LECTURE ①:

§12.1: GEOMETRY OF $\mathbb{R}^3$

• So far, you've studied geometry in 2D space $\mathbb{R}^2$.
distances/angles.

NOTATION:

In symbols "2D space" is often denoted as a Cartesian product

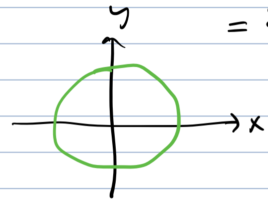
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ \underbrace{(a,b)}_{\text{Collection of all objects}} \mid \underbrace{a,b \in \mathbb{R}}_{\text{which satisfy the condition.}} \}.$$

e.g. (Example)

use of distance in $\mathbb{R}^2$

Consider the unit circle, centered at $(0,0)$

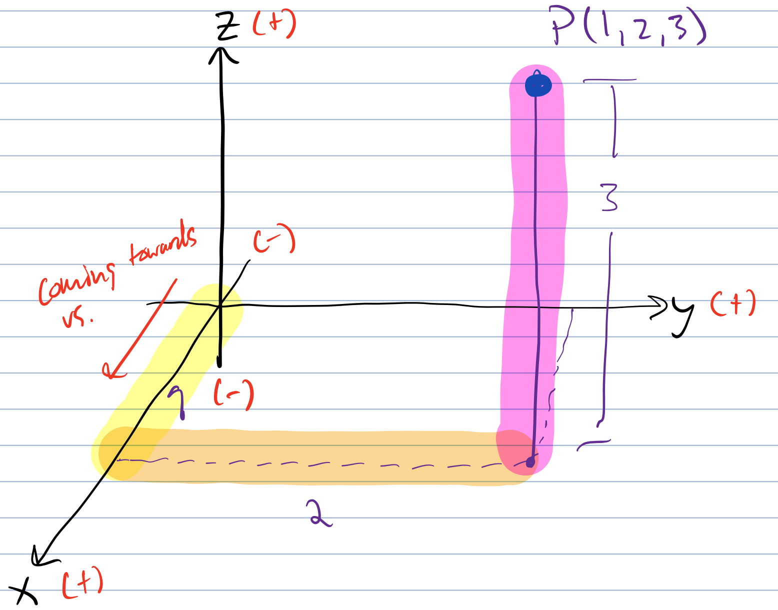
$$= \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}.$$



GOAL: Generalize these ideas (distances, angles, parametrizations, fns, etc....) to 3D (Really, n-dim'l Euclidean space $\mathbb{R}^n$)

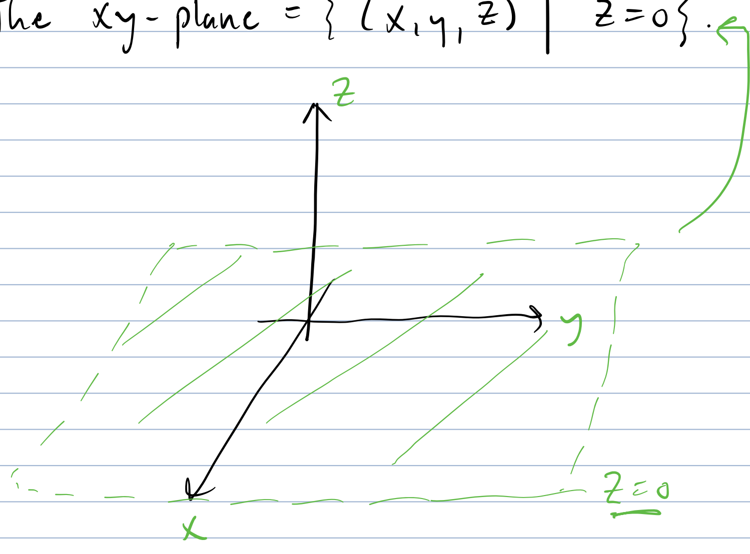
⇝ Then DO CALCULUS!!

Coordinate Axes in $\mathbb{R}^3 = \{ (x,y,z) \mid x,y,z \in \mathbb{R} \}$.

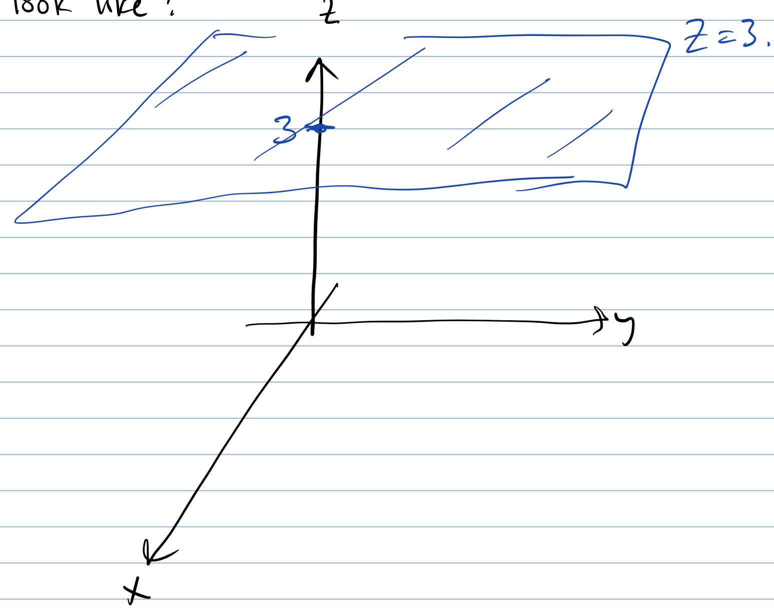


EXAMPLES: (Equations and surfaces in $\mathbb{R}^3$)

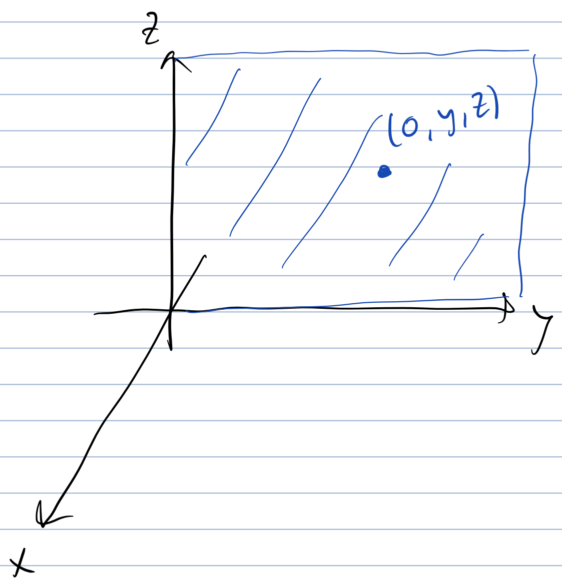
① The xy-plane $= \{(x, y, z) \mid z = 0\}$.



② $z = 3$ look like?

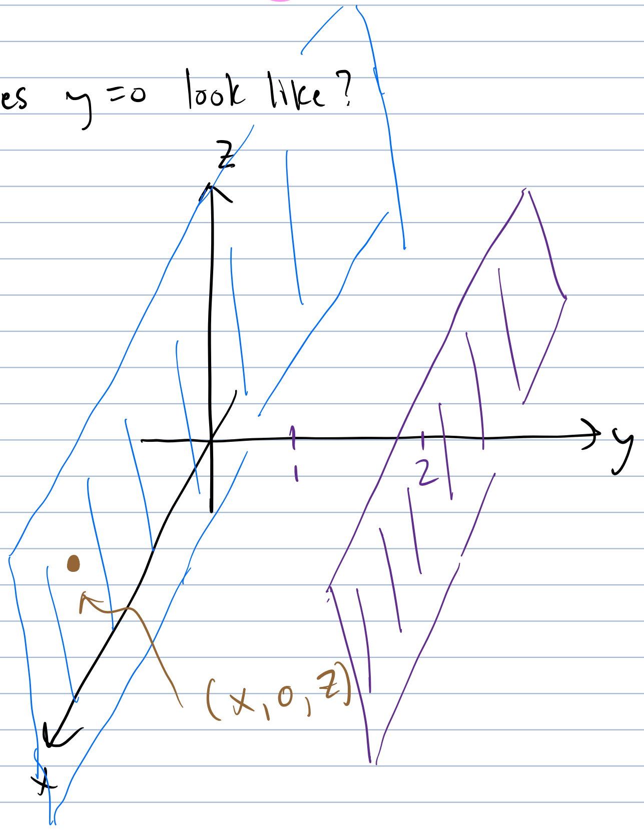


③ What does $x = 0$ look like?



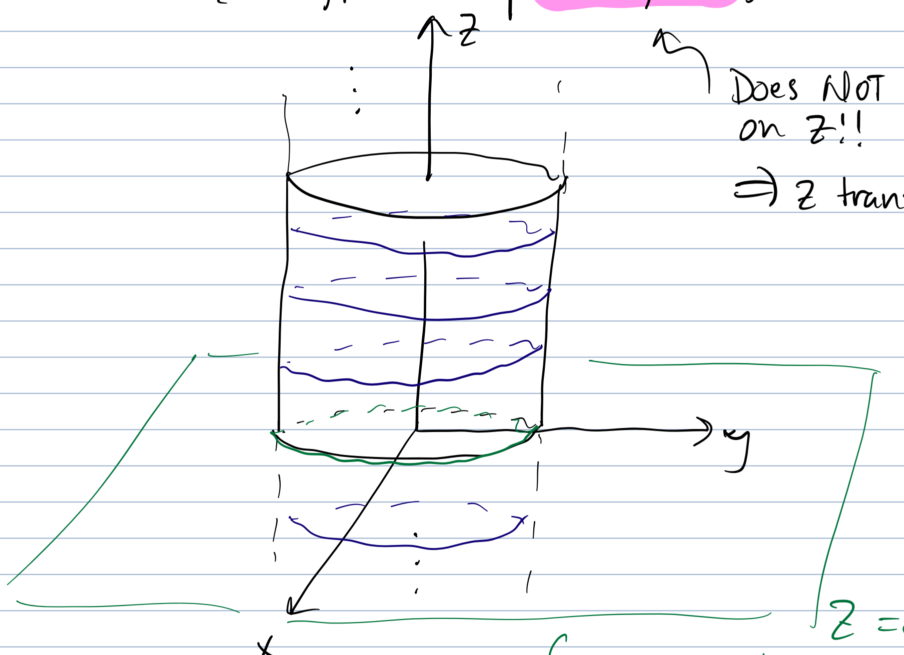
This is the yz-plane.

④ What does $y=0$ look like?



What does $y=2$ look like?

⑤ What does $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1\}$ look like?



Does NOT depend on z!!

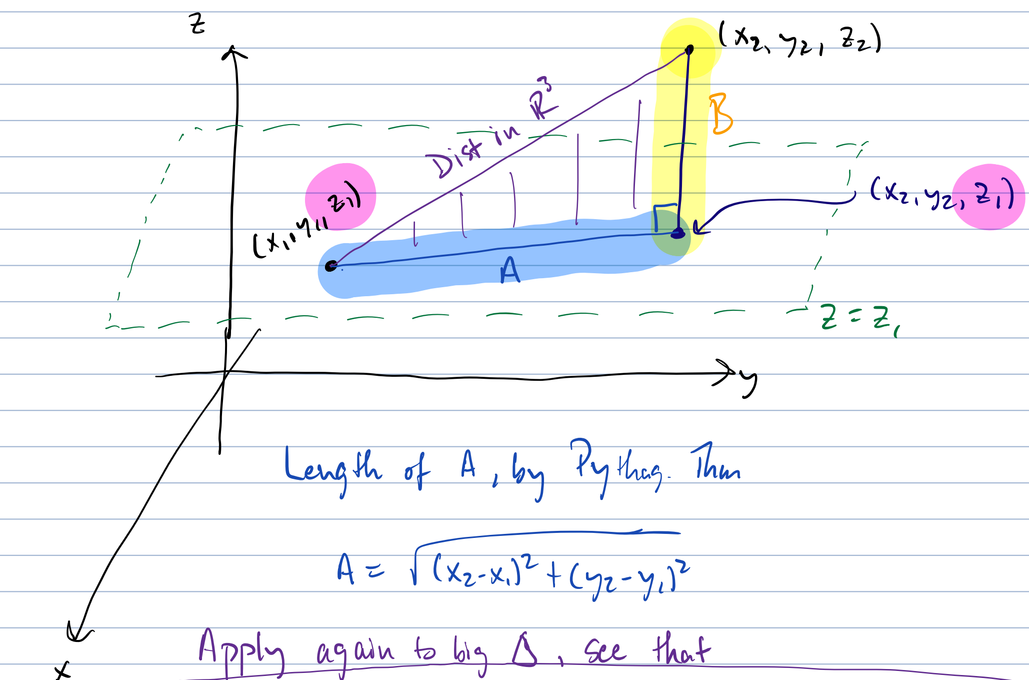
$\Rightarrow$ z translation invariant

$z=0$

$\{(x,y,0) \mid x^2+y^2=1\}$ = unit circle in x-y plane

DISTANCES AND SPHERES

- In $\mathbb{R}^3$, How to measure the distance btw two points?



Length of A, by Pythag. Thm

$$A = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Apply again to big Δ, see that

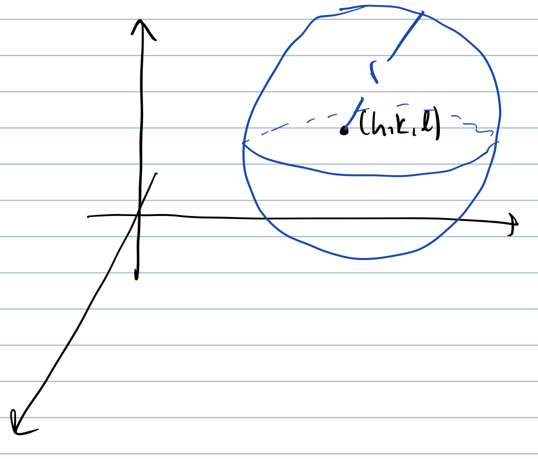
$$\boxed{\mathrm{Dist}\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

⚠ NOT a rigorous proof!! We made lots of implicit assumptions,
(e.g. Said $A \perp B$, but don't really know what this means yet...)
Also, assumed Pythag. holds in 3D

- Logically cleaner to take ✯ as the DEFINITION of distance btw 2 points in 3D, and build up from there.

- Having defined distance in $\mathbb{R}^3$, can make sense of a sphere of radius r centered at $C(h,k,\ell)$

$$\underbrace{S^2}_{\text{two-sphere}} = \{ (x,y,z) \in \mathbb{R}^3 \mid (x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2 \}.$$



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§12.2: VECTORS IN 2D AND 3D

- In 2D: a vector consists of two real #'s,

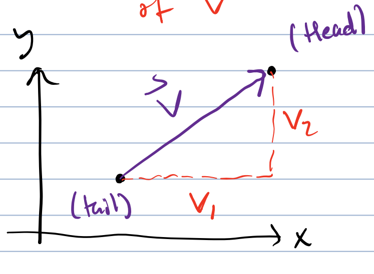
the data of

$$\vec{v} = \langle v_1, v_2 \rangle, \text{ here } v_1, v_2 \text{ are real \#'s.}$$

Vector

The components of $\vec{v}$

Geometric Significance:

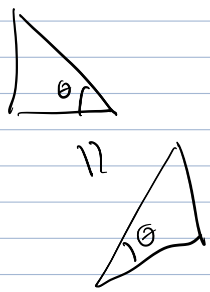


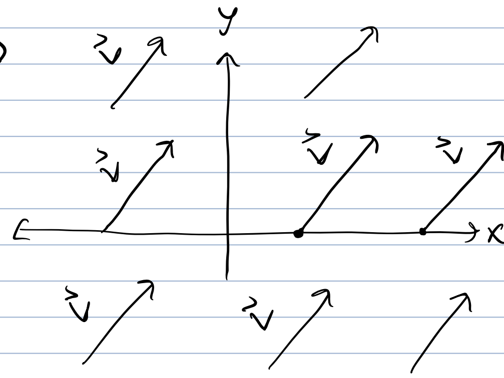
Should think of $\vec{v}$, geometrically, as being an "arrow"

Here, $v_1, v_2 > 0$

⚠ IMPORTANT: The vector $\vec{v}$ only described "up to translations" in the plane.

e.g. $\vec{v} = \langle 1, 2 \rangle$



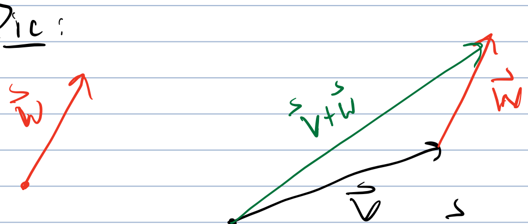


All equivalent vectors... will not distinguish btw them.

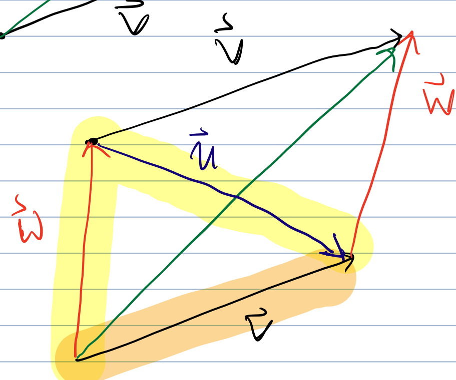
OPERATIONS W/ VECTORS

① Addition: Sps have $\vec{w} = \langle w_1, w_2 \rangle$. Then $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$.

Pic:



② Subtraction:



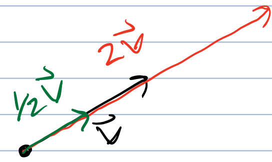
$\vec{w} + \vec{u} = \vec{v}$

$\Rightarrow \vec{u} = \vec{v} - \vec{w}$

③ Multiplication by a scalar: if c is a real #, can define

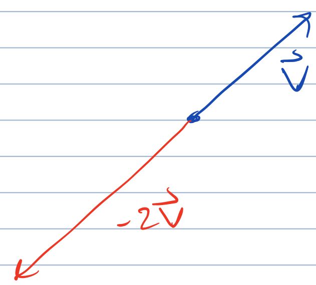
$$c\vec{v} = \langle cv_1, cv_2 \rangle$$

Geometrically:



Notice: if $c>0$, then $c\vec{v}$ points in same direction as $\vec{v}$.

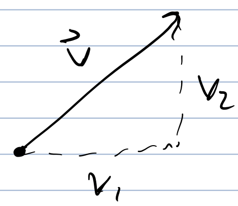
BUT: if $c<0$, then "turn $\vec{v}$ around"



④ Length of a Vector: (Norm, magnitude)

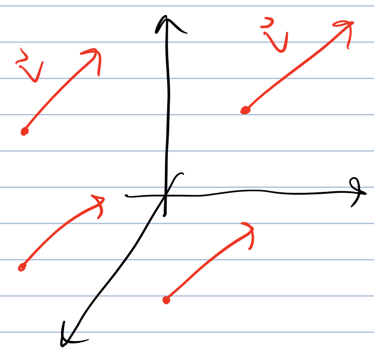
Defined to be, $\vec{v} = \langle v_1, v_2 \rangle$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$$



VECTORS IN 3D:

- A 3D Vector has 3 components $\vec{v} = \langle v_1, v_2, v_3 \rangle$



(up to translations in $\mathbb{R}^3$)

$$\boxed{|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

§12.3: THE DOT PRODUCT

Q: How to multiply two VECTORS?? $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

There are two product operations:

① DOT PRODUCT: INPUT $\longrightarrow$ OUTPUT

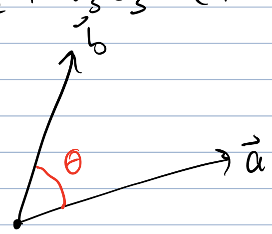
$\vec{a}, \vec{b} \longmapsto$ A Real # (Defined for all dim's)

② CROSS PRODUCT: INPUT $\longrightarrow$ OUTPUT

$\vec{a}, \vec{b} \longmapsto$ ANOTHER VECTOR. (Special to $\mathbb{R}^3$)

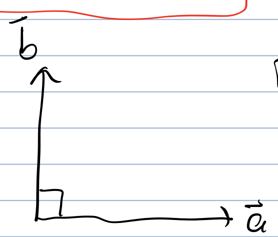
Define: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 \cdot b_2 + a_3 b_3 \quad (+ \cdots + a_n b_n)$

Geometric Interpretation:



Then $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta}$ ★

e.g.



Then $\cos(\theta) = \cos(\pi/2) = 0$

So $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot 0 = 0.$

$\boxed{\text{Define } \vec{a} \perp \vec{b} \text{ to mean } \vec{a} \cdot \vec{b} = 0.}$