LAST TIME: • Limits

TODAY: • Gradient Vector
• Extrema problems

* GRADIENT VECTORS *

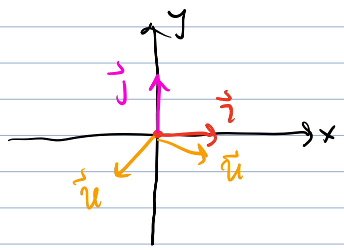
or $\begin{bmatrix}1\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\vec{i} = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{j} = \begin{bmatrix}0\\1\end{bmatrix}$

- Let $z = f(x,y)$ be differentiable fn.
- We've seen how to take derivatives in the "x" and "y" directions.

$\rightsquigarrow \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

In the domain:



Q: How do we take derivatives of $z = f(x,y)$ in OTHER directions $\vec{u}$??

unit vector.

A: Use "Gradient Vector"

Defn: Define the **GRADIENT** of $f(x,y)$, $\vec{\nabla} f(x,y)$, is the vector

$$\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}.$$

In the DOMAIN..

Ex: Find the gradient of $f(x,y) = y \ln x + xy^2$ at $(1,2)$.

Sol:

$$f_x(x,y) = y/x + y^2 \quad \text{and} \quad f_y(x,y) = \ln x + 2xy$$

So that $\nabla f(1,2) = \begin{bmatrix} f_x(1,2) \\ f_y(1,2) \end{bmatrix} = \begin{bmatrix} 2/1 + 2^2 \\ \ln(1) + 2(1)(2) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 6\vec{i} + 4\vec{j}$.

$$\underline{\nabla f(1,2) = 6\vec{i} + 4\vec{j}}.$$

- Back to the Question ....

D: If f is a differentiable fn of x and y, define the DIRECTIONAL DERIVATIVE of f in the direction $\vec{u}$ to be

unit vector !!

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}.$$

Ex: Compute the directional derivative of $f(x,y) = x^2 \sin(2y)$ at $(1, \pi/2)$ in the direction $\vec{v} = 3\vec{i} - 4\vec{j}$.

Sol: Compute $\nabla f(x,y) = \langle 2x\sin(2y),\ 2x^2\cos(2y) \rangle$

$\nabla f(1, \pi/2) = \langle 2(1)\sin(2\pi/2),\ 2(1)\cos(2\pi/2) \rangle$

$= \langle 0, -2 \rangle$

The vector $\vec{v}$, as given, is NOT a unit vector !!

use this.

$\rightsquigarrow$ Quick fix: instead use $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{9+16}} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \dfrac{1}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\left[\ \vec{v} \rightsquigarrow \underbrace{\dfrac{1}{|\vec{v}|}}_{\text{scalar.}} \vec{v} \right.$

$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

$= \sqrt{3^2 + (-4)^2} = \sqrt{9+16}$

$= 5.$

- Now compute

$D_{\vec{v}} f(x,y) = \langle 0, -2 \rangle \cdot \frac{1}{5} \langle 3, -4 \rangle$

$= \frac{1}{5} \left(0 \cdot 3 + (-2)(-4) \right)$

$= 8/5.$ //

↓ More generally: $\nabla f \cdot \left(\frac{\vec{v}}{|\vec{v}|}\right) = \begin{bmatrix} 2x\sin(2y) \\ 2x^2\cos(2y) \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$$= \frac{1}{5}\left(6x\sin(2y) - 8x^2\cos(2y)\right)\Big|_{(1,\pi/2)}.$$

PROPERTIES OF ∇f

THM:

Let $f(x,y)$ be differentiable fn at the point (a,b):

1) if $\nabla f(a,b) = \vec{0}$, then $D_{\hat{u}} f(a,b) = 0$ for any $\vec{u}$.

2) The direction of MAXIMUM increase of f is given by $\nabla f(a,b)$. The max. value of $D_{\hat{u}} f(a,b)$ is $|\nabla f(a,b)|$.

3) The direction of MINIMUM increase of f —// is given by $-\nabla f(a,b)$. —//— is $-|\nabla f(a,b)|$.

4) Provided that $\nabla f(a,b) \neq \vec{0}$, then $\nabla f(a,b)$ is ORTHOGONAL to the level sets of f at (a,b).

Ex: The temperature (in degrees Celsius) on a surface is modeled by

$$T(x,y) = 20 - 4x^2 - y^2.$$

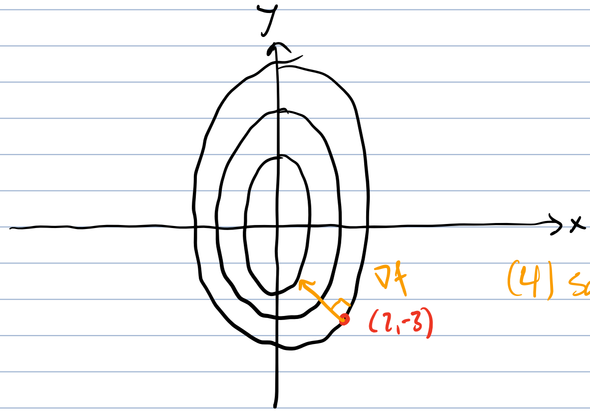
In what direction from $(2,-3)$ does the temperature increase most rapidly?

Sol:

Compute $\nabla f(2,-3)$: $\nabla f(x,y) = \langle -8x, -2y \rangle$

so at $\nabla f(2,-3) = \langle -16, 6 \rangle$.

⇝ So T increases most rapidly at $(2,-3)$ in the $-16\hat{\imath} + 6\hat{\jmath}$ direction.



(4) says ∇f is $\perp$ to level curve

GRADIENT VECTORS AND TANGENT PLANES

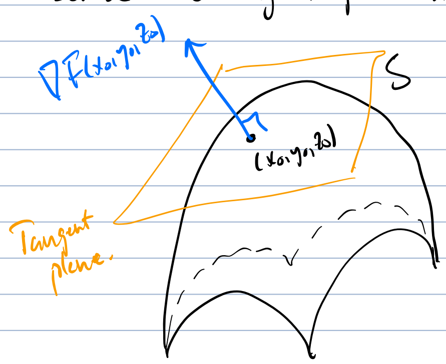
- Mostly, described surfaces in $\mathbb{R}^3$ by $z = f(x,y)$.

- Often convenient to express a surface as the level set of a 3-variable fn.

$$S = \{ (x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0 \}.$$

think: Quadric surface $\underbrace{x^2+y^2+z^2}_{F(x,y,z)} = 0$

- To describe the tangent plane at a point (x_0, y_0, z_0) on S:



THM: if $F(x,y,z)$ is differentiable at (x_0, y_0, z_0), then an equation for the tangent plane to the surface $S = \{ F(x,y,z) = 0 \}$

is given by:

$$F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$$

$F(x,y,z) \rightsquigarrow \nabla F = \langle F_x, F_y, F_z \rangle$

$f(x,y) \rightsquigarrow \nabla f = \langle f_x, f_y \rangle$.

Ex: Find the eqn of tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12 \text{ at } (1,-1,4).$$

Sol: Write eqn as $F(x,y,z) = 0$.

$$\rightsquigarrow \underbrace{z^2 - 2x^2 - 2y^2 - 12}_{F(x,y,z).} = 0.$$

$F_x = -4x \quad F_y = -4y \quad F_z = 2z$

at $(1,-1,4)$: $F_x = -4 \quad F_y = 4 \quad F_z = 8$

So eqn for tangent plane:

$$-4(x-1) + 4(y+1) + 8(z-4) = 0$$

$$x - y - 2z + 6 = 0.$$

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EXTREMA (MAX/MIN) PROBLEMS FOR MULTI-VARIABLE FNS

- Just as in CALC I, we're interested in determining "critical points" of multi-variable functions b/c

[Critical points of $f(x,y)$] $\overset{!!}{\rightsquigarrow}$ [Relative max/min on on a domain D]

⚠ Just b/c have critical point $\not\Rightarrow$ have extrema.

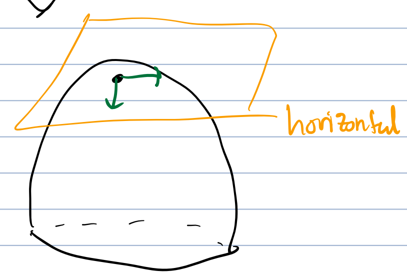
Defn: Let f be differentiable in a domain D. The point (x_0, y_0) is a CRITICAL POINT of $f(x,y)$ if either:

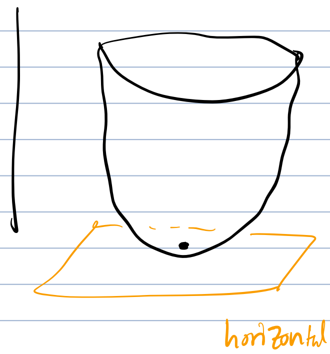
① both $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, or

② either $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ DNE..

e.g.

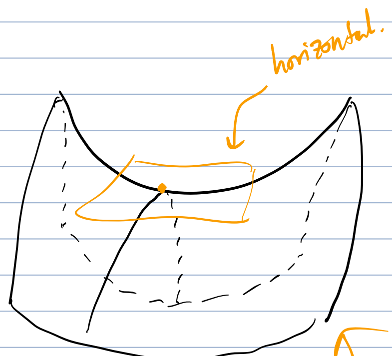
Max

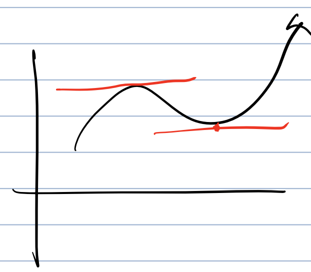


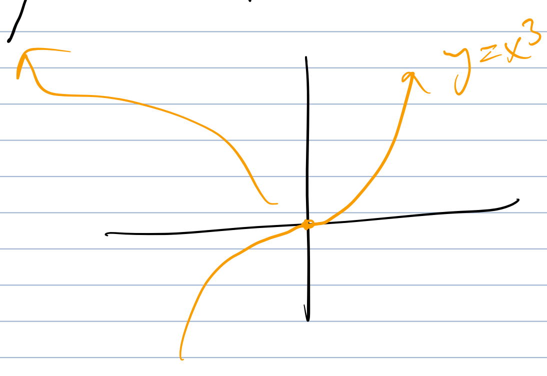


SADDLE:

Pringle chip.







THM: (2^{nd} Partial Derivative Test) $\frac{\partial f}{\partial x}$ can take $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

- Let f have continuous 2^{nd} partial derivatives, and suppose at (a,b) have

$$f_x(a,b) = 0 = f_y(a,b).$$

Consider the quantity $d = f_{xx} f_{yy} - [f_{xy}]^2$

1) if $d > 0$ and $f_{xx} > 0 \rightsquigarrow$ Relative MINIMUM at (a,b)

2) if $d > 0$ and $f_{xx} < 0 \rightsquigarrow$ Relative MAXIMUM

3) if $d < 0$ have SADDLE.

4) if $d = 0 \rightsquigarrow$ NO CONCLUSION.