Suppose in the following questions, μ is a probability measure on $\mathcal{B}(\mathbb{R})$. The Fourier transform of μ is

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x), \qquad t \in \mathbb{R}$$

Denote $F(x) = \mu((-\infty, x])$ as the distribution function of μ. If

$$\mu(B) = \int_B f(x)\, dx, \quad \forall\, B \in \mathcal{B}(\mathbb{R}) \tag{$*$}$$

holds for some $f : \mathbb{R} \mapsto [0, \infty)$ in $L^1(\mathbb{R})$, then

$$\hat{\mu}(t) = \hat{f}(t) = \int_{\mathbb{R}} e^{itx} f(x)\, dx$$

is the Fourier transform of the probability density f.

Problem 1 (HW 8 Problem 1)

If $\hat{\mu}(t) \geq 0$ for all $t \in \mathbb{R}$, then $\hat{\mu} \in L^1(\mathbb{R})$ if and only if $(*)$ holds and $f \in L^\infty(\mathbb{R})$.

If $\hat{\mu} \in L^1(\mathbb{R})$, then by Fourier inversion formula, for all $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \hat{\mu}(t)\, dt \implies |f(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\mu}(t)|\, dt < \infty$$

Since $|f(x)|$ is bounded by a finite constant independent of x, $f \in L^\infty(\mathbb{R})$.

Conversely, suppose μ has a bounded density f. Apply Parseval relation on a Gaussian density with mean zero and variance a^2, we obtain

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixt - a^2t^2/2} \hat{\mu}(t)\, dt = \int_{\mathbb{R}} \frac{e^{-(x-t)^2/(2a^2)}}{\sqrt{2\pi}a}\, d\mu(x)$$

Let $x = 0$, we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-a^2t^2/2} \hat{\mu}(t)\, dt = \int_{\mathbb{R}} \frac{e^{-t^2/(2a^2)}}{\sqrt{2\pi}a}\, d\mu(t) = \int_{\mathbb{R}} \frac{e^{-t^2/(2a^2)}}{\sqrt{2\pi}a} f(t)\, dt \leq M$$

Since $\hat{\mu}(t) \geq 0$, MCT implies that

$$\lim_{a \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-a^2t^2/2} \hat{\mu}(t)\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \lim_{a \to 0} e^{-a^2t^2/2} \hat{\mu}(t)\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(t)\, dt \leq M$$

Therefore, $\hat{\mu} \in L^1(\mathbb{R})$.

Problem 2 (HW 8 Problem 6)

Show that "the atoms of μ can be recovered from the spectrum":

$$\mu(\{x\}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \hat{\mu}(t)\, dt, \quad \forall\, x \in \mathbb{R}$$

The RHS can be reduced to

$$\frac{1}{2T} \int_{-T}^{T} e^{-itx} \hat{\mu}(t)\, dt = \frac{1}{2T} \int_{-T}^{T} e^{-itx} \int_{\mathbb{R}} e^{it\xi}\, d\mu(\xi)\, dt = \int_{\mathbb{R}} \frac{1}{2T} \int_{-T}^{T} e^{it(\xi - x)}\, dt d\mu(\xi)$$

where the second equality is due to Fubini's theorem. It is valid to apply Fubini because $|e^{it(\xi-x)}| \leq 1$ and μ is a probability measure. Since $\sin[(\xi - x)t]$ is odd w.r.t. t and the integral region is $[-T, T]$, we have

$$\frac{1}{2T}\int_{-T}^{T} e^{-itx}\hat{\mu}(t)\, dt = \int_{\mathbb{R}} \frac{1}{2T}\int_{-T}^{T} \cos[(\xi - x)t]\, dtd\mu(\xi)$$

Notice that

$$\left|\frac{1}{2T}\int_{-T}^{T} \cos[(\xi - x)t]\, dt\right| \leq 1, \quad \forall T$$

and if $\xi \neq x$, as $T \to \infty$,

$$\frac{1}{2T}\int_{-T}^{T} \cos[(\xi - x)t]\, dt = \frac{1}{2T}\frac{1}{\xi - x}\sin[t(\xi - x)]\bigg|_{-T}^{T} = \frac{1}{\xi - x}\frac{\sin[T(\xi - x)]}{T} \to 0$$

and if $\xi = x$, then

$$\frac{1}{2T}\int_{-T}^{T} \cos[(\xi - x)t]\, dt \to 1$$

By DCT, we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-itx}\hat{\mu}(t)\, dt = \int_{\mathbb{R}} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \cos[(\xi - x)t]\, dtd\mu(\xi) = \mu(\{x\})$$

Problem 3 (HW 8 Problem 7)

Show that "the total energy in the atoms" equals "the asymptotic energy-per-unit-frequency in the spectrum":

$$\sum_{x\in\mathbb{R}} \mu^2(\{x\}) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\hat{\mu}(t)|^2\, dt$$

Consider two independent random variables X and Y with common distribution μ, then $Z = X - Y$ has probability mass function (a discrete version of convolution)

$$h(y) = P(Z = y) = \sum_{x} \mu\{x\}\mu\{y + x\}$$

The characteristic function of Z is given by $\hat{h}(\xi) = \hat{\mu}(t)\hat{\mu}(-t) = |\hat{\mu}(t)|^2$. Apply the result in Problem 2 on Z,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-ity}|\hat{\mu}(t)|^2\, dt = h(y) = \sum_{x} \mu\{x\}\mu\{y + x\}$$

Let $y = 0$ on both sides, the desired result follows.

Problem 4 (HW 8 Problem 17)

Suppose that X and Y are two real-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that $X - Y$ and X are independent, and $X - Y$ and Y are independent.

(a) Prove that $X - Y$ is a real constant a.e. if both X and Y are square-integrable.

(b) If g is the characteristic function of X, and h the characteristic function of $X - Y$, show

$$g(\xi)(1 - |h(\xi)|^2) = 0, \quad \forall \xi \in \mathbb{R}$$

(c) Show that the function g is uniformly continuous on $\mathbb{R}$.

(d) Conclude from parts (b) and (c), that there is a constant $\epsilon > 0$, such that $|h(\xi)| = 1$ for all $\xi \in (-\epsilon, \epsilon)$.

(e) Show that $X - Y$ is a real constant a.e. without imposing any integrability assumption on X or on Y.

(a) If they are square integrable, then consider

$$\mathrm{Var}(X - Y) = \mathbb{E}[(X - Y)^2] - \mathbb{E}^2[X - Y] = \mathbb{E}[(X - Y)X] - \mathbb{E}[(X - Y)Y] - \mathbb{E}^2[X - Y]$$

Since $X - Y$ are independent of X and Y, and X, Y have the same distribution implies $\mathbb{E}[X - Y] = 0$, so we have

$$\mathrm{Var}(X - Y) = \mathbb{E}[X - Y]\mathbb{E}[X] - \mathbb{E}[X - Y]\mathbb{E}[Y] - \mathbb{E}^2[X - Y] = 0$$

The fact that $\mathrm{Var}(X - Y) = 0$ implies that $X - Y$ is a constant almost everywhere.

(b) Notice that $|h(\xi)|^2 = \mathbb{E}[e^{i\xi(X-Y)}]\mathbb{E}[e^{-i\xi(X-Y)}]$, so we have

$$g(\xi)|h(\xi)|^2 = \mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{i\xi(X-Y)}]\mathbb{E}[e^{-i\xi(X-Y)}]$$

Since X and $X - Y$ are independent, we also have X and $-(X - Y)$ are independent, and

$$\mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{-i\xi(X-Y)}] = \mathbb{E}[e^{i\xi Y}]$$

Thus, we have $g(\xi)|h(\xi)|^2 = \mathbb{E}[e^{i\xi Y}]\mathbb{E}[e^{i\xi(X-Y)}]$. Similarly, Y and $X - Y$ are independent, so

$$g(\xi)|h(\xi)|^2 = \mathbb{E}[e^{i\xi Y}]\mathbb{E}[e^{i\xi(X-Y)}] = \mathbb{E}[e^{i\xi X}] = g(\xi)$$

Thus, we have $g(\xi)(1 - |h(\xi)|^2) = 0$

(c) Consider any sequence $h_n \to 0$ and

$$|g(\xi + h_n) - g(\xi)| = \left| \int_{\mathbb{R}} (e^{i(\xi + h_n)x} - e^{i\xi x})\, d\mu(x) \right| \le \int_{\mathbb{R}} |e^{ih_n x} - 1|\, d\mu(x)$$

Since $|e^{ih_n x} - 1| \le 2$ and $|e^{ih_n x} - 1| \to 0$ as $h_n \to 0$, by DCT, $|g(\xi + h_n) - g(\xi)| \to 0$. Notice that the upper bound does not depend on ξ, so $g(\xi)$ is uniformly continuous.

(d) Notice that $g(0) = 1$ and uniformly continuous, so there is a constant $\epsilon > 0$ such that $g(0) \neq 0$ for all $\xi \in (-\epsilon, \epsilon)$. This implies $|h(\xi)| = 1$ for all $\xi \in (-\epsilon, \epsilon)$.

(e) Let $Z = X - Y$ and consider an independent copy Z' of Z. Then $\mathbb{E}[e^{i\xi(Z-Z')}] = |h(\xi)|^2 = 1$. This shows $\mathbb{E}[1 - \cos(\xi(Z - Z'))] = 0$ a.e., which implies $1 - \cos(\xi(Z - Z')) = 0$ a.e. because $1 - \cos(\xi(Z - Z')) \ge 0$. This shows $\xi(Z - Z') = 2k\pi$ for any $k \in \mathbb{Z}$. If ξ is arbitrarily close to zero, then either $Z - Z' = 0$ or $|Z - Z'|$ is arbitrarily large. Thus, $Z = Z'$ a.e., which means Z is almost surely a constant.

Problem 5 (HW 11 Problem 6)

Suppose that X, Y are independent random variables with common distribution μ which has mean 0 and variance 1. Show that $X + Y$ and $X - Y$ are independent if, and only if, the distribution μ is standard normal. And in this case, the random variables $Z = (X + Y)/\sqrt{2}$ and $W = (X - Y)/\sqrt{2}$ are independent, standard normal.

The "only if" case can be proved following the steps given in the hint:

(a) We first argue that if $X + Y$ and $X - Y$ are independent, then $\varphi(2\xi) = (\varphi(\xi))^3\varphi(-\xi)$. Suppose $\varphi(\xi) = \mathbb{E}[e^{i\xi X}]$, then $\mathbb{E}[e^{i\xi 2X}] = \varphi(2\xi)$. Since $2X = (X + Y) + (X - Y)$ and $X + Y$ and $X - Y$ are independent, we have $\mathbb{E}[e^{i\xi 2X}] = \mathbb{E}[e^{i\xi(X+Y)}]\mathbb{E}[e^{i\xi(X-Y)}]$. Since X and Y are also independent and with the same distribution, $\mathbb{E}[e^{i\xi(X+Y)}] = \mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{i\xi Y}] = (\varphi(\xi))^2$. Since X and $-Y$ are also independent, $\mathbb{E}[e^{i\xi(X-Y)}] = \mathbb{E}[e^{i\xi X}]\mathbb{E}[e^{-i\xi Y}] = \varphi(\xi)\varphi(-\xi)$. Therefore, we have $\varphi(2\xi) = (\varphi(\xi))^3\varphi(-\xi)$.

(b) Then we show that $\varphi(\xi) \neq 0$ for all ξ. Notice that $\varphi(-2\xi) = (\varphi(-\xi))^3\varphi(\xi)$ from the first part and thus $\varphi(2\xi)\varphi(-2\xi) = |\varphi(\xi)|^4$. If there is some ξ_0 such that $\varphi(\xi_0) = 0$, then $0 = \varphi(\xi_0)\varphi(-\xi_0) = |\varphi(\xi_0/2)|^4$ implies that $\varphi(\xi_0/2) = 0$. By induction, one can show $\varphi(\xi_0/2^n) = 0$ for all n. Take $n \to \infty$, by continuity of φ shown in part (b) of Problem 4, $\varphi(\xi_0/2^n) \to \varphi(0) = 1$. This is a contradiction, so $\varphi(\xi) \neq 0$ for all ξ.

(c) Since $\varphi(\xi) \neq 0$ for all ξ, we can define $\rho(\xi) = \varphi(\xi)/\varphi(-\xi)$ and we will show $\rho(\xi) = 1$ for all ξ. Notice that $\rho(2\xi) = (\varphi(\xi))^2/(\varphi(-\xi))^2 = (\rho(\xi))^2$ for any ξ, so we have $\rho(\xi) = (\rho(\xi/2))^2$. By induction, one can show that $\rho(\xi) = (\rho(\xi/2^n))^{2^n}$. Now use the fact that $\varphi(\xi) = 1 - \xi^2/2 + o(\xi^2)$ as $\xi \to 0$, we have $\rho(\xi) = 1 + o(\xi^2)$, as $n \to \infty$,

$$\rho(\xi) = (\rho(\xi/2^n))^{2^n} = \left(1 + o\left(\frac{\xi^2}{4^n}\right)\right)^{2^n} \to 1$$

Thus, $\rho(\xi) = 1$ for all ξ.

(d) Since $\rho(\xi) = 1$, $\varphi(\xi)$ must be a real number, so the result in part (a) reduces to $\varphi(2\xi) = (\varphi(\xi))^4$ for all $\xi \in \mathbb{R}$. Similarly, by induction, we will have $\varphi(\xi) = (\varphi(\xi/2^n))^{4^n}$ for all n. Use $\varphi(\xi) = 1 - \xi^2/2 + o(\xi^2)$ as $\xi \to 0$, and as $n \to \infty$,

$$\varphi(\xi) = (\varphi(\xi/2^n))^{4^n} = \left(1 - \frac{\xi^2}{2 \cdot 4^n} + o\left(\frac{\xi^2}{4^n}\right)\right)^{4^n} \to e^{-\xi^2/2}$$

which by the uniqueness of Fourier inversion, shows that X follows standard normal distribution. Since X and Y have common distribution, Y also follows standard normal.

The "if" part can be proved as follows: If X and Y are standard normal, then they have characteristic function $\mathbb{E}[e^{i\xi X}] = \mathbb{E}[e^{i\xi Y}] = e^{-\xi^2/2}$. Since X and Y are independent, the characteristic function $\varphi_{X+Y}(\xi) = e^{-\xi^2}$. Similarly, $\varphi_{X-Y}(\xi) = e^{-\xi^2}$. To compute the characteristic function $\varphi_{X+Y,X-Y}(\xi,\zeta)$ of joint distribution of $X+Y$ and $X-Y$, consider $\varphi_{X,Y}(\xi,\zeta)$, the joint distribution of X and Y is given by $\mathbb{E}[e^{i(\xi X+\zeta Y)}] = e^{-(\xi^2+\zeta^2)/2}$ because X and Y are independent. Thus,

$$\varphi_{X+Y,X-Y}(\xi,\zeta) = \mathbb{E}[e^{i(\xi(X+Y)+\zeta(X-Y))}] = \mathbb{E}[e^{i(\xi+\zeta)X+i(\xi-\zeta)Y}] = e^{-((\xi+\zeta)^2+(\xi-\zeta)^2)/2} = e^{-(\xi^2+\zeta^2)}$$

Since $\varphi_{X+Y,X-Y}(\xi,\zeta) = \varphi_{X+Y}(\xi)\varphi_{X-Y}(\zeta)$, we conclude that $X + Y$ and $X - Y$ are independent.

Similarly, we can compute $\varphi_W(\xi) = \varphi_{X+Y}(\xi/\sqrt{2}) = e^{-\xi^2/2}$, $\varphi_Z(\xi) = \varphi_{X-Y}(\xi/\sqrt{2}) = e^{-\xi^2/2}$, and $\varphi_{W,Z} = \varphi_{X+Y,X-Y}(\xi/\sqrt{2},\zeta/\sqrt{2}) = e^{-(\xi^2+\zeta^2)/2}$. Thus, by Fourier-Levy theorem for random vector, W and Z are independent and they follows standard normal distribution.