DETERMINANTS IN 2D AND 3D

- Need to review determinants before discussing cross product of two vectors in $\mathbb{R}^3$.
- The **DETERMINANT** is a function

$$\text{Det}: \begin{bmatrix}\text{Square} \\ \text{Matrices}\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Real} \\ \text{Numbers}\end{bmatrix}$$

Really, only interested in 2×2 and 3×3 matrices.

2D:

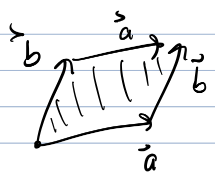
Consider the matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ each entry is a real #.

The determinant is defined to be: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

Means: take det. of what's inside

Geometric Significance: Given $\vec{a} = \langle a_1, a_2 \rangle$
$\vec{b} = \langle b_1, b_2 \rangle$

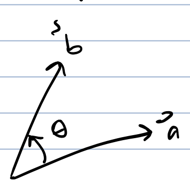
Then $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm \text{Area}(\text{Parallelogram})$



How do we determine the sign?

Ans: "+" if "$\vec{a}$ points to the RIGHT of $\vec{b}$"
"−" if "$\vec{a}$ —— LEFT ——"

Means:

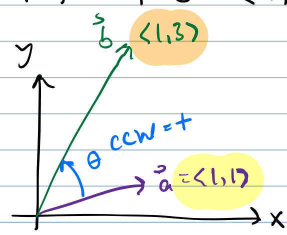


Let θ be the angle $0 \le \theta \le \pi$ FROM $\vec{a}$ TO $\vec{b}$

if θ goes counter-clockwise $\rightsquigarrow$ "+"

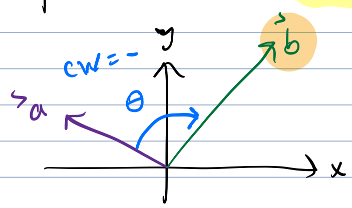
if θ goes clockwise $\longrightarrow$ "−"

Example: $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle 1, 3 \rangle$



$$\text{Det}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (1)(3) - (1)(1) = 3 - 1 = 2 > 0 \checkmark$$

What happens if i choose $\vec{a} = \langle -1, 1 \rangle$ instead?



$$\text{Det}\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = (-1)(3) - (1)(1) = -4 < 0 \checkmark$$

- This is an illustration of the "Right-Hand-Rule" which is a sign convention we'll use throughout.

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3D: Consider the 3x3 matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Its determinant is computed as:

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Don't forget!!

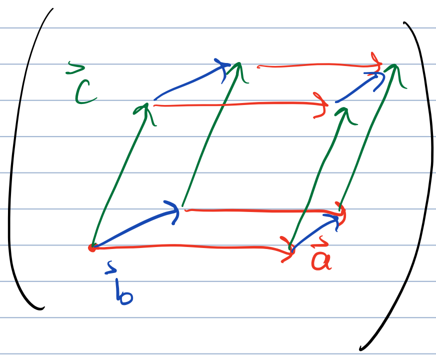
Geometric Interpretation: Given 3 vectors in $\mathbb{R}^3$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

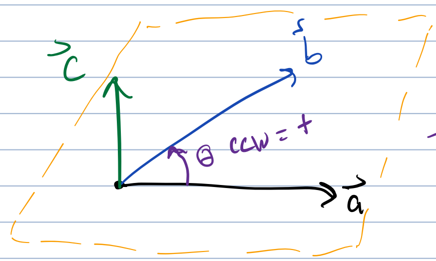
Then the determinant of $\begin{pmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{pmatrix} = \pm \text{Volume}(\text{Parallelopiped})$

$\pm$ Vol  = Det $\begin{vmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{vmatrix}$

How do i determine if the sign is +/− ?

"+" if $\vec{a}, \vec{b}, \vec{c}$ satisfy the RHR

"−" if —— NOT ——



⇝ "+"

- Use Right hand, point your fingers (index, ...) in the direction of $\vec{a}$

  Now curl your fingers in the direction of $\vec{b}$

  ⇝ $\vec{c}$ points in the direction of your thumb.

To see what "−" looks like (i.e. $\vec{a}, \vec{b}, \vec{c}$ do NOT satisfy RHR)