Example: Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ ← oriented surface

where

$\vec{F}(x,y,z) = \langle y, -x, 2z \rangle$ and S is the hemisphere

$S = \{x^2+y^2+z^2 = 4,\ z \geq 0\}$ and oriented DOWNWARD

← The opposite of our usual convention.

S

$z \geq 0$

① Parametrize the surface S: We notice S is the GRAPH of a function

⇝ Use parametrization $\vec{r}(u,v) = \langle u, v, \sqrt{4-u^2-v^2} \rangle$ where $u^2+v^2 \leq 4$

② Associated to a given parametrization is a preferred orientation of S

$$\vec{r}(u,v) \rightsquigarrow \vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Q: Does the orientation of S coming from parametrization agree w/ the given orientation?

If Yes: Keep sign way it is.

If No: change sign

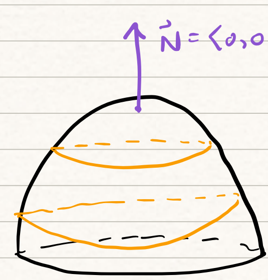
Calculation:

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{-u}{\sqrt{4-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{4-u^2-v^2}} \end{vmatrix} = \begin{bmatrix} \frac{u}{\sqrt{4-u^2-v^2}} \\ \frac{v}{\sqrt{4-u^2-v^2}} \\ 1 \end{bmatrix}$$

Check if orientations match:

Because $(\vec{r}_u \times \vec{r}_v)(u=0, v=0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{N} = \langle 0,0,1 \rangle$



⟹ Orientations do NOT agree!

③ Compute surface integral:

Plug-in the parametrization $\vec{r}(u,v)$ into the vector field $\vec{F}$:

$$\vec{F}(\vec{r}(u,v)) = \begin{array}{r} y \\ -x \\ 2z \end{array} \begin{bmatrix} v \\ -u \\ 2\sqrt{4-u^2-v^2} \end{bmatrix}$$

$y = y(u,v) = v$

$x = x(u,v) = u$

$z = z(u,v) = \sqrt{4-u^2-v^2}$

We're left to calculate the following integral:

$$\text{FLUX} = -\iint_S \vec{F} \cdot d\vec{S} = -\iint_{u^2+v^2 \leq 4} \begin{bmatrix} v \\ -u \\ 2\sqrt{4-u^2-v^2} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} u/\sqrt{4-u^2-v^2} \\ v/\sqrt{4-u^2-v^2} \\ 1 \end{bmatrix}}_{\vec{N}} dA$$

orientation!

$$= -\iint_{u^2+v^2 \leq 4} 2\sqrt{4-u^2-v^2}\, dA$$

$$= \dots$$

$$= \frac{-32\pi}{3} \quad //$$