1. Multiple choice. Clearly mark your answers. No justification is required but may result in partial credit if provided.

   (a) [3 pts] If **F** and **G** are two vector fields which have the same divergence, then **F** − **G** is a constant vector field.

   (a) True.
   (b) False.
   (c) Indeterminable.

   Just b/c $\operatorname{div}\vec{F}=0$ does NOT imply that $\vec{F}$ is constant vector field.

   (b) [3 pts] Every vector field $\mathbf{F}(x,y,z)$ which satisfies the equation $\operatorname{curl}\mathbf{F}(x,y,z)=\vec{0}$ on all of $\mathbb{R}^3$ can be written as $\mathbf{F}=\nabla f$ for some scalar function f.

   (a) True.
   (b) False.
   (c) Indeterminable.

   Satisfies sufficient conditions to be a conservative v.f.

   (c) [3 pts] If $\operatorname{div}\mathbf{F}(x,y,z)=0$ for all (x,y,z) then $\int_C \mathbf{F}\cdot d\mathbf{r}=0$ for any closed curve C.

   (a) True.
   (b) False.
   (c) Indeterminable.

   (d) [3 pts] There is a non-constant function $f(x,y,z)$ such that $\nabla f=\operatorname{curl}(\nabla f)$ everywhere.

   (a) True.
   (b) False.
   (c) Indeterminable.

   FACT: $\operatorname{curl}(\nabla f)=\vec{0}$
   $\Rightarrow \nabla f=\vec{0}$
   $\Rightarrow f_x=0,\ f_y=0,\ f_z=0 \Rightarrow f$ is a constant.

   (e) [3 pts] The vector field $\mathbf{F}(x,y,z)=\langle x^5,x^6,x^7\rangle$ is the curl of another vector field defined on all of $\mathbb{R}^3$.

   (a) True.
   (b) False.
   (c) Indeterminable.

   $\operatorname{div}\vec{F}=5x^4\neq 0$

   (f) [3 pts] If $\mathbf{F}(x,y,z)=\langle\sin(z),\cos(z),0\rangle$, then $\operatorname{curl}(\operatorname{curl}(\operatorname{curl}\mathbf{F}))=\mathbf{F}$.

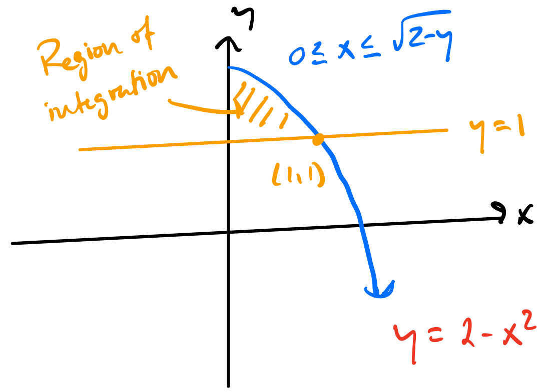
   (a) True.
   (b) False.
   (c) Indeterminable.

2. Consider the double integral

$$\mathbf{I} = \int_1^2 \int_0^{\sqrt{2-y}} \frac{\sin(\pi x)}{1-x^2}\,dx\,dy$$

(a) [4 pts] Sketch the region of integration for **I**.

$1 \leq y \leq 2$ and $0 \leq x \leq \sqrt{2-y} \Rightarrow x = \sqrt{2-y} \Rightarrow x^2 = 2-y \Rightarrow y = 2-x^2$



(b) [4 pts] Express the integral **I** as an iterated integral with the reversed order of integration.

$0 \leq x \leq 1$

$1 \leq y \leq 2-x^2$

$$\rightsquigarrow \int_0^1 \int_1^{2-x^2} \frac{\sin(\pi x)}{1-x^2}\,dy\,dx$$

(c) [4 pts] Determine the value of **I**.

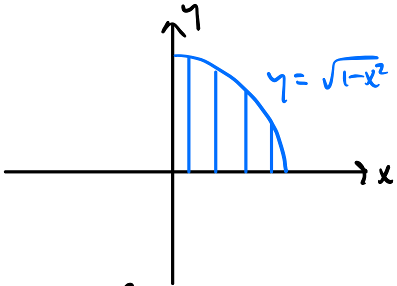
$$= \int_0^1 \frac{\sin(\pi x)}{1-x^2} \cdot y \Big|_1^{2-x^2} dx = \int_0^1 \frac{\sin(\pi x)}{1-x^2} \underbrace{\left[2-x^2-1\right]}_{=1-x^2} dx$$

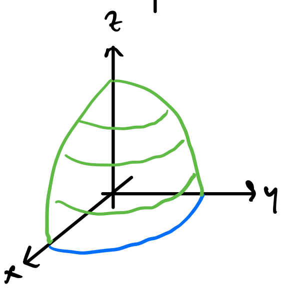
$$= \int_0^1 \sin(\pi x)\,dy = -\frac{\cos(\pi x)}{\pi}\Big|_0^1$$

$$= \frac{-1}{\pi}\left[-1-1\right] = \frac{2}{\pi} \;//$$

3. [8 pts] Evaluate the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{x^2+y^2+z^2} dzdydx$$





$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 \frac{1}{\rho^2}\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\sin\phi\, d\phi = \frac{\pi}{4}\cdot -\cos\phi\Big|_0^{\pi/2} = \frac{\pi}{4}$$

4. Let E be the solid which lies inside both the sphere $x^2+y^2+z^2 = 4$ and the cylinder $(x-1)^2+y^2 = 1$.

(a) [5 pts] Sketch a picture of the solid E.

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(b) [5 pts] Express the volume of E as a triple integral. Do not evaluate your expression.

5. Consider the integral
$$\mathbf{I} = \iint_R \frac{dxdy}{x+y},$$
where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, and $x + y = 4$.

(a) [6 pts] Define $\mathbf{T}$ to be the transformation
$$\mathbf{T} = \begin{cases} x = u - uv \\ y = uv \end{cases}$$

Sketch the region S in the uv-plane which maps onto the region R under the transformation $\mathbf{T}$.

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(b) [4 pts] Compute the Jacobian of the transformation $\mathbf{T}$.

(c) [4 pts] Set up, but do not evaluate an expression for $\mathbf{I}$ as an iterated integral in terms of the variables u and v.

6. Let $\mathbf{F}(x,y) = (xy^2+2y)\mathbf{i} + (x^2y+2x+2)\mathbf{j}$ be a vector field.

(a) [5 pts] Carefully explain why $\mathbf{F}$ is a conservative vector field.

Compute $\text{curl}\,\vec{F} = Q_x - P_y$

$= 2xy + 2 - (2xy+2)$

$= 0$

Since $\text{curl}\,\vec{F} = \vec{0}$ and b/c $\vec{F}$ is defined on a simply-connected domain (all of $\mathbb{R}^2$)

$\Rightarrow \vec{F}$ is conservative.

(b) [5 pts] Find a potential function f such that $\nabla f = \mathbf{F}$.

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} xy^2+2y \\ x^2y+2x+2 \end{bmatrix} \Rightarrow f = \frac{x^2y^2}{2} + 2xy + g(y) \Rightarrow f_y = x^2y + 2x + g'(y)$$

$\Rightarrow g'(y) = 2$

$\Rightarrow g(y) = 2y + C$

$\Rightarrow f(x,y) = \frac{x^2y^2}{2} + 2xy + 2y + C.$ //

(c) [4 pts] Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the path parametrized by $\mathbf{r}(t) = \langle e^t, 1+t\rangle$ for $0 \le t \le 1$.

$\vec{r}(0) = \langle 1, 1\rangle$

$\vec{r}(1) = \langle e, 2\rangle$

By FTC:

$$\int_C \vec{F}\cdot d\vec{r} = f(e,2) - f(1,1) = \frac{e^2\cdot 4}{2} + 2e(2) + 4 - \frac{1}{2} - 2 - 2 \text{ //}$$

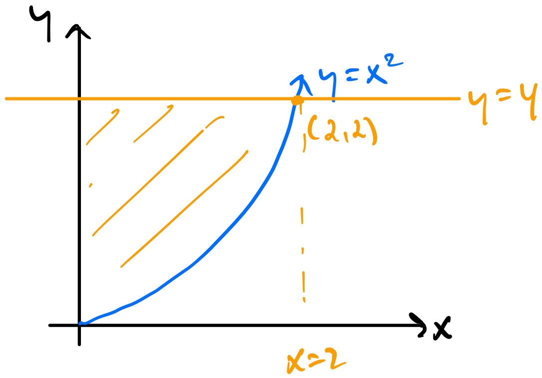
(d) [4 pts] Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a closed curve $\mathbf{r}(t) = \langle 2\sin(t), 2\cos(t)\rangle$ for $0 \le t \le 2\pi$.

$= 0$ b/c $\vec{F}$ is conservative.

7. [10 pts] Evaluate the line integral

$$\int_C (e^x + y^2)dx + (e^y + x^2)dy$$

where C is the positively oriented boundary of the region in the first quadrant bounded by $y = x^2$ and $y = 4$.



Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (Q_x - P_y)\,dA$$

$$= \int_0^2 \int_{x^2}^4 (2x - 2y)\,dy\,dx$$

$$= 2\int_0^2 xy - y^2 \Big|_{x^2}^4 dx$$

$$= 2\int_0^2 4x - 16 - (x^3 - x^4)\,dx$$

$$= 2\int_0^2 (4x - 16 - x^3 + x^4)\,dx$$

$$= 2\left[2x^2 - 16x - \frac{x^4}{4} + \frac{x^5}{5} \Big|_0^2\right]$$

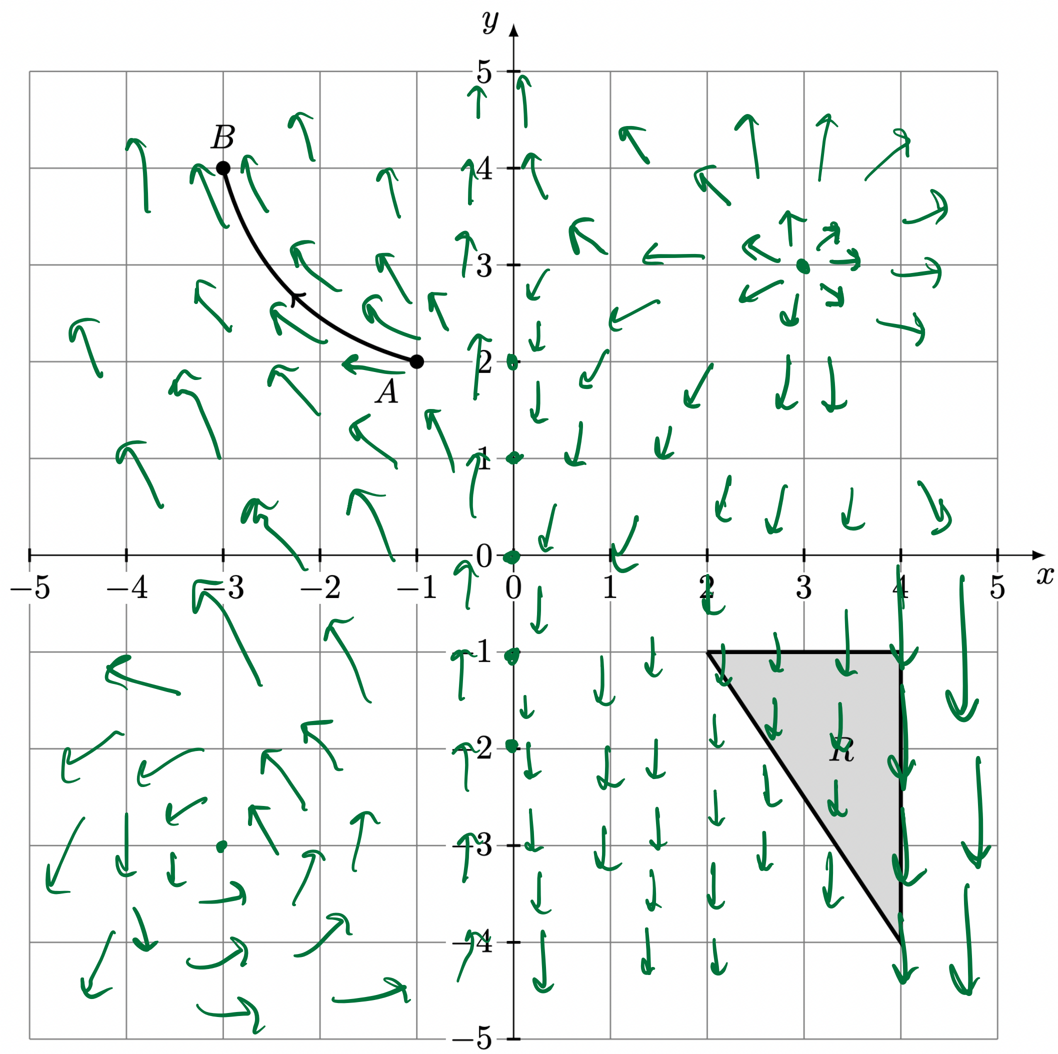
$$= 2\left[2(4) - 16(2) - \frac{2^4}{4} + \frac{2^5}{5}\right]$$

Points earned: ______
out of a possible 10 points

8. (a) [8 pts] On the coordinate axes below, sketch a smooth vector field $\mathbf{F}(x, y)$ which satisfies the following properties (Note: answers may vary):

- At the point $(3, 3)$, the divergence of $\mathbf{F}$ is positive.
- Let C be the path from the point A to the point B drawn below. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive.
- If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, and if R is the triangle region drawn in the fourth quadrant below, then the value of the integral $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ is non-zero.
- At the point $(-3, -3)$, the curl of $\mathbf{F}$ is non-zero.
- Along the y-axis, the vector field vanishes, i.e. $\mathbf{F}(0, y) = \langle 0, 0 \rangle$ for all y.

Hint: To sketch a vector field, you need only draw several representative vectors in the plane. However, the vector field should be *smooth* in that the vectors vary smoothly in the domain.



(b) [2 pts] Comment on whether the vector field $\mathbf{F}$ is conservative or not.

Nope, At the point (-3, -3) there is non zero curl.