Problem 1 (HW 7 Problem 7)

Let $X : \Omega \mapsto [0, \infty)$ be a random variable with distribution μ, distribution function

$$F(x) = P(X \leq x) = \mu((-\infty, x]) = \mu([0, x]) = \int_{\mathbb{R}} I_{[0,x]}(\xi)\, d\mu(\xi)$$

and Laplace transform for $\lambda \in (0, \infty)$:

$$\hat{F}(\lambda) = \mathbb{E}[e^{-\lambda X}] = \int_0^\infty e^{-\lambda\xi}\, d\mu(\xi)$$

Show that the Laplace transform $\hat{F}$ determines the distribution function F, using the strong law of large numbers and the following steps:

(a) Show that the derivatives of this Laplace transform $\hat{F}$ are given by

$$\frac{\partial^k}{\partial\lambda^k}\hat{F}(\lambda) = \hat{F}^{(k)}(\lambda) = \int_0^\infty (-1)^k \xi^k e^{-\lambda\xi}\, d\mu(\xi), \quad \lambda \in (0, \infty),\, k \in \mathbb{N}$$

(b) Suppose that $Z_1(\theta)$, $Z_2(\theta), \cdots$ are independent random variables with the same Poisson distribution of parameter $\theta = \mathbb{E}[Z_k(\theta)]$. Use the weak law of large numbers, to argue that

$$P\left(\frac{1}{n}\sum_{k=1}^n Z_k(\theta) \leq x\right) \to \begin{cases} 1 & \text{if } x > \theta \\ 0 & \text{if } x < \theta \end{cases}$$

and thus

$$\sum_{j \leq nx} e^{-n\theta}\frac{(n\theta)^j}{j!} \to \begin{cases} 1 & \text{if } x > \theta \\ 0 & \text{if } x < \theta \end{cases}$$

What is the above limit, if $x = \theta$?

(c) Conclude that, for any $x \in [0, \infty)$ which is a continuity point of the distribution function F, we have

$$\sum_{j \leq nx} \frac{(-1)^j}{j!} n^j \hat{F}^{(j)}(n) \to F(x)$$

(d) Argue that the distribution function F (thus also the measure μ) is determined uniquely from the Laplace transform $\hat{F}$.

(a) Consider the difference quotient:

$$\frac{\hat{F}(\lambda + h) - \hat{F}(\lambda)}{h} = \int_0^\infty \frac{e^{-(\lambda+h)\xi} - e^{-\lambda\xi}}{h}\, d\mu(\xi), \quad \text{where } \frac{e^{-(\lambda+h)\xi} - e^{-\lambda\xi}}{h} \to -\xi e^{-\lambda\xi}$$

Notice that when h is small enough, we always have

$$\left|\frac{e^{-(\lambda+h)\xi} - e^{-\lambda\xi}}{h}\right| \leq 1 + \xi e^{-\lambda\xi}$$

We claim that $1+\xi e^{-\lambda\xi}$ is integrable. Since μ is a probability measure, 1 is integrable. Since $\lambda > 0$, $\xi e^{-\lambda\xi}$ is nonnegative and vanishes very fast as $\xi \to \infty$, the integral of $\xi e^{-\lambda\xi}$ over $[0,\infty)$ is also finite. Thus, the claim is true and we can apply DCT to conclude

$$\lim_{h\to 0}\frac{\hat{F}(\lambda+h)-\hat{F}(\lambda)}{h}=\int_0^\infty -\xi e^{-\lambda\xi}\,d\mu(\xi)$$

Thus, the desired result is true for $k=1$. Repeat the same argument inductively, based on the fact that $e^{-\lambda\xi}$ is smooth and $\xi^k e^{-\lambda\xi}$ is integrable for any k, DCT implies the desired result is also true for any k.

(b) By weak law of large number, $\bar{Z}_k \to \theta$ in probability, and hence in distribution, so

$$\lim_{n\to\infty} P(\bar{Z}_k \le x) = P(\theta \le x) = \begin{cases} 1 & \text{if } x > \theta \\ 0 & \text{if } x < \theta \end{cases}$$

The case $x=\theta$ is not valid because convergence in distribution is only valid for continuous point, but $x=\theta$ is not continuous. However, consider

$$P(\bar{Z}_k \le \theta) = P(\sqrt{n}(\bar{Z}_k-\theta)/\theta \le 0) \to P(Z \le 0) = \frac{1}{2}$$

where Z follows standard normal distribution by CLT. Since the sum of n i.i.d. Poisson(θ) is Poisson$(n\theta)$, we have

$$P(\bar{Z}_k \le x) = P\left(\sum_{k=1}^n Z_k(\theta) \le nx\right) = \sum_{j\le nx} e^{-n\theta}\frac{(n\theta)^j}{j!}$$

and the desired result follows.

(c) Denote the RHS of (b) as $f(\theta;x)$ and LHS as $f_n(\theta;x)$. Notice that $f_n(\theta;x)\to f(\theta;x)$ a.e. with respect to θ, and $|f_n(\theta;x)|\le 1$. Since F is a probability distribution, 1 is integrable, so by DCT,

$$\int_0^\infty \sum_{j\le nx} e^{-n\theta}\frac{(n\theta)^j}{j!}\,d\mu(\xi) = \int_0^\infty f_n(\theta;x)\,d\mu(\xi) \to \int_0^\infty f(\theta;x)\,d\mu(\xi) = \int_0^x 1\,d\mu(\xi) = F(x)$$

Observe that

$$\int_0^\infty \sum_{j\le nx} e^{-n\theta}\frac{(n\theta)^j}{j!}\,d\mu(\xi) = \sum_{j\le nx}\frac{n^j}{j!}\int_0^\infty e^{-n\theta}\theta^j\,d\mu(\xi) = \sum_{j\le nx}\frac{n^j}{j!}(-1)^j\hat{F}^{(j)}(n)$$

where the last equality follows from the result in (a). Thus, the desired statement follows.

(d) From (c), we know F is uniquely determined from $\hat{F}$ for all continuous points. Since F is a distribution function, it is right continuous with left limit and the discontinuous point is only countably many, so by taking the limit of a sequence of continuous point, every x is uniquely determined for $F(x)$.

Problem 2 (HW 7 Problem 8)

For $s>1$ fixed, let $X:\Omega\mapsto\mathbb{N}$ be a random variable with the "zeta distribution":

$$P(X=n)=\frac{1}{\zeta(s)n^s},\quad n\in\mathbb{N},\quad \zeta(s)=\sum_{n\in\mathbb{N}}\frac{1}{n^s}$$

where $\zeta(s)$ is the Riemann's zeta-function. For each $m\in\mathbb{N}$, consider the event

$$A_m=\{\omega\in\Omega:\ m \text{ is a factor of } X(\omega)\}=\{\omega\in\Omega: X(\omega)=mk(\omega),\ \text{for some } k(\omega)\in\mathbb{N}\}$$

(i) Show that $P(A_m) = 1/m^s$.

(ii) Conclude from (i) that the events in the collection $\{A_p\}_{p\in\mathcal{P}}$ are independent, where $\mathcal{P}$ is the set of prime numbers.

(iii) Use (i) and (ii) to derive the Euler formula

$$\frac{1}{\zeta(s)} = \prod_{p\in\mathcal{P}}\left(1 - \frac{1}{p^s}\right)$$

(i) Notice that by definition, we can compute $P(A_m)$ by

$$P(A_m) = \sum_{k\in\mathbb{N}} \frac{1}{\zeta(s)(mk)^s} = \frac{1}{m^s\zeta(s)}\sum_{k\in\mathbb{N}}\frac{1}{k^s} = \frac{1}{m^s\zeta(s)}\zeta(s) = \frac{1}{m^s}$$

and the desired result follows.

(ii) By definition of independence, it suffices to show that for any finite subsequence $p_1, \ldots, p_K$ of the collection $\{A_p\}_{p\in\mathcal{P}}$, we have $P(A_{p_1}, \ldots, A_{p_K}) = \prod_{i=1}^K P(A_{p_i})$. To compute $P(A_{p_1}, \ldots, A_{p_K})$, consider

$$P(A_{p_1}, \ldots, A_{p_K}) = P(X = mp_1 \cdots p_K) = P(A_{p_1\cdots p_K}) = \frac{1}{(p_1 \cdots p_K)^s} = \prod_{i=1}^K P(A_{p_i})$$

Thus, $\{A_p\}_{p\in\mathcal{P}}$ is an independent collection.

(iii) Note that $P(X = 1) = \frac{1}{\zeta(s)}$ is the LHS of the equation. Observe that

$$P(X = 1) = P(\cap_{p\in\mathcal{P}} A_p^c) = \prod_{p\in\mathcal{P}} P(A_p^c) = \prod_{p\in\mathcal{P}}\left(1 - \frac{1}{p^s}\right)$$

because of the independence proved in (ii) and the formula in (i).