CROSS PRODUCTS

PREVIOUSLY:

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\vec{b} = \langle b_1, b_2, b_3 \rangle = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Vectors in $\mathbb{R}^3$

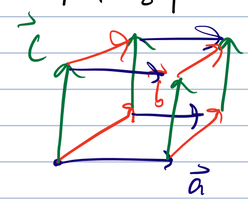
$\vec{c} = \langle c_1, c_2, c_3 \rangle = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Defined the determinant of the 3x3 matrix

$$\operatorname{Det}\begin{pmatrix} - \vec{a} - \\ - \vec{b} - \\ - \vec{c} - \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$= \pm$ Volume (



)

RHR

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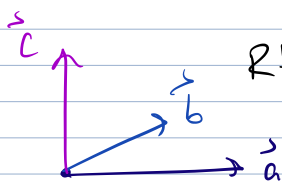

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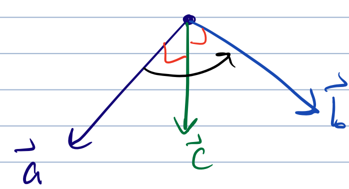


Does NOT satisfy RHR.

- Of particular interest are the UNIT coordinate vectors

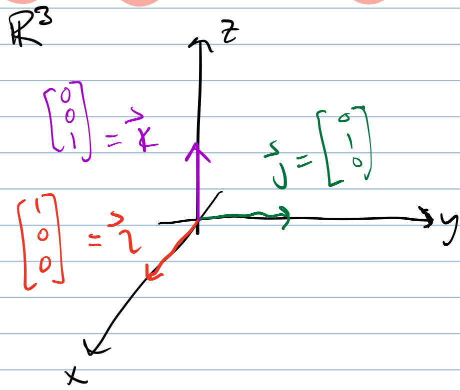
$\vec{i} = \langle 1, 0, 0 \rangle$

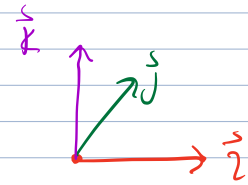
$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

Notice: ① They are unit vectors (i.e. they have length = 1)

② They satisfy the RHR.





Definition of Cross Product: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors in $\mathbb{R}^3$

Define $\underset{\text{cross product}}{\vec{a} \times \vec{b}} :=$ " $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ "

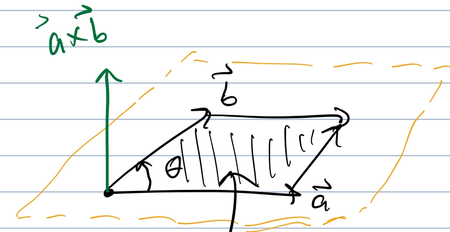
$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

Geometric Interpretation of Cross Product

① $\vec{a} \times \vec{b}$ is always $\perp$ = orthogonal to both $\vec{a}$ and $\vec{b}$

② $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

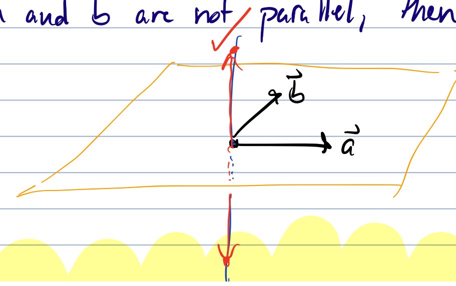
$= \text{Area}\begin{pmatrix}\text{Parallelogram} \\ \text{spanned by } \vec{a}, \vec{b}\end{pmatrix}$



Area = Length of $\vec{a} \times \vec{b}$.

③ $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ satisfy the RHR.

Assuming $\vec{a}, \vec{b} \neq \vec{0}$ and that $\vec{a}$ and $\vec{b}$ are not parallel, then $\vec{a}, \vec{b} \rightsquigarrow$ determine a plane



Proof: ① Observe that if $\vec{c} = \langle c_1, c_2, c_3 \rangle$ a vector in $\mathbb{R}^3$, then

$$\underbrace{\vec{c} \cdot (\vec{a} \times \vec{b})}_{\text{Hw 2}} = \operatorname{Det} \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (*)$$

$\leadsto$ Use this to show that $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{a} \perp \vec{a} \times \vec{b}$ ✓

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{b} \perp \vec{a} \times \vec{b}$

② Let $\vec{c}$ be a $\boxed{\text{unit}}$ vector $\perp$ to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \vec{c}$ satisfy the RHR

Know that $\vec{a} \times \vec{b} = \underbrace{\lambda}_{\text{lambda}} \vec{c}$ Want to show: $\lambda = |\vec{a}|\,|\vec{b}| \sin\theta$

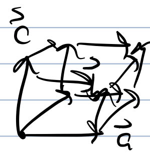
Consider the expression

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{c} \cdot (\lambda \vec{c}) = \lambda\, \vec{c} \cdot \vec{c} = \lambda \underbrace{|\vec{c}|^2}_{|\vec{c}| = 1}$$

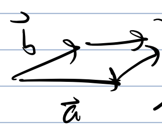
$$= \lambda$$

On the other hand:

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \pm \operatorname{Vol}\Big(\ \Big)$$



$$= \operatorname{Area}\Big(\ \Big)$$



$$= |\vec{a}|\,|\vec{b}| \sin(\theta).$$