The first obstacle in applying the theory of trisections to problems in four-dimensional topology is finding an algebraic context which is flexible enough to accommodate the complicated gluing operations inherent in the theory. We now take a few days to discuss one such setting: simplicial objects in the category $\mathtt{dgCat}_\mathtt{k}$ of dg categories. Afterwards, we'll review applications to knot contact homology, microlocal and perverse sheaves, and wrapped Fukaya categories of cotangent bundles.

1. Simplicial objects

In this section, we introduce notation and recall some basic definitions related to simplicial sets. Our first goal is to show how a trisected four-manifold $(X, \mathsf{T}_X = Z_1 \cup Z_2 \cup Z_3)$ gives rise to simplicial objects in categories of interest.

1.1. **Simplicial objects.** Let Δ denote the simplicial category. Recall that the objects of Δ are the finite ordered sets $[n] := \{0, 1, \ldots, n\}$, $n \geq 0$, and the morphisms are the (weakly) order preserving maps $[n] \to [m]$.

Definition 1.1. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, i.e. $\Delta^{op} \to \mathcal{C}$.

Given a surjective submersion $\pi : Y \to X$ of manifolds, we introduce the following notation; this may seem unnecessarily cumbersome at first, but it opens up a very general and powerful perspective on geometric structures. For $n \in \mathbb{N}$, we define the manifolds

$$\begin{aligned}\check{C}Y_{n-1} :&= Y^{[n]} = Y \times_X Y \times \cdots \times_X Y \\ &= \{(y_0, \ldots, y_{n-1}) \in Y^n | \pi(y_i) = \pi(y_j) \text{ for all } i, j = 0, \ldots, n-1\}.\end{aligned}$$

We define smooth maps $d_i^n : \check{C}Y_n \to \check{C}Y_{n-1}$ and s_i^n where $i \in \{0, \ldots, n-1\}$.

Example 1.2. A simplicial object in the category $\mathtt{Mfld}$ of manifolds and smooth maps is called a *simplicial manifold*. We see from our arguments above that the data $(\check{C}Y, d_i, s_i)$ obtained from any surjective submersion $\pi : Y \to M$ form a simplicial manifold. We call this simplicial manifold the *Cech nerve* of $\pi : Y \to M$.

The simplicial objects in $\mathcal{C}$ form a category whose morphisms are natural transformations of functors. We denote this category by $s\mathcal{C}$. If $X \in \mathsf{Obj}(s\mathcal{C})$ we write $X_n := X([n])$.

The category Δ is generated by two distinguished classes of morphisms $\{\delta^i\}_{0 \leq i \leq n}^{n \geq 1}$ and $\{\sigma^j\}_{0 \leq j \leq n}^{n \geq 0}$, whose images under $X \in s\mathcal{C}$ are called the *face* and *degeneracy maps* of X, respectively. The map $\delta^i : [n-1] \to [n]$ is the (unique) injection that does not contain 'i' in its image; the corresponding face map is denoted by $d_i := X(\delta^i) : X_n \to X_{n-1}$.

Thus, a simplicial object in $s\mathcal{C}$ is determined by a family $X = \{X_n\}_{n \geq 0}$ of objects in $\mathcal{C}$ together with morphisms $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ satisfying some relations.

Example 1.3. Using the covering of X^4 from the trisection T_X, we associate a cosimplicial object in the category of real vector spaces and linear maps. Let $\check{C}Y_n$ be the simplicial manifold associated to T_X. For any $k \in \mathbb{N}_0$, we obtain a family of real vector spaces $\{\Omega^k(X_n)\}_{n \in \mathbb{N}_0}$. The face and degeneracy maps of X induce pullback maps $\partial_i := d_i^* : \Omega^k(X_{n-1}) \to \Omega^k(X_n)$ and $\sigma_i := s_i^* : \Omega^k(X_{n+1}) \to \Omega^k(X_n)$, respectively, which satisfy the cosimplicial identities. More concretely, $(\Omega^k(X), d_i^*, s_i^*)$ is a cosimplicial object in the category of real vector spaces and linear maps. For each $n \in \mathbb{N}_0$, we define the linear maps

$$\delta : \Omega^k(X_n) \to \Omega^k(X_{n+1}) \quad \delta(\omega) = \sum_{i=0}^{n+1} (-1)^i \partial_i(\omega)$$

which make $(\Omega^k(X), \delta)$ into a cochain complex of $\mathbb{R}$-vector spaces. Note that the same construction extends to augmented simplicial manifolds.