Problem 1 (HW5 Exercise 1(i) & (ii))

Given a continuous function $A : [0, \infty) \to \mathbb{R}$, consider the integral equation

$$R(t) = A(t) + \int_0^t \frac{ds}{R(s)}, \quad t \in [0, \infty)$$

with A as its "input" and R as its "output" or "solution". Show that this equation has

(i) at most one non-negative (resp. non-positive) solution for any such function A with $A(0) \geq 0$ (resp., $A(0) \leq 0$) as input;

(ii) exactly one non-negative solution for a Brownian path $A \equiv B$ as input function.

(i) We are going to prove pathwise uniqueness of nonnegative (the nonpositive case shares the same proof) solution for any given $A(t)$. Suppose there are two nonnegative solution R_1 and R_2, then

$$R_1(t) - R_2(t) = \int_0^t \frac{R_2(s) - R_1(s)}{R_1(s)R_2(s)}\, ds, \quad \forall\, t \geq 0$$

Notice that $R_1(s)R_2(s) \geq 0$ for all $s \geq 0$. By letting $R(t) = R_1(t) - R_2(t)$ and taking derivative on both sides,

$$R'(t) = -\frac{R(t)}{R_1(t)R_2(t)}$$

This implies that

$$R(t) = C \exp\left(-\int_0^t \frac{1}{R_1(s)R_2(s)}\, ds\right), \quad \text{for almost every } t \geq 0$$

Notice that the integral is always nonnegative (possibly infinite), so the exponential term is always bounded by 1. Since $R(0) = 0$, the constant $C = 0$. This together with continuity of $R(t)$ shows that $R(t) = 0$ for all $t \geq 0$ and hence $R_1(t) = R_2(t)$ verifies the pathwise uniqueness.

(ii) If $A(t) \triangleq B_t$, then we only need to prove there exists a strong solution of

$$R(t) = B_t + \int_0^t \frac{ds}{R(s)}, \quad t \in [0, \infty) \tag{1}$$

We first notice that to prove there exists a weak solution is easy. Now think of B_t in (1) as any Brownian motion instead of a given Brownian path. Then take three independent standard Brownian motion W_1, W_2, W_3 on a probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $\mathbb{F}^W$. Let $R(t) = \sqrt{W_1^2(t) + W_2^2(t) + W_3^2(t)}$, then by Proposition 3.3.21 in BMSC, there exists a Brownian motion B_t adapted to $\mathbb{F}$ such that $(R(t), B_t)$ satisfies (1). This shows a weak solution exists for (1). Since in (i), we proved the pathwise uniqueness of (1), by Corollary 5.3.23 in BMSC, there exists a strong solution to (1). By pathwise uniqueness, the nonnegative weak solution $R(t)$ we constructed is the unique strong solution.

Problem 2 (HW5 Exercise 2)

Assume that the coefficients $b_i : \mathbb{R}^d \to \mathbb{R}$, $\sigma_{ij} : \mathbb{R}^d \to \mathbb{R}$; $1 \leq i \leq d$, $1 \leq j \leq r$ are meaasurable and bounded on compact subsets of $\mathbb{R}^d$, and let $\mathcal{A}$ be the associated operator

$$(\mathcal{A}f)(x) \triangleq \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}\frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}$$

Let $X = \{X_t, \mathcal{F}_t; t \in [0, \infty)\}$ be a continuous process on some probability space $(\Omega, \mathcal{F}, P)$ and assume that $\{\mathcal{F}_t\}$ satisfies the usual conditions. With $f \in C^2(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$, introduce the processes

$$M_t \triangleq f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds,\ \mathcal{F}_t; \quad t \in [0, \infty)$$
$$\Lambda_t \triangleq e^{-\alpha t} f(X_t) - f(X_0) + \int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s))\,ds,\ \mathcal{F}_t; \quad t \in [0, \infty)$$

and show $M \in \mathcal{M}^{c,loc} \iff \Lambda \in \mathcal{M}^{c,loc}$. If f is bounded away from zero on compact sets and

$$N_t \triangleq f(X_t) \exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}\,ds\right\} - f(X_0),\ \mathcal{F}_t; \quad t \in [0, \infty)$$

then these two conditions are also equivalent to $N \in \mathcal{M}^{c,loc}$.

We first prove $M \in \mathcal{M}^{c,loc} \iff \Lambda \in \mathcal{M}^{c,loc}$. Define a continuous process $C_t = e^{-\alpha t}$, and C_t is of finite variation. By Problem 3.3.12 in BMSC,

$$\begin{aligned}
\int_0^t C_s\,dM_s = C_t M_t - \int_0^t M_s\,dC_s = e^{-\alpha t} M_t + \int_0^t \alpha e^{-\alpha s} M_s\,ds \\
&= e^{-\alpha t} f(X_t) - e^{-\alpha t} f(X_0) - e^{-\alpha t} \int_0^t \mathcal{A}f(X_s)\,ds \\
&+ \int_0^t \alpha e^{-\alpha s} f(X_s)\,ds - f(X_0) \int_0^t \alpha e^{-\alpha s}\,ds - \int_0^t \alpha e^{-\alpha s} \int_0^s \mathcal{A}f(X_u)\,du\,ds \\
&= e^{-\alpha t} f(X_t) - f(X_0) - e^{-\alpha t} \int_0^t \mathcal{A}f(X_s)\,ds + \int_0^t \alpha e^{-\alpha s} f(X_s)\,ds - \int_0^t \alpha e^{-\alpha s} \int_0^s \mathcal{A}f(X_u)\,du\,ds \\
&= e^{-\alpha t} f(X_t) - f(X_0) + \int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s))\,ds = \Lambda_t
\end{aligned}$$

Notice that the second last equality is due to integration by parts for the double integral, i.e.,

$$\begin{aligned}
\int_0^t \alpha e^{-\alpha s} \int_0^s \mathcal{A}f(X_u)\,du\,ds &= \int_0^t \int_0^s \mathcal{A}f(X_u)\,du\,d(-e^{-\alpha s}) \\
&= -e^{-\alpha s} \int_0^s \mathcal{A}f(X_u)\,du \bigg|_0^t + \int_0^t e^{-\alpha s} \mathcal{A}f(X_s)\,ds \\
&= -e^{-\alpha t} \int_0^t \mathcal{A}f(X_s)\,ds + \int_0^t e^{-\alpha s} \mathcal{A}f(X_s)\,ds
\end{aligned}$$

Therefore, Λ_t can be written as a stochastic integral of C_t w.r.t. continuous local martingale M, hence Λ is also in $\mathcal{M}^{c,loc}$. Conversely, replace C_t with $\frac{1}{C_t} = e^{\alpha t}$, and it is still true that $e^{\alpha t}$ is of finite variation, so

$$\begin{aligned}\int_0^t \frac{1}{C_s}\, d\Lambda_s = \int_0^t e^{\alpha s}\, d\Lambda_s &= e^{\alpha t}\Lambda_t - \int_0^t \alpha e^{\alpha s}\Lambda_s\, ds \\ &= f(X_t) - e^{\alpha t} f(X_0) + e^{\alpha t}\int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s))\, ds \\ &- \int_0^t \alpha f(X_s)\, ds + f(X_0)\int_0^t \alpha e^{\alpha s}\, ds - \int_0^t \alpha e^{\alpha s}\int_0^s e^{-\alpha u}(\alpha f(X_u) - \mathcal{A}f(X_u))\, du\, ds \\ &= f(X_t) - f(X_0) + e^{\alpha t}\int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s))\, ds - \int_0^t \alpha f(X_s)\, ds \\ &- \int_0^t \alpha e^{\alpha s}\int_0^s e^{-\alpha u}(\alpha f(X_u) - \mathcal{A}f(X_u))\, du\, ds \\ &= f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds = M_t\end{aligned}$$

Notice that the second last equality is due to integration by parts for the double integral, i.e,

$$\begin{aligned}&\int_0^t \alpha e^{\alpha s}\int_0^s e^{-\alpha u}(\alpha f(X_u) - \mathcal{A}f(X_u))\, du\, ds = \int_0^t \alpha e^{\alpha s}\int_0^s e^{-\alpha u}(\alpha f(X_u) - \mathcal{A}f(X_u))\, du\, ds \\ &= e^{\alpha s}\int_0^s e^{-\alpha u}(\alpha f(X_u) - \mathcal{A}f(X_u))\, du\Big|_0^t - \int_0^t (\alpha f(X_s) - \mathcal{A}f(X_s))\, ds \\ &= e^{\alpha t}\int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s))\, ds - \int_0^t \alpha f(X_s)\, ds + \int_0^t \mathcal{A}f(X_s)\, ds\end{aligned}$$

Thus, M_t can be written as a stochastic integral of $\frac{1}{C_t}$ w.r.t. continuous local martingale Λ, hence M is also in $\mathcal{M}^{c,loc}$.

Now we prove $M \in \mathcal{M}^{c,loc} \iff N \in \mathcal{M}^{c,loc}$. With a bit abuse of notation, we redefine C_t as

$$C_t \triangleq \exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}\, ds\right\}, \quad (C_t)' = \frac{dC_t}{dt} = -C_t\frac{\mathcal{A}f(X_t)}{f(X_t)}$$

Since f is bounded away from zero on compact sets, C_t is of finite variation. By Problem 3.3.12 in BMSC,

$$\begin{aligned}\int_0^t C_s\, dM_s &= C_t M_t - \int_0^t M_s\, dC_s = C_t M_t - \int_0^t M_s (C_s)'\, ds \\ &= C_t f(X_t) - C_t f(X_0) - C_t\int_0^t \mathcal{A}f(X_s)\, ds \\ &+ \int_0^t C_s\mathcal{A}f(X_s)\, ds + f(X_0)\int_0^t (C_s)'\, ds + \int_0^t\int_0^s \mathcal{A}f(X_u)\, du\, (C_s)'\, ds \\ &= C_t f(X_t) - f(X_0) - C_t\int_0^t \mathcal{A}f(X_s)\, ds + \int_0^t C_s\mathcal{A}f(X_s)\, ds + \int_0^t\int_0^s \mathcal{A}f(X_u)\, du\, (C_s)'\, ds \\ &= C_t f(X_t) - f(X_0) = N_t\end{aligned}$$

Notice that the second last equality is due to integration by parts for the double integral, i.e.,

$$\begin{aligned}\int_0^t\int_0^s \mathcal{A}f(X_u)\, du\, (C_s)'\, ds = \int_0^t\int_0^s \mathcal{A}f(X_u)\, du\, dC_s &= C_s\int_0^s \mathcal{A}f(X_u)\, du\Big|_0^t - \int_0^t C_s\mathcal{A}f(X_s)\, ds \\ &= C_t\int_0^t \mathcal{A}f(X_s)\, ds - \int_0^t C_s\mathcal{A}f(X_s)\, ds\end{aligned}$$

Therefore, N_t can be written as a stochastic integral of C_t w.r.t. continuous local martingale M, hence N is also in $\mathcal{M}^{c,loc}$. Conversely, replace C_t by $\frac{1}{C_t}$, and it is still true that $\frac{1}{C_t}$ is of bounded variation, so

$$\begin{aligned}
\int_0^t \frac{1}{C_s}\, dN_s &= \frac{N_t}{C_t} - \int_0^t N_s\, d\left(\frac{1}{C_s}\right) = \frac{N_t}{C_t} + \int_0^t \frac{N_s(C_s)'}{C_s^2}\, ds \\
&= f(X_t) - \frac{f(X_0)}{C_t} + \int_0^t \frac{f(X_s)(C_s)'}{C_s}\, ds - f(X_0)\int_0^t \frac{(C_s)'}{C_s^2}\, ds \\
&= f(X_t) - f(X_0) + \int_0^t \frac{f(X_s)(C_s)'}{C_s}\, ds \\
&= f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\, ds = M_t
\end{aligned}$$

Thus, M_t can be written as a stochastic integral of $\frac{1}{C_t}$ w.r.t. continuous local martingale N, hence M is also in $\mathcal{M}^{c,loc}$.

Problem 3 (HW5 Exercise 3)

Let the function $b : \mathbb{R} \to \mathbb{R}$ be locally square-integrable, i.e., for all $x \in \mathbb{R}$, there exists $\epsilon > 0$ such that $\int_{x-\epsilon}^{x+\epsilon} b^2(y)\, dy < \epsilon$.

(i) Show that, corresponding to every initial distribution μ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, the equation

$$dX_t = b(X_t)dt + dW_t$$

has a weak solution up to an explosion time S, and this solution is unique in the sense of probability law.

(ii) Prove that for every finite $T > 0$, this solution obeys the generalized Girsanov formula,

$$P(X_t \in \Gamma, S > T) = \mathbb{E}\left[\exp\left(\int_0^T b(W_s)\, dW_s - \frac{1}{2}\int_0^T b^2(W_s)\, ds\right) 1_{\{W_t\in\Gamma\}}\right]$$

for every $\Gamma \in \mathcal{B}_T(C[0,\infty))$, the σ-field of (3.19) with $m = 1$. Here W is an one-dimensional Brownian motion with $P(W_0 \in B) = \mu(B)$ for $B \in \mathcal{B}(\mathbb{R})$.

(iii) In particular, conclude that with W as in (ii), the nonnegative supermartingale

$$Z_t = \exp\left\{\int_0^t b(W_s)\, dW_s - \frac{1}{2}\int_0^t b^2(W_s)\, ds\right\}, \quad t \in [0,\infty)$$

is a martingale if and only if $P(S = \infty) = 1$; this is equivalent to $\varphi(\pm\infty) = \infty$, where

$$\varphi(x) = \int_0^x \int_0^y \exp\left\{-2\int_z^y b(u)\, du\right\}\, dz\, dy$$

(i) Here we use the method of removal of drift in Section 5.5 of BMSC. Fix a number c, the scale function is defined by

$$p(x) \triangleq \int_c^x \exp\left\{-2\int_c^\zeta b(\xi)\, d\xi\right\} d\zeta, \quad p'(x) = \exp\left\{-2\int_c^x b(\xi)\, d\xi\right\}$$

Let $q(y)$ be the inverse of $p(x)$, and define $g(y) = p'(q(y))$. According to Engelbert-Schmidt theorem and Proposition 5.5.13 in BMSC, it suffices to show the desired result for a new equation $dY_t = g(Y_t)dW_t$, i.e., $I(g) = Z(g)$, where $Z(g) \triangleq \{x \in \mathbb{R} \,|\, g(x) = 0\}$ and

$$I(g) \triangleq \left\{ x \in \mathbb{R} \,\middle|\, \int_{-\epsilon}^{\epsilon} \frac{dy}{g^2(x+y)} = \infty, \ \forall\, \epsilon > 0 \right\}$$

Notice that $g(y) > 0$ for any $y \in \mathbb{R}$, so $Z(g) = \emptyset$. Also, for any fixed $x \in \mathbb{R}$ and $\epsilon > 0$, $g(x+y)$ is bounded away from zero for all $y \in (-\epsilon, \epsilon)$. Thus, $I(g) = \emptyset$ and hence $I(g) = Z(g)$.

(ii) For each fixed $k \in \mathbb{N}$, define stopping times

$$\tau_k^X \triangleq S_k^X \wedge \inf \left\{ t \geq 0 \,\middle|\, \int_0^t b^2(X_s)\, ds \geq k \right\}, \quad S_k^X = \inf\{t \geq 0 \,|\, |X_k| \geq k\}$$

Similarly, we can define the above stopping times for the Brownian motion W, i.e.,

$$\tau_k^W \triangleq S_k^W \wedge \inf \left\{ t \geq 0 \,\middle|\, \int_0^t b^2(W_s)\, ds \geq k \right\}, \quad S_k^W = \inf\{t \in [0,1] \,|\, |W_t| \geq n\}$$

Notice that $S^X = \lim_{k\to\infty} S_k^X$ is the explosion time of X and $S^W = \lim_{k\to\infty} S_k^W$ is the explosion time of W. By Engelbert-Schmidt 0-1 law, we know for any $T < S_k^X$,

$$\int_0^T b^2(X_t)\, dt < \infty, \quad P - a.e.$$

Therefore, $\{S^X > T\} = \bigcup_{k=1}^{\infty} \{\tau_k^X > T\}$. Similarly, for W, we have $\{S^W > T\} = \bigcup_{k=1}^{\infty} \{\tau_k^W > T\}$. By applying MCT, it suffices to show

$$P(X_t \in \Gamma, \tau_k^X > T) = \mathbb{E}[Z^W_{t \wedge \tau_k^W} 1_{\{W_t \in \Gamma, \tau_k^W > T\}}], \quad \forall\, k \in \mathbb{N} \tag{2}$$

where Z_t^W is given by

$$Z_t^W = \exp\left\{ \int_0^t b(W_s)\, dW_s - \frac{1}{2}\int_0^t b^2(W_s)\, ds \right\}, \quad t \in [0, \infty)$$

By Novikov's condition, since b^2 is locally integrable, we conclude that

$$L^X_{t \wedge \tau_k^X} = \exp\left\{ -\int_0^{t\wedge\tau_k^X} b(X_s)\, dW_s - \frac{1}{2}\int_0^{t\wedge\tau_k^X} b^2(W_s)\, ds \right\}, \quad t \in [0, \infty)$$

is a positive martingale. By Girsanov theorem, this define a new probability measure Q w.r.t. the filtration associated with the weak solution X_t. Also, $dQ = L_t^X dP$ and under Q,

$$\hat{W}_{t\wedge\tau_k^X} = W_{t\wedge\tau_k^X} + \int_0^{t\wedge\tau_k^X} b(X_t)\, dW_t$$

is a Brownian motion with initial distribution μ stopped at τ_k^X. By uniqueness in distribution, $X_{t\wedge\tau_k^X}$ has the same distribution as $\hat{W}_{t\wedge\tau_k^X}$, i.e., it is also a Brownian motion. We also define Z_t^X to be

$$Z_t^X = \exp\left\{ \int_0^t b(X_s)\, d\hat{W}_s - \frac{1}{2}\int_0^t b^2(X_s)\, ds \right\}, \quad t \in [0, \infty)$$

Observe that $L_t^X = \frac{1}{Z_t^X}$, so

$$P(X_t \in \Gamma, \tau_k^X > T) = \mathbb{E}[L^X_{T\wedge\tau_k^X} Z^X_{T\wedge\tau_k^X} 1_{\{X_t\in\Gamma,\tau_k^X>T\}}] = \mathbb{E}^Q[Z^X_{T\wedge\tau_k^X} 1_{\{X_t\in\Gamma,\tau_k^X>T\}}]$$
$$= \mathbb{E}[Z^W_{T\wedge\tau_k^W} 1_{\{W_t\in\Gamma,\tau_k^W>T\}}]$$

This verifies the desired property (2). Applying MCT, we obtain

$$P(X_t \in \Gamma, S^X > T) = \mathbb{E}[Z^W_{T\wedge S^W} 1_{\{W_t\in\Gamma,S^W>T\}}]$$

Notice that $S^W = \infty$ (Brownian motion will never blow up in finite time), so it reduces to the desired formula $P(X_t \in \Gamma, S^X > T) = \mathbb{E}[Z_T^W 1_{\{W_t\in\Gamma\}}]$.

(iii) It is easy to see that, as a nonnegative supermartingale, Z_t is a true martingale if and only if $\mathbb{E}[Z_T] = 1$ for all $T > 0$. Take Γ to be the universal set of the σ-field in the generalized Girsanov formula, it reduces to $P(S > T) = \mathbb{E}[Z_T]$. Since $P(S = \infty) = \infty$ if and only if $P(S > T) = 1$ for all $T > 0$, the desired result follows. Furthermore, from part (i), we could see that both the nondegenerate (ND) and locally integrable (LI) conditions hold, then by Feller's test for explosion (Theorem 5.29, BMSC), $P(S = \infty) = 1$ if and only if the Feller function $v(\pm\infty) = \infty$, i.e.,

$$\varphi(x) = v(x) = \int_0^x \int_0^y \exp\left\{-2\int_z^y b(u)\, du\right\}\, dz\, dy$$

Here we take reference point $c = 0$, $l = -\infty$, $r = +\infty$, and drift coefficient function $\sigma \equiv 1$.

Problem 4 (HW5 Exercise 5)

Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be of class C^2 with bounded second partial derivatives and bounded gradient $\nabla\Phi$, and consider the Smoluchowski equation

$$dX_t = \nabla\Phi(X_t)dt + dW_t, \quad t \in [0,\infty)$$

where W is a standard, $\mathbb{R}^d$-valued Brownian motion. According to Theorems 5.2.5, 5.2.9 and Problem 5.2.12, this equation admits a unique strong solution for every initial distribution on X_0. Show that the measure

$$\mu(dx) = e^{2\Phi(x)}dx \quad \text{on } \mathcal{B}(\mathbb{R}^d)$$

is invariant for Smoluchowski equation, i.e., if $X^{(a)}$ is the unique strong solution with initial condition $X_0^{(a)} = a \in \mathbb{R}^d$, then

$$\mu(A) = \int_{\mathbb{R}^d} P(X_t^{(a)} \in A)\mu(da), \quad \forall\, A \in \mathcal{B}(\mathbb{R}^d)$$

holds for every $t \in [0,\infty)$.

For any $f \in C_b^\infty(\mathbb{R}^d)$, by Ito's rule, we have

$$df(X_t) = \nabla f(X_t)dX_t + \frac{1}{2}\Delta f(X_t)d\langle X\rangle_t = \left(\nabla f\cdot\nabla\Phi + \frac{1}{2}\Delta f\right)dt + (\nabla f)dW_t$$

This implies

$$\frac{d}{dt}\mathbb{E}[f(X_t)] = \frac{\mathbb{E}[df(X_t)]}{dt} = \mathbb{E}\left[\nabla f\cdot\nabla\Phi + \frac{1}{2}\Delta f\right]$$

because $\mathbb{E}[(\nabla f)dW_t] = 0$ (this is just for simplicity, more rigorously it is because $\int_0^t \nabla f \, dW_t$ is a martingale). On the other hand, suppose $p(x,t)$ is the density function of the distribution of X_t, then we also have

$$\frac{d}{dt}\mathbb{E}[f(X_t)] = \frac{d}{dt}\int_{\mathbb{R}^d} f(x)p(x,t)\, dx = \int_{\mathbb{R}^d} f(x)\frac{\partial p(x,t)}{\partial t}\, dx$$

Equate the above two equation, we have

$$\int_{\mathbb{R}^d} f(x)\frac{\partial p(x,t)}{\partial t}\, dx = \int_{\mathbb{R}^d}\left[\nabla f(x)\cdot\nabla\Phi(x) + \frac{1}{2}\Delta f(x)\right] p(x,t)\, dx$$

Using integration by parts for divergence operator,

$$0 = \int_{\mathbb{R}^d} \nabla[f(x)(p(x,t)\nabla\Phi(x))]\, dx = \int_{\mathbb{R}^d} \nabla f(x)\cdot(p(x,t)\nabla\Phi(x))\, dx + \int_{\mathbb{R}^d} f(x)\nabla(p(x,t)\nabla\Phi(x))\, dx$$

because of fundamental theorem of calculus and the fact that $f(x)\nabla\Phi(x)$ vanishes outside a compact region. Similarly, we can obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta f(x)p(x,t)\, dx &= \int_{\mathbb{R}^d} \nabla(\nabla f(x)p(x,t))\, dx - \int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla p(x,t)\, dx = -\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla p(x,t)\, dx \\ &= -\int_{\mathbb{R}^d} \nabla(f(x)\nabla p(x,t))\, dx + \int_{\mathbb{R}^d} f(x)\Delta p(x,t)\, dx = \int_{\mathbb{R}^d} f(x)\Delta p(x,t)\, dx\end{aligned}$$

Therefore, we finally obtain

$$\int_{\mathbb{R}^d} f(x)\frac{\partial p(x,t)}{\partial t}\, dx = \int_{\mathbb{R}^d}\left[-\nabla(p(x,t)\nabla\Phi(x)) + \frac{1}{2}\Delta p(x,t)\right] f(x)\, dx$$

By results in approximation theory, we can conclude that

$$\frac{\partial p(x,t)}{\partial t} = -\nabla(p(x,t)\nabla\Phi(x)) + \frac{1}{2}\Delta p(x,t)$$

Suppose $p(x,t)$ defines an invariant measure, then we require it to be independent of time, i.e., $\frac{\partial p(x,t)}{\partial t} = 0$. Thus, it suffices to check the proposed invariant density $p(x,t) = e^{2\Phi(x)}$ satisfies

$$\nabla(p(x,t)\nabla\Phi(x)) = \frac{1}{2}\Delta p(x,t)$$

Notice that $\nabla p(x,t) = 2p(x,t)\nabla\Phi(x)$, thus the above equality holds and $p(x,t) = e^{2\Phi(x)}$ indeed defines an invariant measure.

Problem 5 (HW5 Exercise 7)

Suppose the function $g : \mathbb{R} \to [c,\infty)$, for some real number $c > 0$, is measurable. Show that the stochastic integral equation

$$Y_t = \int_0^t g(Y_s)\, dW_s, \quad t \in [0,\infty)$$

with W scalar Brownian motion, has a weak solution, which is unique in distribution.

The integral equation is equivalent to $dY_t = g(Y_t)dW_t$. Recall Theorem 5.5.7 in BMSC (Engelbert & Schmidt), the above equation has a weak solution unique in probability law if and only if $I(g) = Z(g)$, where $Z(g) \triangleq \{x \in \mathbb{R} \,|\, g(x) = 0\}$ and

$$I(g) \triangleq \left\{x \in \mathbb{R} \,\middle|\, \int_{-\epsilon}^{\epsilon} \frac{dy}{g^2(x+y)} = \infty,\ \forall\, \epsilon > 0\right\}$$

Since $g(x) \geq c > 0$ for all $x \in \mathbb{R}$, it is easy to see $Z(g) = \emptyset$. Since for all $x \in \mathbb{R}$,

$$\int_{-\epsilon}^{\epsilon} \frac{dy}{g^2(x+y)} \leq \int_{-\epsilon}^{\epsilon} \frac{dy}{c^2} = 2\epsilon c^2 < \infty,$$

we know that $I(g) = \emptyset$. Thus, $I(g) = Z(g)$ and the theorem applies, showing that the equation has a weak solution unique in distribution.

We can also prove the result without using the theorem. A direct proof of weak uniqueness is obtained by verfying that any weak solution of the equation is necessarily a time-changed Brownian motion. From the integral equation we know $Y(t)$ is a continuous local martingale with quadratic variation

$$\langle Y \rangle_t = \int_0^t g^2(Y(s))\, ds \geq c^2 t$$

This implies that $\langle Y \rangle_t = \infty$, and thus DDS applies, showing that $Y(t) = B_{\langle Y \rangle_t}$, where B_t is a Brownian motion adapted to a specific filtration. This shows that $Y(t)$, if exists, is unique in distribution.

A direct proof of existence is obtained by construction. Let B be a standard scalar Brownian motion. Consider the time change

$$C_t = \int_0^t \frac{ds}{g^2(B_s)} \leq \frac{t}{c^2} \tag{3}$$

It is continuous and strictly increasing, so it has an inverse $A_t \geq c^2 t \to \infty$. Notice that

$$A_t = \int_0^{A_t} g^2(B_s)\, dC_s = \int_0^t g^2(B_{A_u})\, du = \int_0^t g^2(Y_u)\, du$$

by defining $Y_t = B_{A_t}$. Notice that Y_t has quadratic variation $\langle Y \rangle_t = A_t$. By local martingale representation, $Y_t = \int_0^t g(Y_s)\, dW_s$ for a Brownian motion W_t adapted to a specific filtration under certain probability measure. This shows that Y_t satisfies the orginal equation, with W_t and its corresponding filtration and probability space.