1. [3 pts] Determine the arc length along the curve

$$\mathbf{r}(t) = \langle \cos(t) + t\sin(t), \sin(t) - t\cos(t), t^2 \rangle$$

from $t = 0$ to $t = \pi/2$.

+1 $\vec{r}'(t) = \langle -\sin(t) + \sin(t) + t\cos(t), \cos(t) - \cos(t) + t\sin(t), 2t \rangle$

$$L = \int_0^{\pi/2} \sqrt{t^2\cos^2 t + t^2 \sin^2 t + 4t^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\, dt = \int_0^{\pi/2} \sqrt{5}\, t\, dt = \frac{\sqrt{5}\, t^2}{2} \Big|_0^{\pi/2} = \frac{\sqrt{5}}{2}\frac{\pi^2}{4} = \frac{\pi^2\sqrt{5}}{8} \;//$$

+2

2. [3 pts] Interestingly, the notion of arc length can be defined in any dimension. A curve in four-dimensional space $\mathbb{R}^4$ is parametrized as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t), w(t) \rangle, \quad a \leq t \leq b.$$

Find the arc length of $\mathbf{r}(t) = \langle t, \ln(t), 1/t, \ln(t) \rangle$ and $1 \leq t \leq 4$.

+1 $\vec{r}'(t) = \langle 1, 1/t, -1/t^2, 1/t \rangle$

$$L = \int_1^4 \sqrt{1^2 + (1/t)^2 + (1/t^2)^2 + 1/t^2}\, dt$$

$$= \int_1^4 \sqrt{1 + 2/t^2 + 1/t^4}\, dt = \int_1^4 \sqrt{(1/t^2 + 1)^2}\, dt = \int_1^4 (1/t^2 + 1)\, dt$$

$$= \left(-\frac{1}{t} + t\right)\Big|_1^4$$

$$= -1/4 + 4 + 1 - 1$$

+2

3. Suppose that the trajectory of a particle in $\mathbb{R}^3$ is described by the vector-valued function $\mathbf{r}(t)$, and let $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$ be its velocity and acceleration vectors, respectively. For each of the following statements, either give a proof or exhibit a counter-example.

No Grade

(a) [2 pts] Let $\mathcal{C}$ be the space-curve in $\mathbb{R}^3$ which is parametrized by $\mathbf{r}(t)$, $-\infty < t < \infty$. If the velocity vector $\mathbf{v}(t)$ is constant, then the curve $\mathcal{C}$ lies entirely in a single plane.

TRUE.

Pf: Since $\vec{v}(t) = \vec{c}$ a constant

$\Rightarrow \vec{r}(t) = \vec{r}_0 + t\vec{c}$ a line which certainly is contained in a plane.

(b) [2 pts] Define the *speed* of the particle at time t to be the length of its velocity vector $s(t) = |\mathbf{v}(t)|$ at time t. If the speed is a constant function, then the curve lies entirely in a plane.

No Grade

FALSE.

Take $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

Then speed is constant but not restricted to be planar.

(c) [2 pts] If the acceleration vector $\mathbf{a}(t)$ is constant, then the curve $\mathcal{C}$ lies entirely in a single plane.

No Grade

TRUE. Since $\vec{a}$ is contained in the plane determined by $\vec{T}$ and $\vec{N}$, it follows that $\vec{v} \times \vec{a}$ is parallel to $\vec{T} \times \vec{N}$.

Moreover: $\dfrac{d(\vec{v} \times \vec{a})}{dt} = \vec{a} \times \vec{a} + \vec{v} \times \dfrac{d\vec{a}}{dt} = \vec{0}$ $\therefore$ $\vec{r}(t)$ is planar.

(d) [2 pts] If the velocity vector $\mathbf{v}(t)$ is *orthogonal* to the acceleration vector $\mathbf{a}(t)$ for all time t, then the curve $\mathcal{C}$ lies entirely in a single plane.

No Grade

FALSE.

Again, consider the helix.

(e) [2 pts] Prove that $\mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0}$ implies $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$, where $\mathbf{c}$ is a constant vector.

No Grade

By taking the derivative:

$$\frac{d}{dt}(\vec{r}(t) \times \vec{v}) = \vec{v} \times \vec{v} + \vec{r} \times \vec{v}'$$
$$= \vec{0} + \vec{0} = \vec{0}$$

$\therefore$ $\vec{r}(t) \times \vec{v}(t)$ a constant vector.

(f) [2 pts] If $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ for all time t, prove that the motion takes place in a plane (i.e. that the space curve parametrized by $\mathbf{r}(t)$ lies entirely in a plane). Consider both $\mathbf{c} = \mathbf{0}$ and $\mathbf{c} \neq \mathbf{0}$.

No Grade

Suppose $\vec{c} \neq \vec{0}$. Then $\vec{c} \perp \vec{v}$ for all time t

$\Rightarrow$ $\vec{c} \cdot \vec{v} = 0$ $\forall$ time t

$\Rightarrow$ $\vec{c} \cdot \int_{t_0}^{t_1} \vec{v} = \int_{t_0}^{t_1} 0$ $\Rightarrow$ $\vec{c} \cdot (\vec{r}(t_0) - \vec{r}(t_1)) = 0$

$\Rightarrow$ Motion $\vec{r}(t)$ is planar.

if $\vec{c} = 0$ Then $\vec{r} \parallel \vec{v}$ so $\vec{r}$ parametrizes a line.

4. Prove that the space curve

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 2t - \cos t + 3 \\ \sin^2 t + 4t \\ \frac{1}{2}(-\cos^2 t + 2\cos t + 1) \end{bmatrix}$$

lies entirely in a single plane. Find an equation for the plane.

- Many ways to solve this.
- You're told the curve lies in a plane $\rightsquigarrow$ the tangent vectors $\vec{r}\,'(t)$ will lie in this plane.

$\rightsquigarrow$ Compute

$$\vec{r}\,'(t) = \begin{bmatrix} 2 + \sin(t) \\ 2\sin(t)\cos(t) + 4 \\ \sin(t)\cos(t) - \sin(t) \end{bmatrix} \rightsquigarrow$$

Evaluate at two times to get two different tangent vectors (they will both lie in the plane)

$$\vec{r}\,'(0) = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{r}\,'(\pi/2) = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

Use these two vectors to make a plane.

Compute $\vec{r}\,'(0) \times \vec{r}\,'(\pi/2)$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ 3 & 4 & -1 \end{vmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \rightsquigarrow$$

Defines $\vec{n}$ for a plane; use $\vec{r}$ to find a point on the plane

(+2) For finding a normal to plane

$$\vec{r}(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow$$

Eqn for plane is

$$-2(x-2) + (1)(y-0) - 2(z-1) = 0$$

$$\Rightarrow -2x + 4 + y - 2z + 2 = 0$$

$$\Rightarrow -2x + y - 2z + 6 = 0$$

CHECK: Need to check that every point of $\vec{r}(t)$ satisfies this equation:

$$-2\big(2t - \cos(t) + 3\big) + \big(\sin^2(t) + 4t\big) - 2(1/2)\big(-\cos^2(t) + 2\cos(t) + 1\big) + 6 \stackrel{?}{=} 0$$

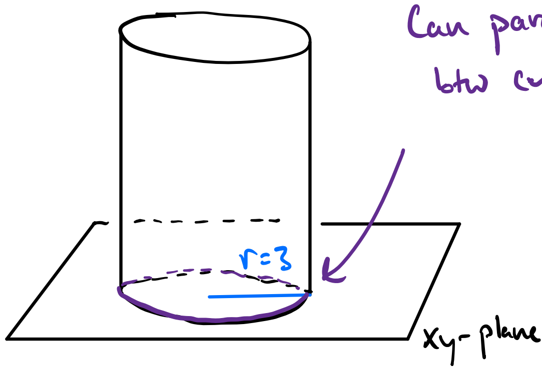
$$\Rightarrow -4t + 2\cos(t) - 6 + \underbrace{\sin^2(t) + 4t + \cos^2(t)}_{=1} - 2\cos(t) - 1 + 6 \stackrel{?}{=} 0$$

$\Rightarrow$ Yes!!

(+2) For showing components of $\vec{r}(t)$ satisfy plane equation or any other justification.

5. [4 pts] Find a vector-valued function, $\mathbf{r}(t)$, that represents the curve of intersection between the cylinder $\{(x, y, z) \mid x^2 + y^2 = 9\}$ and the surface $\{(x, y, z) \mid z = xy\}$.

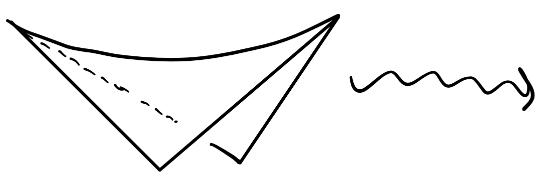
The cylinder looks like:



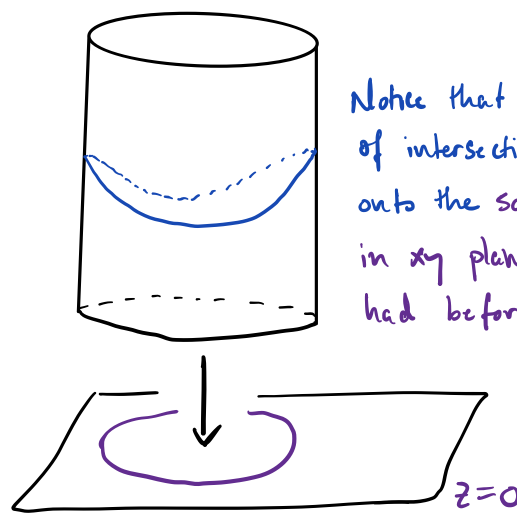
Can parametrize the curve of intersection btw cylinder and $z=0$ by

$= \langle 3\cos(t), 3\sin(t), 0 \rangle, \quad 0 \le t \le 2\pi$

The surface $z=xy$ looks like:



Curve of intersection btw cylinder and surface $z=xy$ looks like:



Notice that curve of intersection projects onto the same circle in xy plane we had before

Therefore, we can "lift" the parametrization of the circle in xy-plane to the cylinder in $\mathbb{R}^3$

$\rightsquigarrow$ Lift comes from the surface $z=xy$.

Solution: $\vec{r}(t) = \begin{bmatrix} 3\cos(t) \\ 3\sin(t) \\ 9\cos(t)\sin(t) \end{bmatrix}, \quad 0 \le t \le 2\pi$ $\rightsquigarrow z = xy$.

+4