Elements of Machine Learning
Exercise Sheet 4
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William LaCroix - wila00001@stud.uni-saarland.de - 7038732
Philipp Hawlitschek - phha00002@stud.unisaarland.de - 7043167

Problem 4 (T, 10 Points). Polynomial regression & Splines.

1. [1pt] In which sense is polynomial regression linear respectively non-linear?

Polynomial regression is linear in the coefficients, not necessarily the input features, thus it is a linear combination of model parameters over polynomial (non-linear) input terms.

2. [2pts] Given a set of basis function $\mathcal{B}_n = \{a_i x^i \| a_i \in \mathbb{R}\}_{i=0}^{n}$ for polynomial regression of degree $n \in \mathbb{N}$, give 4 functions that can not be expressed using $\mathcal{B}_n$ and argue why they can not be expressed?

1) $y = log(x)$ behavior near 0 cannot be captured by polynomial terms
2) $y = exp(x)$ cannot be expressed as a sum of polynomial terms
3) Step functions are not continuous functions, thus cannot be represented by the sum of continous functions (polynomials)
4) Trig functions are periodic and cannot be modelled by (only approximated by) polynomial functions.

3. [1pt] For this exercise consider Figure 1:

a) Determine a spline with appropriate degree that describes the function in Figure 1.

For a linear spline with two knots at $\zeta_1 : x = 1, \zeta_2 : x = 2$, we will use truncated linear base functions:

$$f(x) = -x + 0.5\,(x - \zeta_1)_+ + 0.25\,(x - \zeta_2)_+$$

b) Determine a piece-wise polynomial that describes the function in Figure 1.

$$f(x) = \begin{cases} y=-1.00x+2.0 & 0\leq x\leq 1 \\ y=-0.50x+1.5 & 1\leq x\leq 2 \\ y=-0.25x+1.0 & 2\leq x\leq 3 \end{cases}$$

4. In the lecture we defined cubic splines as functions of the form:

$$f_L(x) = a + bx + cx^2 + dx^3 + e\,(x - \zeta)_+^3$$

Alternatively, cubic splines can also be constructed using piece-wise defined polynomials of degree 3 with the condition that at the knot the polynomial is twice continuously differentiable. With this definition cubic splines take the form:

$$f_I(x) = \begin{cases} \alpha_1+\beta_1 x+\gamma_1 x^2+\delta_1 x^3 & x\leq\zeta \\ \alpha_2+\beta_2 x+\gamma_2 x^2+\delta_2 x^3 & x>\zeta \end{cases}$$

Show that both functional representations of cubic splines express the same function space. For this you have to proof that both representations can be transformed in to each other. In other words, show the following two directions:

a) [2pts] Given a function of the form $f_L(x)$ construct and equivalent $f_I(x)$.

In the region of $x \leq \zeta$, $[a, b, c, d]$ correspond with $[\alpha_1, \beta_1, \gamma_1, \delta_1]$, and in the region $x > \zeta$, they respond to $[\alpha_2, \beta_2, \gamma_2, \delta_2]$ respectively, thus, under the condition that the polynomial is twice continuously differentiable at ζ, (previously the $e\,(x - \zeta)_+^3$ term), we can simply write:

$$f_L(x) = \begin{cases} \alpha_1+\beta_1 x+\gamma_1 x^2+\delta_1 x^3 & x\leq\zeta \\ \alpha_2+\beta_2 x+\gamma_2 x^2+\delta_2 x^3 & x>\zeta \end{cases}$$

b) [3pts] Given a function of the form $f_I(x)$ construct and equivalent $f_L(x)$.

Hint: For the second part exploit that $f_I(x)$ is twice cont. differentiable and $f_L(x)$ can be written as $f_L(x) = \alpha_1 + \beta_1 x + \gamma_1 x^2 + \delta_1 x^3 + (\delta_2 - \delta_1)(x - \zeta)_+^3$.

5. **[1pt] Give an example function, where the continuity requirement is a problem. Use the graph of the function to argue why this is the case.**

An example function might be $f(x) = \frac{1}{x}$, since it is disjoint around 0, growing asymptotically toward $\pm\infty$, depending if 0 is approached from $x > 0$ or from $x < 0$