1. Find the limit, if it exists, or show that the limit does not exist.

(a) [3 pts] $\displaystyle\lim_{(x,y)\to(-3,1)} \frac{x^2y - xy^3}{x - y + 2} = \frac{9\cdot 1 - (-3)(1)^3}{-3-1+2}$

$= \dfrac{9+3}{-2} = \dfrac{12}{-2} = -6$ //

(b) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \text{DNE}$

Approach $(0,0)$ along $\gamma(t) = \langle t, 0\rangle$, $t \to 0^+$

$\rightsquigarrow \displaystyle\lim_{t\to 0^+} \frac{t}{t^2} = \infty.$

DNE.

Approach as $t \to 0^-$ $\rightsquigarrow \displaystyle\lim_{t\to 0^-} \frac{t}{t^2} = -\infty.$

(c) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

Since $\underbrace{\displaystyle\lim_{(x,y)\to(0,0)} \frac{-xy}{x}}_{=0} \le \displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \le \underbrace{\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x}}_{=0}$

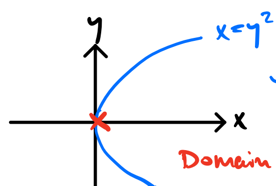
$\longrightarrow$ By squeeze theorem, $\lim = 0$.

(d) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2+y^4} = \text{DNE}$

Since these limits are not equal $\rightarrow$ Limit = DNE.

Approach $(0,0)$ along $\gamma(t) = \langle t, 0\rangle$, $t \to 0^+$

$\displaystyle\lim_{t\to 0^+} \frac{0}{t^2} = 0$



$\rightsquigarrow \gamma(t) = \langle t^2, t\rangle$, $t \to 0^+$

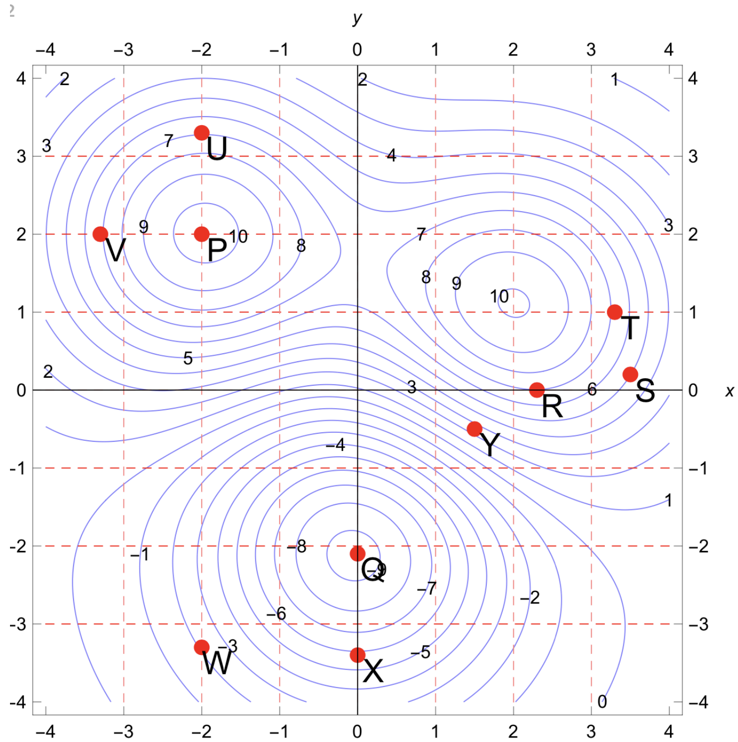
Approach along $x = y^2$

Then

$\displaystyle\lim_{t\to 0^+} \frac{t^4\cos(t)}{2t^4} = \frac{1}{2}.$ //

Domain of $f(x,y)$ is when $x^2+y^4 \neq 0 \Rightarrow$ Domain is $\mathbb{R}^2 - \{(0,0)\}$.

2. A contour plot of the function $f(x, y)$ is shown below.



Answer each of the following questions using a subset of the points $P, Q, \ldots, X$. Some of the questions may have more than one answer–list all that apply. No justification is required.

(a) [3 pts] At which point is the length of the gradient vector ∇f maximal? Y

(b) [3 pts] At which point is $f_x > 0$ and $f_y = 0$? V

(c) [3 pts] At which point is $f_x < 0$ and $f_y > 0$? S

(d) [3 pts] At which point is the directional derivative $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f = 0$ and $f_x \neq 0$? S

(e) [3 pts] At which point does f achieve a global minimum on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$? Q

(f) [3 pts] At wich point is $\nabla f = \vec{0}$ and $f_{xx} < 0$? P

(g) [3 pts] At which point is ∇f parallel to the vector $\mathbf{j}$? R

3. Suppose that three quantities x, y, and z, are constrained by the equation $2x^2 + 3y^2 + z^2 = 20$. This equation describes a surface S as a level set.

   (a) [6 pts] Verify that the point $P(2, 1, 3)$ is a point on S and find an equation for the tangent plane to S at P.

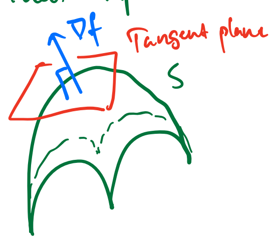
Check:

$$2(2)^2 + 3(1)^2 + (3)^2 = 8 + 3 + 9 = 20 \checkmark.$$

$\therefore$ Equation of tangent plane is:

$$4(x-2) + 3(y-1) + 3(z-3) = 0 \;//$$

Know that ∇f is $\perp$ to tangent plane to S at P.



Compute:

$$\nabla f = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix} \rightsquigarrow \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} = \vec{N}$$

   (b) [5 pts] Near $P(2, 1, 3)$ we can think of z as a function of x and y, $z = f(x, y)$. Approximate the value of z corresponding to $x = 2.2$ and $y = 1.4$.

Plugging into tangent plane:

$$4(2.2-2) + 3(1.4-1) + 3(z-3) = 0$$

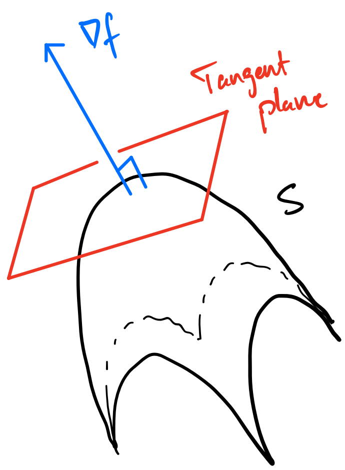
$$\Rightarrow 0.8 + 1.2 + 3z - 9 = 0$$

$$\Rightarrow 3z = 11$$

$$\Rightarrow z = 11/3 = 3.\overline{66}$$

∇f always $\perp$ to level sets!

   (c) [5 pts] Find parametric equations for a line ℓ which is orthogonal to the surface S and which passes through the point $P(2, 1, 3)$.



Direction vector for normal line is ∇f.

$$\rightsquigarrow \vec{r}(t) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}, \quad -\infty < t < \infty.$$

4. [8 pts] Find the critical points of the function $f(x,y) = x^4 + 2y^2 - 4xy$, and classify each as a local maximum, local minimum, or saddle point.

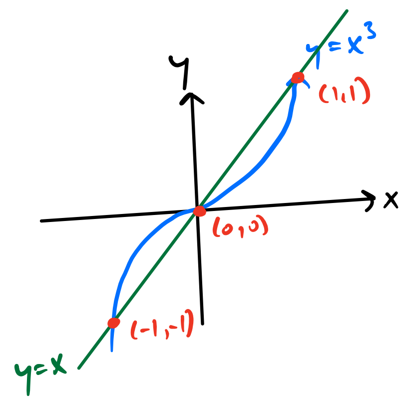
To find the critical points:
solve the equation $\nabla f = 0$

$$\rightsquigarrow \nabla f = \begin{bmatrix} 4x^3 - 4y \\ 4y - 4x \end{bmatrix} = 4 \begin{bmatrix} x^3 - y \\ y - x \end{bmatrix} \begin{matrix} ① \\ ② \end{matrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solutions to the set of equations:

$$\begin{cases} x^3 - y = 0 \\ y - x = 0 \end{cases}$$

→ correspond to the geometric intersections between these two curves



⇝ Can see that there are 3 solutions
i.e. 3 critical points
$(-1,-1)$, $(0,0)$, and $(1,1)$

CRITICAL POINT	SIGN OF DET	SIGN OF F_{xx}	TYPE OF CRIT POINT
$(-1,-1)$	$\begin{vmatrix} 12 & -4 \\ -4 & 4 \end{vmatrix} > 0$ +	$f_{xx} > 0$ $\overset{\shortparallel}{12}$	Local Min
$(0,0)$	$\begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16$	/	SADDLE
$(1,1)$	$\begin{vmatrix} 12 & 4 \\ 4 & 4 \end{vmatrix} > 0$	$f_{xx} > 0$	Local Min.