University of Virginia | **MATH 2310** |
Department of Mathematics | **Homework 2** | SOLUTIONS | Fall 2022

1. Consider the points $P(3,1,1), Q(4,1,2)$, and $R(4,4,1)$ in $\mathbb{R}^3$.

   (a) [3 pts] Find an equation for the plane containing the points P, Q, and R.

   Compute $\vec{PQ} = \text{"}Q-P\text{"} = \begin{bmatrix}4\\1\\2\end{bmatrix} - \begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

   and

   $\vec{QR} = \begin{bmatrix}4\\4\\1\end{bmatrix} - \begin{bmatrix}4\\1\\2\end{bmatrix} = \begin{bmatrix}0\\3\\-1\end{bmatrix}$

   Compute $\vec{PQ} \times \vec{QR} = \text{Det}\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 1\\ 0 & 3 & -1\end{vmatrix}$ +1

   $= \vec{i}\begin{vmatrix}0 & 1\\3 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}1 & 0\\0 & 3\end{vmatrix}$

   $= \begin{bmatrix}-3\\1\\3\end{bmatrix}$ $\therefore$ $\boxed{-3(x-3)+(y-1)+3(z-1)=0 \text{ is equation of plane}}$ +2

   (b) [2 pts] Find the area of the triangle with vertices P, Q, and R.

   Area of $\Delta = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\|$ +1

   $= \frac{1}{2}\sqrt{(-3)^2+1^2+3^2} = \frac{1}{2}\sqrt{9+1+9}$

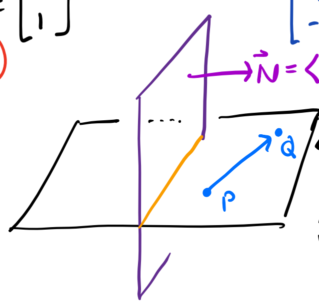
   $= \sqrt{19}/2$ +1

2. [4 pts] Find an equation of the plane which passes through the points $(2,2,1)$ and $(-1,1,-1)$ and is perpendicular to the plane $2x-3y+z=3$.

   Since passes thru points $P=(2,2,1)$ and $Q=(-1,1,-1)$

   $\vec{N} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$ +1 $\rightsquigarrow$ $\vec{PQ} = \begin{bmatrix}-1\\1\\-1\end{bmatrix} - \begin{bmatrix}2\\2\\1\end{bmatrix} = \begin{bmatrix}-3\\-1\\-2\end{bmatrix}$ lies in the plane +1

   $\vec{N} = \langle 2,-3,1\rangle$ Normal to given plane

   Desired plane

   See that Normal to desired plane given by

   $\vec{PQ}\times\vec{N} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & -1 & -2\\ 2 & -3 & 1\end{vmatrix} = \begin{bmatrix}-1-6\\-(-3+4)\\9+2\end{bmatrix} = \begin{bmatrix}-7\\-1\\11\end{bmatrix}$

   So equation of desired plane given by

   $-7(x-2)-(y-2)+11(z-1)=0$ +2

3. [4 pts] Find a set of parametric equations for the line of intersection of the planes:

$$\begin{array}{c} R \\ S \end{array}\begin{cases}6x-3y+z=5\\-x+y+5z=5\end{cases}$$

Normal direction to plane $\vec{R} = \langle 6,-3,1\rangle$

$\vec{S} = \langle -1,1,5\rangle$

So line of intersection is parallel to $\vec{R}\times\vec{S}$ +2

$\rightsquigarrow \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & -3 & 1\\ -1 & 1 & 5\end{vmatrix} = \begin{bmatrix}-15-1\\-(30+1)\\6-3\end{bmatrix} = \begin{bmatrix}-16\\-31\\3\end{bmatrix}$

To find a point on line of intersection, solve system of equations .... one solution is $\vec{r}_0 = \begin{bmatrix}20/3\\35/3\\0\end{bmatrix}$

One possibility is:

$\vec{r}(t) = \begin{bmatrix}20/3\\35/3\\0\end{bmatrix} + t\begin{bmatrix}-16\\-31\\3\end{bmatrix}$

+2

4. [4 pts] Given the line L through $(1, 2, 3)$ parallel to the vector $\langle 1, 1, 1\rangle$, and given a point $(2, 3, 5)$ which is not on L. Find a Cartesian equation for the plane M through $(2, 3, 5)$ which contains every point on L.

Plane contains $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\rightarrow \vec{N} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{bmatrix} 2-1 \\ -(2-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\rightsquigarrow$ Plane given by $(x-2) - (y-3) = 0$

$\rightsquigarrow x - y - 5 = 0.$

5. [3 pts] If the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ satisfy $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{w} \neq \mathbf{0}$, but $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{0}$, then it follows that

(a) The plane spanned by $\mathbf{u}$ and $\mathbf{v}$ is orthogonal to the one spanned by $\mathbf{u}$ and $\mathbf{w}$.

(b) $\mathbf{v} \perp \mathbf{w}$.

(c) $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$.

(d) $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane.

6. [3 pts] Determine whether the parametrizations

$$\mathbf{r}_1(t) = \langle 3, -1, 4\rangle + t\langle 8, 12, -6\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 11, 11, -2\rangle + t\langle 4, 6, -3\rangle$$

describe the same line. If they do, show why. If they don't, show why not.

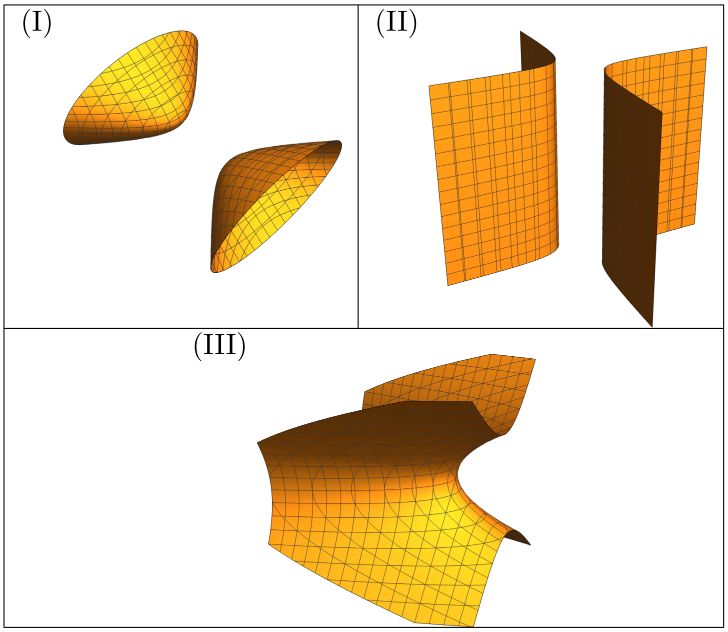
Yes. $\vec{r}_1(t) = \begin{bmatrix} 3 + 8t \\ -1 + 12t \\ 4 - 6t \end{bmatrix}$ $\vec{r}_2(t) = \begin{bmatrix} 11 + 4t \\ 11 + 6t \\ -2 - 3t \end{bmatrix}$

$\therefore \vec{r}_2(t) - \vec{r}_1(t) = \vec{0}$

when $t = 2$

$\rightsquigarrow$ Since $\vec{v}_1 \parallel \vec{v}_2$ and intersection btw $\vec{r}_1$ and $\vec{r}_2$ is non empty $\Rightarrow \vec{r}_1$ and $\vec{r}_2$ parametrize same line.

7. [4 pts] Match the contour surfaces with their equations. Enter O if there's no match.



$g(x,y,z) =$	O, (I), (II), (III)
$x^2 - y^2 + z^2 = 1$	O
$x^2 - y^2 = 1$	II
$x^4 + z = 1$	O
$x^2 + y - z^2 = 1$	III

SADDLE!