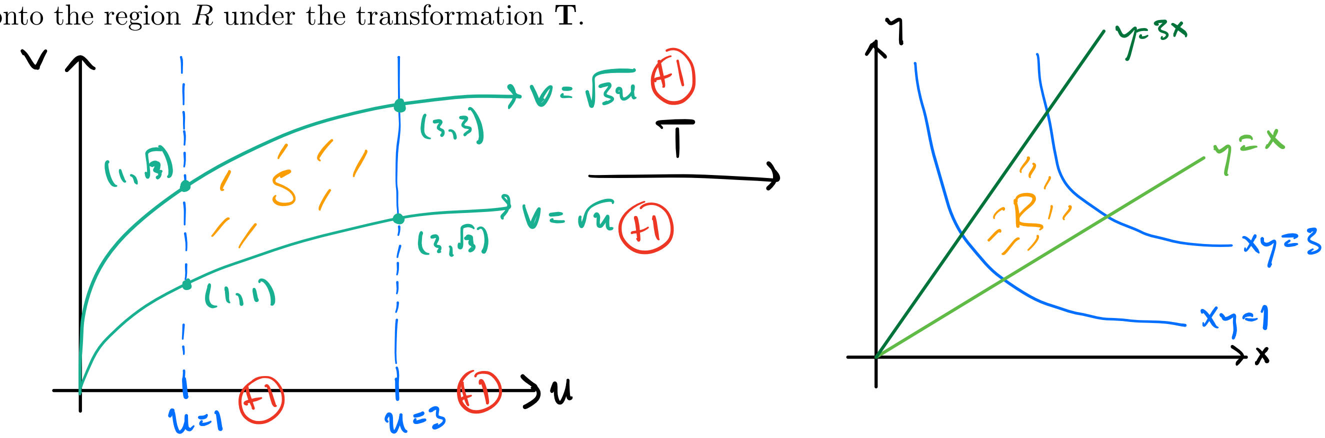
$$\mathbf{T} = \begin{cases} x = \dfrac{u}{v} \\ y = v \end{cases}$$

$y = x \Rightarrow \frac{u}{v} = v$
$\Rightarrow u = v^2$ and $3u = v^2$

$xy = 1 \Rightarrow \left(\frac{u}{v}\right) v = 1 \Rightarrow u = 1$

Also:
$xy = 3 \Rightarrow u = 3$

Sketch the region R in the xy-plane. Also, sketch the region S in the uv-plane which maps onto the region R under the transformation $\mathbf{T}$.



(b) [3 pts] Compute the Jacobian of the transformation $\mathbf{T}$.

$$\text{Jac} = \text{Det}\begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v} - 0 = 1/v \quad (+3)$$

(c) [4 pts] Set up, but do not evaluate, an expression for $\mathbf{I}$ as an iterated integral in terms of the variables u and v.

$$(+2)\ \boxed{\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}}} \underbrace{\left(\frac{u}{v}\right)(v)^3}_{(+1)} \left(\frac{1}{v}\right) dv\, du$$

(+1) For magnification factor

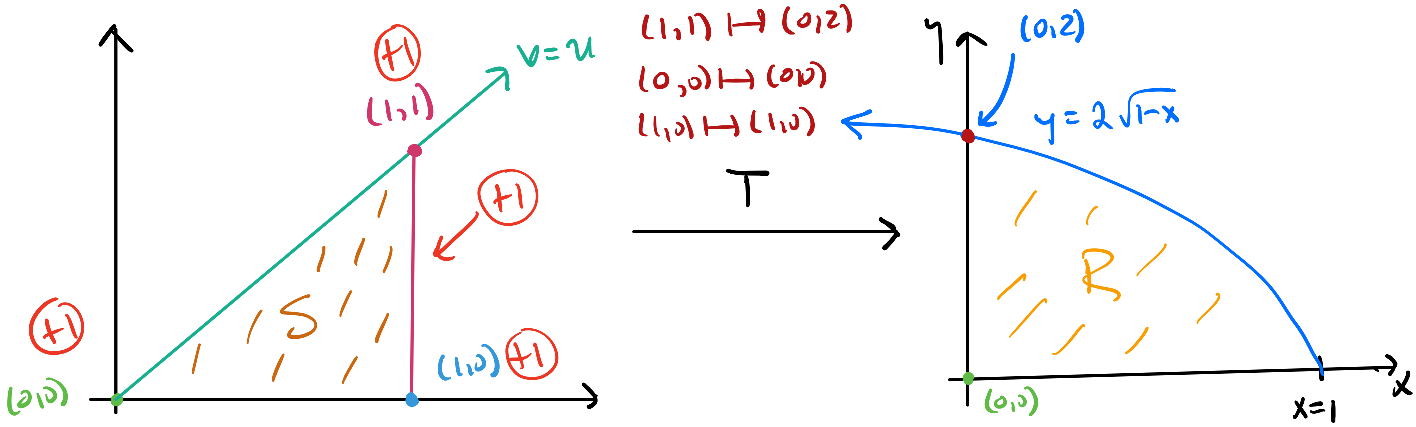
2. Consider the double integral

$$\mathbf{I} = \int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx,$$

and define the transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{T} = \begin{cases} x &= u^2 - v^2 \\ y &= 2uv \end{cases}$$

(a) [4 pts] Sketch the region R of integration in the xy-plane. Also, sketch the region of integration S in the uv-plane which maps onto the region R under the coordinate transformation $\mathbf{T}$.



(b) [3 pts] Compute the Jacobian of the transformation $\mathbf{T}$.

Not Graded

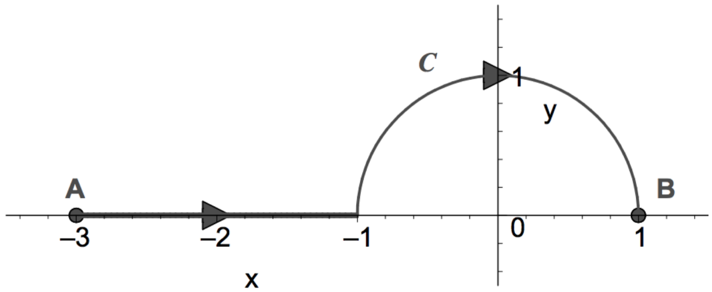
(c) [4 pts] Use the transformation $\mathbf{T}$ to evaluate the integral $\mathbf{I}$.

Not Graded.

3. In this problem we consider the two vector fields

$$\mathbf{F}(x,y) = \langle -y, x\rangle \quad \text{and} \quad \mathbf{G}(x,y) = \langle \cos(x) + y, x - 1\rangle,$$

and the curve C from the point $A(-3,0)$ to the point $B(1,0)$ that first goes along the x-axis, and then follows along the unit circle (see the picture below).



(a) [4 pts] Carefully explain whether either of the vector fields $\mathbf{F}$ or $\mathbf{G}$ is conservative or not.

+2 $\vec{F}$ is NOT conservative b/c curl $\vec{F} = Q_x - P_y = 1 - (-1) = 2 \neq 0$

$\vec{G}$ is conservative b/c:

+1 ① Defined on a simply connected domain

+1 ② curl $\vec{G} = \vec{0}$.

(b) [3 pts] Compute the work done by the vector field $\mathbf{F}$ in moving a particle along C.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= 0 - \pi$$

$$= -\pi$$ +3

(c) [3 pts] Compute the work done by the vector field $\mathbf{G}$ in moving a particle along C.

Use Fundamental Thm of Calculus....

Find potential function

Then $\int_C \vec{G}\cdot d\vec{r} = g((1,0)) - g((-3,0))$

+3

4. Consider the vector field

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

which is defined everywhere on $\mathbb{R}^2$ except the origin.

(a) [3 pts] Let C be the unit circle $x^2+y^2=1$ in $\mathbb{R}^2$ oriented counterclockwise. Parametrize the closed loop C and evaluate the integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Not Graded.

(b) [2 pts] Write $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. Compute the partial derivatives Q_x and P_y.

+2 For participation.

(c) [3 pts] Decide whether or not $\mathbf{F}$ is a conservative vector field on the domain $\mathbb{R}^2 - \{(0,0)\}$. Explain your reasoning.

(d) [3 pts] Consider the function $f(x,y) = \arctan\left(\frac{y}{x}\right)$. Show that the gradient vector field of f is the vector field $\mathbf{F}$ shown above (i.e. show that $\nabla f = \mathbf{F}$). Does this contradict your answers to part (a) and (c)? Explain why or why not.

(e) [2 pts] Use part (d) to compute the line integral $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}$ where C_2 is the circle of radius $r=1$ and center $(0,3)$.