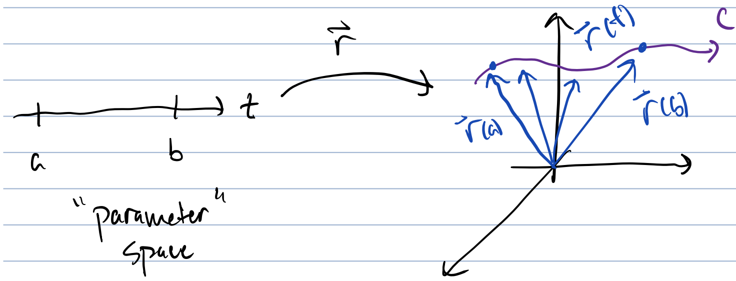
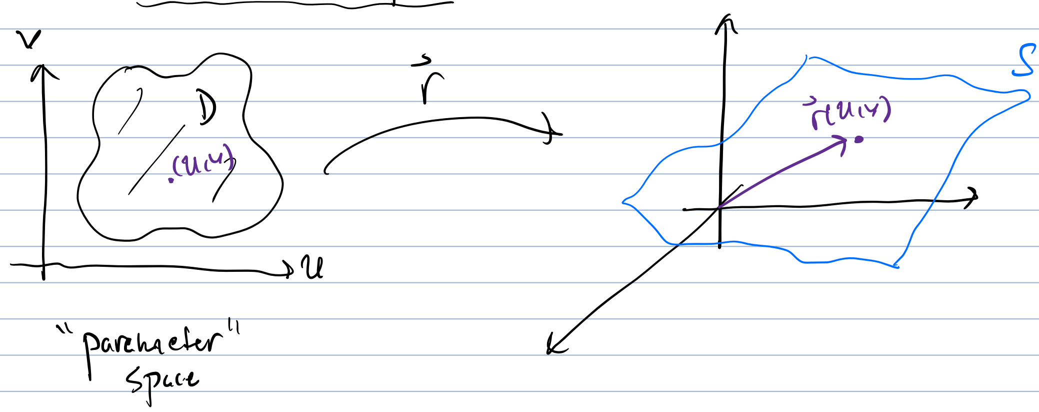
Parametrized curve



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

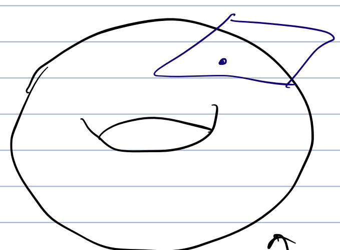
Parametrized Surfaces



$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle.$

- Parametrizations open up new surfaces we can think about.

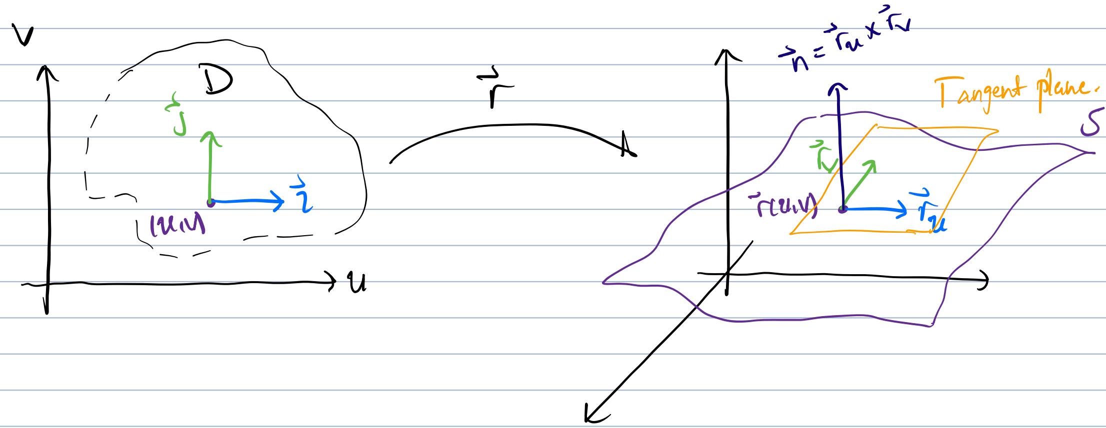
(e.g. Torus



)

Not $z = f(x,y)$.

• Easy to describe tangent planes to parametrized surfaces:



• TWO Types of integrals over surface S.

① "integration w/ respect to surface area"

$$\iint_S f \underbrace{dS}_{\substack{\text{surface}\\ \text{area elt.}\\ dA.}}, \quad \text{where } f = f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}.$$

② $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}$ a 3D v.f. defined on S.

Type ①: $$\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

Ex: Compute the surface area of the part of the plane $x+y+z=2$ cut out by the cylinder $x^2+y^2=4$

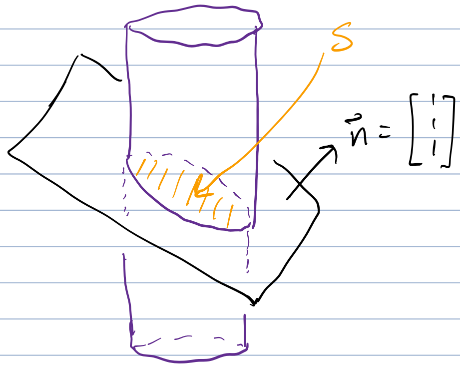
Sol: STRATEGY: ① Parametrize S.

$f(x,y,z) = 1.$

② $$A(S) = \iint_S 1\, dS = \iint_D 1 \cdot |\vec{r}_u \times \vec{r}_v|\, dA.$$

Compare $$\mathrm{vol}(E) = \iiint_E dV.$$

Pic:



① Parametrize:

$\vec{r}(u,v) = \langle u, v, 2-u-v \rangle$

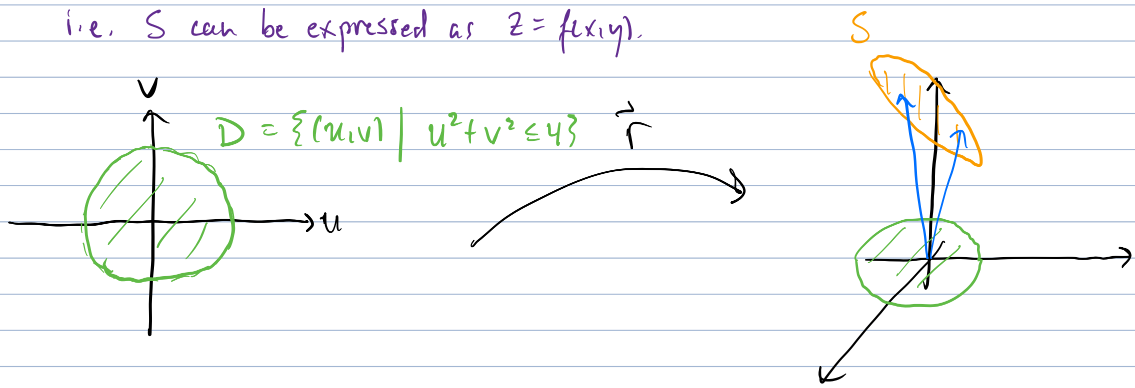
$x = u$
$y = v$
$z = 2-u-v$

WHERE

$u^2+v^2 \leq 4.$

Notice: S is the "graph" of a function

i.e. S can be expressed as $z = f(x,y)$.



$\vec{r}_u = \langle 1, 0, -1 \rangle$, $\vec{r}_v = \langle 0, 1, -1 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{bmatrix}1\\1\\1\end{bmatrix} \longrightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{3}.$

So:

$$A(S) = \iint_{u^2+v^2\leq 4} \sqrt{3}\, du\, dv = \sqrt{3}\, 4\pi. //$$

* ———————————— *

Type ②: Interested in $\iint_S \vec{F} \cdot d\vec{S}$ w/ $\vec{F}$ a vector field.

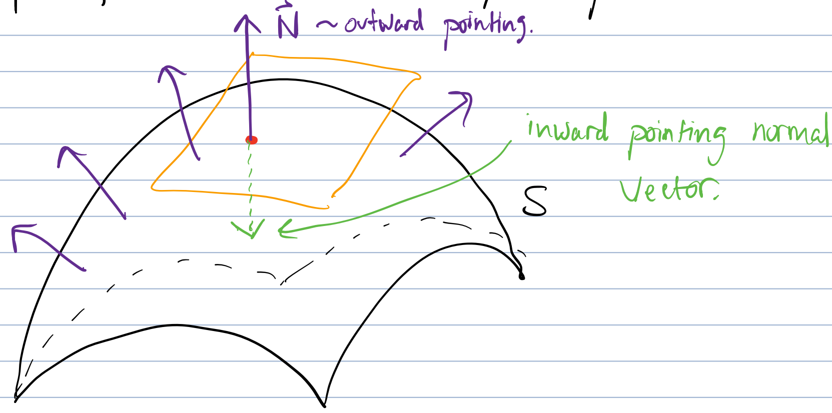
→ Did an example last time.

DISCUSSION: ORIENTATION OF SURFACE S.

- The integral $\iint_S \vec{F} \cdot d\vec{S}$ depends on orientation of S.
- Essentially, orientation tells you how to assign "+/−" to $\iint_S \vec{F} \cdot d\vec{S}$.

Defn: An ORIENTATION of a surface S is a choice of normal vector at each point, which varies continuously as you move along S.

S is oriented by the outward pointing normal vector.



NOTICE: There are precisely two choices for normal direction

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{S is oriented by outward pointing normal vector field.}} = - \underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{S is oriented by inward pointing v.f.}}$$

Ex: Type ② Compute the flux over the sphere S given by

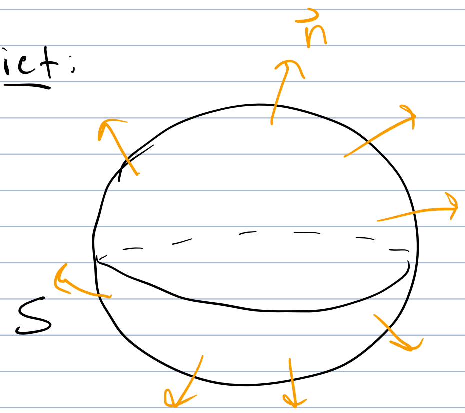
$$= \iint_S \vec{F} \cdot d\vec{S}$$

$x^2+y^2+z^2=a^2$ where $\vec{F}$ is the vector field

$$\vec{F} = \frac{K \cdot q\, \vec{r}}{|\vec{r}|^3}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Assume S is oriented outward.

Pict:



$0 \le u \le \pi$, $0 \le v \le 2\pi$

Sol: Parametrize S by $\vec{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u \rangle$

Different parametrization of sphere!!

$\vec{r}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u \rangle$

$\vec{r}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \dots = a^2 \begin{bmatrix} \sin^2 u\cos v \\ \sin^2 u\sin v \\ \sin u\cos u \end{bmatrix}$$

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

$$\vec{F}(x,y,z) = \underset{\substack{\uparrow \\ \text{constants}}}{kq} \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\|x\vec{i} + y\vec{j} + z\vec{k}\|^3} \right)$$

$$\cdots = \frac{kq}{a^3} \left((a\sin u\cos v)\vec{i} + (a\sin u\sin v)\vec{j} + (a\cos u)\vec{k} \right)$$

- Next compute: $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = \cdots = kq\sin u$

- Finally: compute $\iint_S \vec{F} \cdot d\vec{S} = \iint_D (kq\sin u)\, du\, dv$

$$= \int_0^{2\pi} \int_0^{\pi} kq\sin u\, du\, dv$$

$$= 4\pi kq \;//$$

- If $\vec{E}$ is an electric field ("$\vec{F}$")

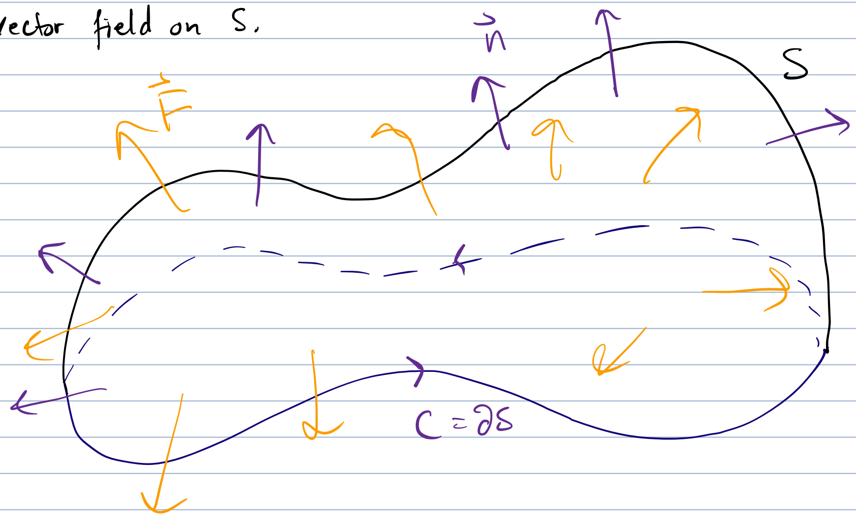
$$\iint_S \vec{E} \cdot \vec{N}\, dS = 4\pi kq \quad \text{"}\underline{\text{Gauss's Law}}\text{"}.$$

* ——————————— *

§ 16.8 : STOKE'S THM.

THM : (STOKES)

Let S be an oriented surface is bounded by a simple closed curve in $\mathbb{R}^3$ w/ positive orientation. Let $\vec{F}$ be smooth Vector field on S.



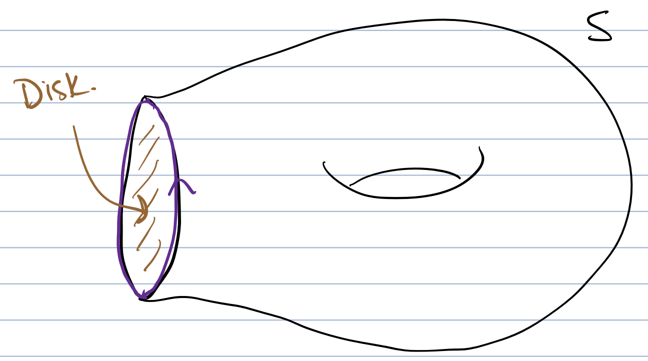
THEN:

$$\boxed{\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}.}$$

IMPORTANT OBSERVATION

① Astounding fact: The RHS does not reference the interior of S at all.

i.e.



$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_{\text{Disk}} (\nabla \times \vec{F}) \cdot d\vec{S}$$

MORAL: Instead of integrating over a complicated surface S can integrate over much simpler surface which has same boundary.