§14.2: LIMITS AND CONTINUITY

- Let f be a function on a domain $D \subseteq \mathbb{R}^2$ (i.e. $f: D \to \mathbb{R}$)

Defn: " $\lim_{(x,y)\to(a,b)} f(x,y) = L$ " MEANS:

For every $\varepsilon > 0$, there exists a $\delta > 0$, such that if (x,y) in D and $(x,y) \neq (a,b)$ and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

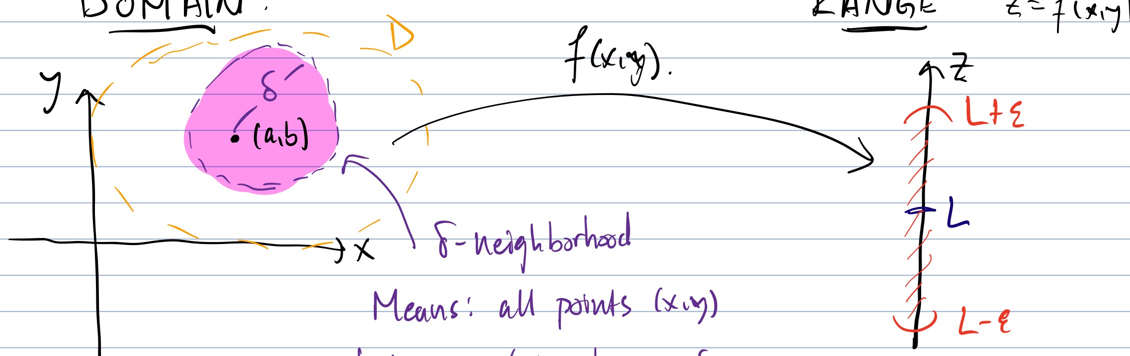
Intuition: Think of ε as being an "error tolerance"

(i.e. suppose i want to guarantee that $f(x,y)$ is w/ in distance ε of L)

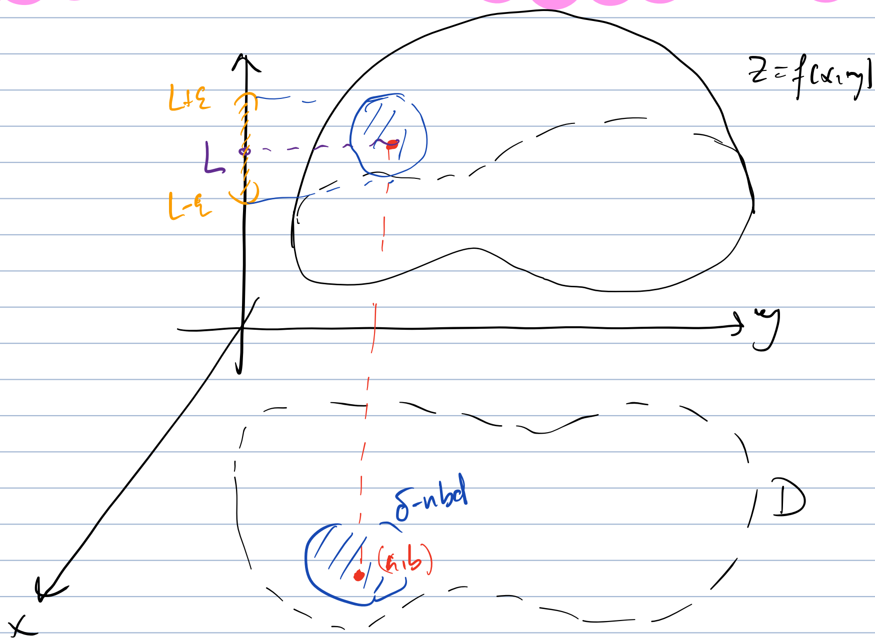
The definition says i can guarantee this by requiring (x,y) be sufficiently close to (a,b)

$\rightsquigarrow$ δ: " how close (x,y) needs to be to (a,b) to guarantee that $f(x,y)$ is w/ in error tolerance of L "

DOMAIN: RANGE $z = f(x,y)$



Means: all points (x,y) which are w/ in distance δ of (a,b).



Ex: Prove that $\lim\limits_{(x,y)\to(4,-1)} 2\cdot x = 4$

Pf: Let $f(x,y) = x$ and $L = 4$. Need to show that for each $\varepsilon > 0$, there exists a δ-nbd of $(4,-1)$ such that

$$|f(x,y) - L| < \varepsilon \iff |x-4| < \varepsilon \quad \text{whenever } (x,y) \neq (4,-1)$$
lies in δ-nbd of $(4,-1)$.

- Can first observe from

$$0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta \quad \text{it follows that}$$

$$|f(x,y) - 4| = |x-4| = \sqrt{(x-4)^2} \leq \sqrt{(x-4)^2 + (y+1)^2} < \delta.$$

So can choose $\delta = \varepsilon$, and the limit is verified. $\blacksquare$

$\delta = \varepsilon/2$.

end of proof.

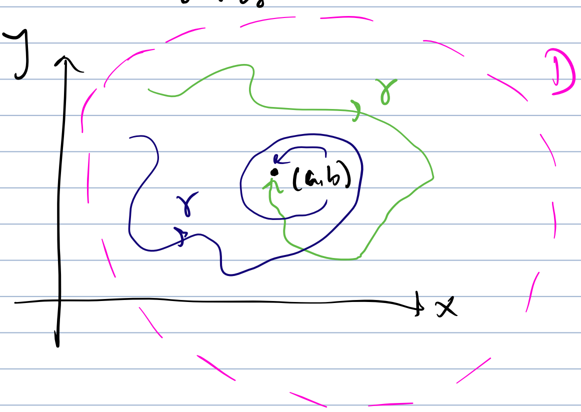
PROVING LIMITS DON'T EXIST

IMPORTANT PROPERTY:

- For the limit $\lim_{(x,y)\to(a,b)} f(x,y) \overset{= L}{}$ to exist means, in particular, that

  if i choose ANY path $\underset{\overset{\parallel}{(x(t),y(t))}}{\gamma(t)}$ in the domain D which approaches (a,b) then

$$\lim_{t\to t_0} f(x(t),y(t)) = L.$$



- Often helpful: find two paths for which limits DON'T agree.

  $\Longrightarrow$ Conclude $\lim_{(x,y)\to(a,b)} f(x,y) = \text{DNE}.$

Ex: Prove that $\lim_{(x,y)\to(0,0)} f(x,y) = \text{DNE}$ where

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}.$$

IDEA: Approach origin $(0,0)$ in two ways.

① Approach $(0,0)$ along x-axis $\rightsquigarrow$ fix $y=0$ and vary x.

Path: $\gamma(t) = \langle t, 0 \rangle$

$$\lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{t^2}{t^2} = \lim_{t\to 0} 1 = 1.$$

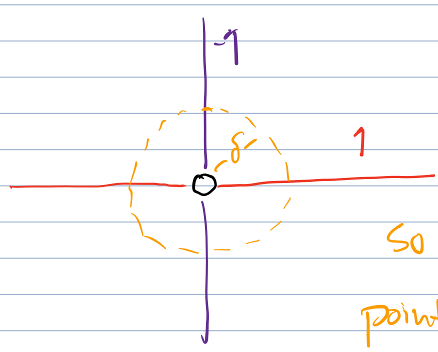
② Approach $(0,0)$ along y-axis $\rightsquigarrow$ fix $x=0$ and vary y.

Path: $\gamma(t) = \langle 0, t \rangle$

$$\lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{-t^2}{t^2} = \lim_{t\to 0} -1 = -1$$

Because these limits don't agree,

$$\Longrightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \text{DNE}.$$



So every δ-nbd of $(0,0)$ contains points for which $f=1$ and $f=-1$

$\Rightarrow$ Limit DNE.

Ex: Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$ exist?

A: No. Check: Approach origin along BOTH x and y axes

$\rightsquigarrow$ get 0.

BUT: if approach origin along line $y=x$, see that

$$r(t) = \langle t, t \rangle$$

Then

$$\lim_{t \to 0} f(t,t) = \lim_{t \to 0} \frac{t^2}{t^4 + t^4} = \lim_{t \to 0} \frac{1}{2t^2} \longrightarrow \infty.$$

$\Rightarrow$ Limit = DNE.