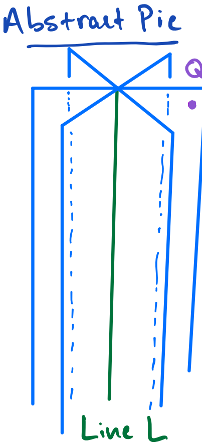
University of Virginia **MATH 2310**
Department of Mathematics **Homework 2** Spring 2023

1. [4 pts] Given the line L through $\overbrace{(1,2,3)}^{P}$ parallel to the vector $\langle 1,1,1\rangle$, and given a point $\overbrace{(2,3,5)}^{Q}$ which is not on L. Find a Cartesian equation for the plane M through $(2,3,5)$ which contains every point on L.



Many planes which contain the same line L

To specify one of them, need to add requirement of an additional point.

The vector $\vec{PQ} =$ "$Q-P$" $= \begin{bmatrix}2\\3\\5\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$

Normal direction for the plane computed as:

$\vec{N} = \vec{PQ} \times \vec{v} = \langle 1,1,2\rangle \times \langle 1,1,1\rangle = \text{Det} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 1-2 \\ -(1-2) \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \vec{N}$

Direction Vector given in Question

So, equation of plane given by

$-(x-1) + (y-2) = 0$

$\rightsquigarrow 1 - x + y - 2 = 0$

$\boxed{\rightsquigarrow y - x - 1 = 0}$ Answer!

CHECK: Does L lie entirely in the plane?

The line L is parametrized by $\vec{r}(t) = \begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix}$, $-\infty < t < \infty$

$\rightsquigarrow$ Plug in the components $\left.\begin{matrix} x(t) = 1+t \\ y(t) = 2+t \\ z(t) = 3+t \end{matrix}\right\}$ into plane equation

$\rightsquigarrow y(t) - x(t) - 3 \overset{?}{=} 0$

$\rightsquigarrow (2+t) - (1+t) - 1 = 0$ ✓

2. [3 pts] If the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ satisfy $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u}\times\mathbf{w} \neq \mathbf{0}$, but $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{w}) = \mathbf{0}$, then it follows that

   (a) The plane spanned by $\mathbf{u}$ and $\mathbf{v}$ is orthogonal to the one spanned by $\mathbf{u}$ and $\mathbf{w}$.

   (b) $\mathbf{v} \perp \mathbf{w}$.

   (c) $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$.

   (d) $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane.

The Question asks: Which of these answer choices is NECESSARILY True.

For counter-examples to (b), (c), and (d) consider the vectors

$\vec{u} = \hat{i}$ $\rightsquigarrow \vec{u} \times \vec{v} = \hat{i} \times (\hat{i} + \hat{j}) = \hat{k}$

$\vec{v} = \hat{i} + \hat{j}$

$\vec{w} = \hat{i} + \hat{k}$ $\rightsquigarrow \vec{u} \times \vec{w} = \hat{j}$

so $(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{w}) = 0$

BUT! $\left.\begin{matrix} \vec{v}\cdot\vec{w} = 1 \neq 0 \\ \vec{u}\cdot\vec{w} = 1 \neq 0 \end{matrix}\right]$ Not $\perp$ and $\vec{u}, \vec{v}, \vec{w}$ do not lie in a plane!

3. [3 pts] Determine whether the parametrizations

$$\mathbf{r}_1(t) = \langle 3,-1,4\rangle + t\langle 8,12,-6\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 11,11,-2\rangle + t\langle 4,6,-3\rangle$$

describe the same line. If they do, show why. If they don't, show why not.

Solve the system of equations:

$\vec{r}_1(t) \overset{?}{=} \vec{r}_2(s)$

$\rightsquigarrow \begin{bmatrix} 3+8t \\ -1+12t \\ 4-6t \end{bmatrix} \overset{?}{=} \begin{bmatrix} 11+4s \\ 11+6s \\ -2-3s \end{bmatrix}$

"x"-equation

$\rightsquigarrow 3 + 8t = 11 + 4s \rightsquigarrow 8t - 8 = 4s \rightsquigarrow \boxed{s = 2t - 2}$

Plug this in to other eqns and verify

"y"-equation:

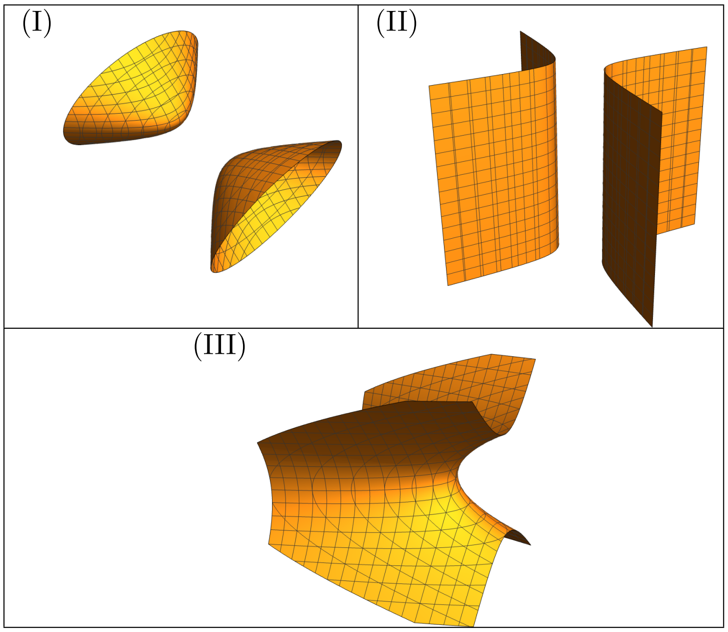
$11 + 6(2t-2) = 11 + 12t - 12 = 12t - 1$ ✓ Matches

"z"-equation:

$-2 - 3(2t-2) = -2 - 6t + 6 = 4 - 6t$ ✓ Matches

Conclusion: Yes, $\vec{r}_1$ and $\vec{r}_2$ parametrize the same line.

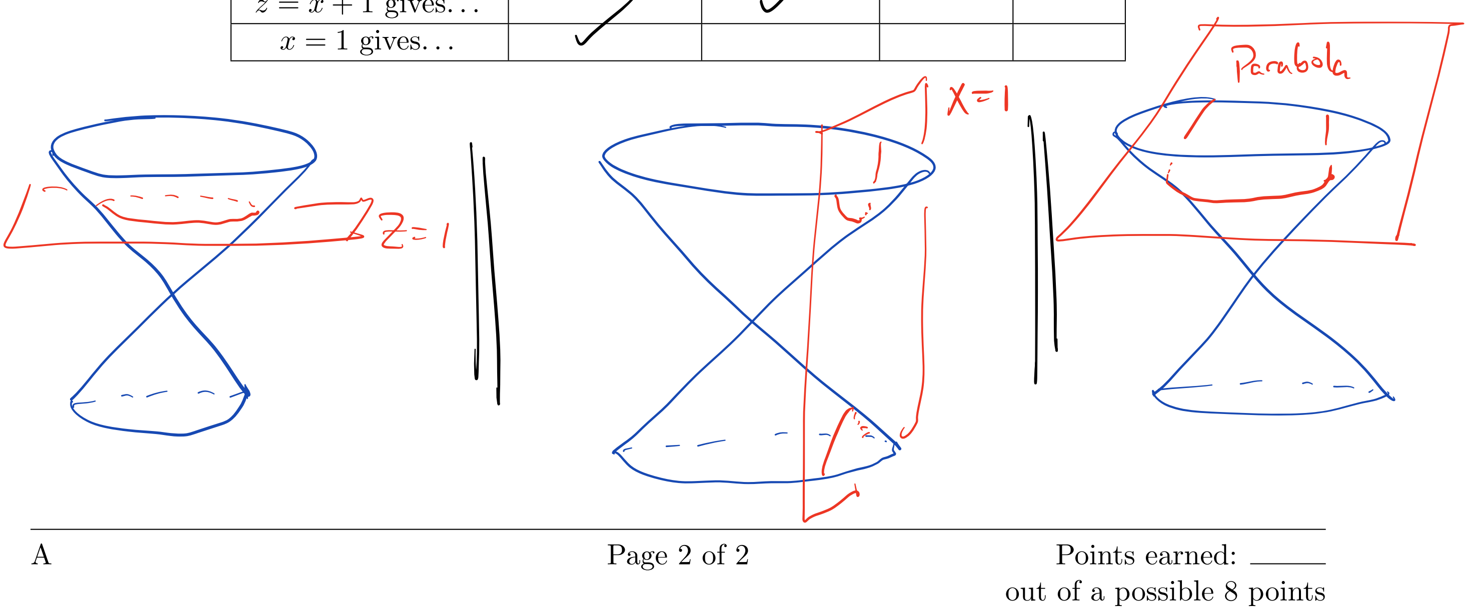
4. [4 pts] Match the contour surfaces with their equations. Enter O if there's no match.



$g(x,y,z) =$	O, (I), (II), (III)
$x^2 - y^2 + z^2 = 1$	O
$x^2 - y^2 = 1$	II
$x^4 + z = 1$	O
$x^2 + y - z^2 = 1$	III

5. [4 pts] The intersection of a plane with the cone $S = \{(x,y,z) : x^2 + y^2 - z^2 = 0\}$ is called a **conic section**. What curve do we get? In each row check only one box.

Intersect S with...	hyperbola(s)	parabola(s)	circle(s)	line(s)
$z = 1$ gives...			✓	
$z = x$ gives...				✓
$z = x + 1$ gives...		✓		
$x = 1$ gives...	✓			



Points earned: ______
out of a possible 8 points