1. [3 pts] If $\mathbf{a} = \langle 2, -1, 2\rangle$ and $\mathbf{b} = \langle 1, -1, 2\rangle$, find a non-zero vector $\mathbf{c}$ such that $\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = 0$.

Two approaches:

① Compute $\vec{c} = \vec{a}\times\vec{b}$. By geometric property of cross product, know that both $\vec{a} \perp \vec{a}\times\vec{b}$ and $\vec{b} \perp \vec{a}\times\vec{b}$.

② More Rudimentary solution: Let $\vec{c} = \langle c_1, c_2, c_3\rangle$ and solve the system of equations

$$\begin{cases} \vec{a}\cdot\vec{c} = 0 \\ \vec{b}\cdot\vec{c} = 0 \end{cases} \rightsquigarrow \begin{cases} 2c_1 - c_2 + 2c_3 = 0 \\ c_1 - c_2 + 2c_3 = 0 \end{cases}$$

$\rightsquigarrow$ A system of two equations and 3 unknowns

one possibility is $\vec{c} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

2. [3 pts] Determine the projection vector $\mathrm{proj}_{\mathbf{a}}(\mathbf{b})$ of $\mathbf{b}$ onto $\mathbf{a}$ where $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 1, 2, 2\rangle$.

Projection vector is computed as:

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

$\rightsquigarrow$ First: $\vec{a}\cdot\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1+4+6 = 11$

$\rightsquigarrow$ Second: $|\vec{a}|^2 = \vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2$
$= 1+4+9$
$= 14$

Conclusion:

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{11}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3. [3 pts] If the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\|\mathrm{proj}_{\mathbf{a}}(\mathbf{b})\| = 1$, determine the value of $\|\mathrm{proj}_{2\mathbf{a}}(3\mathbf{b})\|$.

Know:

$$|\mathrm{proj}_{\vec{a}}(\vec{b})| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = 1$$

Using the same formula:

$$|\mathrm{proj}_{2\vec{a}}(3\vec{b})| = \frac{(2\vec{a})\cdot(3\vec{b})}{|2\vec{a}|} = \frac{6}{2}\overbrace{\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)}^{\overset{1}{\|}}$$

$= 3$ //

4. True/False. If the statement is true, give an explanation why you think so. If a statement is false, provide a counter-example.

   (a) [3 pts] The cross product of two unit vectors is a unit vector.

FALSE

Suppose $\vec{a}, \vec{b}$ are unit vectors so that $|\vec{a}| = 1 = |\vec{b}|$.

Then the magnitude of cross product computed as:

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$
$$= 1 \cdot 1 \sin\theta$$
$$= \sin\theta$$

But this is not always $= 1$!!

Counter-example:

$\vec{a} = \vec{i}$

$\vec{b} = -\vec{i}$

Then $\vec{a} \times \vec{b} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

   (b) [3 pts] If $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

TRUE.

Again, we know that

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$
$$= |c\vec{v}|\,|\vec{v}| \sin\theta$$
$$= |c|\,|\vec{v}|^2 \sin\theta$$

But since $\vec{u} = c\vec{v}$, know that $\theta = 0$ or $\theta = \pi$

In either case,

know that $\sin(0) = 0$ and $\sin(\pi) = 0$

$\Longrightarrow |\vec{u} \times \vec{v}| = 0 \Rightarrow \vec{u} \times \vec{v} = \vec{0}$.

   (c) [3 pts] If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all non-zero vectors in space and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

FALSE!

Consider the counter example of $\vec{u} = \vec{i}$, $\vec{v} = \vec{j}$ and $\vec{w} = \vec{k}$

Then $\vec{i} \cdot \vec{j} = 0 = \vec{i} \cdot \vec{k}$

But $\vec{j} \neq \vec{k}$!

   (d) [3 pts] The vector equation

$$\langle x, y, z \rangle \times \langle 1, 1, 1 \rangle = \langle 0, 1, 0 \rangle$$

   has a solution in $\mathbb{R}^3$.

FALSE

Want to find a solution to the system of equations

$$\text{Det} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y - z \\ -(x - z) \\ x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} y = z \\ -(x-z) = 1 \Rightarrow y - x = 1 \\ x = y \end{array}$$

cannot happen!

No Solution

5. Consider the points $P(3,1,1), Q(4,1,2)$, and $R(4,4,1)$ in $\mathbb{R}^3$.

(a) [3 pts] Find an equation for the plane containing the points P, Q, and R.

Compute $\vec{PQ} = \text{"}Q-P\text{"} = \begin{bmatrix}4\\1\\2\end{bmatrix} - \begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

and

$\vec{QR} = \begin{bmatrix}4\\4\\1\end{bmatrix} - \begin{bmatrix}4\\1\\2\end{bmatrix} = \begin{bmatrix}0\\3\\-1\end{bmatrix}$

Compute $\vec{PQ} \times \vec{QR} = \text{Det}\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 1\\ 0 & 3 & -1\end{vmatrix}$

$= \vec{i}\begin{vmatrix}0 & 1\\3 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}1 & 0\\0 & 3\end{vmatrix}$

$= \begin{bmatrix}-3\\1\\3\end{bmatrix}$ $\therefore$ $-3(x-3)+(y-1)+3(z-1)=0$ is equation of plane

(b) [2 pts] Find the area of the triangle with vertices P, Q, and R.

Area of $\Delta = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\|$

$= \frac{1}{2}\sqrt{(-3)^2+1^2+3^2} = \frac{1}{2}\sqrt{9+1+9}$

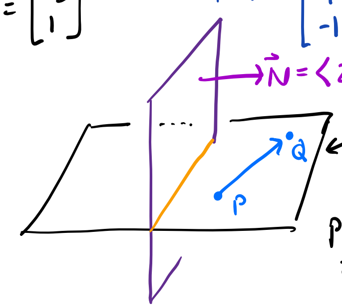
$= \sqrt{19}/2$

6. [4 pts] Find an equation of the plane which passes through the points $(2,2,1)$ and $(-1,1,-1)$ and is perpendicular to the plane $2x-3y+z=3$.

Since passes thru points $P=(2,2,1)$ and $Q=(-1,1,-1)$

$\vec{N} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$ $\rightsquigarrow$ $\vec{PQ} = \begin{bmatrix}-1\\1\\-1\end{bmatrix} - \begin{bmatrix}2\\2\\1\end{bmatrix} = \begin{bmatrix}-3\\-1\\-2\end{bmatrix}$ lies in the plane

$\rightarrow \vec{N} = \langle 2,-3,1\rangle$ Normal to given plane



Desired plane

See that Normal to desired plane given by

$\vec{PQ}\times\vec{N} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & -1 & -2\\ 2 & -3 & 1\end{vmatrix} = \begin{bmatrix}-1-6\\-(-3+4)\\9+2\end{bmatrix} = \begin{bmatrix}-7\\-1\\11\end{bmatrix}$

So equation of desired plane given by

$-7(x-2)-(y-2)+11(z-1)=0$

7. [4 pts] Find a set of parametric equations for the line of intersection of the planes:

$$\begin{matrix} R \\ S \end{matrix}\begin{cases} 6x - 3y + z = 5 \\ -x + y + 5z = 5 \end{cases}$$

Normal direction to plane $\vec{R} = \langle 6, -3, 1 \rangle$
$\vec{S} = \langle -1, 1, 5 \rangle$

So line of intersection is parallel to $\vec{R} \times \vec{S}$

$$\rightsquigarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \begin{bmatrix} -15-1 \\ -(30+1) \\ 6-3 \end{bmatrix} = \begin{bmatrix} -16 \\ -31 \\ 3 \end{bmatrix}$$

One possibility is:

$$\vec{r}(t) = \begin{bmatrix} 20/3 \\ 35/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -16 \\ -31 \\ 3 \end{bmatrix} //$$

To find a point on line of intersection, solve system of equations .... one solution is $\vec{r}_0 = \begin{bmatrix} 20/3 \\ 35/3 \\ 0 \end{bmatrix}$

8. [4 pts] Find an equation of the plane containing both the point $P(1, -1, 5)$ and the line L parametrized by:

$$\mathbf{r}(t) = \begin{cases} x(t) = 1 + 2t \\ y(t) = -1 + 3t \\ z(t) = 4 + t \end{cases}$$

Since P contains the line parametrized by $\vec{r}(t) \to$ P contains the direction vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Also, plane P contains the vector $\vec{w} = \vec{r}(0) - \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\rightsquigarrow$ $\vec{N}$ for plane P given by $\vec{N} = \vec{v} \times \vec{w} = \text{Det} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$

$\rightsquigarrow$ Equation of plane is:

$$-3(x-1) + 2(y+1) = 0 //$$

9. Suppose that a plane P passes through the points $(4,2,1)$ and $(-3,5,7)$, and suppose that P is parallel to the z-axis. Find an equation for the plane P.

Since plane P is parallel to z-axis, it contains the vector $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Also, P contains the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -6 \end{bmatrix}$

So Normal direction to plane given by

$$\vec{N} = \vec{v} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} -3 \\ -7 \\ 0 \end{bmatrix}$$

So equation of plane given by

$$3(x-4) + 7(y-2) = 0 //$$