• In our last lecture, defined the notion of a critical point for 2-variable fn, and introduced 2^{nd} Derivative Test.

D:

A point (a,b) is called a CRITICAL POINT of $f(x,y)$ if either:

(★) ① $f_x(a,b) = f_y(a,b) = 0$. ⇝ "Horizontal Tangent", or

② one of $f_x(a,b)$ or $f_y(a,b)$ DNE.

THM: (2^{nd} DERIVATIVES TEST)

Suppose the 2^{nd} partial derivatives are continuous on a disk w/ center (a,b) and sps $f_x(a,b) = f_y(a,b) = 0$. Let

$$D = D(a,b) := \operatorname{Det}\begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

$$= f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

THEN:

- if $D > 0$, and $f_{xx}(a,b) > 0 \Rightarrow$ local min. ····· both eigenvalues +
- if $D > 0$, and $f_{xx}(a,b) < 0 \Rightarrow$ local max. ····· both eigenvalues −
- if $D < 0 \Longrightarrow$ SADDLE. ····· +/−
- $D = 0 \Longrightarrow$ Test is inconclusive!! ····· 0 an eigenvalue.

Ex: Find and classify the critical points of

$$f(x,y) = x^5 + y^4 - 5x - 32y - 3$$

(Domain: All of $\mathbb{R}^2$
$-\infty < x < \infty$
$-\infty < y < \infty$.)

Is there a global max? Global min?

Sol:

$f_x = 5x^4 - 5 \rightsquigarrow f_x = 0 \Rightarrow x = \pm 1.$

$f_y = 4y^3 - 32 \rightsquigarrow f_y = 0 \Rightarrow y = 2$

Critical points are: $(-1,2)$ and $(1,2)$.

$f_{xx} = 20x^3$

$f_{yy} = 12y^2$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(5x^4 - 5\right) = 0.$

CHECK:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(4y^3 - 32\right) = 0.$$

Analyze $(-1,2)$:

$$D = \operatorname{Det}\begin{vmatrix} -20 & 0 \\ 0 & 48 \end{vmatrix} < 0. \Longrightarrow \text{A SADDLE.}$$

Since $D<0$ and $f_{xx}(-1,2) = -20 < 0$

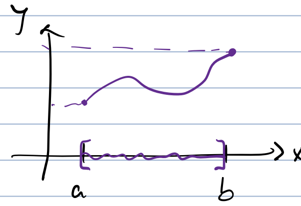
Analyze $(1,2)$:

$$D = \operatorname{Det}\begin{vmatrix} 20 & 0 \\ 0 & 48 \end{vmatrix} > 0 \Longrightarrow \text{LOCAL MIN.}$$

Since $D>0$ and $f_{xx}(1,2) > 0$

• B/c have **No local max**, and b/c $f(x,y)$ is defined on all of $\mathbb{R}^2$

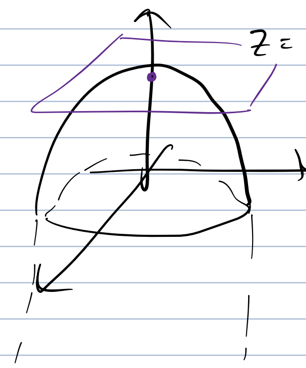
$\Longrightarrow$ No GLOBAL MAX.



$f : [a,b] \to \mathbb{R}$

closed (Meaning: includes its bdry)

$f : D \longrightarrow \mathbb{R}.$



$z = -(x^2+y^2)$

Q: Is there a global min?

Consider $f(x,0) = x^5 - 5x - 3$ ⇝ Notice this fn can reach arbitrarily Negative Values.

$\Longrightarrow$ No global Min.

* ———————————————— *

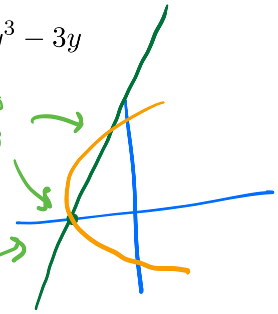
6. Find and classify all critical points of the function

$$f(x, y) = \frac{5}{2}x^2 - xy + 15x + \frac{1}{75}y^3 - 3y$$

Solve system of equations:

$$\nabla f = \vec{0} \Rightarrow \begin{cases} 5x - y + 15 = 0 \\ -x + y^2/25 - 3 = 0 \end{cases} \rightsquigarrow$$

Two solns



$\rightsquigarrow$ $y = 0$ and $x = -3 \rightsquigarrow (-3, 0)$

And

$y = 5$ and $x = -2 \rightsquigarrow (-2, 5)$

Use 2nd Derivative test to classify crit. pts.

CRITICAL POINT	SIGN OF D	SIGN OF f_{xx}	TYPE OF CRIT POINT
$(-3, 0)$	$\begin{vmatrix} 5 & -1 \\ -1 & 0 \end{vmatrix} \rightarrow = -1 < 0$		SADDLE
$(-2, 5)$	$\begin{vmatrix} 5 & -1 \\ -1 & 2/5 \end{vmatrix} = 1 > 0$	$f_{xx} = 5 > 0$	LOCAL MINIMUM.