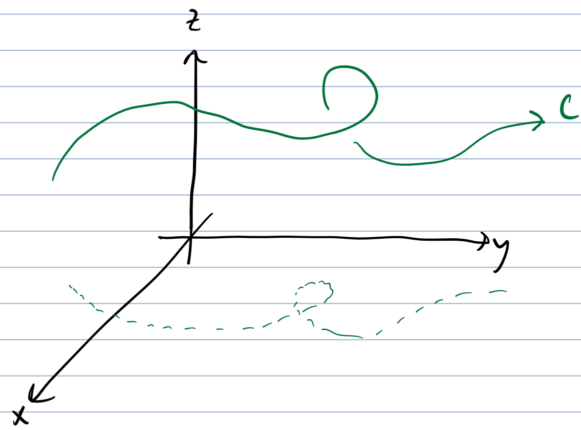
§13.1: Parametrized Curves in $\mathbb{R}^3$

· Interested in studying parametrized "space curves"



The curve C is described by a vector valued fn

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$ "position at time t."

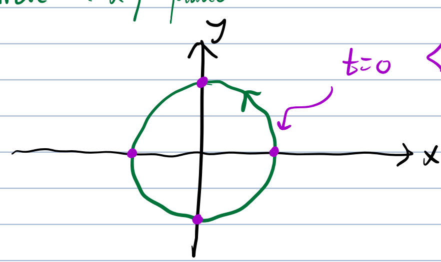
Don't forget about domain!!

Ex: Sketch the curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ where $-\infty < t < \infty$.

So particle increases in height at unit speed.

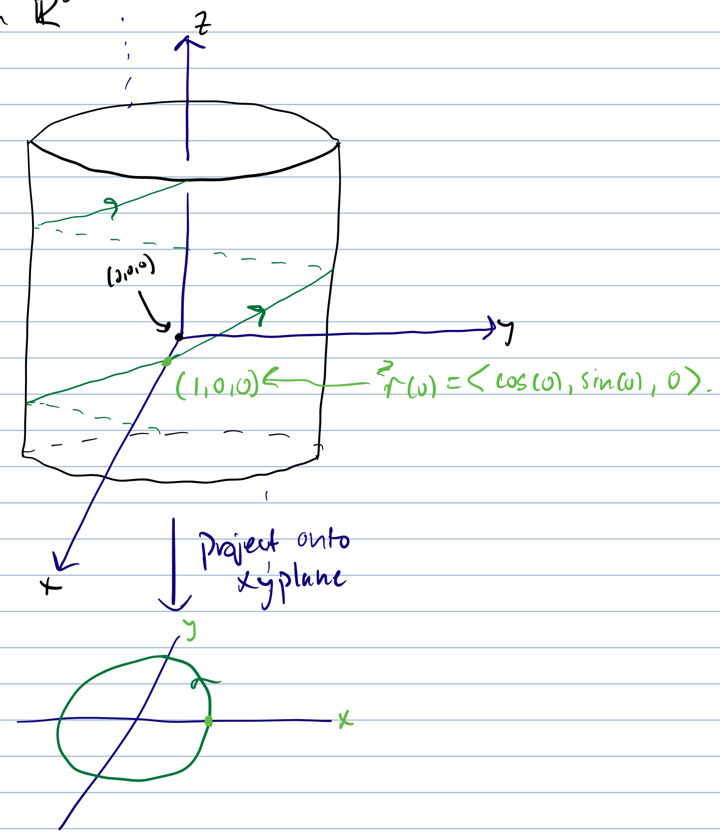
Forget about z-component (for a moment) and consider only xy-components

$\rightsquigarrow$ get unit circle in xy-plane



$t=0$ $\langle \cos(0), \sin(0) \rangle$

Draw the whole curve in $\mathbb{R}^3$



⚠ The same curve can have many parametrizations!!

e.g. Consider

$\vec{r}_1(t) = \langle 1+t,\ 2+4t,\ -3-3t \rangle$

$\vec{r}_2(s) = \langle -2s^3,\ -2-8s^3,\ 6s^3 \rangle$.

These two parametrizations describe the same curve.

Q : How to show?

⤳ look at the first components; set them equal:

$1+t = -2s^3 \rightsquigarrow$ solve for t : $\boxed{t = -1-2s^3}$ Plug this into 2^{nd} component

② : In $\vec{r}_1$: $2+4t \rightsquigarrow 2+4(-1-2s^3) = 2-4-8s^3$

$= -2-8s^3$

③ : In $\vec{r}_1$: $-3-3t \rightsquigarrow -3-3(-1-2s^3) = -3+3+6s^3$

$= 6s^3$

CONCLUSION: $\vec{r}_1(t) = \vec{r}_2(t = u(s)) \rightarrow$ so $\vec{r}_1$ and $\vec{r}_2$ describe the same curve. //

- Important idea: Velocity Vector (at time t)

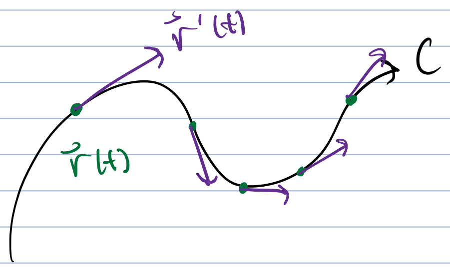
Defined to be

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$ provided this limit exists.

Not hard to check. $= \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$

GEOMETRIC INTERPRETATION:

- $\vec{r}\,'(t)$ = points in the direction tangent to the curve
- Length of $\vec{r}\,'(t) = |\vec{r}\,'(t)|$ = Speed at time t.



Usually denote $\vec{v}(t) = \vec{r}\,'(t)$

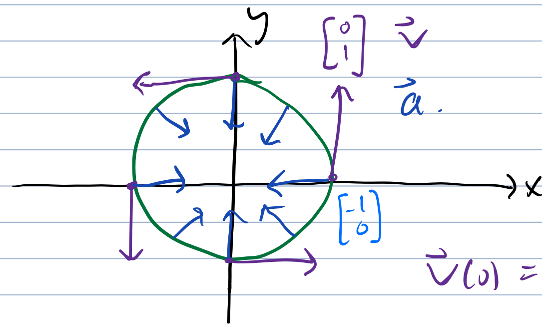
- Also have an acceleration vector $\rightarrow \vec{a}(t) = \vec{r}\,''(t)$.

Ex: Return to helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$, $-\infty < t < \infty$.

Then $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle = \vec{v}(t)$

$\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle = \vec{a}(t)$.

In xy-plane :

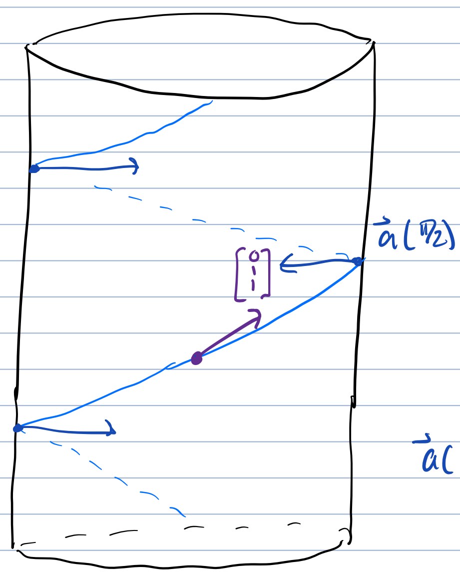


$\vec{v}(0) = \langle -\sin(0), \cos(0) \rangle$

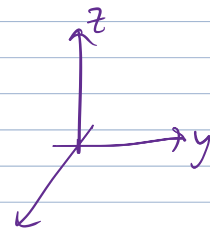
$= \langle 0, 1 \rangle$

$\vec{a}(0) = \langle -1, 0 \rangle$

Now, include the last component :



$\vec{v}(0) = \langle 0, 1, 1 \rangle$



$\vec{a}(\pi/2) = \langle 0, -1, 0 \rangle$