1. [3 pts] Determine the arc length along the curve

$$\mathbf{r}(t) = \langle \cos(t) + t\sin(t), \sin(t) - t\cos(t), t^2 \rangle$$

from $t = 0$ to $t = \pi/2$.

$$\rightsquigarrow \vec{r}\,'(t) = \langle -\sin(t) + \sin(t) + t\cos(t), \cos(t) - \cos(t) + t\sin(t), 2t \rangle$$

$$L = \int_0^{\pi/2} \sqrt{t^2\cos^2 t + t^2 \sin^2 t + 4t^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\, dt = \int_0^{\pi/2} \sqrt{5}\, t\, dt = \frac{\sqrt{5}\, t^2}{2}\Big|_0^{\pi/2} = \frac{\sqrt{5}}{2}\frac{\pi^2}{4} = \frac{\pi^2\sqrt{5}}{8} \,//$$

2. [3 pts] Interestingly, the notion of arc length can be defined in any dimension. A curve in four-dimensional space $\mathbb{R}^4$ is parametrized as

$$\mathbf{r}(t) = \langle x(t), y(t), z(t), w(t) \rangle, \quad a \le t \le b.$$

Find the arc length of $\mathbf{r}(t) = \langle t, \ln(t), 1/t, \ln(t) \rangle$ and $1 \le t \le 4$.

$$\vec{r}\,'(t) = \langle 1, 1/t, -1/t^2, 1/t \rangle$$

$$L = \int_1^4 \sqrt{1^2 + (1/t)^2 + (1/t^2)^2 + 1/t^2}\, dt$$

$$= \int_1^4 \sqrt{1 + 2/t^2 + 1/t^4}\, dt = \int_1^4 \sqrt{(1/t^2 + 1)^2}\, dt = \int_1^4 (1/t^2 + 1)\, dt$$

$$= \left(-\frac{1}{t} + t\right)\Big|_1^4$$

$$= -1/4 + 4 + 1 - 1$$

3. Suppose that the trajectory of a particle in $\mathbb{R}^3$ is described by the vector-valued function $\mathbf{r}(t)$, and let $\mathbf{v}(t) = \mathbf{r}'(t)$ and $\mathbf{a}(t) = \mathbf{r}''(t)$ be its velocity and acceleration vectors, respectively. For each of the following statements, either give a proof or exhibit a counter-example.

   (a) [2 pts] Let $\mathcal{C}$ be the space-curve in $\mathbb{R}^3$ which is parametrized by $\mathbf{r}(t)$, $-\infty < t < \infty$. If the velocity vector $\mathbf{v}(t)$ is constant, then the curve $\mathcal{C}$ lies entirely in a plane.

   TRUE.

   Pf: Since $\vec{v}(t) = \vec{c}$ a constant

   $\Rightarrow \vec{r}(t) = \vec{r}_0 + t\vec{c}$ a line which certainly is contained in a plane.

   (b) [2 pts] Define the *speed* of the particle at time t to be the length of its velocity vector $s(t) = |\mathbf{v}(t)|$ at time t. If the speed is a constant function, then the curve lies entirely in a plane.

   FALSE.

   Take $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

   Then speed is constant but not restricted to be planar.

   (c) [2 pts] If the acceleration vector $\mathbf{a}(t)$ is constant, then the curve $\mathcal{C}$ lies entirely in a single plane.

   TRUE. Since $\vec{a}$ is contained in the plane determined by $\vec{T}$ and $\vec{N}$, it follows that $\vec{v} \times \vec{a}$ is parallel to $\vec{T} \times \vec{N}$.

   Moreover: $\dfrac{d(\vec{v} \times \vec{a})}{dt} = \vec{a} \times \vec{a} + \vec{v} \times \dfrac{d\vec{a}}{dt} = \vec{0}$ $\therefore$ $\vec{r}(t)$ is planar.

   (d) [2 pts] If the velocity vector $\mathbf{v}(t)$ is *orthogonal* to the acceleration vector $\mathbf{a}(t)$ for all time t, then the curve $\mathcal{C}$ lies entirely in a single plane.

   FALSE.

   Again, consider the helix.

(e) [2 pts] Prove that $\mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0}$ implies $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$, where $\mathbf{c}$ is a constant vector.

By taking the derivative:

$$\frac{d}{dt}(\vec{r}(t) \times \vec{v}) = \vec{v} \times \vec{v} + \vec{r} \times \vec{v}'$$
$$= \vec{0} + \vec{0} = \vec{0} \quad \therefore \ \vec{r}(t) \times \vec{v}(t) \text{ a constant vector.}$$

(f) [2 pts] If $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ for all time t, prove that the motion takes place in a plane (i.e. that the space curve parametrized by $\mathbf{r}(t)$ lies entirely in a plane). Consider both $\mathbf{c} = \mathbf{0}$ and $\mathbf{c} \neq \mathbf{0}$.

Suppose $\vec{c} \neq \vec{0}$. Then $\vec{c} \perp \vec{v}$ for all time t

$\Rightarrow \vec{c} \cdot \vec{v} = 0 \ \forall$ time t

$\Rightarrow \vec{c} \cdot \int_{t_0}^{t_1} \vec{v} = \int_{t_0}^{t_1} 0 \Rightarrow \vec{c} \cdot (\vec{r}(t_0) - \vec{r}(t_1)) = 0$

$\Rightarrow$ Motion $\vec{r}(t)$ is planar.

if $\vec{c} = 0$ Then $\vec{r} \parallel \vec{v}$ so $\vec{r}$ parametrizes a line.

4. [4 pts] Find a vector-valued function, $\mathbf{r}(t)$, that represents the curve of intersection between the cylinder $\{(x, y, z) \mid x^2 + y^2 = 9\}$ and the surface $\{(x, y, z) \mid z = xy\}$.

Can parametrize the circle $\{(x,y) \mid x^2+y^2=9\}$ in the xy-plane

as $\langle \underbrace{3\cos(t)}_{=x(t)}, \underbrace{3\sin(t)}_{=y(t)} \rangle$

$\rightsquigarrow$ Plugging this parametrization into Quadric equation gives parametrization for curve of intersection

$$\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 9\cos(t)\cdot\sin(t) \rangle, \quad 0 \leq t \leq 2\pi.$$