Problem 1 (HW 3 Problem 3)

Let $R = \{R_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a Bessel process with dimension $d \ge 2$ starting at $r > 0$, and define $m = \inf_{t\in[0,\infty)} R_t$.

(i) Show that if $d = 2$, then $m = 0$ a.e.

(ii) Show that if $d \ge 3$, then m has the beta distribution $P(m \le c) = (c/r)^{d-2}$ for $0 \le c \le r$.

(iii) Show that if $r \ge 0$ and $d \ge 3$, then $P(\lim_{t\to\infty} R_t = \infty) = 1$.

(i) Let $T_l = \inf\{t \ge 0 \,|\, R_t = l\}$ and $S_L = \inf\{t \ge 0 \,|\, R_t = L\}$ where $l < r < L$, and $\tau = T_l \wedge S_L \wedge n$. By similar argument in the proof of Proposition 3.3.22 in BMSC,

$$P\left(S_L < \infty, \forall\, L \ge 0 \text{ and } \lim_{L\to\infty} S_L = \infty\right) = 1$$

Applying Ito's rule to $\log R_t$ and by Proposition 3.3.21,

$$\log R_\tau = \log r + \int_0^\tau \frac{1}{R_s}\, dB_s$$

because log is C^2 over (l, L). Since τ is a bounded stopping time and $\frac{1}{R_s}$ is bounded over $(0, \tau)$, by optional sampling theorem

$$0 = \mathbb{E}\left[\int_0^\tau \frac{1}{R_s}\, dB_s\right] = \mathbb{E}[\log R_\tau] - \log r$$

This implies that

$$\log r = \mathbb{E}[\log R_\tau] = \log l \cdot P(T_l \le S_L \wedge n) + \log L \cdot P(S_L \le T_l \wedge n) + \mathbb{E}[(\log R_n)1_{\{n<S_L\wedge T_l\}}]$$

Let $n \to \infty$, since $P(n < S_l \wedge T_k) \to 0$ and $\log R_n$ is bounded in $[\log l, \log L]$, we have

$$\log r = \log l \cdot P(T_l \le S_L) + \log L \cdot P(S_L \le T_l)$$

Combined with the fact that $P(T_l \le S_L) + P(S_L \le T_l) = 1$, we have

$$P(T_l \le S_L) = \frac{\log L - \log r}{\log L - \log l}$$

Let $L \to \infty$, we have $P(T_l < \infty) = 1$. Note that $P(m \le l) \ge P(T_l < \infty) = 1$, so by letting $l \to 0$, $P(m \le 0) = 1$. Since $m \ge 0$ by definition, we have $m = 0$ almost surely.

(ii) By exactly the same argument but applying Ito's rule to R_t^{2-d}, we obtain

$$P(T_l \le S_L) = \frac{L^{2-d} - r^{2-d}}{L^{2-d} - l^{d-2}}$$

Let $L \to \infty$, we have $P(T_l < \infty) = (\frac{l}{r})^{d-2}$ and thus $P(m \le l) \ge (\frac{l}{r})^{d-2}$. Notice that for any $\epsilon > 0$, $P(m \le l) \le P(T_{l+\epsilon} < \infty)$. By continuity, $P(T_{l+\epsilon} < \infty) \to P(T_l < \infty)$ as $\epsilon \to 0$. This shows $P(m \le l) = (\frac{l}{r})^{d-2}$.

(iii) Define events $E_n = \{R_t > n, \forall\, t \ge T_{n^3}\}$. By strong Markov property and the result in (ii),

$$P(E_n^c) = P(R_t \le n \text{ for some } t > T_{n^3}) = P(m \le n \,|\, R_0 = n^3) = (1/n^2)^{d-2}$$

Thus, $\sum_{n=1}^\infty P(E_n^c) < \infty$, and by Borel-Cantelli lemma, $P(E_n^c, i.o.) = 0$. This shows that there exists N such that for $n \ge N$, $P(E_n) = 1$. Again, since $P(T_{n^3} < \infty) = 1$, $\lim_{t\to\infty} R_t = \infty$ almost surely.

Problem 2 (HW 3 Problem 5)

Let $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a standard, one-dimensional Brownian motion, and let T be a stopping time of $\{\mathcal{F}_t\}$ with $\mathbb{E}[\sqrt{T}] < \infty$. Prove the Wald identities $\mathbb{E}[W_T] = 0$ and $\mathbb{E}[W_T^2] = \mathbb{E}[T]$.

Note that W is a martingale, so $\{W_{T\wedge t}, \mathcal{F}_t; 0 \le t < \infty\}$ is also a martingale. Thus, $\mathbb{E}[W_{T\wedge t}] = \mathbb{E}[W_0] = 0$. Also $W^2 - t$ is a martingale, so $\{W_{T\wedge t}^2 - T\wedge t, \mathcal{F}_t; 0 \le t < \infty\}$ is also a martingale. Thus, $\mathbb{E}[W_{T\wedge t}^2 - T\wedge t] = 0$ implies that $\mathbb{E}[W_{T\wedge t}^2] = \mathbb{E}[T\wedge t]$. By BDG inequality with $p = 1$, there exists a constant C such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |W_t|\right] \le C\mathbb{E}\left[\langle W\rangle_T^{1/2}\right] = C\mathbb{E}\left[\sqrt{T}\right] < \infty$$

Also, $\mathbb{E}[\sqrt{T}] < \infty$ impiles $P(T < \infty) = 1$, so $W_{T\wedge t} \to W_T$ a..e.. Thus, using $\sup_{0\le t\le T} |W_t|$ as dominating function of $W_{T\wedge t}$, by DCT we have $W_{T\wedge t} \to W_T$ in L^1. This shows $\mathbb{E}[W_T] = \lim_{t\to\infty} \mathbb{E}[W_{T\wedge t}] = 0$.

Since $W_{T\wedge t} \to W_T$ in L^1, $W_{T\wedge t}$ is a martingale with last element W_T, i.e., $W_{T\wedge t} = \mathbb{E}[W_T \,|\, \mathcal{F}_t]$. By Jensen's inequality for conditional expectation,

$$W_{T\wedge t}^2 = (\mathbb{E}[W_T \,|\, \mathcal{F}_t])^2 \le \mathbb{E}[W_T^2 \,|\, \mathcal{F}_t]$$

Taking expectation on both sides, $\mathbb{E}[W_{T\wedge t}^2] \le \mathbb{E}[W_T^2]$ for all $t \ge 0$. Also, by Fatou's lemma, we have

$$\mathbb{E}\left[W_T^2\right] \le \liminf_{t\to\infty} \mathbb{E}[W_{T\wedge t}^2] \le \limsup_{t\to\infty} \mathbb{E}[W_{T\wedge t}^2] \le \mathbb{E}[W_T^2]$$

Thus, all inequalities become equalities and $\lim_{t\to\infty} \mathbb{E}[W_{T\wedge t}^2] = E[W_T^2]$. By MCT, $\mathbb{E}[T\wedge t] \to \mathbb{E}[T]$. Thus, by taking limit on both sides of $\mathbb{E}[W_{T\wedge t}^2] = \mathbb{E}[T\wedge t]$, we obtain the Wald identities.

Problem 3 (HW 3 Problem 7)

Suppose $X = X(0) + M + A$ is a continuous semimartingale; and H, $\{H^{(n)}\}_{n\in\mathbb{N}}$ progressively measurable and locally bounded (e.g., continuous-path) processes. Assume that there exists a progressively measurable process $K \ge 0$, so that

- $H^{(n)}(t,\omega) \to H(t,\omega)$ as $n\to\infty$ for every $t \in [0,T]$
- $|H^{(n)}(t,\omega)| \le K(t,\omega)$, for every $(t,n) \in [0,T]\times\mathbb{N}$
- $\int_0^T K^2(t,\omega)\, d\langle M\rangle_t(\omega) + \int_0^T K(t,\omega)\, dA^{(\pm)}(t,\omega) < \infty$

hold for a.e. $\omega \in \Omega$. Show that $I_T^X(H^{(n)}) \to I_T^X(H)$ as $n\to\infty$ in probability.

Let $V_t = X(0) + A_t$, then by the usual DCT, we have $I_T^V(H^{(n)}) \to I_T^V(H)$ a.e. because V_t is of bounded variation. Thus, it suffices to show that $I_T^M(H^{(n)}) \to I_T^M(H)$. For each $j \ge 1$, define a stopping time

$$T_j = \inf\left\{ s \ge 0 \,\middle|\, \int_0^s K_u^2\, d\langle M\rangle_u \ge j \right\} \wedge t$$

By the third property, $P(T_j = T) \to 1$ as $j\to\infty$. Thus, fix j and n, we have

$$\begin{aligned}
P(|I_T^M(H^{(n)}) - I_T^M(H)| > \delta) &= P(|I_T^M(H^{(n)}) - I_T^M(H)| > \delta, T_j = T) + P(|I_T^M(H^{(n)}) - I_T^M(H)| > \delta, T_j < T) \\
&\le P(|I_{T_j}^M(H^{(n)}) - I_{T_j}^M(H)| > \delta) + P(T_j < T) \\
&\le \frac{1}{\delta^2}\mathbb{E}[|I_{T_j}^M(H^{(n)}) - I_{T_j}^M(H)|^2] + P(T_j < T) \\
&= \frac{1}{\delta^2}\mathbb{E}[I_{T_j}^{\langle M\rangle}((H^{(n)} - H)^2)] + P(T_j < T)
\end{aligned}$$

Notice that $I_{T_j}^{\langle M\rangle}((H^{(n)}-H)^2)$ is a Lebesgue integral, so by the usual DCT on $(H^{(n)}-H)^2$, it converges to 0 a.e.. Using $I_{T_j}^{\langle M\rangle}(K^2)\le j$ as dominating function, by usual DCT on $I_{T_j}^{\langle M\rangle}((H^{(n)}-H)^2)$, we have $\mathbb{E}[I_{T_j}^{\langle M\rangle}((H^{(n)}-H)^2)]\to 0$. Thus, by taking $n\to\infty$, we have $\lim_{n\to\infty}P(|I_T^M(H^{(n)})-I_T^M(H)|>\delta)\le P(T_j<T)$ for all j. Take $j\to\infty$, we have $\lim_{n\to\infty}P(|I_T^M(H^{(n)})-I_T^M(H)|>\delta)=0$.

Problem 4 (HW 3 Problem 8)

Let T be a stopping time of the filtration $\{\mathcal{F}_t^W\}$ with $P(T<\infty)=1$. A necessary and sufficient condition for the validity of the Wald identity $\mathbb{E}[\exp(\mu W_T-\frac{1}{2}\mu^2 T)]=1$, where μ is a given real number, is that $P^{(\mu)}(T<\infty)=1$. In particular, if $b\in\mathbb{R}$ and $\mu b<0$, then this condition holds for the stopping time $S_b\triangleq\inf\{t\ge 0; W_t-\mu t=b\}$.

Notice that $\{T<\infty\}=\bigcup_{k=1}^\infty\{T\le k\}$. Since $P^{(\mu)}$ is a probability measure on $(\Omega,\mathcal{F}_\infty^W)$, it is continuous from below, i.e., $P^{(\mu)}(T<\infty)=\lim_{k\to\infty}P^{(\mu)}(T\le k)$. Since $\{T\le k\}\in\mathcal{F}_k^W$, by BMSC, $P^{(\mu)}(T\le k)=\mathbb{E}[1_{\{T\le k\}}Z_k]$, where $Z_t=\exp(\mu W_t-\frac{1}{2}\mu^2 t)$ is a continuous martingale. Thus,

$$P^{(\mu)}(T<\infty)=1\iff\lim_{k\to\infty}\mathbb{E}[1_{\{T\le k\}}Z_k]=1$$

Notice that by optional sampling theorem, $Z_{T\wedge t}$ is also a continuous martingale because $T\wedge t$ is a bounded stopping time. Since $\mathbb{E}[Z_{T\wedge t}]=\mathbb{E}[Z_t]=1$ for all $t\ge 0$,

$$1=\mathbb{E}[Z_0]=\mathbb{E}[Z_{T\wedge t}]=\mathbb{E}[1_{\{T\le t\}}Z_T]+\mathbb{E}[1_{\{T>t\}}Z_t]=\mathbb{E}[1_{\{T\le t\}}Z_T]+1-\mathbb{E}[1_{\{T\le t\}}Z_t]$$

This implies that $\mathbb{E}[1_{\{T\le t\}}Z_T]=\mathbb{E}[1_{\{T\le t\}}Z_t]$ for all $t\ge 0$. Therefore,

$$P^{(\mu)}(T<\infty)=1\iff\lim_{k\to\infty}\mathbb{E}[1_{\{T\le k\}}Z_T]=1$$

Notice that $1_{\{T\le k\}}Z_T$ is nonnegative and increasing in k, so by MCT, $\mathbb{E}[1_{\{T\le k\}}Z_T]\to\mathbb{E}[1_{\{T<\infty\}}Z_T]$ as $k\to\infty$. Since $P(T<\infty)=1$, we have $\mathbb{E}[1_{\{T<\infty\}}Z_T]=\mathbb{E}[Z_T]$. Thus,

$$P^{(\mu)}(T<\infty)=1\iff\mathbb{E}[Z_T]=1$$

and the desired necessary and sufficient statement follows.

Note that under measure $P^{(\mu)}$, $\tilde{W}_t=W_t-\mu t$ is a Brownian motion. Thus, $S_b=\inf\{t\ge 0\,|\,\tilde{W}_t=b\}$ and by the known result from the usual Brownian motion, $P^{(\mu)}(S_b<\infty)=1$. It suffices to show $P(S_b<\infty)=1$. Consider

$$\begin{aligned}P(S_b\in dt)&=\mathbb{E}[1_{\{S_b\in dt\}}]=\mathbb{E}[1_{\{S_b\in dt\}}]=\mathbb{E}^{P^{(\mu)}}[1_{\{S_b\in dt\}}e^{-\mu W_t+\frac{1}{2}\mu^2 t}]\\&=\mathbb{E}^{P^{(\mu)}}[1_{\{S_b\in dt\}}e^{-\mu\tilde{W}_t-\frac{1}{2}\mu^2 t}]=e^{-\mu b-\frac{1}{2}\mu^2 t}\mathbb{E}^{P^{(\mu)}}[1_{\{S_b\in dt\}}]=e^{-\mu b-\frac{1}{2}\mu^2 t}P^{(\mu)}(S_b\in dt)\end{aligned}$$

Thus, we have

$$P(S_b\le t)=\int_0^t e^{-\mu b-\frac{1}{2}\mu^2 t}P^{(\mu)}(S_b\in dt)$$

By letting $t\to\infty$, we have

$$P(S_b<\infty)=e^{-\mu b}\mathbb{E}^{P^{(\mu)}}[e^{-\frac{1}{2}\mu^2 S_b}]$$

Again by the result for usual Brownian motion, $\mathbb{E}^{P^{(\mu)}}[e^{-\frac{1}{2}\mu^2 S_b}]=e^{-|\mu b|}$, so

$$P(S_b<\infty)=e^{-\mu b-|\mu b|}=\begin{cases}1 & \text{if } \mu b<0\\ e^{-2\mu b} & \text{if } \mu b\ge 0\end{cases}$$

Problem 5 (HW 3 Problem 12)

Suppose M, N are local martingales with continuous paths with $M_0 = N_0 = 0$, $\langle M \rangle \equiv \langle N \rangle =: A$, $A_\infty = \infty$, and $\langle M, N \rangle \equiv 0$. Show that the Brownian Motions β, γ in the DDS representations of these two local martingales $M_t = \beta_{A_t}$, $N_t = \gamma_{A_t}$, $0 \le t < \infty$ are independent.

Define a stopping time T_s for each $s \ge 0$ by $T_s = \inf\{t \ge 0 \,|\, A_t \ge s\}$. Then we have $\beta_s = M_{T_s}$ and $\gamma_s = N_{T_s}$. By the proof of DDS theorem, we know that β and γ are adapted to the same filtration, denoted by $\mathcal{G}_s = \mathcal{F}_{T_s}$. Since $\langle M, N \rangle \equiv 0$, by the uniqueness of Doob-Meyer decomposition, $M_t N_t$ is a continuous local martingale. By Problem 5.24 in Chapter 1 of BMSC, $M_{t \wedge T_n} N_{t \wedge T_n}$ is a uniformly integrable martingale for every $n \in \mathbb{N}^+$. By optional sampling theorem, for any $s \le u \le n$, we have

$$\mathbb{E}[\beta_u \gamma_u \,|\, \mathcal{G}_s] = \mathbb{E}[M_{T_u} N_{T_u} \,|\, \mathcal{F}_{T_s}] = \mathbb{E}[M_{T_u \wedge T_n} N_{T_u \wedge T_n} \,|\, \mathcal{F}_{T_s}] = M_{T_s \wedge T_n} N_{T_s \wedge T_n} = M_{T_s} N_{T_s} = \beta_s \gamma_s$$

Thus, $\beta_u \gamma_u$ is a $\mathcal{G}$-martingale and $\langle \beta_u, \gamma_u \rangle \equiv 0$. By Levy's characterization of Brownian motion, (β, γ) is a 2-dimensional Brownian motion w.r.t. $\mathcal{G}$, and hence β is independent of γ.