

1. [3 pts] Determine the arc length along the curve

$$\mathbf{r}(t) = \langle \cos(t) + t \sin(t), \sin(t) - t \cos(t), t^2 \rangle$$

from $t = 0$ to $t = \pi/2$.

2. [3 pts] Interestingly, the notion of arc length can be defined in any dimension. A curve in four-dimensional space \mathbb{R}^4 is parametrized as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t), w(t) \rangle, \quad a \leq t \leq b.$$

Find the arc length of $\mathbf{r}(t) = \langle t, \ln(t), 1/t, \ln(t) \rangle$ and $1 \leq t \leq 4$.

3. Suppose that the trajectory of a particle in \mathbb{R}^3 is described by the vector-valued function $\mathbf{r}(t)$, and let $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$ be its velocity and acceleration vectors, respectively. For each of the following statements, either give a proof or exhibit a counter-example.
- (a) [2 pts] Let \mathcal{C} be the space-curve in \mathbb{R}^3 which is parametrized by $\mathbf{r}(t)$, $-\infty < t < \infty$. If the velocity vector $\mathbf{v}(t)$ is constant, then the curve \mathcal{C} lies entirely in a plane.
- (b) [2 pts] Define the *speed* of the particle at time t to be the length of its velocity vector $s(t) = |\mathbf{v}(t)|$ at time t . If the speed is a constant function, then the curve lies entirely in a plane.
- (c) [2 pts] If the acceleration vector $\mathbf{a}(t)$ is constant, then the curve \mathcal{C} lies entirely in a single plane.
- (d) [2 pts] If the velocity vector $\mathbf{v}(t)$ is *orthogonal* to the acceleration vector $\mathbf{a}(t)$ for all time t , then the curve \mathcal{C} lies entirely in a single plane.

(e) [2 pts] Prove that $\mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0}$ implies $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$, where \mathbf{c} is a constant vector.

(f) [2 pts] If $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ for all time t , prove that the motion takes place in a plane (i.e. that the space curve parametrized by $\mathbf{r}(t)$ lies entirely in a plane). Consider both $\mathbf{c} = \mathbf{0}$ and $\mathbf{c} \neq \mathbf{0}$.

4. [4 pts] Find a vector-valued function, $\mathbf{r}(t)$, that represents the curve of intersection between the cylinder $\{(x, y, z) \mid x^2 + y^2 = 9\}$ and the surface $\{(x, y, z) \mid z = xy\}$.