- Yesterday, we introduced the geometry of $\mathbb{R}^3$ and vectors

Sections 12.1 and 12.2 in textbook.

Today: Discuss Dot product and cross product. //

Section 12.3: DOT PRODUCT

- A natural question to ask is:

Q: How can we multiply two vectors?

- There are two operations which "multiply" vectors

① Dot product (defined for two vectors in any dimension)

② Cross product (only defined for two vectors in $\mathbb{R}^3$)

Dot Product:

[Two vectors of the same dimension $\vec{v} = \langle v_1, \dots, v_n \rangle$, $\vec{w} = \langle w_1, \dots, w_n \rangle$] $\longrightarrow$ [A scalar i.e. A real number]

(Input) (output)

The dot product of $\vec{v} = \langle v_1, \dots, v_n \rangle$ and $\vec{w} = \langle w_1, \dots, w_n \rangle$ is defined to be

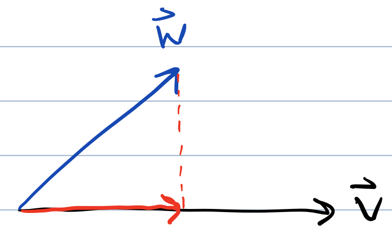
$$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Example: if $\vec{v} = \langle -3, 1, 0 \rangle$ and $\vec{w} = \langle 4, -1, 2 \rangle$

Then $\vec{v} \cdot \vec{w} = (-3)(4) + (1)(-1) + (0)(2) = -12 - 1 + 0$
$= -13$ //

Geometric Significance of Dot Product

Given two vectors $\vec{v}$ and $\vec{w}$, can "project $\vec{w}$ onto $\vec{v}$"



The red arrow is the "projection of $\vec{w}$ onto $\vec{v}$", and is denoted $\text{proj}_{\vec{v}}(\vec{w})$

NOTICE: The vector $\text{proj}_{\vec{v}}(\vec{w})$ is a scalar multiple of $\vec{v}$, i.e.

$$\text{proj}_{\vec{v}}(\vec{w}) = c\,\vec{v}$$

A scalar.

FACT: if $\vec{v}$ is a unit vector (so that $|\vec{v}| = 1$), then the scalar c is

$$c = \vec{v} \cdot \vec{w}$$

This fact gives the geometric significance of dot product

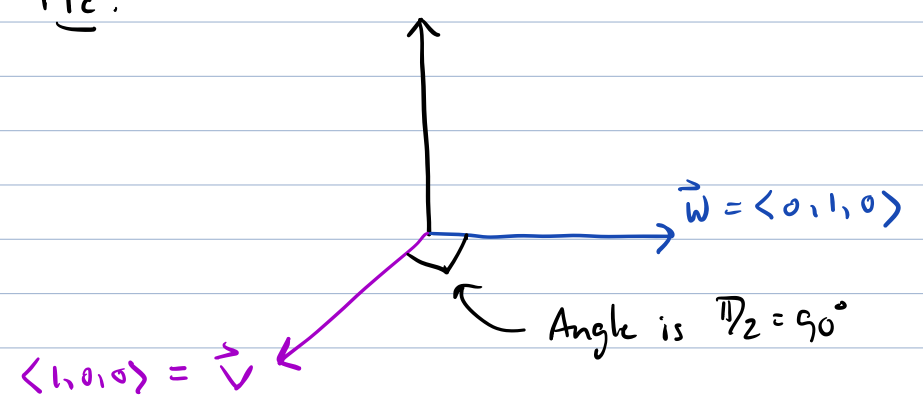
i.e.

The dot product $\vec{v}\cdot\vec{w}$ is a measure of how much of $\vec{w}$ projects onto $\vec{v}$

For instance, consider $\vec{v} = \langle 1,0,0\rangle$ and $\vec{w} = \langle 0,1,0\rangle$

Then $\vec{v}\cdot\vec{w} = 0$ means that projection of $\vec{w}$ onto $\vec{v}$ is the zero vector.

Pic:



Because of this observation, make a definition

Def: Define two vectors $\vec{v}$ and $\vec{w}$ to be **ORTHOGONAL** if $\vec{v}\cdot\vec{w} = 0$.

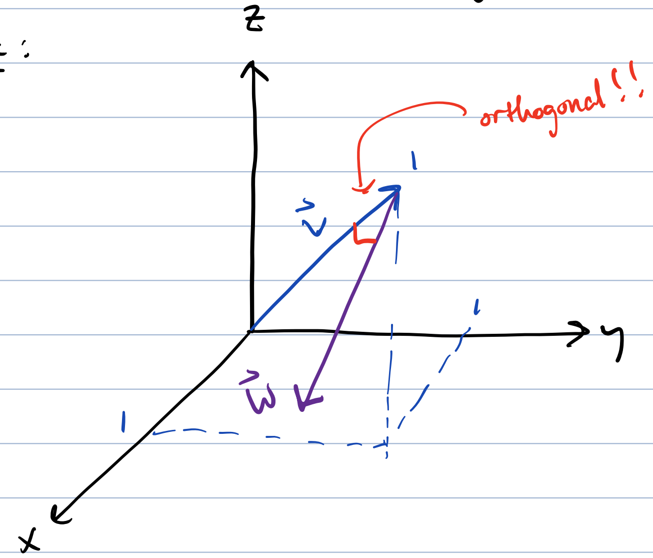
i.e. perpendicular, normal, etc...

Example: Show that the vectors $\vec{v} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \langle 1, 0, -1 \rangle$ are orthogonal.

Sol:

Simply compute
$$\vec{v} \cdot \vec{w} = \langle 1, 1, 1 \rangle \cdot \langle 1, 0, -1 \rangle$$
$$= (1)(1) + (1)(0) + (1)(-1)$$
$$= 0$$

Pic:



Final Comments and Remarks:

① The dot product can also be computed as

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos \theta$$, where θ is angle btw $\vec{v}$ and $\vec{w}$

Certainly, $\vec{v} \perp \vec{w} \iff \theta = \pi/2 \iff \vec{v} \cdot \vec{w} = 0$.

② Your textbook distinguishes between the

- SCALAR projection of $\vec{w}$ onto $\vec{v}$, and
- VECTOR projection of $\vec{w}$ onto $\vec{v}$

This is the <u>vector</u> $\text{proj}_{\vec{v}}(\vec{w})$ and can be computed as:

$$\text{proj}_{\vec{v}}(\vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \cdot \vec{v}$$

Notice: if $\vec{v}$ is a unit vector, so $|\vec{v}| = 1$, then

$$\text{proj}_{\vec{v}}(\vec{w}) = |\vec{v} \cdot \vec{w}|\, \vec{v}$$

This is the <u>Length</u> of the vector $\text{proj}_{\vec{v}}(\vec{w})$ and can be computed as

$$|\text{proj}_{\vec{v}}(\vec{w})| = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|}$$