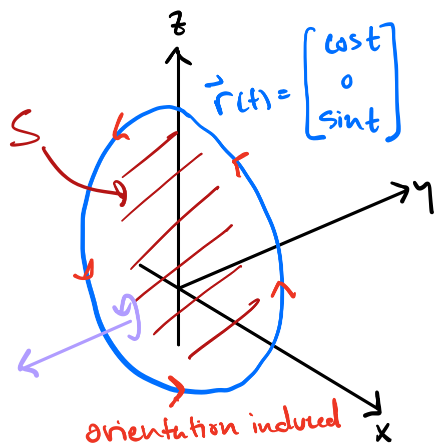
1. [10 pts] Let C be the space curve parametrized by $\mathbf{r}(t) = \langle \cos t, 0, \sin t\rangle$ for $0 \le t \le 2\pi$, and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = \langle \sin(x^3) + z^3, \sin(y^3), \sin(z^3) - x^3\rangle.$$

Compute the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$.



Apply Stokes' Theorem

$$\underbrace{\oint_{\partial S} \vec{F}\cdot d\vec{r}}_{\text{Asked to compute this}} = \underbrace{\iint_S (\nabla\times\vec{F})\cdot d\vec{S}}_{\text{Calculate this instead.}}$$

To this end, consider the surface S given by

$$S = \{(x,0,z) \mid x^2+z^2 \le 1\}$$

A subset of xz-plane.

Orient the surface S via the vector $\vec{n} = -\vec{j} = \langle 0,-1,0\rangle$

To apply Stokes, first compute

$$\nabla\times\vec{F} = \operatorname{Det}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ \sin(x^3)+z^3 & \sin(y^3) & \sin(z^3)-x^3 \end{vmatrix}$$

$$= \begin{bmatrix} 0-0 \\ -(-3x^2-3z^2) \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3x^2+3z^2 \\ 0 \end{bmatrix} = \nabla\times\vec{F}$$

Finally, calculate RHS of Stokes' Thm!

$$\text{RHS} = \iint_S \underbrace{\begin{bmatrix} 0 \\ 3x^2+3z^2 \\ 0 \end{bmatrix}}_{\nabla\times\vec{F}} \cdot \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{\vec{n}} \, dS$$

$$= -3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta \quad \text{Use Polar coordinates.}$$

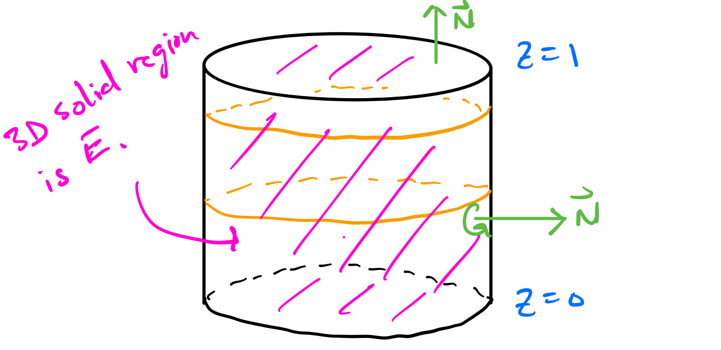
$$= -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta$$

$$= -3(2\pi)(1/4) = -3\pi/2.$$

2. [10 pts] Calculate the double integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle x, y + z^3, e^y \rangle$ and S is the boundary of the solid region E determined by $E = \{(x, y, z) \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}$. Orient S by the outward pointing unit normal field.



Apply Divergence Theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

Compute

$$\operatorname{div} \vec{F} = 1 + 1 + 0 = 2$$

By Divergence Thm, the flux of the v.f. $\vec{F}$ across $\underbrace{S = \partial E}_{\text{The cylinder is the boundary of 3D solid region}}$, can be computed as:

$$\underbrace{\int_0^{2\pi}\int_0^1\int_0^1}_{\text{Description of } E \text{ in cylindrical coordinates}} 2r\,dz\,dr\,d\theta = 2\pi \int_0^1 2r\,dr = 2\pi \left[r^2 \Big|_0^1 \right]$$

$$= 2\pi$$

$$= 2\,\underbrace{(\text{Volume of inside the cylinder})}_{\begin{aligned} &= \text{width} \cdot \text{height} \\ &= \pi r^2 \cdot h \\ &= \pi. \end{aligned}}$$