MULTIVARIABLE CHAIN RULE (§14.5)

- In one-variable calculus, if $y = f(x)$, $x = g(t)$, then using function composition, $y = f(g(t))$ ⤳ y is indirectly a fn of t.

  And: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ "Chain Rule"

- Now, sps $z = f(x,y)$ and $x = x(t)$, $y = y(t)$.

  Again by composition, $z = f(x(t), y(t))$ can be viewed indirectly as a fn of t.

  Moreover: $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ ⟵ First Version of chain Rule.

Ex: Compute $\frac{dw}{dt}$ at $t=0$ where $w = x^2 y - y^2$ and $x(t) = \sin(t)$, $y(t) = e^t$

Sol: Using the chain rule: $\frac{\partial w}{\partial x} = 2 \underset{\sin(t)}{x} \underset{e^t}{y}$ $\qquad \frac{\partial w}{\partial y} = \underset{\sin(t)}{x^2} - 2\underset{e^t}{y}$

and $\frac{dx}{dt} = \cos(t)$ $\qquad \frac{dy}{dt} = e^t$

So, at $t=0$: $\left.\frac{dx}{dt}\right|_{t=0} = \cos(0) = 1$ $\qquad \left.\frac{dy}{dt}\right|_{t=0} = e^0 = 1.$

- Finally: $\left.\frac{dw}{dt}\right|_{t=0} = \underbrace{2\sin(0)e^{(0)}}_{\left.\frac{\partial w}{\partial x}\right|_{t=0} = 0} \cdot (1) + \underbrace{(\sin^2(0) - 2e^{(0)})(1)}_{= -2.}$

$= -2.$ //

CHECK: What if i simply substitute $x(t) = \sin(t)$, $y(t) = e^t$ into eqn for w first and then take d/dt: i.e. $w = \underbrace{\sin^2(t)}_{x}\underbrace{e^t}_{y} - \underbrace{e^{2t}}_{y}$

Show you get the same answer!

⤳ Now take $\left.\frac{dw}{dt}\right|_{t=0}$

BUT!! Something New happens in the multi-variable case....

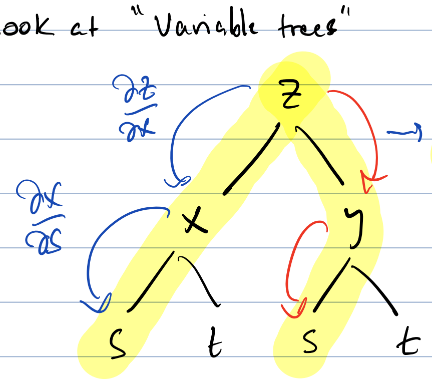
- Sps have $z = f(x,y)$ AND $x = x(s,t)$, $y = y(s,t)$

Now, both fns of two variables..

- Again, indirectly, $z = f(x(s,t), y(s,t))$ is a fn of (s,t).

Q: How do we compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$??

Look at "Variable trees"



$$\longrightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Ex: Consider $w = xy + yz + xz$ $\quad\Big\|\quad$ $x = s\cos(t)$ — $x(s,t)$

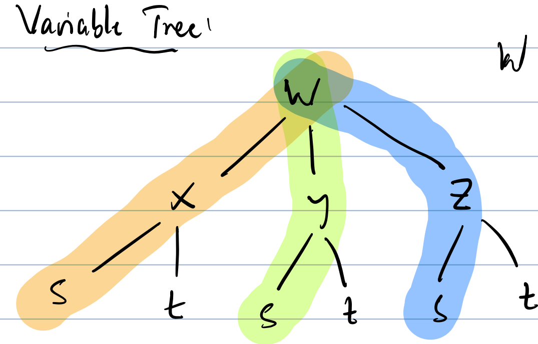
$w(x,y,z)$ $\qquad$ $y = s\cdot\sin(t)$ — $y(s,t)$

$z = t$ — $z(s,t)$.

Compute $\left.\frac{\partial w}{\partial s}\right|_{(s=1,\ t=2\pi)}$.

Variable Tree!

!!!



$w = xy + yz + xz$

So, if want to compute $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \cancelto{0}{\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}}$

So:

$$\frac{\partial w}{\partial x} = y+z \qquad \frac{\partial w}{\partial y} = x+z \qquad \frac{\partial w}{\partial z} = y+x$$

$$\begin{cases} x(s,t) = s\cdot\cos(t) \\ y(s,t) = s\cdot\sin(t) \\ z(s,t) = t \end{cases} \rightarrow \quad \frac{\partial x}{\partial s} = \cos(t) \qquad \frac{\partial y}{\partial s} = \sin(t) \qquad \boxed{\frac{\partial z}{\partial s} = 0}$$

Finally:

$$\left.\frac{\partial w}{\partial s}\right|_{(s=1,\, t=2\pi)} = \big((1)\sin(2\pi) + 2\pi\big)\cos(2\pi) + \underbrace{\big((1)\cos(2\pi) + 2\pi\big)\overset{0}{\sin(2\pi)}}_{=0}$$

$$= 2\pi$$