Problem 1 (HW4 Exercise 1)

For given measurable, locally bounded functions $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, initial position $\xi \in \mathbb{R}$, and standard Brownian motion W, consider the integral equation

$$X(t) = \xi + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW_s, \quad t \in [0, \infty) \tag{1}$$

(i) If X_1 and X_2 are solutions of this equation, defined on the same filtered probability space as W, show that $X_1 \vee X_2$ is also a solution if and only if the local time at the origin of the continuous semimartingale $X_1 - X_2$ is identically equal to zero, i.e., $L^{X_1-X_2}(\cdot, 0) \equiv 0$.

(ii) If uniqueness in distribution holds for (1), and if $L^{X_1-X_2}(\cdot, 0) \equiv 0$ holds for any pair X_1, X_2 of solutions defined on the same probability space as W, then pathwise uniqueness holds for (1) as well.

(i) Notice that $X_1 \vee X_2 = X_2 + (X_1 - X_2)^+$. Apply second Tanaka's formula at level $a = 0$ to the process $X_1 - X_2$, we obtain

$$\begin{aligned}
(X_1 - X_2)^+ &= (X_1(0) - X_2(0))^+ + \int_0^t 1_{\{X_1(s)-X_2(s)>0\}}\, d[X_1(s) - X_2(s)] + \frac{1}{2} L^{X_1-X_2}(t, 0) \\
&= (X_1(0) - X_2(0))^+ + \int_0^t 1_{\{X_1(s)>X_2(s)\}} (b(s, X_1(s)) - b(s, X_2(s)))\, ds \\
&\quad + \int_0^t 1_{\{X_1(s)>X_2(s)\}} (\sigma(s, X_1(s)) - \sigma(s, X_2(s)))\, dW_s + \frac{1}{2} L^{X_1-X_2}(t, 0)
\end{aligned}$$

Since X_1 and X_2 are solutions of this equation, we know $X_1(0) = X_2(0) = \xi$, and

$$\begin{aligned}
X_1 \vee X_2 &= X_2 + (X_1 - X_2)^+ \\
&= \xi + \int_0^t b(s, X_2(s))\, ds + \int_0^t 1_{\{X_1(s)>X_2(s)\}} (b(s, X_1(s)) - b(s, X_2(s)))\, ds \\
&\quad + \int_0^t \sigma(s, X_2(s))\, dW_s + \int_0^t 1_{\{X_1(s)>X_2(s)\}} (\sigma(s, X_1(s)) - \sigma(s, X_2(s)))\, dW_s + \frac{1}{2} L^{X_1-X_2}(t, 0) \\
&= \xi + \int_0^t 1_{\{X_1(s)>X_2(s)\}} b(s, X_1(s)) + 1_{\{X_1(s)\le X_2(s)\}} b(s, X_2(s))\, ds \\
&\quad + \int_0^t 1_{\{X_1(s)>X_2(s)\}} \sigma(s, X_1(s)) + 1_{\{X_1(s)\le X_2(s)\}} \sigma(s, X_2(s)))\, dW_s + \frac{1}{2} L^{X_1-X_2}(t, 0) \\
&= \xi + \int_0^t b(s, (X_1 \vee X_2)(s))\, ds + \int_0^t \sigma(s, (X_1 \vee X_2)(s))\, dW_s + \frac{1}{2} L^{X_1-X_2}(t, 0)
\end{aligned}$$

Therefore, $X_1 \vee X_2$ is a solution if and only if $L^{X_1-X_2}(t, 0) = 0$ for all $t \ge 0$.

(ii) Notice that $X_1 \wedge X_2 = X_1 - (X_1 - X_2)^+$, so by similar argument as in (i), we can show that $X_1 \wedge X_2$ is also a solution if and only if $L^{X_1-X_2}(t, 0) = 0$ for all $t \ge 0$. Now we assume $L^{X_1-X_2}(t, 0) = 0$, so both $X_1 \vee X_2$ and $X_1 \wedge X_2$ are solutions. Since $X_1 \vee X_2 - X_1 \wedge X_2 = |X_1 - X_2|$, and by uniqueness in distribution, we have

$$\mathbb{E}[|X_1 - X_2|] = \mathbb{E}[X_1 \vee X_2 - X_1 \wedge X_2] = \mathbb{E}[X_1 \vee X_2] - \mathbb{E}[X_1 \wedge X_2] = 0$$

This shows that $X_1(t) - X_2(t) = 0$ for each fixed t almost surely.

Problem 2 (HW4 Exercise 2)

Let $a_1 < a_2 < \cdots < a_n$ be real numbers, and denote $D = \{a_1, \ldots, a_n\}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and f' and f'' exist and are continuous on $\mathbb{R} \setminus D$, and the limits

$$f'(a_k\pm) \triangleq \lim_{x \to a_k\pm} f'(x), \quad f''(a_k\pm) = \lim_{x \to a_k\pm} f''(x)$$

exist and are finite. Show that f is the difference of two convex functions and, for every $z \in \mathbb{R}$,

$$f(W_t) = f(z) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, ds + \sum_{k=1}^n L_t(a_k)[f'(a_k+) - f'(a_k-)],\ t \in [0, \infty),\ a.s.\ P^z$$

Define $f''_+(x) \triangleq \max\{f''(x), 0\}$ and $f''_-(x) \triangleq \max\{-f''(x), 0\}$, then $f''(x) = f''_+(x) - f''_-(x)$ for all $x \in \mathbb{R}$. Choose any $a \in \mathbb{R} \setminus D$. Let $g_1(x)$ and $h_1(x)$ be defined by

$$g_1(x) \triangleq f'(a) + \int_a^x f''_+(u)\, du + \sum_{k=1}^n [f'(a_k+) - f'(a_k-)]^+ 1_{[a_k,\infty)}(x)$$

$$h_1(x) \triangleq \int_a^x f''_-(u)\, du + \sum_{k=1}^n [f'(a_k+) - f'(a_k-)]^- 1_{[a_k,\infty)}(x)$$

Then it is easy to see that $g_1(x)$ and $h_1(x)$ are both increasing function and $f'(x) = g_1(x) - h_1(x)$ for any $x \in \mathbb{R} \setminus D$ because over this region,

$$f'(x) = f'(a) + \int_a^x f(u)\, du + \sum_{k=1}^n [f'(a_k+) - f'(a_k-)] 1_{[a_k,\infty)}(x)$$

Furthermore, let $g(x)$ and $h(x)$ be defined as

$$g(x) \triangleq f(a) + \int_a^x g_1(u)\, du, \quad h(x) \triangleq \int_a^x h_1(u)\, du$$

and obviously $f(x) = g(x) - h(x)$ where $g(x)$ and $h(x)$ are convex.

By generalized Ito's formula, we have

$$f(W_t) = f(z) + \int_0^t D^- f(W_s)\, dW_s + \int_{\mathbb{R}} L_t(y)\mu(dy)$$

Since the discontinuous point of $f'(x)$ is finite and W_s is continuous,

$$\int_0^t D^- f(W_s)\, dW_s = \int_0^t f'(W_s)\, dW_s$$

By definition of measure μ and occupation density formula,

$$\begin{aligned}\int_{\mathbb{R}} L_t(y)\mu(dy) &= \int_{\mathbb{R}} L_t(y) f''(y)\, dy + \sum_{k=1}^n L_t(a_k)(f'(a_k+) - f'(a_k-)) \\ &= \frac{1}{2}\int_0^t f''(W_s)\, ds + \sum_{k=1}^n L_t(a_k)(f'(a_k+) - f'(a_k-))\end{aligned}$$

Combine the above two results, we obtain the desired equality.

Problem 3 (HW4 Exercise 3)

Suppose that the Borel-measurable function $f : \mathbb{R} \to [0, \infty)$ satisfies $m\{f(y) > 0\} > 0$. Show that

$$P^x \left[\int_0^\infty f(W_s)\, ds = \infty \right] = 1$$

holds for every $x \in \mathbb{R}$. Assume further that f has compact support, and consider the sequence of continuous process

$$X_t^{(n)} \triangleq \frac{1}{\sqrt{n}} \int_0^{nt} f(W_s)\, ds, \quad t \in [0, \infty),\ n \geq 1$$

Prove $X^{(n)} \to X$ in distribution under P^0 as $n \to \infty$, where $X_t \triangleq 2\|f\|_1 L_t(0)$ and $\|f\|_1 \triangleq \int_{\mathbb{R}} f(y)\, dy > 0$.

Let $A = \{f(y) > 0\}$. By occupation density formula, we have

$$\int_0^t f(W_s)\, ds = 2 \int_{\mathbb{R}} f(a) L_t(a)\, da = 2 \int_A f(a) L_t(a)\, da$$

Since $f(a) > 0$ on A and $L_t(a)$ is positive and increasing in t, by MCT, we have

$$\int_0^\infty f(W_s)\, ds = 2 \int_A f(a) \left(\lim_{t \to \infty} L_t(a) \right) da$$

Notice that $L_t(a) \to \infty$ for any fixed a as $t \to \infty$. This is because a standard Brownian motion will eventually hit any $K < 0$ and Tanaka formula gives

$$L_t(a) = |W_t - a| - |x - a| - \int_0^t \operatorname{sgn}(W_s - a)\, ds \geq -|x - a| - \int_0^t \operatorname{sgn}(W_s - a)\, ds$$

where $\int_0^t \operatorname{sgn}(W_s - a)\, ds$ is equal in distribution to a standard Brownian motion. Thus, for $a \in A$, since $f(a) > 0$, we have $f(a) L_\infty(a) = \infty$ with $m(A) > 0$, and the desired result follows.

Similarly, by occupation density formula, we have

$$X_t^{(n)} = 2 \int_{\mathbb{R}} f(y) \frac{L_{nt}(y)}{\sqrt{n}}\, dy$$

By scaling property of Brownian local time, we have $\frac{L_{nt}(y)}{\sqrt{n}} = L_t(\frac{y}{\sqrt{n}})$. By continuity of $L_t(a)$ in a, we have $\frac{L_{nt}(y)}{\sqrt{n}} \to L_t(0)$ as $n \to \infty$ for all $y \in \mathbb{R}$. First consider the case when $\|f\|_1 < \infty$, since f has compact support C, together with continuity, $\sup_{a \in C} L_t(a)$ is almost surely finite. Thus, for each fixed t, by applying DCT on the Lebesgue integral with respect to y, $X_t^{(n)} \to 2\|f\|_1 L_t(0)$ almost surely. Thus, $X_t^{(n)} \to 2\|f\|_1 L_t(0)$ for any rational t almost surely. By right continuity in t, $X_t^{(n)} \to 2\|f\|_1 L_t(0)$ for all $t \in [0, \infty)$ almost surely. This implies $X^{(n)} \to X$ in distribution. If $\|f\|_1 = \infty$, then $L_t(0) > 0$ almost surely for each fixed t, and thus $X_t = \infty$. By Fatous' lemma,

$$\infty = \int_{\mathbb{R}} f(y) L_t(0)\, dy \leq \liminf_{n \to \infty} X_t^{(n)}$$

which can still imply $X_t^{(n)} \to \infty$ almost surely. Note that $L_t(0) > 0$ almost surely for each fixed t because $2L_t(0)$ is equal to $|B_t|$ in distribution and $|B_t| > 0$ almost surely for each t.

To verify the scaling property, consider the approximation of local time

$$\begin{aligned}\frac{1}{\sqrt{n}} L_{nt}(y) &= \lim_{\epsilon \to 0} \frac{1}{4\epsilon\sqrt{n}} \int_0^{nt} 1_{\{|W_s - y| \leq \epsilon\}}\, ds = \lim_{\epsilon \to 0} \frac{\sqrt{n}}{4\epsilon} \int_0^t 1_{\{|W_{nu} - y| \leq \epsilon\}}\, du \\ &= \lim_{\epsilon \to 0} \frac{\sqrt{n}}{4\epsilon} \int_0^t 1_{\{|\frac{W_{nu}}{\sqrt{n}} - \frac{y}{\sqrt{n}}| \leq \frac{\epsilon}{\sqrt{n}}\}}\, du = \lim_{\delta \to 0} \frac{1}{4\delta} \int_0^t 1_{\{|\tilde{W}_u - \frac{y}{\sqrt{n}}| \leq \delta\}}\, du = L_t\left(\frac{y}{\sqrt{n}}\right)\end{aligned}$$

This completes the proof.

Problem 4 (HW4 Exercise 4)

Let X be a continuous semimartingale with canonical decomposition $X_t = X_0 + M_t + V_t$. Show that for every $a \in \mathbb{R}$,

$$\int_0^\infty 1_{\{a\}}(X_s)\, d\langle M\rangle_s = 0, \quad a.s.\ P$$

For every $t > 0$, define A_t to be

$$A_t \triangleq \int_0^t 1_{\{a\}}(X_s)\, d\langle M\rangle_s = \int_0^t 1_{\{a\}}(X_s)\, d\langle X\rangle_s$$

Then it suffices to show that $A_\infty = 0$. The occupation density formula gives

$$\int_0^t 1_{\{a\}}(X_s)\, d\langle X\rangle_s = 2\int_{-\infty}^\infty 1_{\{a\}}(y) L_t(y)\, dy$$

Since $L_t(y)$ is right continuous in y, and the integrand is only nonzero at a single point $y = a$, we have $A_t = 0$ for all t. Note that the integrand is nonnegative, by MCT, we know $A_\infty = 0$ and the desired result follows.

Problem 5 (HW4 Exercise 5)

Let X be a continuous semimartingale with canonical decomposition, and suppose that there exists a Borel-measurable function $k : (0,\infty) \to (0,\infty)$ such that $\int_0^\epsilon (du/k(u)) = \infty$, for all $\epsilon > 0$, but for every $t \in (0,\infty)$, we have

$$\int_0^t \frac{d\langle M\rangle_s}{k(X_s)} 1_{\{X_s>0\}} < \infty, \quad a.s.\ P$$

Then the local time $\Lambda(0)$ of X at the origin is identically zero, almost surely.

By right continuity of $\Lambda_t(a)$ in a, we have $\lim_{a\to 0+} \Lambda_t(a) = \Lambda_t(0)$ for all $t \geq 0$ almost surely. Thus, there exists $C(\omega) > 0$ and $\epsilon(\omega) > 0$ such that $\Lambda_t(a) \geq C(\omega)\Lambda_t(0)$ for all $a \in (0, \epsilon(\omega))$, $t \geq 0$ and almost every ω. By occupation density formula, we have

$$\int_0^t \frac{d\langle M\rangle_s}{k(X_s)} 1_{\{X_s>0\}} = 2\int_{\mathbb{R}} \frac{\Lambda_t(y)}{k(y)}\, dy \geq 2\int_0^\epsilon \frac{\Lambda_t(y)}{k(y)}\, dy \geq 2C(\omega)L_t(0)\int_0^\epsilon \frac{1}{k(y)}\, dy$$

Since $C(\omega) > 0$ and $\int_0^\epsilon \frac{1}{k(y)}\, dy = \infty$, we must have $\Lambda_t(0) = 0$ for all $t \geq 0$ almost surely to guarantee their product to be finite. Therefore, the desired result follows.

Problem 6 (HW4 Exercise 6)

Consider a continuous local martingale M and denote $S_t \triangleq \max_{s\in[0,t]} M_s$, $L_t \triangleq 2\Lambda_t(0)$. Suppose now that $f : \mathbb{R}^3 \to \mathbb{R}$ is a function of class $C^2(\mathbb{R}^3)$ which satisfies $f_t + \frac{1}{2} f_{xx} = 0$ in $\mathbb{R}^3$ and $f_x(t,0,y) + f_y(t,0,y) = 0$ for every $(t,y) \in \mathbb{R}^2$. Show that

(i) the processes $f(\langle M\rangle_t, |M_t|, L_t)$ and $f(\langle M\rangle_t, S_t - M_t, S_t)$ are local martingales.

(ii) the process $(S_t - M_t)^2 - \langle M\rangle_t$ is a local martingale.

(iii) for every real-valued function g of class $C^1(\mathbb{R})$, the processes

$$g(L_t) - |M_t| g'(L_t) \quad \text{and} \quad g(S_t) - (S_t - M_t) g'(S_t)$$

are local martingales.

(i) By Ito's formula and Tanaka's formula, together with the fact that $\langle L\rangle_t \equiv 0$ and $\langle M, L\rangle_t \equiv 0$ we have

$$f(\langle M\rangle_t, |M_t|, L_t) = f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_t(\langle M\rangle_s, |M_s|, L_s)\, d\langle M\rangle_s + \int_0^t f_x(\langle M\rangle_s, |M_s|, L_s)\, d|M_s|$$
$$+ \int_0^t f_y(\langle M\rangle_s, |M_s|, L_s)\, dL_s + \frac{1}{2}\int_0^t f_{xx}(\langle M\rangle_s, |M_s|, L_s)\, d\langle |M|\rangle_s$$

Notice that by Tanaka's formula, $\langle |M|\rangle_s = \langle M\rangle_s$ and since $f_t + \frac{1}{2} f_{xx} = 0$,

$$f(\langle M\rangle_t, |M_t|, L_t) = f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_y(\langle M\rangle_s, |M_s|, L_s)\, dL_s + \int_0^t f_x(\langle M\rangle_s, |M_s|, L_s)\, d|M_s|$$

Again, by Tanaka's formula, $d|M_s| = dL_s + \operatorname{sgn}(M_s) dM_s$, we have

$$\begin{aligned} f(\langle M\rangle_t, |M_t|, L_t) &= f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_x(\langle M\rangle_s, |M_s|, L_s)\, dM_s \\ &\quad + \int_0^t [f_x(\langle M\rangle_s, |M_s|, L_s) + f_y(\langle M\rangle_s, |M_s|, L_s)]\, dL_s \end{aligned}$$

Since dL_s is only supported on $M_t = 0$, combined with $f_x(t, 0, y) + f_y(t, 0, y) = 0$, we finally obtain

$$\begin{aligned} f(\langle M\rangle_t, |M_t|, L_t) &= f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_x(\langle M\rangle_s, |M_s|, L_s)\, dM_s \\ &\quad + \int_0^t [f_x(\langle M\rangle_s, 0, L_s) + f_y(\langle M\rangle_s, 0, L_s)] 1_{\{M_s = 0\}}\, dL_s \\ &= f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_x(\langle M\rangle_s, |M_s|, L_s)\, dM_s \end{aligned}$$

This implies that $f(\langle M\rangle_t, |M_t|, L_t)$ is a continuous local martingale.

Similarly, we apply Ito's formula and Tanaka's formula to $f(\langle M\rangle_t, S_t - M_t, S_t)$. Consider

$$\begin{aligned} f(\langle M\rangle_t, S_t - M_t, S_t) &= f(\langle M\rangle_0, |M_0|, L_0) + \int_0^t f_t(\langle M\rangle_u, S_u - M_u, S_u)\, d\langle M\rangle_u \\ &\quad + \int_0^t f_x(\langle M\rangle_u, S_u - M_u, S_u)\, d(S_u - M_u) + \int_0^t f_y(\langle M\rangle_u, S_u - M_u, S_u)\, dS_u \\ &\quad + \frac{1}{2}\int_0^t f_{xx}(\langle M\rangle_u, S_u - M_u, S_u)\, d\langle S - M\rangle_u \\ &= f(\langle M\rangle_0, |M_0|, L_0) - \int_0^t f_x(\langle M\rangle_u, S_u - M_u, S_u)\, dM_u \\ &\quad + \int_0^t [f_x(\langle M\rangle_u, S_u - M_u, S_u) + f_y(\langle M\rangle_u, S_u - M_u, S_u)]\, dS_u \\ &\quad + \int_0^t \left[f_t(\langle M\rangle_u, S_u - M_u, S_u) + \frac{1}{2} f_{xx}(\langle M\rangle_u, S_u - M_u, S_u)\right] d\langle M\rangle_u \end{aligned}$$

Notice that dS_u is supported only on $S_u - M_u = 0$ because S_u is the running maximum of M_u and they are continuous process. Thus, by the assumptions,

$$f(\langle M\rangle_t, S_t - M_t, S_t) = f(\langle M\rangle_0, |M_0|, L_0) - \int_0^t f_x(\langle M\rangle_u, S_u - M_u, S_u)\, dM_u$$

This verifies that $f(\langle M\rangle_t, S_t - M_t, S_t)$ is a continuous local martingale.

(ii) Let $f(t,x,y) = x^2 - t$, then it is easy to see that $f_t = -1$, $f_{xx} = 2$, $f_x(t,x,y) = 2x$ and $f_y = 0$. Thus, $f_t + \frac{1}{2}f_{xx} = 0$ and $f_x(t,0,y) + f_y(t,0,y) = 0$ are satisfied for any (t,y). By result in (i), the process $(S_t - M_t)^2 - \langle M\rangle_t = f(\langle M\rangle_t, S_t - M_t, S_t)$ is a local martingale.

(iii) By using Theorem 4.2 in Chapter VI of "Continuous Martingales and Brownian Motion" (Daniel Revuz & Marc Yor),

$$g'(L_t)|M_t| = g'(L_0)|M_0| + \int_0^t g'(L_s)\, d|M_s| = g'(L_0)|M_0| + \int_0^t g'(L_s)\, dL_s + \int_0^t g'(L_s)\mathrm{sgn}(M_s)\, dM_s$$

Also notice that L_t is an increasing process, so

$$g(L_s) = \int_0^t g'(L_s)\, dL_s$$

The above two equations imply that

$$g(L_s) - g'(L_t)|M_t| = -g'(L_0)|M_0| - \int_0^t g'(L_s)\mathrm{sgn}(M_s)\, dM_s$$

Since M_t is a continuous local martingale, we conclude that $g(L_s) - g'(L_t)|M_t|$ is also a continuous local martingale. Notice that we use the theorem by letting $K_t = g'(L_t)$ and $Y = |M_t|$ and $X = M_t$. In this case the condition $Y_{d_t} = |X_{d_t}| = 0$ holds by continuity and $d_t \triangleq \inf\{s > t \,|\, X_s = 0\}$. Also, K_t is locally bounded because $g'(L_t)$ is continuous. Finally, we use the fact that $L_{g_t} = L_t$ because $g_t \triangleq \sup\{s < t \,|\, X_s = 0\}$ and L_t will only increase over the set $\{X_s = 0\}$.

Similarly, let $K_t = g'(S_t)$ and $X = Y = S_t - M_t$. In this case, S_t is a continuous increasing process, so

$$g(S_t) = \int_0^t g'(S_u)\, dS_u$$

Also, $g'(S_0)(S_0 - M_0) = 0$. Furthermore, $X_{d_t} = 0$ by continuity and definition. Finally, $S_{g_t} = S_t$ because S_t only increases when $X_t = 0$ and g_t is the last time X_t hit 0. Thus, we have

$$g(S_t) - g'(S_t)(S_t - M_t) = \int_0^t g'(S_u)\, dM_u$$

Thus, $g(S_t) - g'(S_t)(S_t - M_t)$ is a continuous local martingale.

Problem 7 (HW4 Exercise 7)

For a nonnegative continuous semimartingale X with canonical decomposition with $X_0 = 0$, the following are equivalent:

(i) V is flat off $\{t \geq 0;\ X_t = 0\}$.

(ii) The process $\int_0^t 1_{\{X_s \neq 0\}}\, dX_s$ where $t \in [0,\infty)$ is a continuous local martingale.

(iii) There exists a continuous local martingale N such that $X_t = \max_{s\in[0,t]} N_s - N_t$.

1. Proof of (i) $\iff$ (ii). By Exercise 4, we know that

$$\int_0^t 1_{\{X_s=0\}} d\langle M\rangle_s \equiv 0 \implies \int_0^t 1_{\{X_s=0\}} dM_s \equiv 0 \implies M_t = \int_0^t 1_{\{X_s\neq 0\}} dM_s$$

Given (i), we have

$$\int_0^t 1_{\{X_s\neq 0\}} dX_s = \int_0^t 1_{\{X_s\neq 0\}} dM_s + \int_0^t 1_{\{X_s\neq 0\}} dV_s = M_t$$

which implies (ii). Conversely, given (ii), the difference of two continuous local martingale

$$\int_0^t 1_{\{X_s\neq 0\}} dV_s = \int_0^t 1_{\{X_s\neq 0\}} dX_s - \int_0^t 1_{\{X_s\neq 0\}} dM_s = \int_0^t 1_{\{X_s\neq 0\}} dX_s - M_t$$

is again a continuous local martingale. However, this continuous local martingale is also of finite variation, so it is a constant. Since when $t = 0$, it is zero, we know it is a constant 0, i.e., (i) is verified.

2. Proof of (ii) $\implies$ (iii). By two Tanaka's formula and nonnegativity of X_t, we have

$$X_t = \int_0^t 1_{\{X_s\neq 0\}}\, dX_s + L_t(0), \quad L_t(0) = \int_0^t 1_{\{X_s=0\}}\, dX_s = \int_0^t 1_{\{X_s=0\}}\, dV_s$$

Consider using Lemma 3.6.14 in BMSC (The Skorokhod equation). Let $z = 0$ and $y(t) \triangleq \int_0^t 1_{\{X_s\neq 0\}}\, dX_s$ a continuous function (by (ii)) with $y(0) = 0$. Since $L_t(0)$ here satisfies all three properties in Lemma 3.6.14, by uniqueness,

$$L_t(0) = k(t) = 0 \vee \max_{0\le s\le t}(-y(s)) = \max_{0\le s\le t}(-y(s))$$

where the last equation is because $y(0) = 0$ and hence $\max_{0\le s\le t}(-y(s)) \ge 0$. Therefore,

$$X_t = \max_{0\le s\le t}(-y(s)) + y(t) \triangleq \max_{0\le s\le t} N_s - N_t$$

if we define $N_t = -y(t)$. Note that N_t is also a continuous local martingale because $y(t)$ is.

3. Proof of (iii) $\implies$ (i). Since there exists a continuous local martingale N_t such that $X_t = \sup_{s\in[0,t]} N_s - N_t$, and $\sup_{s\in[0,t]} N_s$ is continuous and of finite variation, by uniqueness of canonical decomposition of continuous semimartingale, $-N_t = M_t$ and $\sup_{s\in[0,t]} N_s = V_t$. Let $y(t) = -N_t$, then y is a continuous function with $y(0) = 0$. Since $V_t = k(t) = 0 \vee \max_{s\in[0,t]}(-y(s))$ defined in Lemma 3.6.14 and $x(t) = X_t$, by property (iii), V_t is flat off $\{t \ge \mid X_t = 0\}$.