TODAY: CURL AND DIVERGENCE (§16.5).

- Fix $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ a vector field on $\mathbb{R}^3$.

Define two New operations:

① CURL:

$$\text{Curl}(\vec{F}) := \nabla \times \vec{F} = \text{Det}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

② DIVERGENCE

$$\text{Div}(\vec{F}) := \nabla \cdot \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

NOTICE:

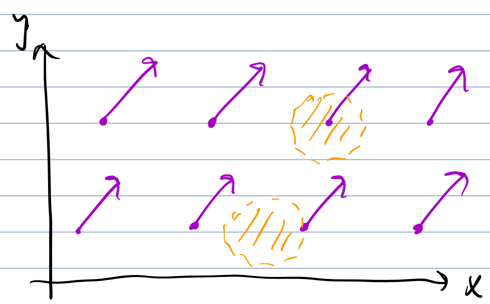
	INPUT		OUTPUT
CURL	[A vector field $\vec{F}$]	$\longmapsto$	[Another v.f. $\nabla \times \vec{F}$]
DIV	[A vector field $\vec{F}$]	$\longmapsto$	[A "Scalar" field i.e. a function of 3-variables]

QUALITATIVE ANALYSIS

- **CURL**: A measure of "vorticity" of a vector field at a point.
  ⇝ Measures the "rotation component of a motion"
  ⟶ How much "twisting" is happening at a point.

DIV: A measure of expansion/retraction (+/- dilation) of a v.f. at a point.

Ex: ① Constant v.f. $\vec{F} = \langle a, b \rangle$



ASIDE: We only defined curl/div. for 3D v.f.'s. In these examples/exercises we take for granted that a 2D v.f. can be seen as a 3D v.f. by placing a zero in 3rd component
$\vec{F}^{2D} = \langle a, b \rangle \rightsquigarrow \vec{F}^{3D} = \langle a, b, 0 \rangle$.

$\text{Curl}(\vec{F}) = \vec{0}$

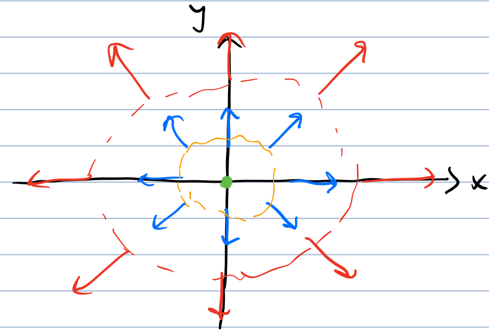
Can see this by

$$\text{Det}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a & b & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}.$$

$$\text{Div}(\vec{F}) = \nabla \cdot \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = 0$$

the zero fn, i.e. $f(x,y,z) = 0$.

② Radial vector field $\vec{F} = \langle x, y \rangle = \langle x, y, 0 \rangle$.



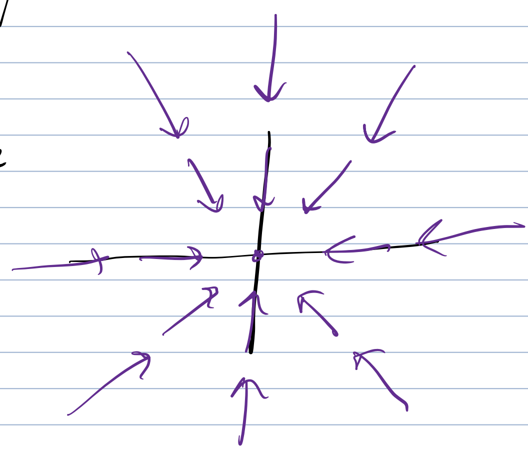
$\text{Div}(\vec{F}) > 0$

Compute $\text{curl}(\vec{F}) = \text{Det}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ -(0-0) \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$.

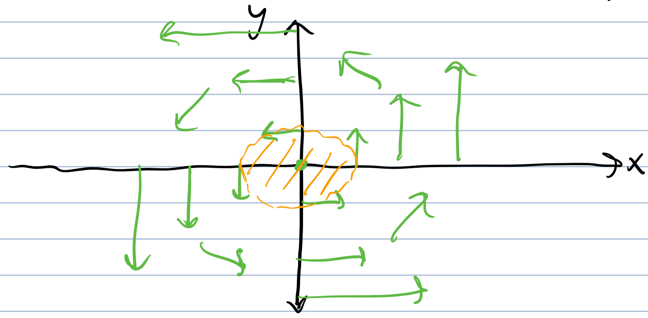
Not dependent (x, y, z).

Compute $\text{Div}(\vec{F}) = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 1 + 1 + 0 = 2. > 0$

$\underline{\text{Div}(\vec{F}) < 0}$: Looks like



③ Consider the v.f. $\vec{F} = \langle -y, x \rangle. = \langle -y, x, 0 \rangle$.



Compute $\operatorname{curl}(\vec{F}) = \operatorname{Det}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\vec{k}$$

curl($\vec{F}$) points in the direction of the axis of rotation (RHR) and the magnitude measures how intense the twisting action is.

Compute Div($\vec{F}$):

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = \overbrace{0}^{\text{scalar.}}$$

<u>THM</u>: If f is a scalar field (i.e. a smooth fn of 3 variables), THEN

$$\operatorname{curl}(\nabla f) = \vec{0}.$$

<u>i.e.</u>: [If $\vec{F}$ is a CONSERVATIVE v.f.] $\Longrightarrow$ [$\operatorname{curl}(\vec{F}) = \vec{0}$]

An excellent way to show that a 3D v.f. is NOT conservative. is to show that $\nabla \times \vec{F} \neq \vec{0}$.

Q: is the converse true?

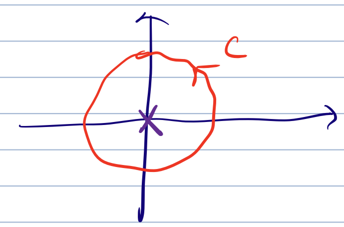
i.e. $\left[\begin{array}{l}\text{Suppose } \vec{F} \text{ a v.f.} \\ \text{w/ } \nabla \times \vec{F} = \vec{0}\end{array}\right] \longrightarrow \left[\vec{F} \text{ is conservative}\right]$?

A: If Domain D is SIMPLY-CONNECTED then yes.

If D NOT simply-connected you cannot conclude this!!

e.g. $\vec{F}(x,y,z) = \dfrac{-y}{x^2+y^2}\,\vec{i} + \dfrac{x}{x^2+y^2}\,\vec{j} + 0\vec{k}$

Can check $\text{curl}(\vec{F}) = \vec{0}$ but $\vec{F}$ NOT conservative!!



$\oint_C \vec{F}\cdot d\vec{r} \neq 0.$

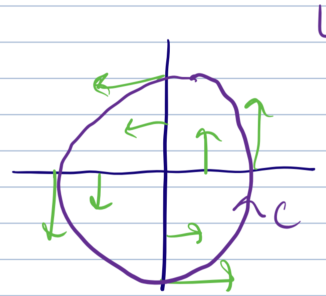
Domain of $\vec{F}$ is $\mathbb{R}^2 - \{(0,0)\}$ NOT S.C.

For example ③: $\vec{F} = \langle -y, x, 0 \rangle$

$\text{curl}(\vec{F}) = 2\vec{k} \neq \vec{0}. \Rightarrow \vec{F}$ is NOT conservative.

Another way to see this:

Loop c is pos. oriented (i.e. ccw).



unit circle

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \cdot \underbrace{\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}}_{\vec{r}\,'(t)} dt$$

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad 0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{\sin^2(t) + \cos^2(t)}_{=1} \, dt$$

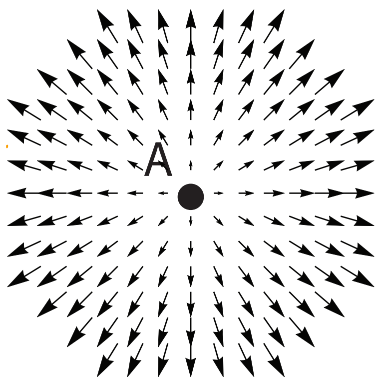
$$= 2\pi \neq 0.$$

THM: For any vector field $\vec{F}$ on $\mathbb{R}^3$, have

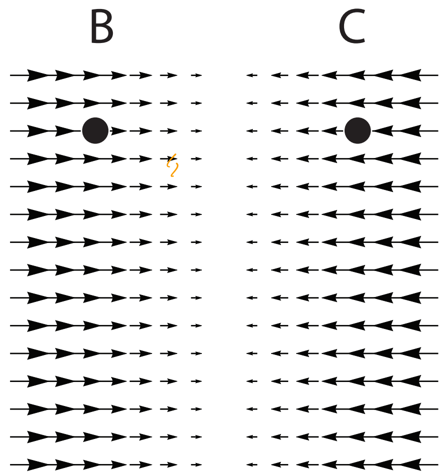
$$\mathrm{Div}\left(\mathrm{curl}(\vec{F}) \right) = 0. \quad \left(\begin{array}{c} \text{As a scalar} \\ \text{fn} \end{array} \right).$$

* ———————————— *

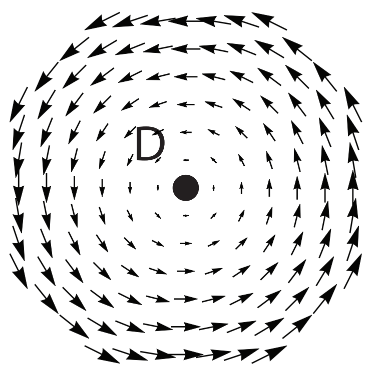
6. (10 points; No partial points) Consider the following vector fields $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$. Answer the following questions. You do not need to provide an explanation.



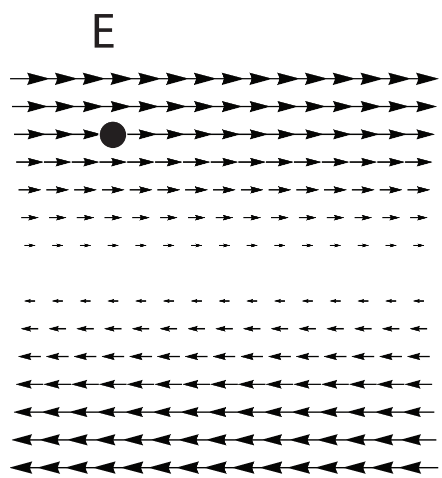
(a)



(b)



(c)



(d)

(i) Is $\vec{\nabla} \cdot \vec{F}$ at point A in picture (a) 0, positive, or negative? It is Positive.

(ii) Is $\vec{\nabla} \cdot \vec{F}$ at point B in picture (b) 0, positive, or negative? It is Neg.

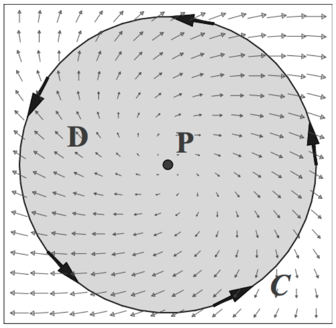
(iii) Is $\vec{\nabla} \cdot \vec{F}$ at point C in picture (b) 0, positive, or negative? It is Neg.

For the next two questions, consider the three-dimensional vector fields $\vec{G}(x,y,z) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ made out of the vector field $P(x,y)\vec{i} + Q(x,y)\vec{j}$ in Picture (c) and (d). In other words, $\vec{G}(x,y,z)$ is given as Picture (c) and (d) on every plane parallel to the xy-plane.

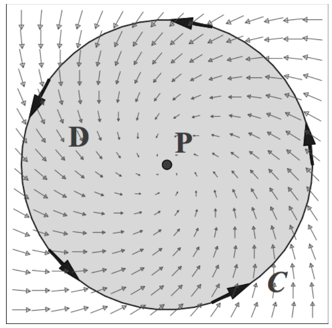
(iv) Is $(\vec{\nabla} \times \vec{G}) \cdot \vec{k}$ at point D in picture (c) 0, positive, or negative? It is Positive.

(v) Is $(\vec{\nabla} \times \vec{G}) \cdot \vec{k}$ at point E in picture (d) 0, positive, or negative? It is Negative.

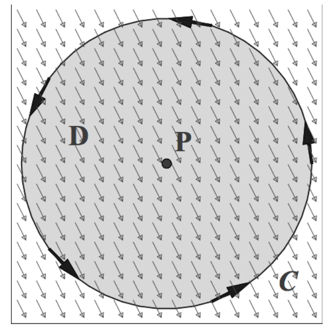
30. Below are six pictures of a vector field $\mathbf{F}$, region D and its oriented boundary C, and a point P inside D. For each of the given properties, indicate all plots that have that property.



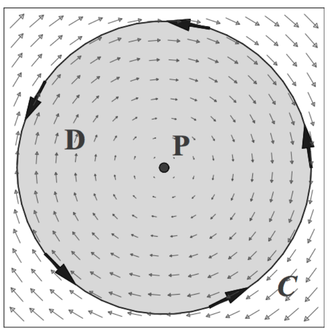
(a)



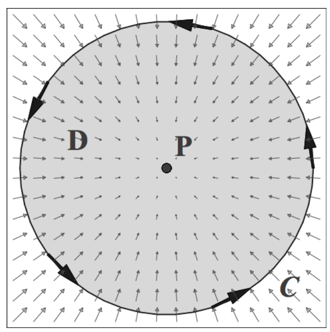
(b)



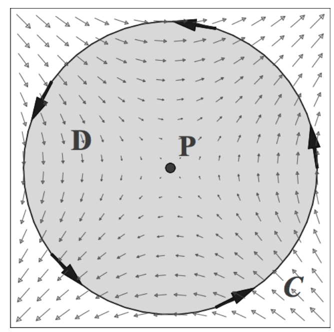
(c)



(d)



(e)



(f)

(a) $\mathsf{curl}\mathbf{F}$ at P is positive.

B

(b) The circulation of $\mathbf{F}$ around C is positive.

B

(c) The circulation of $\mathbf{F}$ around C is negative.

A, D

(d) The flux of $\mathbf{F}$ across C is negative.

B, E.

(e) $\mathbf{F}$ can be a gradient vector field.

C, E, F.

DISCUSSION

(A): Can see that P is a source, so $\operatorname{div}(\vec{F})(P) > 0$, flux of $\vec{F}$ across C is +, field spins CW $\rightsquigarrow$ $\operatorname{curl}(\vec{F})(P) < 0$, and $\oint_C \vec{F}\cdot d\vec{r} < 0$. Therefore, $\vec{F}$ cannot be a gradient v.f.

(B): See that P is a SINK, so $\operatorname{div}(\vec{F})(P) < 0$, flux across C is negative, field spins CCW $\Rightarrow \operatorname{curl}(\vec{F})(P) > 0$, and $\oint_C \vec{F}\cdot d\vec{r} > 0 \Rightarrow \vec{F}$ cannot be gradient vector field.

(C): This looks like a constant v.f. Constant v.f. have no curl and no divergence and are gradient v.f.'s.

(D): Looks like pure rotation, spins CW $\Rightarrow \operatorname{curl}\vec{F} < 0$, $\oint_C \vec{F}\cdot d\vec{r} < 0$, and $\operatorname{div}\vec{F} = 0 =$ Flux across C.

$\Downarrow$ Cannot be gradient v.f.

(E): See that P is a sink, so $\operatorname{div}(\vec{F})(P) < 0$, and flux across C is negative. This field has no spin, so its $\operatorname{curl}\vec{F} = 0 = \oint_C \vec{F}\cdot d\vec{r}$. Looks like it may be a gradient v.f.

(F): By symmetry, both $\oint_C \vec{F}\cdot d\vec{r}$ and flux of $\vec{F}$ across C are zero (because contributions from different parts of the curve cancel each other out)

See that $\operatorname{curl}\vec{F} = 0 = \operatorname{div}\vec{F}$.

Looks like it can be a gradient v.f. (it is.... for what function?)