1. Multiple choice.

(a) [3 pts] If $\text{div}\,\mathbf{F}(x,y,z) = 0$ for all (x,y,z) then $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed curve C.

(i) True.
(ii) False.
(iii) Indeterminable.

If $\text{curl}\,\vec{F} = \vec{0}$ for all (x,y,z) in $\mathbb{R}^3$, then statement is TRUE.

(b) [3 pts] Which of the following apply to the vector field $\mathbf{F} = (x^2 - y^2)\mathbf{i} - 2xy\mathbf{j} + z^2\mathbf{k}$?

(i) Its divergence vanishes.
(ii) It is conservative and its curl vanishes.
(iii) It is conservative, but its curl does not vanish.
(iv) Its curl vanishes, but it is not conservative.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2-y^2 & -2xy & z^2 \end{vmatrix} = \begin{bmatrix} 0-0 \\ 0 \\ -2y-(2y) \end{bmatrix} = \vec{0}$$

Domain of $\vec{F}$ is all of $\mathbb{R}^3$ which is simply-connected.

(c) [3 pts] There is a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\text{curl}\,\mathbf{F} = \langle x\sin(y), \cos(y), 6z - xy\rangle$.

(i) True.
(ii) False.
(iii) Indeterminable.

Recall:
$\text{div}\,\text{curl}\,\vec{F} = 0$

So compute

$$\text{div}\,\langle x\sin(y), \cos(y), 6z - xy\rangle = \sin(y) - \sin(y) + 6 = 6 \neq 0$$

(d) [3 pts] If $\mathbf{F}(x,y,z) = \langle \sin(z), \cos(z), 0\rangle$, then $\text{curl}(\text{curl}(\text{curl}\,\mathbf{F})) = \mathbf{F}$.

(a) True.
(b) False.
(c) Indeterminable.

This is a v.f. whose curl is itself!
Like e^x is its own derivative!

$f(x) = e^x$

$$\text{curl} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \sin z & \cos z & 0 \end{vmatrix} = \begin{bmatrix} \sin z \\ -(-\cos z) \\ 0 \end{bmatrix} = \begin{bmatrix} \sin z \\ \cos z \\ 0 \end{bmatrix}$$

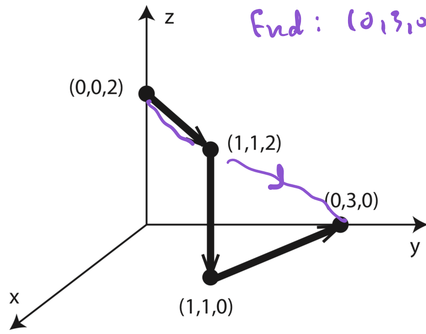
2. [5 pts] Let

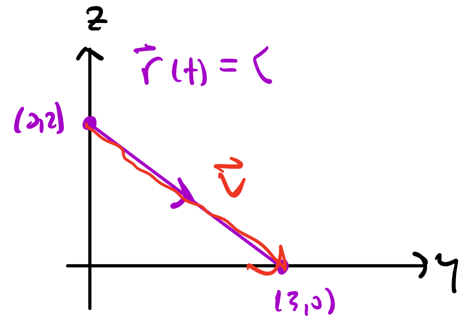
$$\mathbf{F}(x,y,z) = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z \rangle.$$

Evaluate the work done by the vector field $\mathbf{F}$ in moving a particle along the following curve from the point $(0,0,2)$ to the point $(0,3,0)$.

Start: $(0,0,2)$

End: $(0,3,0)$





Need a point on the line and slope

↑ Direction vector.

Strategy: Show $\vec{F}$ is a CONSERVATIVE V.F.

⤳ Once you know $\vec{F}$ is conservative, there's two possible ways forward

① Use independence of path and $\int_C \vec{F}\cdot d\vec{r}$ along a more convenient path.

② Use FTC, then need to find a potential function and then

$$\int_C \vec{F}\cdot d\vec{r} = f(\text{end}) - f(\text{start}).$$

$$\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\ell(t) = \vec{r}_0 + t\vec{v}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\ell(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ and } \ell(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

3. [5 pts] Consider the vector field $\mathbf{F}(x,y) = \langle -y^3 + \sin(\cos x), x^3 + y^{2^{y^2+2022}} \rangle$. Calculate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle centered at the origin oriented counter-clockwise.

Final parametrization is:

$$\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \quad 0 \le t \le 1$$

$$= \begin{bmatrix} 0 \\ 3t \\ 2-2t \end{bmatrix}, \ 0 \le t \le 1$$

Use Green's Thm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\,dA$$

$$= \iint_R (3x^2 + 3y^2)\,dA$$

Polar

$$= \int_0^{2\pi}\int_0^1 3r^3\,dr\,d\theta$$

$$= 6\pi\left[\frac{r^4}{4}\Big|_0^1\right] = 3\pi/2.$$

To show $\vec{F}$ conservative, Need two conditions to be satisfied

① curl $\vec{F} = \vec{0}$

② The domain of $\vec{F}$ is simply-connected.

Domain of $\vec{F}$ is all of $\mathbb{R}^3$ which is simply-connected.

Compute curl $\vec{F}$ = Det $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2yz-3y & x^3z-3x & x^3y+2z \end{vmatrix}$

$$= \begin{bmatrix} x^3 - x^3 = 0 \\ -(3x^2y - 3x^2y) = 0 \\ 3x^2z-3-(3x^2z-3) = 0 \end{bmatrix} = \vec{0}$$

$\vec{F}$ is conservative.

Parametrization: $\vec{r}(t) = \begin{bmatrix} 0 \\ 3t \\ 2-2t \end{bmatrix}, \ 0 \le t \le 1$

$\rightsquigarrow \vec{r}\,'(t) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \ 0 \le t \le 1$

Compute: $\vec{F} = \langle 3x^2yz - 3y, \ x^3z - 3x, \ x^3y + 2z \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \begin{bmatrix} -3(3t) \\ 0 \\ 2(2-2t) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} dt = \int_0^1 -4(2-2t)\,dt$$

$$= -4\int_0^1 2-2t\,dt$$

$$= -4\left[2t - t^2 \Big|_0^1 \right]$$

$$= -4\left[2-1 \right] = -4.$$

Another Approach:

Find the potential function for $\vec{F}$, i.e. $\nabla f = \vec{F}$.

To do so, solve system of differential equations

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 3x^2yz - 3y \\ x^3z - 3x \\ x^3y + 2z \end{bmatrix}$$

①:

$$f_x = 3x^2yz - 3y \xrightarrow[\text{derivative}]{x\text{-anti-}} \underbrace{f(x,y,z) = x^3yz - 3yx + g(y,z)}_{\text{Our candidate soln}}$$

②:

$$f_y = x^3z - 3x + g_y \overset{\text{set equal}}{=} x^3z - 3x \Rightarrow g_y = 0 \Rightarrow g(y,z) \text{ a function of only } g(z).$$

So now:

$$f(x,y,z) = x^3yz - 3yx + g(z)$$

③:

$$f_z = x^3y - g'(z) \overset{\text{set equal}}{=} x^3y + 2z \Rightarrow g'(z) = 2z \Rightarrow g(z) = z^2 + C.$$

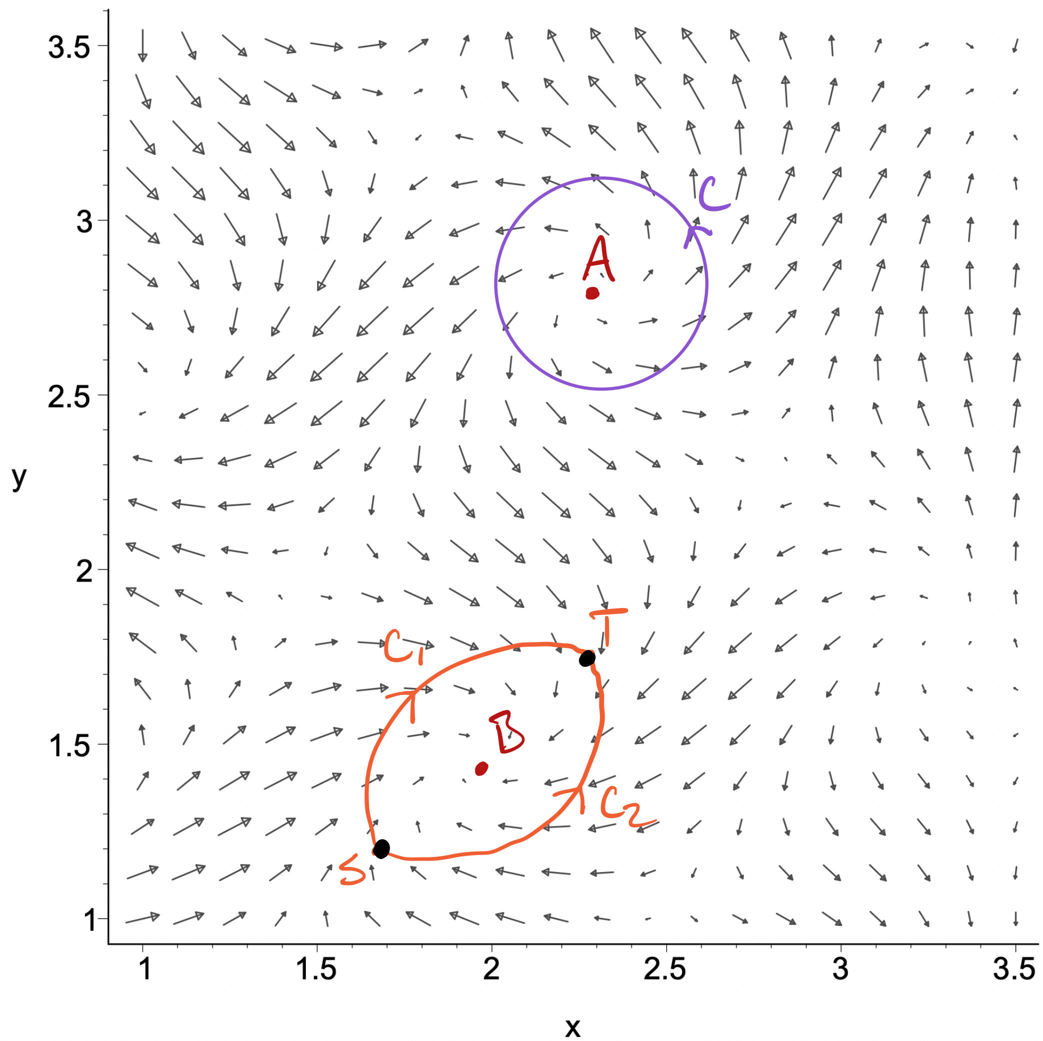
$$\rightarrow f(x,y,z) = x^3yz - 3yx + z^2 + \cancel{C}^{0}$$

By FTC:

$$\int_C \vec{F} \cdot d\vec{r} = f(0,3,0) - f(0,0,2)$$

$$= -4.$$

4. The plot of a vector field $\mathbf{F}(x, y)$ is drawn below.



(a) [2 pts] Mark a point A on the plot at which $\mathsf{curl}\,\mathbf{F} > 0$.

(b) [2 pts] Mark a point B on the plot at which $\mathsf{div}\,\mathbf{F} < 0$.

(c) [2 pts] Sketch a closed curve C such that the work done by $\mathbf{F}$ along C is positive.

(d) [3 pts] Mark two points on the plot S and T and two curves C_1 and C_2 from S to T such that the work done by $\mathbf{F}$ in moving an object along those paths is positive for C_1 and negative for C_2.

(e) [2 pts] Can the vector field $\mathbf{F}$ be a gradient field? In the space below explain why or why not.

No b/c at point A have curl $\vec{F} \neq 0$.