1. Consider the integral

$$\mathbf{I} = \iint_R \frac{dxdy}{x+y},$$

where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, and $x + y = 4$.

$y=0$
$\Rightarrow uv=0$
$\Rightarrow u=0$ or $v=0$

(a) [6 pts] Define $\mathbf{T}$ to be the transformation $y=1-x$ $y=4-x$

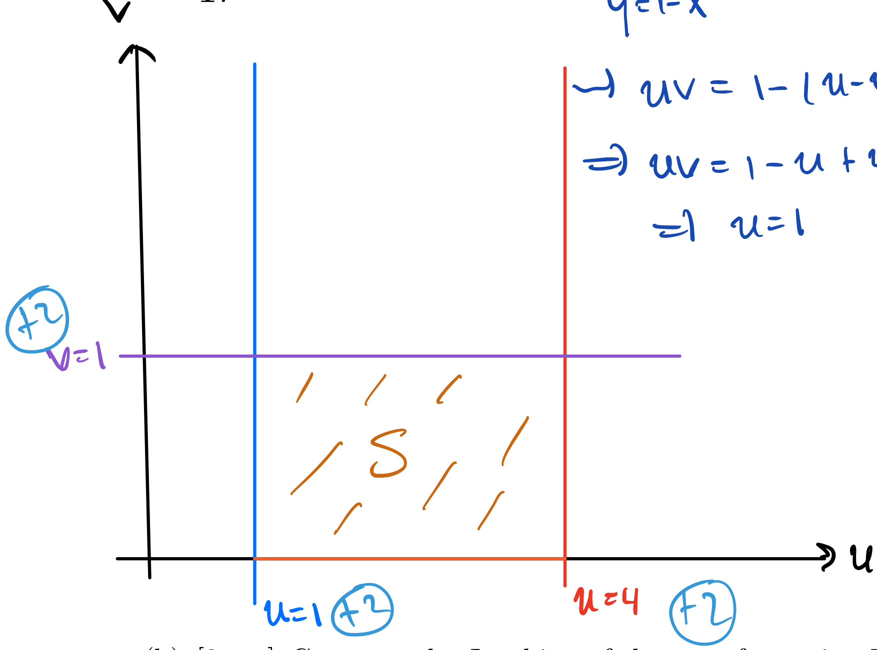
$u=0$ | $v=0$
$x=0$ | $x=u$

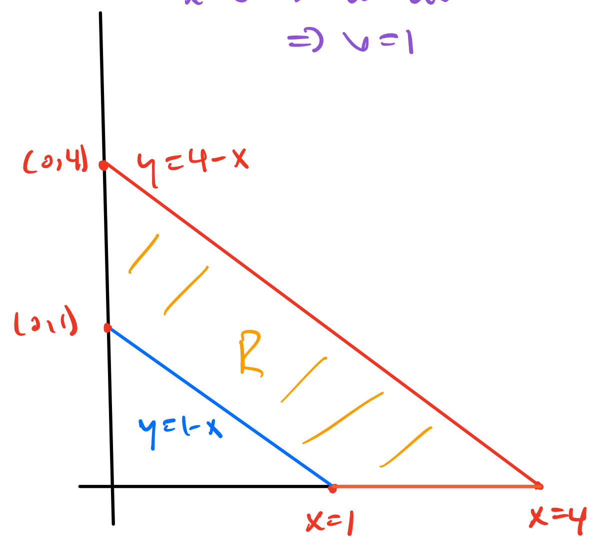
$$\mathbf{T} = \begin{cases} x = u - uv \\ y = uv \end{cases}$$

Sketch the region S in the uv-plane which maps onto the region R under the transformation $\mathbf{T}$.

$y=1-x$
$\rightarrow uv = 1-(u-uv)$
$\Rightarrow uv = 1-u+uv$
$\Rightarrow u=1$

$x=0 \Rightarrow u=uv$
$\Rightarrow v=1$





(b) [3 pts] Compute the Jacobian of the transformation $\mathbf{T}$.

$$J = \text{Det}\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v)+uv$$
$$= u - uv + uv$$
$$= u.$$

+1 +2

(c) [4 pts] Set up, but do not evaluate an expression for $\mathbf{I}$ as an iterated integral in terms of the variables u and v.

$$I = \int_0^1 \int_1^4 \frac{u\,du\,dv}{u} = du\,dv$$

+2 +1

2. [4 pts] Compute the work done by the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{x+1}{(x+1)^2+y^2}\,,\ \frac{y}{(x+1)^2+y^2} \right\rangle$$

in moving a particle along the line segment $1 \le y \le 4$ on the y-axis, oriented upward.

$\vec{r}(t) = \langle 0, t \rangle$, $1 \le t \le 4$ $\qquad u = 1+t^2$

$\Rightarrow du = 2t$

$\vec{F}(0,t) = \left\langle \frac{1}{1+t^2}, \frac{t}{1+t^2} \right\rangle$

$$\int_1^4 \left\langle \frac{1}{1+t^2}, \frac{t}{1+t^2} \right\rangle \cdot \langle 0, 1 \rangle\, dt = \int_1^4 \frac{t}{1+t^2}\, dt = \int \frac{1}{2} \cdot \frac{du}{u}$$

$$= \frac{\ln(1+t^2)}{2} \Bigg|_1^4 = \frac{1}{2}\left[\ln(17) - \ln(2)\right].$$

3. Consider the vector field

$$\mathbf{F}(x,y) = \left\langle 4x\ln(y)\,,\ \frac{2x^2-1}{y} \right\rangle$$

defined on the domain $D = \{(x,y) \mid y > 0\}$.

(a) [6 pts] Carefully explain whether or not $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is conservative, then find a potential function $f(x,y)$ defined on D.

① Domain is  A simply-connected domain.

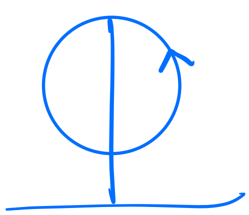
② $Q_x = \frac{\partial}{\partial x}\left(\frac{2x^2-1}{y}\right) = \frac{4x}{y}$

$P_y = \frac{\partial}{\partial x}\left(4x \ln y\right) = \frac{4x}{y}$

equal!

$\Longrightarrow$ ① + ② imply that $\vec{F}$ is conservative!

(b) [3 pts] Determine the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the circle centered at $(0,3)$ with radius $r = 1$. Justify your answer.



$\oint_C \vec{F} \cdot d\vec{r} = 0$ b/c conservative!