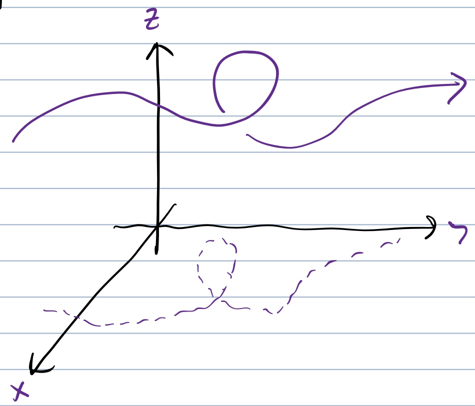
LAST TIME: Discussed vector-valued functions

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ Visualized as curves in $\mathbb{R}^3$

"space-curves"



• Thinking of $\vec{r}(t)$ as describing the position vector of a particle as it moves in $\mathbb{R}^3$, defined

VELOCITY VECTOR: $\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$ provided it exists.

ACCELERATION VECTOR : $\vec{a}(t) = \vec{r}\,''(t)$.

Q: Have two ways of multiplying vectors, product rule?

THM: (Properties of Derivatives) "Scalar function"

Let $\vec{r}(t)$ be differentiable, and let f be a differentiable real-valued function

1) $$\frac{d}{dt}\Big(\overbrace{f(t)\vec{r}(t)}^{\text{Another vector-valued fn}}\Big) = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$$

2) $$\frac{d}{dt}\Big(\overbrace{\vec{r}(t)\cdot\vec{s}(t)}^{\text{A scalar fn}}\Big) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t).$$

3) $$\frac{d}{dt}\Big(\overbrace{\vec{r}(s)\times\vec{s}(t)}^{\text{A vector-valued fn}}\Big) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t).$$

4) Chain rule for "reparametrization"

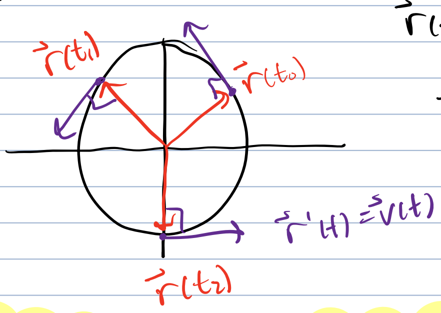
$$\frac{d}{dt}\Big(\vec{r}(f(t))\Big) = \vec{r}\,'(f(t))\,f'(t).$$

INTERESTING CONSEQUENCE

(*) (5) If $\vec{r}(t)\cdot\vec{r}(t) = C$, a constant. Then

$$\vec{r}(t)\cdot\vec{r}\,'(t) = 0.$$

Familiar in $\mathbb{R}^2$

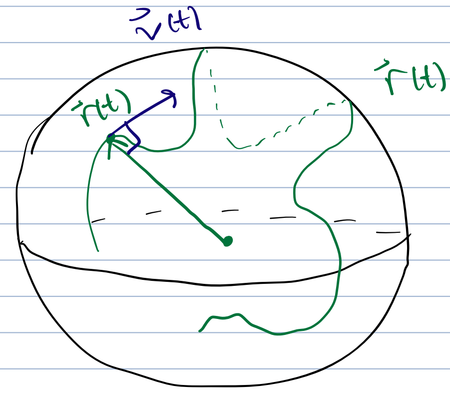


$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = C$

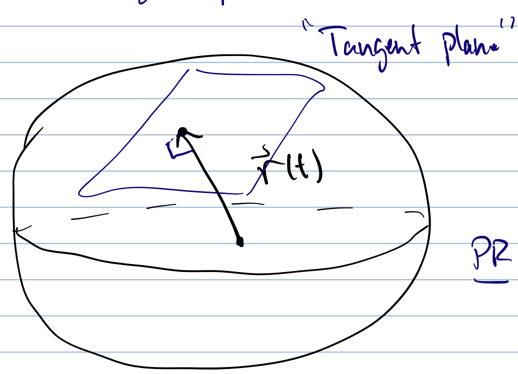
$\rightsquigarrow$ Then $\vec{r}(t)$ lies on a circle

$r = \sqrt{C}$.

· In fact: true in higher dimensions as well.



Really: The entire tangent plane is $\perp$ to $\vec{r}$.



PR: $\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$

Proof of (5): Notice that $\vec{r}(t)\cdot\vec{r}\,'(t) = \dfrac{1}{2}\dfrac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right)$

$$= \frac{1}{2}\frac{d}{dt}\left(c\right)$$

by assumption, this is a constant.

$$= 0.$$

☺

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CALCULUS OF VECTOR-VALUED FNS

ARC LENGTH: Define the LENGTH of $\vec{r}(t)$, $a \le t \le b$, to be

$$L = \int_a^b \underbrace{|\vec{r}\,'(t)|}_{} \, dt$$

SPEED = Norm of $\vec{v}(t) = \vec{r}\,'(t)$

INTEGRATION OF VECTOR-VALUED FNS

- If $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, define

$$\int_a^b \vec{r}(t)\, dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

⚠ Not the "area under the curve"

⇝ This is a VECTOR!!

FUNDAMENTAL THM OF CALCULUS

$$\boxed{\int_a^b \vec{r}\,'(t)\, dt = \vec{r}(b) - \vec{r}(a)}$$

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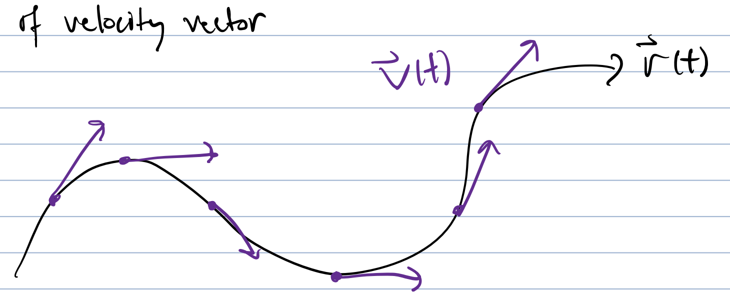
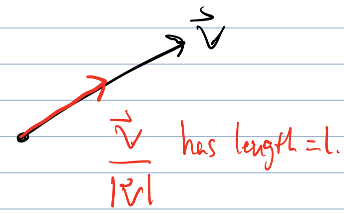
TANGENT AND NORMAL VECTORS

- Have notion of velocity vector

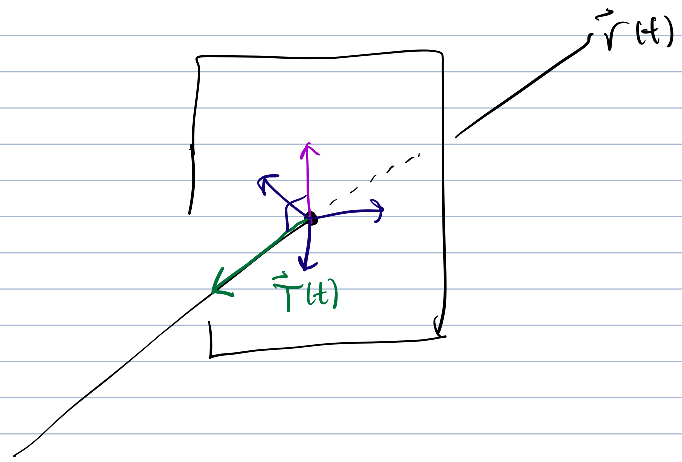


- Define the UNIT TANGENT VECTOR to be

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}, \quad \text{provided } \vec{r}\,'(t) \neq \vec{0}.$$



- Notice: there are infinitely many vectors $\perp$ to $\vec{T}(t)$



- One SPECIAL direction that is $\perp$ to $\vec{T}(t)$, given by $\vec{T}\,'(t)$

Follows by (5) of previous thm b/c

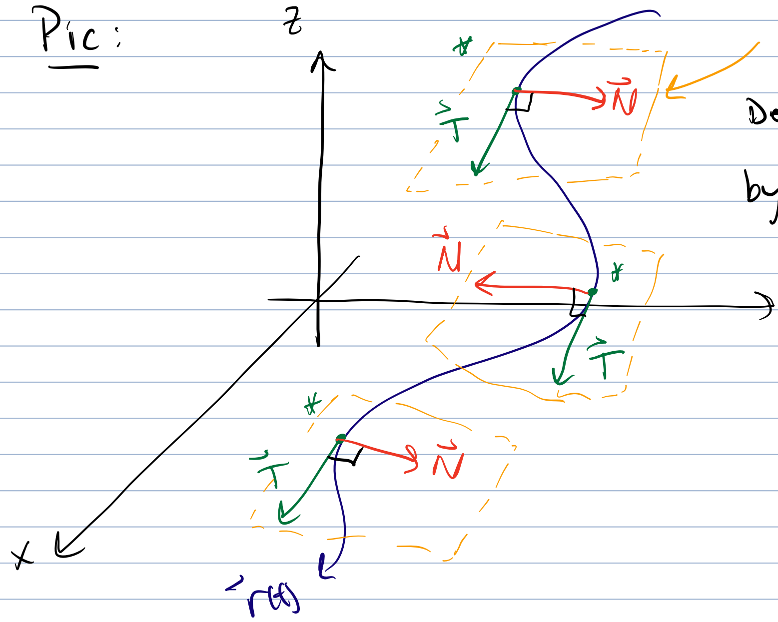
$$\vec{T}(t) \cdot \vec{T}(t) = |\vec{T}(t)|^2 = 1$$

$$\overset{(5)}{\Longrightarrow} \vec{T}(t) \cdot \vec{T}\,'(t) = 0.$$

Define the PRINCIPAL UNIT NORMAL VECTOR to be

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

Pic:

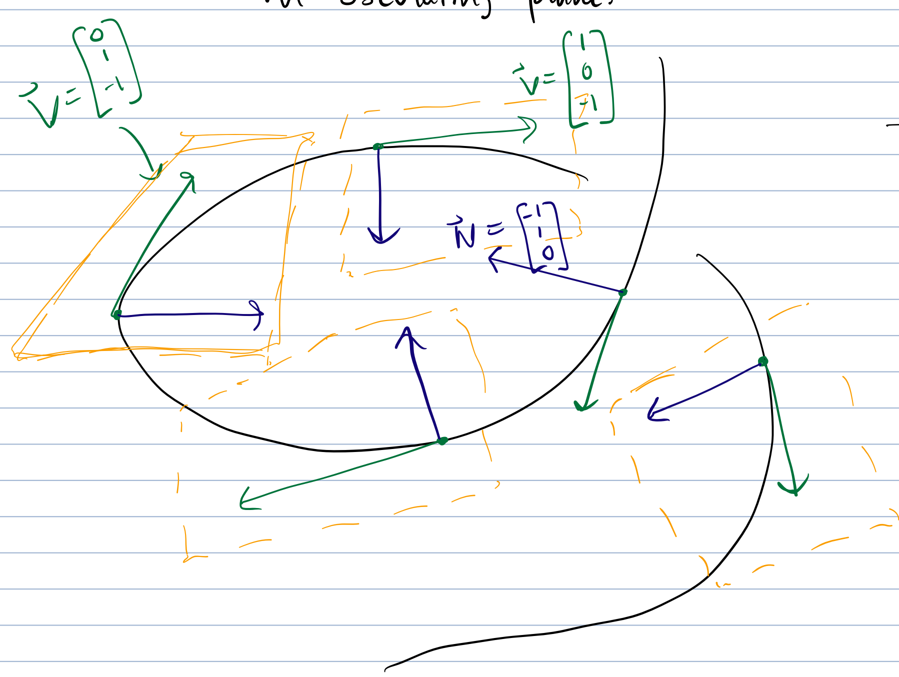


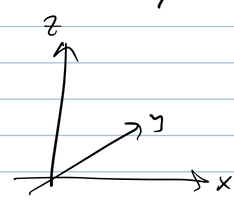
OSCULATING PLANE

Define the plane spanned by $\vec{T}, \vec{N}$ to be the osculating plane

"the tangent planes of best fit"

- Thinking again of $\vec{r}(t)$ as describing the position vector of a particle in $\mathbb{R}^3$ $\rightsquigarrow$ The velocity and acceleration vectors are always contained in osculating plane.





THM: If $\vec{r}(t)$ is the position vector for a smooth curve C and $\vec{N}(t)$ exists, then the acceleration vector $\vec{a}(t)$ lies in the plane determined by $\vec{T}$ and $\vec{N}$ (i.e. the osculating plane).

⤳ Can express acceleration as

$$\vec{a}(t) = \underbrace{a_T}_{\text{Tangent component}} \vec{T}(t) + \underbrace{a_N}_{\text{Normal component.}} \vec{N}(t)$$

Proof: Since $\vec{v}(t) = |\vec{v}(t)|\, \vec{T}(t)$.

Differentiating both sides:

$$\vec{a}(t) = \vec{v}'(t) = \frac{d}{dt}|\vec{v}(t)|\,\vec{T}(t) + |\vec{v}(t)|\,\vec{T}'(t)$$

$$= \frac{d}{dt}|\vec{v}(t)|\,\vec{T}(t) + |\vec{v}(t)|\,\frac{d}{dt}\vec{T}(t)\left(\frac{\|\vec{T}'(t)\|}{\|\vec{T}'(t)\|}\right)$$

$$= \underbrace{\frac{d}{dt}|v(t)|}_{\text{A function}}\,\vec{T}(t) + \underbrace{|\vec{v}(t)|\,|\vec{T}'(t)|}_{\text{A function}}\,\vec{N}(t)$$

A function $\cdot\, \vec{T}(t)$

A function $\cdot\, \vec{N}(t)$.