1. Multiple choice. Clearly mark your answers. No justification is required but may result in partial credit if provided.

   (a) [3 pts] If $|\mathbf{v} \times \mathbf{w}| = 0$ for two unit vectors $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

   (a) True.
   (b) False.
   (c) Indeterminable.

   $|\vec{v}\times\vec{w}| = \vec{0} \Rightarrow |\vec{v}||\vec{w}|\sin\theta = 0$

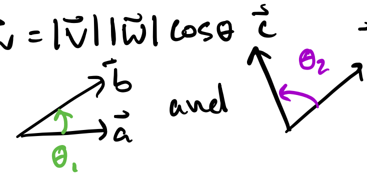
   $\Rightarrow \sin\theta = 0$

   $\Rightarrow \theta = 0$ oR $\theta = \pi \Rightarrow \vec{v} = \vec{w}$ or $\vec{v} = -\vec{w}$.

   (b) [3 pts] If $\mathbf{a} \cdot \mathbf{b} > 0$, and $\mathbf{b} \cdot \mathbf{c} > 0$, then $\mathbf{a} \cdot \mathbf{c} > 0$.

   (a) True.
   (b) False.
   (c) Indeterminable.

   $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$

   $\Rightarrow$ 

   but $\theta_1 + \theta_2 > \frac{\pi}{2} \Rightarrow \vec{a}\cdot\vec{c} < 0$

   i.e. angles less than $\frac{\pi}{2}$ can add up to be greater than $\frac{\pi}{2}$!

   (c) [3 pts] If the acceleration of a curve $\mathbf{r}(t)$ is zero for all time t and the velocity $\mathbf{v}(t)$ is nonzero at time $t = 0$, then the curve parametrized by $\mathbf{r}(t)$ is a line.

   (a) True.
   (b) False.
   (c) Indeterminable.

   If $\vec{a}(t) = 0$ for all time t, then Fundamental theorem of Calculus says $\vec{v}(t)$ is a constant.

   (d) [3 pts] Let $f(x, y) = xye^{xy}$. Which of the following is a unit vector pointing in the direction of the maximal rate of increase at the point $(1, 1)$?

   (a) $\langle 2e, 2e\rangle$
   (b) $\langle 1, 1\rangle$
   (c) $\frac{1}{\sqrt{2}}\langle 1, 1\rangle$
   (d) $\frac{1}{\sqrt{2}}\langle -1, -1\rangle$
   (e) None of the above.

   Gradient of f always points in direction of max increase!

   $$\nabla f = \begin{bmatrix} ye^{xy} + y^2xe^{xy} \\ xe^{xy} + x^2ye^{xy} \end{bmatrix} \overset{(1,1)}{\rightsquigarrow} = \underbrace{\begin{bmatrix} 2e \\ 2e \end{bmatrix}}_{\text{Not a unit vector!}} = 2e\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   $\rightarrow$ Points in same direction as $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (e) [3 pts] If a function $f(x, y, z)$ has gradient satisfying $|\nabla f(x, y, z)| = 1$ everywhere, then the level surface $f(x, y, z) = 1$ is a sphere.

   (a) True.
   (b) False.
   (c) Indeterminable.

   (f) [3 pts] Find the area of the triangle with vertices $(4, 2, 2)$, $(3, 3, 1)$, and $(5, 5, 1)$.

   (a) 0
   (b) 4
   (c) $\sqrt{3}$
   (d) $\sqrt{6}$

   $$\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

   $$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ -1 & -3 & 1 \end{vmatrix} = \begin{bmatrix} -1+3 \\ -(1+1) \\ -3-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} \rightarrow \frac{1}{2}\sqrt{4+4+16} = \frac{1}{2}\sqrt{24} = \frac{1}{2}2\cdot\sqrt{6} = \sqrt{6}$$

(g) [4 pts] Which of the following best describes the critical points of the function

$$f(x, y) = x + \frac{y^3}{3} + \frac{y^2}{2} - \frac{x^2}{2}? \longrightarrow \nabla f = \begin{bmatrix} 1-x \\ y(y+1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x=1, y=0 \\ x=1, y=-1 \end{matrix}$$

$$d = \begin{vmatrix} -1 & 0 \\ 0 & 2y+1 \end{vmatrix} = -2y-1$$

$y=0 \rightarrow d<0$ saddle

$y=-1 \rightarrow d>0$

$f_{xx}<0$ max.

(a) One critical point.

(b) 2 critical points. One local minimum and one local maximum.

(c) 2 critical points. One saddle point and one local maximum.

(d) 2 critical points. One saddle point and one local minimum.

(e) 3 critical points. One saddle point, one local maximum, and one local minimum.

(h) [4 pts] Consider the space curve $\mathbf{r}(t) = \langle 2\cos t, e^t, t \rangle$, $-\infty < t < \infty$, which of the following points lies on the tangent line to the curve at the point $(2, 1, 0)$?

(a) $(\cancel{1}, 1, 1)$

(b) $(2, 1, 1)$

(c) $(\cancel{1}, 2, 0)$

$t=1$ (d) $(2, 2, 1)$

(e) $(\cancel{0}, 1, 2)$

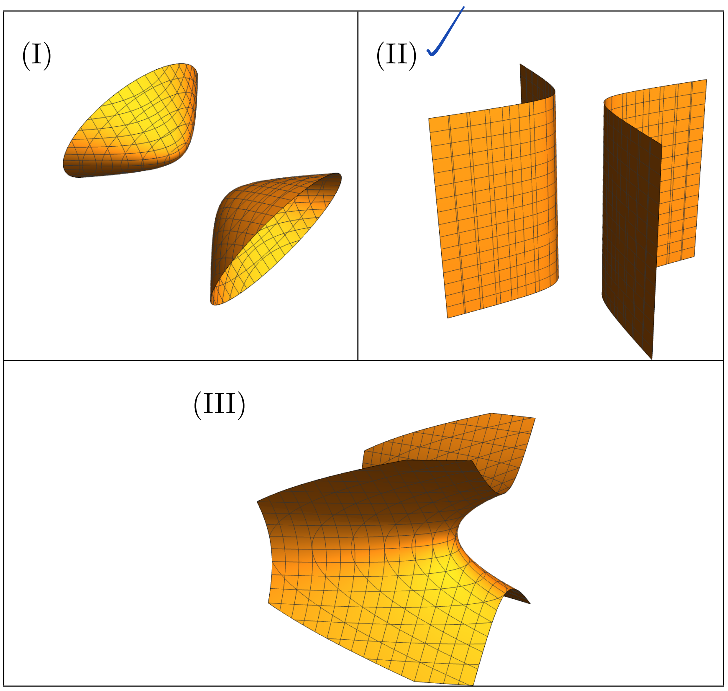
$\vec{r}\,'(t) = \langle -2\sin(t), e^t, 1 \rangle$

When $\vec{r} = \langle 2, 1, 0 \rangle \rightarrow t = 0$

So tangent line is:

$$l(t) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1+t \\ t \end{bmatrix}$$

2. [4 pts] Match the contour surfaces with their equations. Enter O if there's no match.



$g(x, y, z) =$	O, (I), (II), (III)
$x^2 - y^2 + z^2 = 1$	O
$x^2 - y^2 = 1$	II
$x^4 + z = 1$	O
$x^2 + y - z^2 = 1$	III

3. A leaf tumbles down along the curve

$$\mathbf{r}(t) = \langle t^2\cos(t), t^2\sin(t), 16-2t\rangle$$

in space.

(a) [4 pts] What is the speed of the leaf at time $t = \pi$?

Speed $= |\vec{v}(t)|$ where $\vec{v}(t) = \langle 2t\cos(t) - t^2\sin(t), 2t\sin(t) + t^2\cos(t), -2\rangle$

$\rightsquigarrow \vec{v}(\pi) = \langle 2\pi, 0, -2\rangle$

$\rightsquigarrow |\vec{v}(\pi)| = \sqrt{4\pi^2+4}$ //

(b) [4 pts] Find the distance the leaf travels along $\mathbf{r}(t)$ from $t = -8$ to $t = 8$.

Arc length formula:

$$\text{Distance} = \int_{-8}^{8} \sqrt{4t^2\cos^2 t - 4t^3\cos(t)\sin(t) + t^4\sin^2(t) + 4t^2\sin^2(t) + \cdots + 4t^3\cos(t)\sin(t) + t^4\cos^2(t) + 4}$$

factor

cancel

factor

$$= \int_{-8}^{8} \sqrt{t^4 + 4t^2 + 4}\, dt$$

$$= \int_{-8}^{8} \sqrt{(t^2+2)^2}\, dt = \int_{-8}^{8} t^2 + 2\, dt = \frac{t^3}{3}\Big|_{-8}^{8} + 2t\Big|_{-8}^{8}$$

$$= \frac{2}{3}(512) + 32 \text{ //}$$

Points earned: ______
out of a possible 8 points

4. For each of the following, determine whether the limit exists. If so, compute the limit. If not, explain why.

(a) [5 pts] $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2ye^y}{x^4+3y^2}$

Approach $(0,0)$ along the path $y=x^2$, parametrized by $\gamma(t)=\langle t,t^2\rangle$, $t\to0^+$

Then

$$\lim_{t\to0^+} f(\gamma(t)) = \lim_{t\to0^+} \frac{t^2\cdot t^2e^{t^2}}{t^4+3t^4} = \lim_{t\to0^+} \frac{t^4e^{t^2}}{4t^4} = \lim_{t\to0^+} \frac{e^{t^2}}{4} = 1/4$$

However, if we approach $(0,0)$ along the positive x-axis, parametrized by $\gamma(t)=\langle t,0\rangle$, $t\to0^+$

Then

$$\lim_{t\to0^+} f(\gamma(t)) = 0.$$

Since these limits do not agree $\Rightarrow$ $\boxed{\text{Limit} = \text{DNE}}$

(b) [5 pts] $\lim\limits_{(x,y)\to(0,0)} \boxed{\dfrac{y^2(1-\cos(2x))}{x^4+y^2}} = f(x,y)$

Notice: $f(x,y)\geq 0$ for all values (x,y) in domain of f.

Also, $x^4+y^2\geq y^2$ for all (x,y) in $\mathbb{R}^2$.

Therefore:

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} f(x,y) \leq \lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{y^2} = 0.$$

By squeeze thm, $\boxed{\text{LIMIT} = 0}$

(c) [5 pts] $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

Let $\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right]$ Then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to0} f(r\cos\theta, r\sin\theta)$

"$0\cdot\infty$"

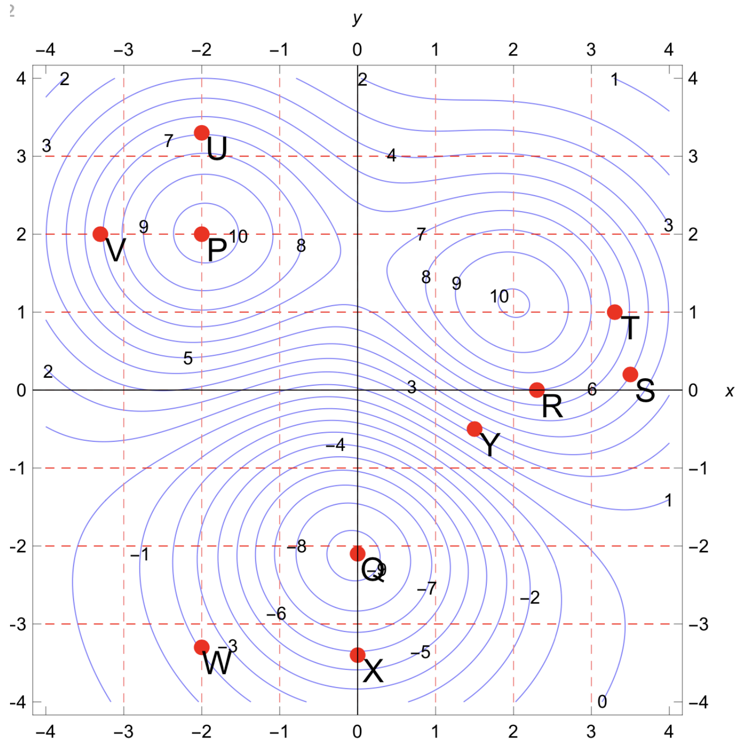
$$= \lim_{r\to0} r^2\ln(r^2) \overset{\text{L.H.R.}}{=\!=} \lim_{r\to0} \frac{\ln(r^2)}{\frac{1}{r^2}} = \lim_{r\to0} \frac{1/r^2\cdot 2r}{-2r^{-3}}$$

$$= \lim_{r\to0} -\frac{r^{-1}}{r^{-3}}$$

$$= \lim_{r\to0} -r^2 = 0. //$$

Points earned: ______
out of a possible 15 points

5. A contour plot of the function $f(x, y)$ is shown below.



Answer each of the following questions using a subset of the points $P, Q, \ldots, X$. Some of the questions may have more than one answer–list all that apply. No justification is required.

(a) [3 pts] At which point is the length of the gradient vector ∇f maximal? ______

(b) [3 pts] At which point is $f_x > 0$ and $f_y = 0$? ______

Hw Problem.

(c) [3 pts] At which point is $f_x < 0$ and $f_y > 0$? ______

(d) [3 pts] At which point is the directional derivative $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f = 0$ and $f_x \neq 0$? ______

(e) [3 pts] At which point does f achieve a global minimum on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$? ______

(f) [3 pts] At wich point is $\nabla f = \vec{0}$ and $f_{xx} < 0$? ______

(g) [3 pts] At which point is ∇f parallel to the vector $\mathbf{j}$? ______

6. Suppose that three quantities x, y, and z, are constrained by the equation $2x^2 + 3y^2 + z^2 = 20$. This equation describes a surface S as a level set.

   (a) [6 pts] Verify that the point $P(2, 1, 3)$ is a point on S and find an equation for the tangent plane to S at P.

   Plug-in coordinates of P into equation: $2(2)^2 + 3(1)^2 + (3)^2$
   $= 8 + 3 + 9 = 20 \checkmark$.

   $\rightsquigarrow$ Normal vector given by $\nabla f(x,y)$

   $$\nabla f = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix} \xrightarrow{P(2,1,3)} \nabla f(2,1,3) = \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} = \vec{N}$$

   Equation of plane: $4(x-2) + 3(y-1) + 3(z-3) = 0$ //

   (b) [5 pts] Near $P(2, 1, 3)$ we can think of z as a function of x and y, $z = f(x, y)$. Approximate the value of z corresponding to $x = 2.2$ and $y = 1.4$.

   Plug-in values of x and y and solve for z

   $\rightarrow 4(2.2-2) + 3(1.4-1) + 3(z-3) = 0$

   $\Rightarrow 4(.2) + 3(.4) + 3z - 9 = 0$

   $\Rightarrow .8 + 1.2 + 3z - 9 = 0$

   $\Rightarrow 3z = 7$

   $\Rightarrow z \simeq 7/3$

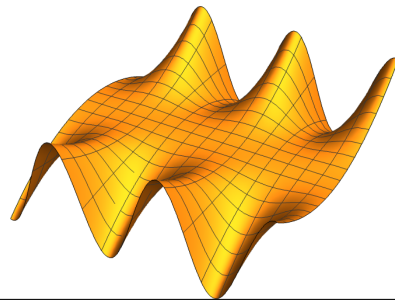
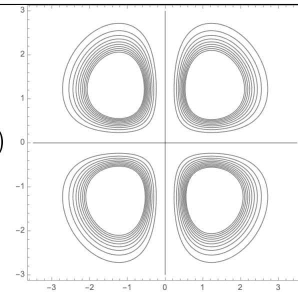
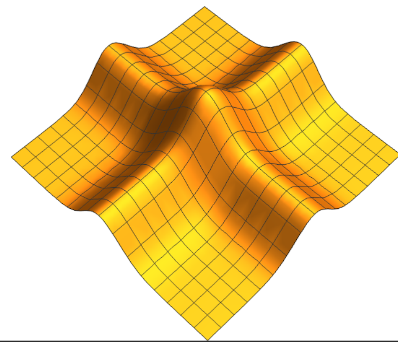
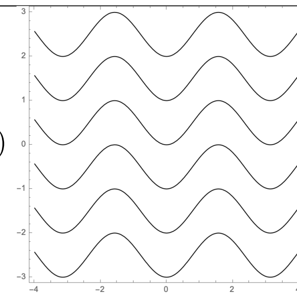
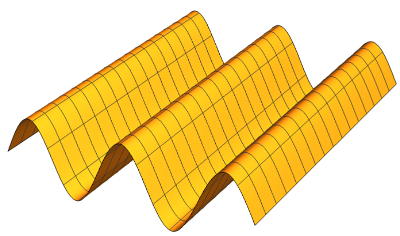
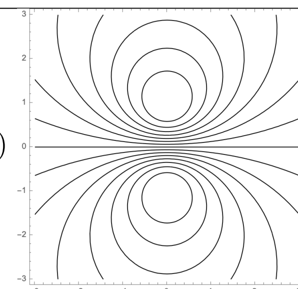
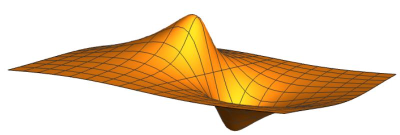
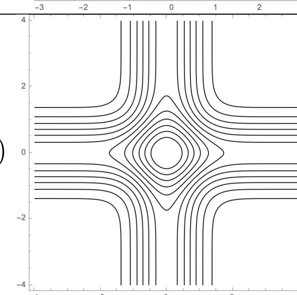
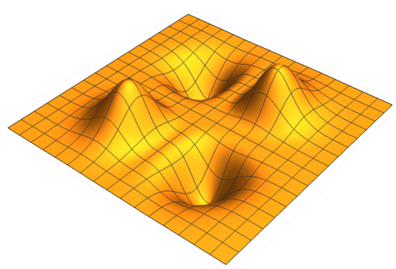
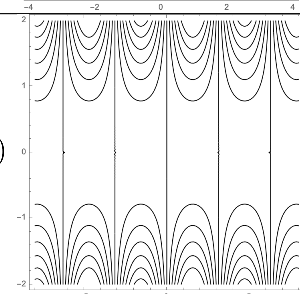
   (c) [5 pts] Find parametric equations for a line ℓ which is orthogonal to the surface S and which passes through the point $P(2, 1, 3)$.

   Again, the gradient is $\perp$ to the surface S

   $\rightsquigarrow$ Normal line parametrized as:

   $$\vec{r}(t) = \vec{r}_0 + t\vec{N}$$
   $$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}, \quad -\infty < t < \infty.$$ //

7. [10 pts] Consider the following table of functions $f(x, y)$, their graphs, and their level curves. On the following page, fill out the table provided by matching the functions with their graphs and level curves.

(a) $x^3y^3\exp\left(-x^2-y^2\right)$	(A) 	(I) 
(b) $\dfrac{-10y}{x^2+y^2+1}$	(B) 	(II) 
(c) $\cos(x)^2+y$	(C) 	(III) 
(d) $y^2\sin(x)$	(D) 	(V) 
(e) $e^{-x^2}+e^{-y^2}$	(E) 	(VI) 

Points earned: ______
out of a possible 10 points

By filling in the table below, correctly match these functions, graphs, and level curves.

Function	**Graph**	**Level Curves**
$x^3y^3\exp\left(-x^2-y^2\right)$	E	I
$\dfrac{-10y}{x^2+y^2+1}$	D	III
$\cos(x)^2+y$	C	II
$y^2\sin(x)$	A	VI
$e^{-x^2}+e^{-y^2}$	B	V