Equations of Line Example (Two intersecting planes)

Example:

Find the intersection of the planes $\underbrace{x+2y+z=0}_{P_1}$ and $\underbrace{x-3y-z=0}_{P_2}$

Sol:

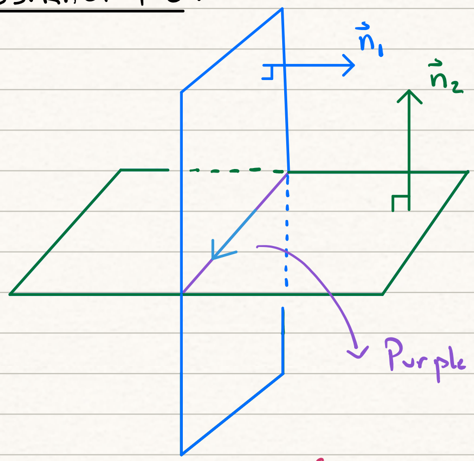
Notice: A normal vector to the first plane P_1 is given by:

$$\vec{n}_1 = \langle 1, 2, 1 \rangle$$

A normal vector to the second plane P_2 is given by:

$$\vec{n}_2 = \langle 1, -3, -1 \rangle$$

ABSTRACT PIC:



Purple line = [Line of intersection between the two planes P_1 and P_2]

Goal: Want to find a parametrization $\vec{r}(t)$ for this line of intersection

⤳ Need:

1. A point on the line $\vec{r}_0$
2. A direction vector $\vec{v}$ so that

Line given by $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

for $-\infty < t < \infty$

Can see from the picture that a good candidate for direction vector is:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \text{Det} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & -3 & -1 \end{vmatrix}$$

$$= (-2+3)\vec{i} - (-1-1)\vec{j} + (-3-2)\vec{k}$$

$$= \vec{i} + 2\vec{j} - 5\vec{k}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

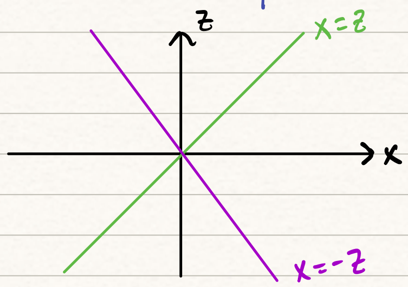
$$= \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \quad \text{Direction vector } \vec{v}$$

Next, Need to find any point on the intersection of two planes

i.e. Find at least one solution to the system of equations

$$\begin{cases} x + 2y + z = 0 \\ x - 3y - z = 0 \end{cases}$$

$\rightsquigarrow$ if we set $y = 0$ (amounts to looking at what happens in xz-plane)



Get a solution at $x = 0$, $y = 0$, $z = 0$ $\rightsquigarrow$ $\vec{r}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

CONCLUSION:

A parametrization of the line of intersection btw these two planes is given by:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} t \\ 2t \\ -5t \end{bmatrix} \quad \text{for } -\infty < t < \infty.$$

There are many different parametrizations of this line! Your answer might look different!

Check: We can double check that our answer is correct by plugging in our solution to both equations

Plane 1: Plug in $x(t) = t$, $y(t) = 2t$, $z(t) = -5t$ into equation for first plane

$$x + 2y + z = 0 \rightsquigarrow t + 2(2t) + (-5t) = 5t - 5t = 0$$

$\rightsquigarrow$ Means: For every value of t, the equation for plane 1 is satisfied!

Plane 2: -//-

$$x - 3y - z = 0 \rightsquigarrow t - 3(2t) - (-5t) = \underbrace{t - 6t + 5t = 0}_{\text{Equation is always satisfied, regardless of } t.}$$

Conclusion: See that the components of the line parametrized by $\vec{r}(t) = \langle t, 2t, -5t \rangle$ satisfies BOTH planar equations for all time t!

$\Longrightarrow$ $\vec{r}(t)$ parametrizes the line of intersection between Planes 1 and 2. //