1. Find the limit, if it exists, or show that the limit does not exist. Fully justify your answer.

(a) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+y^4}$ = DNE

Proof:

Approach the origin along positive x-axis: $\gamma(t) = \langle t, 0\rangle$, $t \to 0^+$

Then $\displaystyle\lim_{t\to 0^+} f(\gamma(t)) = \lim_{t\to 0^+} \frac{0}{t^4+0} = 0$

Approach the origin along the line $y = x$: $\gamma(t) = \langle t, t\rangle$, $t \to 0^+$

Then

$\displaystyle\lim_{t\to 0^+} f(\gamma(t)) = \lim_{t\to 0^+} \frac{t^4}{2t^4} = 1/2$

Because these do not agree $\Longrightarrow$ Limit = DNE.

(b) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) = 0$

Proof:

Notice that $-1 \le \sin\left(\frac{1}{x^2+y^2}\right) \le 1$

These inequalities imply that:

$$\underbrace{\lim_{(x,y)\to(0,0)} -xy}_{=\,0} \le \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) \le \underbrace{\lim_{(x,y)\to(0,0)} xy}_{=\,0}$$

By squeeze theorem $\Rightarrow$ lim = 0. //

(c) [3 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4+5y^2}$ = DNE

Approach the origin (0,0) along the positive x-axis: $\gamma(t) = \langle t, 0\rangle$, $t \to 0^+$

Then $\displaystyle\lim_{t\to 0^+} f(\gamma(t)) = \lim_{t\to 0^+} \frac{0}{t^4} = 0.$

Approach the origin (0,0) along the parabola $y = x^2$: $\gamma(t) = \langle t, t^2\rangle$, $t \to 0$

Then

$\displaystyle\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0} \frac{(t^2)(t^2)e^{t^2}}{t^4+5(t^2)^2} = \lim_{t\to 0} \frac{t^4}{6t^4} = 1/6$

Since these limits are not the same $\Longrightarrow$ Limit = DNE //

2. [6 pts] Find an equation of the tangent plane to the surface $z = \ln(x - 9y)$ at the point $(10, 1, 0)$.

For a surface expressed as the graph of a surface (i.e. $z = f(x,y)$)
Know that equation of tangent plane given by

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$\rightsquigarrow$ Compute:

$\frac{\partial f}{\partial x} = \frac{1}{x-9y} \rightsquigarrow$ Evaluate at the point $(x,y) = (10,1) \rightsquigarrow \frac{1}{10-9} = 1$

$\frac{\partial f}{\partial y} = \frac{-9}{x-9y} \rightsquigarrow$ // $\longrightarrow \frac{-9}{10-9} = -9$

$\therefore$ Equation of tangent plane is:

$$z - 0 = (x - 10) - 9(y - 1) \;//$$