BASICS OF CYCLIC HOMOLOGY I

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1. Introduction

The theory of trisections suggests that one can study smooth four-manifold topology via loops of Morse functions on a closed, oriented surface [Gay19]. In this post, we introduce *cyclic homology* as a versatile tool for studying smooth four-manifolds through this lens.

1.1. Overview of cyclic homology.

Fix k to be a field of characteristic zero, and let A a unital, not necessarily commutative, k-algebra. We can associate to A its cyclic cohomology $\mathsf{HC}^*(A)$ and its cyclic homology $\mathsf{HC}_*(A)$ [Con85, LQ84]. In [10], another 'cyclic homology theory' $\mathsf{HC}^-_*(A)$ theory was introduced, and we'll quickly outline its role. Cyclic cohomology $\mathsf{HC}^*(A)$ has a periodicity operator $\mathsf{HC}^n(A) \to \mathsf{HC}^{n+2}(A)$ which makes $\mathsf{HC}^*(A)$ into a module over the polynomial ring $k[u]$. This is because the cyclic cohomology of the ground field $\mathsf{HC}^*(k) \cong k[u]$ where u has degree 2 and the action of $\mathsf{HC}^*(k)$ agrees with the periodicity operator. It is natural, then, to regard $k[u]$ as the coefficient ring for cyclic cohomology and to make cyclic homology a module over $k[u]$ using the periodicity operator $\mathsf{HC}_n(A) \to \mathsf{HC}_{n-2}(A)$. It also seems natural to seek a homology theory which is dual to $\mathsf{HC}^*(A)$ over the ring $k[u]$. This cannot be $\mathsf{HC}_*(A)$ since every element of $\mathsf{HC}_*(A)$ will be u-torsion. This dual theory is $\mathsf{HC}^-_*(A)$.
The groups $\mathsf{HC}^-_*(A)$ become modules over $k[u]$ where the action of u corresponds to the periodicity operator $\mathsf{HC}^-_n(A) \to \mathsf{HC}^-_{n-2}(A)$. By localizing at u, there is also a periodic theory $\mathsf{HC}^\infty_*(A)$, and the three theories fit into a long exact sequence

$$\cdots \to \mathsf{HC}_{*+1}(A) \xrightarrow{\delta} \mathsf{HC}^-_*(A) \to \mathsf{HC}^\infty_*(A) \to HC_*(A) \to \cdots \tag{1}$$

Taking into account the product operations in each of the theories, we find that the coefficient rings for the various theories are as follows:

$$\begin{aligned}
\mathsf{HC}^*(k) &= k[u] \text{ as rings; here the degree of } u \text{ is } 2\\
\mathsf{HC}^-_*(k) &= k[u] \text{as rings; here the degree of } u \text{ is } -2\\
\mathsf{HC}^\infty_*(k) &= k[u, u^1] \text{ as rings; here the degree of } u \text{ is } -2\\
\mathsf{HC}_*(k) &= k[u, u^{-1}]/u \cdot k[u] \text{ as modules over } k[u]\text{; here the degree of } u \text{ is } -2
\end{aligned}$$

2. Overview of cyclic homology

Cyclic homology is perhaps best understood as being a refinement of another homology theory known as *Hochschild homology*. We begin by defining the latter and exploring the relationship between the two homology theories. The connection with K-theory will be postponed until the next section.

Date: June 6, 2022.

2.1. **Hochschild homology.** Let k denote a commutative ring, and let A be a k-algebra. Usually, k is either $\mathbb{Z}$, in which A is just an arbitrary ring, or a field of characteristic zero.
View A as a bimodule over itself, and consider the module $\mathsf{CH}_n(A) := A^{\otimes(n+1)}$. The *Hochschild boundary* is the k-linear map $b : \mathsf{CH}_n(A) \to \mathsf{CH}_{n-1}(A)$
(2)

$$(a_0, \ldots, a_n) \mapsto (a_0 \cdot a_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i (a_0, a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_n) + (-1)^n (a_n \cdot a_0, a_1, \ldots, a_{n-1})$$

For convenience, we compute a few examples in low degree:

$$b : \mathsf{CH}_1(A) \to \mathsf{CH}_0(A) \qquad (a_0, a_1) \mapsto a_0 \cdot a_1 - a_1 \cdot a_0$$
$$b : \mathsf{CH}_2(A) \to \mathsf{CH}_1(A) \qquad (a_0, a_1, a_2) \mapsto (a_0 \cdot a_1, a_2) - (a_0, a_1 \cdot a_2) + (a_2 \cdot a_0, a_1)$$

Lemma 2.1. $b \circ b = 0$.

As a consequence of Lemma 2.1, we get the *Hochschild complex*

$$\cdots \xrightarrow{b} \mathsf{CH}_n(A) \xrightarrow{b} \mathsf{CH}_{n-1}(A) \to \cdots \to \mathsf{CH}_1(A) \to \mathsf{CH}_0(A) \to 0 \tag{3}$$

Define the nth *Hochschild homology group* of the unital k-algebra A with coefficients in itself to be the nth homology group of the Hochschild complex (3) and is denoted by $\mathsf{HH}_n(A)$. As usual, the direct sum $\bigoplus_{n\geqslant 0} \mathsf{HH}_n(A)$ will be denoted by $\mathsf{HH}_*(A)$.

Example 1. As an elementary example, take $A = k$ to be the ground field. Then $\mathsf{HH}_*(k)$ is computed as the homology of the complex

$$\cdots \to 0 \to k \xrightarrow{1} k \xrightarrow{0} \cdots \xrightarrow{1} k \to 0 \tag{4}$$

It follows that

$$\mathsf{HH}_*(k) = \begin{cases} k & * = 0 \\ 0 & * = \text{else} \end{cases} \tag{5}$$

Example 2. Here's an example which arises naturally in the context of trisected four-manifolds. Let $(X, T_X = X_1 \cup X_2 \cup X_3)$ be a trisected four-manifold. Consider the algebra of differential forms $\Omega(U_1 \cap U_2)$ as a left module over $\Omega(U_2)$. Similarly, view $\Omega(U_2 \cap U_3)$ as a right module over $\Omega(U_2)$. Extending this viewpoint to the other double intersections, we form

$$A = \Omega(U_3 \cap U_1) \otimes_{\Omega(U_3)} \Omega(U_2 \cap U_3) \otimes_{\Omega(U_2)} \Omega(U_1 \cap U_2)$$

We wish to compute the Hochschild homology of A.

Proposition 2.2. *Let k be a commutative ring and let A be a k-algebra which is projective as a module over k (of course, this is automatic if k is a field). Then the Hochschild homology is isomorphic to*

$$\mathsf{HH}_i(A) \cong \mathsf{Tor}_i^{R\otimes_k R^{op}}(R, R), \tag{6}$$

where R^{op} denotes the opposite ring of R, and we identify $R - R$ bimodules with left $R \otimes_k R^{op}$-modules.

Example 3. Let $R = k[t]$ be a polynomial ring in one variable. This is free over k (with basis monomials t^i), hence certainly k-projective. Also $R = R^{op}$ since R is commutative, and so $R \otimes_k R^{op} \cong k[t, s]$. As a $k[t, s]$-module, R can be recovered as $k[t, s]/(t - s)$. Hence,

$$k[t, s] \xrightarrow{(t-s)} k[t, s] \to R \to 0$$

is a $R \otimes_k R^{op}$-projective resolution of R, and thus

$$\mathsf{HH}_*(R) \cong \begin{cases} R & * = 0, 1 \\ 0 & * > 1 \end{cases}$$

It is helpful to visualize a basis for the nth Hochschild chain group as comprising a circle of algebra elements:

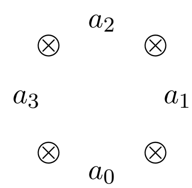

FIGURE 1. A depiction of a basis element for $CH_3(A)$.

References

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