MAT3220 Additional Exercises: Convexity

李肖鹏 (116010114)

Due date: Tomorrow

Question 1. Why a real symmetric matrix will always have real (as opposed to complex) eigenvalues?

Suppose λ_0 is an eigenvalue of real symmetric matrix A, then there exists a nonzero eigenvector $\vec{a}$ such that

$$A\vec{a} = \lambda_0 \vec{a} \tag{1}$$

Take complex conjugate on both sides of (1), we have

$$\overline{A\vec{a}} = \overline{\lambda_0 \vec{a}} \Longrightarrow \overline{A} \cdot \overline{\vec{a}} = \overline{\lambda_0} \cdot \overline{\vec{a}} \Longrightarrow A \cdot \overline{\vec{a}} = \overline{\lambda_0} \cdot \overline{\vec{a}} \tag{2}$$

Also, take transpose on both sides of (1), we have

$$(A\vec{a})^{\mathrm{T}} = (\lambda_0 \vec{a})^{\mathrm{T}} \Longrightarrow \vec{a}^{\mathrm{T}} A^{\mathrm{T}} = \lambda_0 \vec{a}^{\mathrm{T}} \Longrightarrow \vec{a}^{\mathrm{T}} A = \lambda_0 \vec{a}^{\mathrm{T}} \tag{3}$$

Multiply $\vec{a}^{\mathrm{T}}$ on the left on both sides of (2), and multiply $\overline{\vec{a}}$ on the right on both sides of (3), we have

$$\vec{a}^{\mathrm{T}} A \overline{\vec{a}} = \overline{\lambda_0} \vec{a}^{\mathrm{T}} \overline{\vec{a}} \quad \text{and} \quad \vec{a}^{\mathrm{T}} A \overline{\vec{a}} = \lambda_0 \vec{a}^{\mathrm{T}} \overline{\vec{a}}$$

Hence, we conclude that

$$(\lambda_0 - \overline{\lambda_0}) \|\vec{a}\|_2^2 = 0$$

Since $\vec{a} \neq \vec{0}$, $\|\vec{a}\|_2 \neq 0$, then $\lambda_0 = \overline{\lambda_0}$, meaning that $\lambda_0 \in \mathbb{R}$.

Question 2. Prove the following Cauchy-Schwarz inequality, i.e., for any $\vec{u}, \vec{v} \in \mathbb{R}^n$, we have

$$\vec{u}^{\mathrm{T}} \vec{v} \leq \|\vec{u}\|_2 \cdot \|\vec{v}\|_2$$

Consider the following inequality,

$$\begin{aligned} 0 \leq \|\vec{u} - \lambda\vec{v}\|_2^2 &= (\vec{u} - \lambda\vec{v})^{\mathrm{T}}(\vec{u} - \lambda\vec{v}) \\ &= (\vec{u}^{\mathrm{T}} - \lambda\vec{v}^{\mathrm{T}})(\vec{u} - \lambda\vec{v}) \\ &= \|\vec{u}\|_2^2 - 2\lambda\vec{u}^{\mathrm{T}}\vec{v} + \lambda^2\|\vec{v}\|_2^2 \end{aligned}$$

Since for any λ,

$$f(\lambda) = \|\vec{u}\|_2^2 - 2\lambda\vec{u}^{\mathrm{T}}\vec{v} + \lambda^2\|\vec{v}\|_2^2 \geq 0$$

We have

$$\Delta = 4(\vec{u}^{\mathrm{T}}\vec{v})^2 - 4\|\vec{u}\|_2^2\|\vec{v}\|_2^2 \leq 0$$

We will finally conclude that

$$\vec{u}^{\mathrm{T}}\vec{v} \leq \|\vec{u}\|_2 \cdot \|\vec{v}\|_2$$

Question 3. Use the Cauchy-Schwarz inequality to prove the so-called triangle inequality for the Euclidean norm,

$$\|\vec{x} + \vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

To prove $\|\vec{x} + \vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$, we only need to prove

$$(\vec{x} + \vec{y})^{\mathrm{T}}(\vec{x} + \vec{y}) \leq \vec{x}^{\mathrm{T}}\vec{x} + 2\|\vec{x}\|_2\|\vec{y}\|_2 + \vec{y}^{\mathrm{T}}\vec{y}$$

But the left hand side is just

$$\vec{x}^{\mathrm{T}}\vec{x} + 2\vec{x}^{\mathrm{T}}\vec{y} + \vec{y}^{\mathrm{T}}\vec{y}$$

By Cauchy-Schwarz inequality, $2\vec{x}^{\mathrm{T}}\vec{y} \leq 2\|\vec{x}\|_2\|\vec{y}\|_2$, hence, we finish the proof.

Question 4. For a square matrix, $A \in \mathbb{R}^{n\times n}$, its *trace* is $\mathrm{tr}\,(A) = \sum_{i=1}^{n} a_{ii}$. Prove that for any $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{m\times n}$, we have $\mathrm{tr}\,(XY^{\mathrm{T}}) = \mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$.

Consider the (i,i)-th entry of XY^{T}, if we denote X_i as the i-th row of X, and Y_i as the i-th row of Y, then we have

$$(XY^{\mathrm{T}})_{i,i} = X_iY_i^{\mathrm{T}} = \sum_{j=1}^{n} X_{ij}Y_{ij}$$

Hence, the trace of XY^{T} can be computed by

$$\mathrm{tr}\,(XY^{\mathrm{T}}) = \sum_{i=1}^{m} X_iY_i^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$$

Similarly, consider the (i,i)-th entry of YX^{T}, we have

$$(YX^{\mathrm{T}})_{i,i} = Y_iX_i^{\mathrm{T}} = \sum_{j=1}^{n} Y_{ij}X_{ij}$$

Hence, the trace of YX^{T} can be computed by

$$\mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m} Y_iX_i^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}X_{ij}$$

In conclusion,

$$\mathrm{tr}\,(XY^{\mathrm{T}}) = \mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$$

Question 5. Let $X \in \mathbb{R}^{m\times n}$ be a real matrix. The so-called Frobenius norm of X is defined as

$$\|X\|_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2\right)^{1/2}$$

and its spectrum norm is defined as $\|X\|_2 := \left(\lambda_{\max}(X^{\mathrm{T}}X)\right)^{1/2}$. Prove that both $\|\cdot\|_F$ and $\|\cdot\|_2$ are indeed matrix norms.

We first prove $\|\cdot\|_F$ is matrix norms, by checking whether it satisfies the five defining properties. For property (1), it is obvious that $\|\cdot\|_F \geq 0$. For property (2), if $\|X\|_F = 0$, we can derive that all X_{ij}^2 are equal to zero, meaning that X is zero matrix. For property (3),

$$\begin{aligned}
\|\alpha X\|_F &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(\alpha X_{ij})^2\right)^{1/2} \\
&= \left(\alpha^2\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} \\
&= |\alpha|\left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} = |\alpha|\|X\|_F
\end{aligned}$$

For property (4), to prove $\|X+Y\|_F \leq \|X\|_F + \|Y\|_F$, we only need to prove

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}+Y_{ij})^2 \leq \left[\left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} + \left(\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij}^2\right)^{1/2}\right]^2$$

which is equivalent to say

$$\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}Y_{ij} \leq \left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij}^2\right)^{1/2}$$

However, this is exactly Cauchy-Schwarz inequality, so the proof of property (4) is finished. For property (5),

$$\begin{aligned}
\|XY\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}X_{ik}Y_{kj}\right)^2 \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}X_{ik}^2\sum_{k=1}^{n}Y_{kj}^2\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}^2\left(\sum_{j=1}^{n}\sum_{k=1}^{n}Y_{kj}^2\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}^2\|Y\|_F^2 \\
&= \|X\|_F^2\|Y\|_F^2
\end{aligned}$$

Hence, $\|\cdot\|_F$ is matrix norm.

Then we prove $\|\cdot\|_2$ is matrix norm. For property (1), since $X^{\mathrm{T}}X$ is always positive semi-definite, so all of its eigenvalues are non-negative, hence $\|X\|_2 := \left(\lambda_{\max}(X^{\mathrm{T}}X)\right)^{1/2} \geq 0$. For property (2), if $\|X\|_2 = 0$, we can derive that all eigenvalues of $X^{\mathrm{T}}X$ are zero, but since it is symmetric, so it must be zero matrix. If $X^{\mathrm{T}}X$ is zero matrix, consider its (i,i)-th entry,

$$(X^{\mathrm{T}}X)_{ii} = X_i^{\mathrm{T}}X_i = 0 \Longrightarrow X_i = \vec{0}$$

where X_i denote the i-th column of X. It is obvious that X is zero matrix, and we finish the proof of property (2). For property (3), we have

$$\|\alpha X\|_2 := \left(\lambda_{\max}(\alpha^2 X^{\mathrm{T}} X)\right)^{1/2} = \left(\alpha^2 \lambda_{\max}(X^{\mathrm{T}} X)\right)^{1/2} = |\alpha| \|X\|_2$$

For property (4), we only need to prove,

$$\left(\lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y))\right)^{1/2} \leq \left(\lambda_{\max}(X^{\mathrm{T}} X)\right)^{1/2} + \left(\lambda_{\max}(Y^{\mathrm{T}} Y)\right)^{1/2}$$

Let $\mu = \lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y))$, then we can take a unit eigenvetor $\vec{v}$ corresponding to μ, i.e.,

$$(X+Y)^{\mathrm{T}}(X+Y)\vec{v} = \mu \vec{v}, \quad \|\vec{v}\|_2 = 1$$

Then, we know

$$\begin{aligned}
\mu &= \vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} + \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} + 2(X\vec{v})^{\mathrm{T}}(Y\vec{v}) \\
&\leq \vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} + \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} + 2\|X\vec{v}\|_2 \|Y\vec{v}\|_2 \\
&= (\|X\vec{v}\|_2 + \|Y\vec{v}\|_2)^2 = \left(\sqrt{\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v}} + \sqrt{\vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v}}\right)^2
\end{aligned}$$

Since $X^{\mathrm{T}}X$ is a real symmetric matrix, according to spectral decomposition, there exists orthogonal matrix T, such that $T^{-1}X^{\mathrm{T}}XT = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n$ is the eigenvalues of $X^{\mathrm{T}}X$. For any vector $\vec{v}$, suppose $(T^{\mathrm{T}}\vec{v})^{\mathrm{T}} = (w_1, \ldots, w_n)$, then

$$\begin{aligned}
\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} &= \vec{v}^{\mathrm{T}} T \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) T^{-1} \vec{v} = (T^{\mathrm{T}}\vec{v})^{\mathrm{T}} \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(T^{\mathrm{T}}\vec{v}) \\
&= (w_1, \ldots, w_n) \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(w_1, \ldots, w_n)^{\mathrm{T}} = \lambda_1 w_1^2 + \ldots + \lambda_n w_n^2 \\
&\leq \lambda_1 (w_1^2 + \ldots + w_n^2) = \lambda_1 \|T^{\mathrm{T}}\vec{v}\|_2^2 = \lambda_1 \|\vec{v}\|_2^2 = \lambda_1
\end{aligned}$$

Hence, $\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} \leq \lambda_{\max}(X^{\mathrm{T}}X)$. Similarly, we have $\vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} \leq \lambda_{\max}(Y^{\mathrm{T}}Y)$. Therefore, we have

$$\lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y)) \leq \left((\lambda_{\max}(X^{\mathrm{T}}X))^{1/2} + (\lambda_{\max}(Y^{\mathrm{T}}Y))^{1/2}\right)^2$$

which proves property (4). For property (5), we need to prove

$$\mu = \lambda_{\max}((XY)^{\mathrm{T}}(XY)) \leq \lambda_{\max}(X^{\mathrm{T}}X)\lambda_{\max}(Y^{\mathrm{T}}Y)$$

Similar to property (4), we will obtain

$$\begin{aligned}
\mu = \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} X^{\mathrm{T}} X Y \vec{v} &\leq \lambda_{\max}(X^{\mathrm{T}}X) \|Y\vec{v}\|_2^2 \\
&\leq \lambda_{\max}(X^{\mathrm{T}}X)\lambda_{\max}(Y^{\mathrm{T}}Y)\|\vec{v}\|_2^2 = \lambda_{\max}(X^{\mathrm{T}}X)\lambda_{\max}(Y^{\mathrm{T}}Y)
\end{aligned}$$

Hence, $\|\cdot\|_2$ is matrix norm.

Question 6. Prove that for any $X \in \mathbb{R}^{m\times n}$ and $\vec{y} \in \mathbb{R}^m$,

$$\|X\vec{y}\|_2 \leq \|X\|_2 \cdot \|\vec{y}\|_2$$

Actually, we have already prove this during the proof of Question 5. Since $X^{\mathrm{T}}X$ is a real symmetric matrix, according to spectral decomposition, there exists orthogonal matrix T, such that

$T^{-1}X^{\mathrm{T}}XT = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n$ is the eigenvalues of $X^{\mathrm{T}}X$. For any vector $\vec{y}$, suppose $(T^{\mathrm{T}}\vec{y})^{\mathrm{T}} = (w_1, \ldots, w_n)$, then

$$\begin{aligned}
\vec{y}^{\mathrm{T}}X^{\mathrm{T}}X\vec{y} &= \vec{y}^{\mathrm{T}}T\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)T^{-1}\vec{y} = (T^{\mathrm{T}}\vec{y})^{\mathrm{T}}\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(T^{\mathrm{T}}\vec{y}) \\
&= (w_1, \ldots, w_n)\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(w_1, \ldots, w_n)^{\mathrm{T}} = \lambda_1 w_1^2 + \ldots + \lambda_n w_n^2 \\
&\leq \lambda_1(w_1^2 + \ldots + w_n^2) = \lambda_1\|T^{\mathrm{T}}\vec{y}\|_2^2 = \lambda_1\|\vec{y}\|_2^2
\end{aligned}$$

However, by definition, $\lambda_1 = \lambda_{\max}(X^{\mathrm{T}}X) = \|X\|_2^2$, we then conclude that

$$\|X\vec{y}\|_2 \leq \|X\|_2 \cdot \|\vec{y}\|_2$$

Question 7. Prove that for any X, it holds that $\|X\|_2 \leq \|X\|_F$.

Use the same method as we did in Question 4, we can obtain

$$\mathrm{tr}\,(X^{\mathrm{T}}X) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2$$

Since $X^{\mathrm{T}}X$ is positive semi-definite matrix, all of its eigenvalue is nonegative, so the trace of it is larger than or equal to the largest eigenvalue of it, i.e.,

$$\mathrm{tr}\,(X^{\mathrm{T}}X) \geq \lambda_{\max}\,(X^{\mathrm{T}}X)$$

Therefore,

$$\|X\|_2 = \left(\lambda_{\max}\,(X^{\mathrm{T}}X)\right)^{1/2} \leq \left(\mathrm{tr}\,(X^{\mathrm{T}}X)\right)^{1/2} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2\right)^{1/2} = \|X\|_F$$

Question 8. Compute the gradient of the quartic function

$$f(x) = (\vec{x}^{\mathrm{T}}A\vec{x})^2$$

where $A \in \mathcal{S}^n$.

First, we know that the derivative of the quadratic form with respect to vector $\vec{x}$ is given by (assuming that A is symmetric)

$$\nabla_{\vec{x}}\left(\vec{x}^{\mathrm{T}}A\vec{x}\right) = 2A\vec{x}$$

Hence, by chain rule, we have

$$\nabla_{\vec{x}} f(\vec{x}) = 2\vec{x}^{\mathrm{T}}A\vec{x}(2A\vec{x}) = 4(\vec{x}^{\mathrm{T}}A\vec{x})A\vec{x}$$

Question 9. Compute the Hessian matrix of the quartic function

$$f(x) = (\vec{x}^{\mathrm{T}}A\vec{x})^2$$

where $A \in \mathcal{S}^n$.

We can see that the hessian matrix is given by

$$\nabla^2_{\vec{x}} f(\vec{x}) = \nabla_{\vec{x}}(4(\vec{x}^{\mathrm{T}} A\vec{x})A\vec{x})$$

Therefore, we have

$$\nabla^2_{\vec{x}} f(\vec{x}) = 4(A\vec{x})(A\vec{x})^{\mathrm{T}} + 8(\vec{x}^{\mathrm{T}} A\vec{x})A$$

Question 10. Prove that if $h(\vec{x})$ is twice continuously differentiable, then that $h(\vec{x})$ is convex in $\mathbb{R}^n$ is equivalent to $\nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$.

We first claim that $h(\vec{x})$ is convex in $\mathbb{R}^n$ if and only if for any $\vec{x}, \vec{y}$, we have

$$h(\vec{y}) \geq h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x})$$

If so, suppose $H_h(\vec{z}) = \nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$, by Taylor expansion, we have

$$h(\vec{y}) = h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x}) + \frac{1}{2}\left[(\vec{y} - \vec{x})^{\mathrm{T}} H_h(\vec{z})(\vec{y} - \vec{x})\right]$$

for some $\vec{z} \in [\vec{x}, \vec{y}]$. Therefore, we obtain

$$h(\vec{y}) \geq h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x})$$

By our claim, we can conclude that $h(\vec{x})$ is convex.

If we suppose $h(\vec{x})$ is convex, then for any $\vec{x}$ and $\vec{d}$, some $\lambda > 0$ will yield $\vec{x} + \lambda\vec{d}$. By Taylor expansion, we have

$$h(\vec{x} + \lambda\vec{d}) = h(\vec{x}) + \lambda\nabla h(\vec{x})^{\mathrm{T}}\vec{d} + \frac{\lambda^2}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + o(\|\lambda\vec{d}\|^2)$$

From our claim, we have

$$h(\vec{x} + \lambda\vec{d}) \geq h(\vec{x}) + \lambda\nabla h(\vec{x})^{\mathrm{T}}\vec{d}$$

Hence, we have

$$\frac{\lambda^2}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + o(\|\lambda\vec{d}\|^2) \geq 0$$

which implies

$$\frac{1}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + \|\vec{d}\|^2 o(1) \geq 0$$

Take $\lambda \to 0$, we conclude that $\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} \geq 0$, which means $H_h(\vec{x})$ is positive semi-definite for all $\vec{x}$. Thus, that $h(\vec{x})$ is convex in $\mathbb{R}^n$ is equivalent to $\nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$.

Now we prove our claim. First assume h is convex, and let $\vec{z} = \lambda\vec{y} + (1-\lambda)\vec{x}$ for some $\vec{x}, \vec{y}$ and $\lambda \in [0, 1]$. Since h is convex, we have

$$h(\vec{z}) = h(\lambda\vec{y} + (1-\lambda)\vec{x}) \leq \lambda h(\vec{y}) + (1-\lambda)h(\vec{x})$$

and therefore,

$$h(\vec{z}) - h(\vec{x}) \leq \lambda h(\vec{y}) + (1-\lambda)h(\vec{x}) - h(\vec{x}) = \lambda h(\vec{y}) - \lambda h(\vec{x})$$

Since we know

$$\nabla h(\vec{x})^{\mathrm{T}}\vec{d} = \lim_{\lambda\to 0+} \frac{h(\vec{x}+\lambda\vec{d}) - h(\vec{x})}{\lambda}$$

and therefore,

$$\nabla h(\vec{x})^{\mathrm{T}}(\vec{y}-\vec{x}) = \lim_{\lambda\to 0+} \frac{h(\vec{x}+\lambda(\vec{y}-\vec{x})) - h(\vec{x})}{\lambda} = \lim_{\lambda\to 0+} \frac{h(\vec{z}) - h(\vec{x})}{\lambda} \le h(\vec{y}) - h(\vec{x})$$

Now we assume $h(\vec{y}) \ge h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y}-\vec{x})$ for any $\vec{x}, \vec{y}$. Let $\vec{z} = \lambda\vec{y} + (1-\lambda)\vec{x}$, we have

$$h(\vec{y}) \ge h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\vec{y}-\vec{z}) \tag{1}$$

$$h(\vec{x}) \ge h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\vec{x}-\vec{z}) \tag{2}$$

Therefore, we have

$$\begin{aligned}
\lambda h(\vec{y}) + (1-\lambda)h(\vec{x}) &\ge \lambda h(\vec{z}) + \lambda\nabla h(\vec{z})^{\mathrm{T}}(\vec{y}-\vec{z}) + (1-\lambda)h(\vec{z}) + (1-\lambda)\nabla h(\vec{z})^{\mathrm{T}}(\vec{x}-\vec{z}) \\
&= h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\lambda\vec{y} - \lambda\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}((1-\lambda)\vec{x} - (1-\lambda)\vec{z}) \\
&= h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\lambda\vec{y} + (1-\lambda)\vec{x} - \vec{z}) \\
&= h(\vec{z}) = h(\lambda\vec{y} + (1-\lambda)\vec{x})
\end{aligned}$$

Hence, we conclude that h is convex. Therefore, we finish the proof of our claim.

Question 11. Prove that $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ is a concave function in $\mathbb{R}^n_{++}$.

Let $f(\vec{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$, and we need to compute the hessian matrix of $f(\vec{x})$. First we have

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{f(\vec{x})}{n x_i} \quad \text{for all} \quad i = 1, \dots, n$$

Then we compute the second-order partial derivative, we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) = \frac{f(\vec{x})}{n^2 x_i x_j}, \text{ for } i \ne j; \qquad \frac{\partial^2 f}{\partial x_i^2}(\vec{x}) = \frac{f(\vec{x})}{n^2 x_i^2}(1-n)$$

Therefore, we check the quadratic form of arbitrary vector $\vec{u} = (u_1, u_2, \dots, u_n)^{\mathrm{T}}$.

$$\begin{aligned}
\vec{u}^{\mathrm{T}} H_f(\vec{x})\vec{u} = \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}u_i u_j &= \frac{f(\vec{x})}{n^2}\left(\sum_{i=1}^{n}\frac{1-n}{x_i^2}u_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n}\frac{1}{x_i x_j}u_i u_j\right) \\
&= \frac{f(\vec{x})}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{u_i u_j}{x_i x_j} - n\sum_{i=1}^{n}\frac{u_i^2}{x_i^2}\right) \\
&= \frac{f(\vec{x})}{n^2}\left[\left(\sum_{i=1}^{n}\frac{u_i}{x_i}\cdot 1\right)^2 - \left(\sum_{i=1}^{n}1^2\right)\left(\sum_{i=1}^{n}\left(\frac{u_i}{x_i}\right)^2\right)\right] \\
&\le \frac{f(\vec{x})}{n^2}\cdot 0 = 0
\end{aligned}$$

By what we proved previously, if the hessian matrix $H_f(\vec{x})$ is negative semi-definite, then f is concave function in $\mathbb{R}^n_{++}$.

Question 12. Prove that
$$\frac{x_1^n}{x_2x_3\cdots x_n}$$
is a convex function in $\mathbb{R}^n_{++}$.

Let
$$f(\vec{x}) = \frac{x_1^n}{x_2x_3\cdots x_n}, \quad g(\vec{x}) = \ln f(\vec{x}) = n\ln x_1 - \sum_{i=2}^{n}\ln x_i$$
Then, we can compute
$$\nabla f(\vec{x}) = f(\vec{x})\nabla g(\vec{x}), \text{ where } \nabla g(\vec{x}) = \begin{bmatrix} \frac{n}{x_1} & -\frac{1}{x_2} & \cdots & -\frac{1}{x_n} \end{bmatrix}^{\mathrm{T}}$$
Also, by chain rule, we have
$$\nabla^2 f(\vec{x}) = f(\vec{x})\left(\nabla g(\vec{x})\nabla g(\vec{x})^{\mathrm{T}} + \nabla^2 g(\vec{x})\right), \text{ where } \nabla^2 g(\vec{x}) = \begin{bmatrix} -\frac{n}{x_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n^2} \end{bmatrix}$$
For any vector $\vec{u} \in \mathbb{R}^n$, we have
$$\begin{aligned}
\vec{u}^{\mathrm{T}}\nabla^2 f(\vec{x})\vec{u} &= f(\vec{x})\left[-n\left(\frac{u_1}{x_1}\right)^2 + \sum_{i=2}^{n}\left(\frac{u_i}{x_i}\right)^2 + \left(n\frac{u_1}{x_1} - \sum_{i=2}^{n}\frac{u_i}{x_i}\right)^2\right] \\
&= f(\vec{x})\left[-n\left(\frac{u_1}{x_1}\right)^2 + \sum_{i=2}^{n}\left(\frac{u_i}{x_i}\right)^2 + \left((n-1)\frac{u_1}{x_1} - \sum_{i=2}^{n}\frac{u_i}{x_i}\right)^2 + (2n-1)\frac{u_1^2}{x_1^2} - 2\frac{u_1}{x_1}\sum_{i=2}^{n}\frac{u_i}{x_i}\right] \\
&= f(\vec{x})\left[\sum_{i=2}^{n}\left(\frac{u_1}{x_1}\right)^2 - \sum_{i=2}^{n}2\frac{u_1}{x_1}\frac{u_i}{x_i} + \sum_{i=2}^{n}\left(\frac{u_i}{x_i}\right)^2 + \left((n-1)\frac{u_1}{x_1} - \sum_{i=2}^{n}\frac{u_i}{x_i}\right)^2\right] \\
&= f(\vec{x})\left[\sum_{i=2}^{n}\left(\frac{u_1}{x_1} - \frac{u_i}{x_i}\right)^2 + \left((n-1)\frac{u_1}{x_1} - \sum_{i=2}^{n}\frac{u_i}{x_i}\right)^2\right] \geq 0
\end{aligned}$$
Hence, the Hessian of $f(\vec{x})$ is always positive semi-definite, which implies that $f(\vec{x})$ is a convex function on $\mathbb{R}^n_{++}$.

Question 13. Consider $X \in S^{n\times n}$, and so X has n real eigenvalues as we discussed before. Let them be
$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$$
Prove that $\lambda_1(X)$ is a convex function.

First we prove a lemma. Suppose $f_\gamma : X \to \mathbb{R}$ is a family of convex functions, with $\gamma \in A$, some index set, and let $f(x) = \sup_{\gamma\in A} f_\gamma(x)$. Then, for any fixed $\alpha \in A$, $\lambda \in [0,1]$,
$$\begin{aligned}
f_\alpha(\lambda x + (1-\lambda)y) &\leq \lambda f_\alpha(x) + (1-\lambda)f_\alpha(y) \\
&\leq \sup_{\gamma\in A}(\lambda f_\gamma(x) + (1-\lambda)f_\gamma(y)) \\
&\leq \lambda \sup_{\gamma\in A} f_\gamma(x) + (1-\lambda)\sup_{\gamma\in A} f_\gamma(y) \\
&= \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

By taking the supremum of the left hand side, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Hence, $f(x)$ is also convex.

From Question 5, we know that for any unit vector $\vec{v}$, if X is symmetric matrix, then $\vec{v}^{\mathrm{T}} X \vec{v} \leq \lambda_1$, and when $\vec{v}$ is the unit eigenvector corresponding to λ_1, the maximum value λ_1 can be obtained. Thus, we could consider

$$\lambda_1(X) = \sup_{\|\vec{v}\|_2=1} g_{\vec{v}}(X), \quad \text{where } \; g_{\vec{v}}(X) = \vec{v}^{\mathrm{T}} X \vec{v}$$

For any fixed $\vec{v}$, $g_{\vec{v}}(X)$ is linear with respect to X, hence convex. By the lemma we proved just now, the supreme of it, that is, $\lambda(X)$, must be convex.

Question 14. Prove that

$$\ln\left(\sum_{i=1}^{n} e^{x_i}\right)$$

is a convex function.

Let $f(\vec{x})$ denote the original function, then we can compute

$$\nabla f(\vec{x}) = \frac{1}{\sum_{i=1}^{n} e^{x_i}} \begin{bmatrix} e^{x_1} & \cdots & e^{x_n} \end{bmatrix}^{\mathrm{T}}$$

and denote $H = \nabla^2 f(\vec{x})$, we have

$$\hat{H} = \left(\sum_{k=1}^{n} e^{x_k}\right)^2 [H]_{ij} = \begin{cases} e^{x_i} \sum_{k=1}^{n} e^{x_k} - e^{x_i+x_j} & \text{when } i = j \\ -\, e^{x_i+x_j} & \text{when } i \neq j \end{cases}$$

We only need to prove $\hat{H}$ is positive semi-definite matrix. For any $\vec{u} \in \mathbb{R}$, we have

$$\begin{aligned} \vec{u}^{\mathrm{T}} \hat{H} \vec{u} &= \sum_{i=1}^{n} \sum_{j=1}^{n} [\hat{H}]_{ij} u_i u_j \\ &= \left(\sum_{i=1}^{n} e^{x_i} u_i^2\right) \cdot \left(\sum_{i=1}^{n} e^{x_i}\right) - \sum_{i,j=1}^{n} e^{x_i} e^{x_j} u_i u_j \\ &= \left(\sum_{i=1}^{n} e^{x_i} u_i^2\right) \cdot \left(\sum_{i=1}^{n} e^{x_i}\right) - \left(\sum_{i=1}^{n} e^{x_i} u_i\right)^2 \geq 0 \end{aligned}$$

where the last line holds by Cauchy-Schwarz inequality. Hence, $\hat{H}$ is positive semi-definite, which means H is PSD, and f is a convex function.

Question 15. Suppose that $f(\vec{x}) \geq 0$ is convex for $\vec{x} \in S$, and $g(\vec{x}) > 0$ is concave for $\vec{x} \in S$. Prove that

$$\frac{f(\vec{x})}{g(\vec{x})}$$

is a quasi-convex function.

We only need to prove that for all a, the level set (when $g(\vec{x}) > 0$)

$$L_a = \left\{ \vec{x} \in S \;\middle|\; \frac{f(\vec{x})}{g(\vec{x})} < a \right\} = \{\vec{x} \in S \,|\, f(\vec{x}) < ag(\vec{x})\}$$

is a convex set. Take any two elements $\vec{x}, \vec{y}$ in L_a, we have

$$f(\vec{x}) < ag(\vec{x}), \quad f(\vec{y}) < ag(\vec{y})$$

Therefore, since f is convex, g is concave, we have for $\lambda \in (0, 1)$,

$$\begin{aligned} f(\lambda\vec{x} + (1-\lambda)\vec{y}) &\le \lambda f(\vec{x}) + (1-\lambda) f(\vec{y}) \\ &< \lambda a g(\vec{x}) + (1-\lambda) a g(\vec{y}) \\ &\le a g(\lambda\vec{x} + (1-\lambda)\vec{y}) \end{aligned}$$

Hence, $\lambda\vec{x} + (1-\lambda)\vec{y} \in L_a$, which means L_a is a convex set, and

$$\frac{f(\vec{x})}{g(\vec{x})}$$

is quasi-convex.

Question 16. Show that

$$\frac{\vec{a}^{\mathrm{T}}\vec{x} + b}{\vec{c}^{\mathrm{T}}\vec{x} + d}$$

is quasi-linear in $\{\vec{x} \,|\, \vec{c}^{\mathrm{T}}\vec{x} + d > 0\}$.

Let $f(\vec{x})$ denote the original function, we tend to prove both $f(\vec{x})$ and $-f(\vec{x})$ are quasi-convex. Consider the level set of $f(\vec{x})$,

$$\begin{aligned} S_\alpha &= \left\{ \vec{x} \;\middle|\; \vec{c}^{\mathrm{T}}\vec{x} + d > 0,\ \frac{\vec{a}^{\mathrm{T}}\vec{x} + b}{\vec{c}^{\mathrm{T}}\vec{x} + d} \le \alpha \right\} \\ &= \left\{ \vec{x} \;\middle|\; \vec{c}^{\mathrm{T}}\vec{x} + d > 0 \right\} \cap \left\{ \vec{x} \;\middle|\; \vec{a}^{\mathrm{T}}\vec{x} + b \le \alpha(\vec{c}^{\mathrm{T}}\vec{x} + d) \right\} \\ &= \left\{ \vec{x} \;\middle|\; \vec{c}^{\mathrm{T}}\vec{x} + d > 0 \right\} \cap \left\{ \vec{x} \;\middle|\; (\vec{a} - \alpha\vec{c})^{\mathrm{T}}\vec{x} \le \alpha d - b \right\} \\ &= S_\alpha^{(1)} \cap S_\alpha^{(2)} \end{aligned}$$

Since $S_\alpha^{(1)}$ and $S_\alpha^{(2)}$ are both half spaces, so they are both convex, and the intersection of two convex sets are convex, so S_α is convex, which shows $f(\vec{x})$ is quasi-convex.

Similarly, we can show that the level set of $-f(\vec{x})$ can also be written as the intersection of two half spaces, which are convex, so $-f(\vec{x})$ is also quasi-convex. Therefore, $f(\vec{x})$ is quasi-linear.

Question 17. Suppose that $f(\vec{x})$ is convex for $x \in S$, and $g(\vec{x}) > 0$ is concave for $\vec{x} \in S$. Prove that

$$\frac{[f(\vec{x})]^2}{g(\vec{x})}$$

is a convex function.

First we prove a lemma, for $a, b, c, d \in \mathbb{R}$ and $c, d > 0$,

$$\frac{(a+b)^2}{c+d} \le \frac{a^2}{c} + \frac{b^2}{d}$$

This is indeed true because

$$\begin{aligned}\frac{(a+b)^2}{c+d}-\left(\frac{a^2}{c}+\frac{b^2}{d}\right)&=\frac{(a+b)^2cd-(a^2d+b^2c)(c+d)}{(c+d)cd}\\&=\frac{(a^2cd+2abcd+b^2cd)-(a^2dc+b^2c^2+a^2d^2+b^2cd)}{(c+d)cd}\\&=\frac{-(bc-ad)^2}{(c+d)cd}\le 0\end{aligned}$$

Let $h(\vec{x})=[f(\vec{x})]^2/g(\vec{x})$, then for $\lambda\in(0,1)$, we have

$$\begin{aligned}h(\lambda\vec{x}+(1-\lambda)\vec{y})&=\frac{[f(\lambda\vec{x}+(1-\lambda)\vec{y})]^2}{g(\lambda\vec{x}+(1-\lambda)\vec{y})}\\&\le\frac{[\lambda f(\vec{x})+(1-\lambda)f(\vec{y})]^2}{\lambda g(\vec{x})+(1-\lambda)g(\vec{y})}\\&\le\frac{\lambda^2[f(\vec{x})]^2}{\lambda g(\vec{x})}+\frac{(1-\lambda)^2[f(\vec{y})]^2}{(1-\lambda)g(\vec{y})}&&\text{(By lemma)}\\&=\lambda h(\vec{x})+(1-\lambda)h(\vec{y})\end{aligned}$$

Hence, $h(\vec{x})$ is a convex function.

Question 18. Prove that $\prod_{i=1}^n x_i$ is quasi-concave in $\mathbb{R}^n_{++}$.

To prove $\prod_{i=1}^n x_i$ is quasi-concave, we only need to prove that the level set

$$S_\alpha=\left\{\vec{x}\in\mathbb{R}^n_{++}\;\middle|\;\prod_{i=1}^n x_i\ge\alpha\right\}$$

is convex for any α (because the domain of the function is convex). If $\alpha\le 0$, then the level set is reduced to be $S_\alpha=\mathbb{R}^n_{++}$, which is obviously convex. If $\alpha>0$, then S_α is equivalent to

$$S_\alpha=\left\{\vec{x}\in\mathbb{R}^n_{++}\;\middle|\;\sum_{i=1}^n\ln x_i\ge\ln\alpha\right\}$$

Consider any $\vec{x},\vec{y}\in S_\alpha$, and $\lambda\in[0,1]$, it is easy to know $\lambda\vec{x}+(1-\lambda)\vec{y}\in\mathbb{R}^n_{++}$. Also, since $\sum_{i=1}^n\ln x_i\ge\ln\alpha$ and $\sum_{i=1}^n\ln y_i\ge\ln\alpha$, we have

$$\begin{aligned}\sum_{i=1}^n\ln(\lambda x_i+(1-\lambda)y_i)&\ge\sum_{i=1}^n(\lambda\ln x_i+(1-\lambda)\ln y_i)\\&\ge\lambda\ln\alpha+(1-\lambda)\ln\alpha=\ln\alpha\end{aligned}$$

Therefore, $\lambda\vec{x}+(1-\lambda)\vec{y}\in S_\alpha$, which shows that S_α is convex.

Question 19. Show that $S:=\{\vec{x}\mid\|\vec{x}-\vec{a}\|_2\le\|\vec{x}-\vec{b}\|_2\}$ is a convex region. Further prove that $\|\vec{x}-\vec{a}\|_2/\|\vec{x}-\vec{b}\|_2$ is quasi-convex in S.

Consider the set S, we have

$$\begin{aligned}\{\vec{x}\mid\|\vec{x}-\vec{a}\|_2\le\|\vec{x}-\vec{b}\|_2\}&=\{\vec{x}\mid\vec{x}^{\mathrm{T}}\vec{x}-2\vec{a}^{\mathrm{T}}\vec{x}+\vec{a}^{\mathrm{T}}\vec{a}\le\vec{x}^{\mathrm{T}}\vec{x}-2\vec{b}^{\mathrm{T}}\vec{x}+\vec{b}^{\mathrm{T}}\vec{b}\}\\&=\{\vec{x}\mid 2(\vec{b}-\vec{a})^{\mathrm{T}}\vec{x}\le\vec{b}^{\mathrm{T}}\vec{b}-\vec{a}^{\mathrm{T}}\vec{a}\}\end{aligned}$$

which shows that S is a half-space. It is very easy to show by definition that a half-space is convex, and hence S is convex.

Next we need to prove the level set of $\|\vec{x}-\vec{a}\|_2/\|\vec{x}-\vec{b}\|_2$, which is given by

$$S_\alpha = \{\vec{x}\in S \,|\, \|\vec{x}-\vec{a}\|_2/\|\vec{x}-\vec{b}\|_2 \le \alpha\}$$

is convex for all α. If $\alpha < 0$, then S_α is empty set, hence trivially convex. If $\alpha \ge 1$, then $S_\alpha = S$, which we have proved is convex, so we only need to consider the case when $\alpha \in [0,1)$. In this case, S_α is equivalent to

$$\{\vec{x}\in S \,|\, (1-\alpha^2)\vec{x}^{\mathrm{T}}\vec{x} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{x} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a}\}$$

Take $\vec{x}$ and $\vec{y}$ in S_α, we have

$$(1-\alpha^2)\vec{x}^{\mathrm{T}}\vec{x} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{x} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{1}$$

$$(1-\alpha^2)\vec{y}^{\mathrm{T}}\vec{y} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{y} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{2}$$

Multiply (1) by λ and (2) by $(1-\lambda)$, then consider the sum of them, for $\lambda\in[0.1]$, we have

$$(1-\alpha^2)\big[\lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y}\big] + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{$*$}$$

Since

$$\begin{aligned}
&\lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y} \ge (\lambda\vec{x}+(1-\lambda)\vec{y})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\
\iff\ & \lambda(1-\lambda)\vec{x}^{\mathrm{T}}\vec{x} + \lambda(1-\lambda)\vec{y}^{\mathrm{T}}\vec{y} \ge 2\lambda(1-\lambda)\vec{x}^{\mathrm{T}}\vec{y}
\end{aligned}$$

which is obviously true, and since $1-\alpha^2>0$, we can obtain

$$\begin{aligned}
&(1-\alpha^2)(\lambda\vec{x}+(1-\lambda)\vec{y})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\
\le\ & (1-\alpha^2)\big[\lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y}\big] + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\
\le\ & \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a}
\end{aligned}$$

which means $\lambda\vec{x}+(1-\lambda)\vec{y}\in S_\alpha$, and we conclude that S_α is convex, and the function is quasi-convex.

Question 20. Prove that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

is a log-concave function.

We need to prove that $g(x)=\ln\Phi(x)$ is concave function. Consider the first-order derivative of it, we have

$$g'(x) = \frac{e^{-x^2/2}}{\int_{-\infty}^{x} e^{-t^2/2}\,dt}$$

Then consider the second-order derivative of it, we have

$$g''(x) = e^{-x^2/2}\frac{-x\int_{-\infty}^{x} e^{-t^2/2}\,dt - e^{-x^2/2}}{\left(\int_{-\infty}^{x} e^{-t^2/2}\,dt\right)^2}$$

Let

$$h(x) = -x \int_{-\infty}^{x} e^{-t^2/2}\, dt - e^{-x^2/2}$$

we consider the monotonicity and limit of it. Compute

$$h'(x) = -x \int_{-\infty}^{x} e^{-t^2/2}\, dt - e^{-x^2/2} < 0$$

We know that $h(x)$ is strictly decreasing, the the supremum of it is its limit as $t \to -\infty$, however,

$$\lim_{x\to-\infty} \left[-x \int_{-\infty}^{x} e^{-t^2/2}\, dt - e^{-x^2/2} \right] = \lim_{x\to-\infty} \frac{\int_{-\infty}^{x} e^{-t^2/2}\, dt}{-x^{-1}} = \lim_{x\to-\infty} \frac{e^{-x^2/2}}{x^{-2}} = 0$$

Therefore, $h(x) < 0$ for all $x \in \mathbb{R}$, and we know that $g''(x) < 0$, which shows $g(x)$ is concave.

Question 21. Suppose $Q \in S_{++}^{n\times n}$. Prove that

$$2\vec{x}^{\mathrm{T}}\vec{y} \leq \vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y}$$

for any $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Since Q is positive definite matrix, there exists orthogonal matrix P such that

$$\vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y} = \vec{x}^{\mathrm{T}}P^{\mathrm{T}}DP\vec{x} + \vec{y}^{\mathrm{T}}PD^{-1}P^{\mathrm{T}}\vec{y} = \vec{\bar{x}}^{\mathrm{T}}D\vec{\bar{x}} + \vec{\bar{y}}^{\mathrm{T}}D^{\mathrm{T}}\vec{\bar{y}}$$

If we suppose $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\vec{\bar{x}} = (\bar{x}_1, \ldots, \bar{x}_n)^{\mathrm{T}}$, and $\vec{\bar{y}} = (\bar{y}_1, \ldots, \bar{y}_n)^{\mathrm{T}}$, since all $\lambda_i > 0$, we have

$$\begin{aligned}
\vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y} &= \lambda_1\bar{x}_1^2 + \ldots + \lambda_n\bar{x}_n^2 + \lambda^{-1}\bar{y}_1^2 + \ldots + \lambda^{-1}\bar{y}_n^2 \\
&\geq 2(\bar{x}_1\bar{y}_1 + \ldots + \bar{x}_n\bar{y}_n) \\
&= 2(P\vec{x})^{\mathrm{T}}P^{\mathrm{T}}\vec{y} = 2\vec{x}^{\mathrm{T}}PP^{\mathrm{T}}\vec{y} \\
&= 2\vec{x}^{\mathrm{T}}I_n\vec{y} = 2\vec{x}^{\mathrm{T}}\vec{y}
\end{aligned}$$

Hence, we finish the proof.

Question 22. Suppose $0 < p < 1$. Show that

$$\left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

is a concave function in $\mathbb{R}_{++}^n$.

Let $f(\vec{x})$ denote the original function, and $g(\vec{x}) = \ln f(\vec{x})$, we have

$$[\nabla g(\vec{x})]_i = \frac{1}{\sum_{k=1}^{n} x_k^p} x_i^{p-1}$$

and

$$[\nabla^2 g(\vec{x})]_{ij} = \begin{cases} \frac{1}{\left(\sum_{k=1}^{n} x_k^p\right)^2} \left[(p-1)x_i^{p-2}\sum_{k=1}^{n} x_k^p - px_i^{p-1}x_j^{p-1} \right] & \text{when } i = j \\ \frac{1}{\left(\sum_{k=1}^{n} x_k^p\right)^2} \left[-px_i^{p-1}x_j^{p-1} \right] & \text{when } i \neq j \end{cases}$$

Since we know

$$\nabla^2 f(\vec{x}) = f(\vec{x}) \left[\nabla g(\vec{x})\nabla g(\vec{x}^{\mathrm{T}}) + \nabla^2 g(\vec{x})\right]$$

If we let $\bar{H} = f(\vec{x})^2 \nabla^2 f(\vec{x})$, we only need to check $\bar{H}$ is negative semi-definite, then we can conclude that $f(\vec{x})$ is concave function. Take any vector $\vec{u}$, we consider for $1 - p > 0$,

$$\begin{aligned}
\vec{u}^{\mathrm{T}} \bar{H} \vec{u} &= (1-p) \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{p-1} x_j^{p-1} u_i u_j + (p-1) \left(\sum_{k=1}^{n} x_k^{p-2} u_k^2\right) \left(\sum_{k=1}^{n} x_k^{p}\right) \\
&= (1-p) \left[\left(\sum_{i=1}^{n} x_i^{p-1} u_i\right)^2 - \left(\sum_{k=1}^{n} x_k^{p-2} u_k^2\right) \left(\sum_{k=1}^{n} x_k^{p}\right)\right] \\
&\leq 0
\end{aligned}$$

Therefore, $\bar{H}$ is negative semi-definite, and $f(\vec{x})$ is concave function in $\mathbb{R}^n_{++}$.

Question 23. If $f(\vec{x})$ is twice continuously differentiable and quasi-convex, then for any $\vec{x} \in \mathrm{dom}(f)$,

$$\vec{d}^{\mathrm{T}} \nabla f(\vec{x}) = 0 \Longrightarrow \vec{d}^{\mathrm{T}} \nabla^2 f(\vec{x}) \vec{d} \geq 0$$

Suppose for some $\vec{x}$, $\vec{d}^{\mathrm{T}} \nabla^2 f(\vec{x}) \vec{d} < 0$ under that condition. Let $h(t) = f(\vec{x} + t\vec{d})$, then $h'(0) = \vec{d}^{\mathrm{T}} \nabla f(\vec{x}) = 0$ and $h''(0) = \vec{d}^{\mathrm{T}} \nabla^2 f(\vec{x}) \vec{d} < 0$. Then in a small neighborhood $(-\delta, \delta)$, 0 is a local maximum of $h(t)$. Then, we will have $h(0) > \max\{h(t_1), h(-t_1)\}$ for some $0 \neq t_1 \in (-\delta, \delta)$. Now we consider the level set S_α of $f(\vec{x})$, let $\alpha = \max\{h(t_1), h(-t_1)\}$, then $h(t_1) = f(\vec{x} + t_1\vec{d})$ and $h(-t_1) = f(\vec{x} - t_1\vec{d})$ are both in S_α, but their convex combination $h(0) = f(\vec{x})$ is not in S_α, so f is not quasi-convex at least in that small neighborhood. Contradiction shows that our assumption is wrong, and $\vec{d}^{\mathrm{T}} \nabla^2 f(\vec{x}) \vec{d} \geq 0$ for all $\vec{x}$.

Question 24. If the condition in Question 23 holds, then there must exist some real value α such that

$$\nabla^2 f(\vec{x}) + \alpha \nabla f(\vec{x}) (\nabla f(\vec{x}))^{\mathrm{T}} \succeq 0$$

Also, the Hessian matrix of a quasi-convex function can have at most one negative eigenvalue

We first prove that the hessian matrix of quasi-convex function can never have two or more negative eigenvalues. If it does have, then take any two negative of them λ_1 and λ_2, with corresponding eigenvector $\vec{v}_1$ and $\vec{v}_2$. Since for symmetric matrix, it has orthogonal eigenbasis, we have $\vec{v}_1 \perp \vec{v}_2$. Let $\vec{u} = \nabla f(\vec{x})$, the orthogonal complement space of $\vec{u}$ has dimension $n-1$, but span$\{\vec{v}_1, \vec{v}_2\}$ has dimension 2, so the intersection of them always contains nontrivial vector $\vec{d}$. Therefore, $\vec{d}^{\mathrm{T}} \vec{u} = 0$, but if we consider $H = \nabla^2 f(\vec{x})$, we have

$$\begin{aligned}
\vec{d}^{\mathrm{T}} H \vec{d} &= \vec{d}^{\mathrm{T}} H (a\vec{v}_1 + b\vec{v}_2) \\
&= \vec{d}^{\mathrm{T}} (\lambda_1 a \vec{v}_1 + \lambda_2 b \vec{v}_2) \\
&= (a\vec{v}_1 + b\vec{v}_2)^{\mathrm{T}} (\lambda_1 a \vec{v}_1 + \lambda_2 b \vec{v}_2) \\
&= \lambda_1 a^2 \|\vec{v}_1\|_2^2 + \lambda_2 b^2 \|\vec{v}_2\|_2^2 < 0
\end{aligned}$$

which contradicts to what we proved in Question 23.

If H is PSD, then we are done by choosing $\alpha = 0$. If H has exactly one negative eigenvalue, $\lambda_1 < 0$, so H is indefinite matrix. We now prove a more general theorem as follows

Theorem [Finsler]. *For symmetric matrix $A, B \in \mathbb{R}^{n\times n}$ with B indefinite, if $\vec{x}^{\mathrm{T}} B \vec{x} = 0 \Longrightarrow \vec{x}^{\mathrm{T}} A \vec{x} \geq 0$, then $A + tB$ is positive semidefinite for some $t \in \mathbb{R}$.*

Proof. Define two sets as follows

$$F_1 = \{t \in \mathbb{R} \mid \vec{x} \in \mathbb{R}^n,\ \vec{x}^{\mathrm{T}}(-B)\vec{x} \geq 0 \Longrightarrow \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0\}$$

$$F_2 = \{t \in \mathbb{R} \mid \vec{x} \in \mathbb{R}^n,\ \vec{x}^{\mathrm{T}} B \vec{x} \geq 0 \Longrightarrow \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0\}$$

where $A(t) = A + tB$. If there exists real number $t_0 \in F_1 \cap F_2$, then $A(t_0)$ is positive semidefinite. Thus, we need to show $F_1 \cap F_2 \neq \varnothing$.

From our assumption, we have for $t \in \mathbb{R}$,

$$\vec{x}^{\mathrm{T}} B \vec{x} = 0 \Longrightarrow \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0$$

which implies $E(t) \subset C \cup D$, where

$$E(t) = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^{\mathrm{T}} A(t) \vec{x} < 0\},\ C = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^{\mathrm{T}} B \vec{x} > 0\},\ D = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^{\mathrm{T}} B \vec{x} < 0\}$$

The set $E(t)$ consists of at most two connected components (This is not trivial, you can consider the canonical form of quadratic form $\vec{x}^{\mathrm{T}} A(t) \vec{x} = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_r^2$, when there is only one negative term, $E(t)$ will be disconnected and has only two connected components; when the number of negative term is larger than or equal to 2, $E(t)$ will be connected), and these two components are symmetric (though each single component is not symmetric) with respect to the origin; the sets C and D, whose union is disconnected, are also symmetric (here C and D itself is symmetric) with respect to origin. Since we can easily check that any connected component(s) of $E(t)$ must be contained in C or D, the whole set $E(t)$ is contained in C or D for each fixed t. Therefore, for any $t \in \mathbb{R}$, $t \in F_1$ or $t \in F_2$, and this means $F_1 \cup F_2 = \mathbb{R}$.

Since B is indefinite, It is easy to show that F_1 and F_2 are nonempty sets. Also, since quadratic function is always continuous, so F_1 and F_2 can be shown to be closed set easily. In this way, we can conclude that $F_1 \cap F_2 \neq \varnothing$. This just means there exists a t, no matter what the result of $\vec{x}^{\mathrm{T}} B \vec{x}$ is, we always have $\vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0$, meaning that $A(t) \succeq 0$.

□

Then let $B = \vec{u}\vec{u}^{\mathrm{T}}$ and $A = H$ in the above theorem, we can directly obtain what we need to prove.

Question 25. For $X \in \mathcal{S}^{n\times n}$, its eigenvalues are denoted to be

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_{n-1}(X) \geq \lambda_n(X)$$

Let $1 \leq k \leq n$. Consider

$$f(X) := \sum_{i=1}^{k} \lambda_i(X)$$

Prove that $f(X)$ is a convex function. You could first show that

$$f(X) = \sup\{\text{tr}\,(U^{\text{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\text{T}}U = I_k\}$$

If we prove that

$$f(X) = \sup\{\text{tr}\,(U^{\text{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\text{T}}U = I_k\} \tag{$*$}$$

then $f(X)$ is obviously convex, because it can be regared as the supremum of $g(X) = \text{tr}\,(U^{\text{T}}XU)$, which is linear with respect to X (trace function is linear, and $U^{\text{T}}XU$ is also linear). Since linear function is convex, so the supremum of it must be convex. Thus, it suffices to prove $(*)$ is correct.

Take the eigen-decomposition of $X = Q^{\text{T}}DQ$, where Q is orthogonal matrix and D is diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of X as its diagonal entries. If we let $\overline{U} = QU$ for all $U^{\text{T}}U = I_k$, then

$$\overline{U}^{\text{T}}\overline{U} = U^{\text{T}}Q^{\text{T}}QU = U^{\text{T}}I_nU = U^{\text{T}}U = I_k$$

Thus, we have

$$\left\{\text{tr}\,\left(U^{\text{T}}XU\right) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\text{T}}U = I_k\right\} = \left\{\text{tr}\,\left(\overline{U}^{\text{T}}D\overline{U}\right) \,|\, \overline{U} \in \mathbb{R}^{n\times k},\ \overline{U}^{\text{T}}\overline{U} = I_k\right\}$$

If we denote the I-th row of $\overline{U}$ to be $\vec{\overline{U}}_i$, and the j-th entry of $\vec{\overline{U}}_i$ to be $\overline{U}_{ij}$, then we have

$$\text{tr}\,\left(\overline{U}^{\text{T}}D\overline{U}\right) = \text{tr}\,\left(D\overline{U}\,\overline{U}^{\text{T}}\right) = \sum_{i=1}^{n}\lambda_i\left\|\vec{\overline{U}}_i\right\|_2^2$$

Since $\text{tr}\,\left(\overline{U}^{\text{T}}\overline{U}\right) = \text{tr}\,(I_k) = k$, we have $\sum_{i=1}^{n}\left\|\vec{\overline{U}}_i\right\|_2^2 = k$. Also, notice that $\overline{U}$ is a $n\times k$ matrix whose $k \le n$ columns form an orthonormal set of vectors in $\mathbb{R}^n$, hence linearly independent. Thus, we can extend it to a basis of $\mathbb{R}^n$, and by applying Gram-Schmidt process, we can obtain an orthonormal basis of $\mathbb{R}^n$ including all k columns of $\overline{U}$. In other words, we have extended the orginal $\overline{U}$ to a larger orthogonal matrix $\widetilde{U} = \left[\overline{U}, \overline{V}\right]$. Therefore, if we denote $\vec{\overline{V}}_i$ as the i-th row of $\overline{V}$

$$\left\|\vec{\overline{U}}_i\right\|_2^2 + \left\|\vec{\overline{V}}_i\right\|_2^2 = 1 \Longrightarrow \left\|\vec{\overline{U}}_i\right\|_2^2 \le 1$$

Therefore, if we consider the weighted average of $\left\|\vec{\overline{U}}_i\right\|_2^2$, i.e., $\sum_{i=1}^{n}\lambda_i\left\|\vec{\overline{U}}_i\right\|_2^2$, to maximize it, we should assign the maximum value to the maximum weight. However, each weight can be at most 1, and we have k units in total, hence, the maximized case is that we allocate 1 to the largest k weights, i.e.,

$$\sum_{i=1}^{n}\lambda_i\left\|\vec{\overline{U}}_i\right\|_2^2 \le \sum_{i=1}^{k}\lambda_i$$

If we choose the k columns of U to be k eigenvectors of X, then we have $\text{tr}\,(U^{\text{T}}XU) = \lambda_1 + \cdots + \lambda_k$. Therefore,

$$\sup\{\text{tr}\,(U^{\text{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\text{T}}U = I_k\} = \lambda_1 + \cdots + \lambda_k = f(X)$$

and the proof is finished.

Question 26. A function $f : \mathbb{R}^n_{++} \mapsto \mathbb{R}$

$$h(\vec{x}) = cx_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$$

with $c > 0$ and $\lambda \in \mathbb{R}^n$ is called a *monomial*. Sum of monomials, $f(\vec{x}) = \sum_{i=1}^{k} h_i(\vec{x})$, is called a *posynomial*.

The so-called *geometric programming* problem is as follows,

$$
\begin{aligned}
(G) \quad \min_{\vec{x}} \quad & f_0(\vec{x}) \\
s.t. \quad & f_i(\vec{x}) \leq 1, \; i = 1, 2, \ldots, m \\
& h_j(\vec{x}) = 1, \; j = 1, 2, \ldots, p
\end{aligned}
$$

where $f_i(\vec{x})$ are posynomials ($i = 1, 2, \ldots, m$), and $h_j(\vec{x})$ are monomials ($j = 1, 2, \ldots, p$). Show that (G) can be formulated as convex optimization through a variable transformation.

First, we clarify some notations,

$$h_j(\vec{x}) = c_j x_1^{\lambda_{j,1}} x_2^{\lambda_{j,2}} \cdots x_n^{\lambda_{j,n}}, \; j = 1, \ldots, p$$

Similarly,

$$f_i(\vec{x}) = \sum_{k=1}^{a_i} h_k^{(i)}(\vec{x}), \; i = 0, 1, \ldots, m, \; h_k^{(i)}(\vec{x}) = c_k^{(i)} x_1^{\lambda_{k,1}^{(i)}} x_2^{\lambda_{k,2}^{(i)}} \cdots x_n^{\lambda_{k,n}^{(i)}}$$

Take $x_t = e^{y_t}$ for $t = 1, \ldots, n$, the reformulation is

$$
\begin{aligned}
(G_1) \quad \min_{\vec{y}} \quad & \sum_{k=1}^{a_0} c_k^{(0)} \exp\left\{ \sum_{t=1}^{n} \lambda_{k,t}^{(0)} y_t \right\} \\
s.t. \quad & \sum_{k=1}^{a_i} c_k^{(i)} \exp\left\{ \sum_{t=1}^{n} \lambda_{k,t}^{(i)} y_t \right\} \leq 1, \; i = 1, 2, \ldots, m \\
& c_j \exp\left\{ \sum_{t=1}^{n} \lambda_{j,t} y_t \right\} = 1, \; j = 1, 2, \ldots, p
\end{aligned}
$$

To simplify it, we have

$$
\begin{aligned}
(G_2) \quad \min_{\vec{y}} \quad & \ln\left\{ \sum_{k=1}^{a_0} c_k^{(0)} \exp\left\{ \sum_{t=1}^{n} \lambda_{k,t}^{(0)} y_t \right\} \right\} \\
s.t. \quad & \ln\left\{ \sum_{k=1}^{a_i} c_k^{(i)} \exp\left\{ \sum_{t=1}^{n} \lambda_{k,t}^{(i)} y_t \right\} \right\} \leq 0, \; i = 1, 2, \ldots, m \\
& \sum_{t=1}^{n} \lambda_{j,t} y_t = -\ln c_j, \; j = 1, 2, \ldots, p
\end{aligned}
$$

From Question 14, we have known that the log-sum-exponential function $\ln\left(\sum_{t=1}^{k} e^{y_t}\right)$ is convex, since all $c_t > 0$ are positive, this result can be easily generalized to the function $\ln\left(\sum_{t=1}^{k} c_t e^{y_t}\right)$. The objective function and inequality constraints of (G_2) can be regarded as the composite of log-sum-exponential and affine function, so they are all convex. The equality constraints are all affine functions, so (G_2) is a convex problem.

Question 27. Formulate the following L_4-norm approximation problem as QCQP,

$$\min_{\vec{x}} \quad \|A\vec{x} - b\|_4 = \left(\sum_{i=1}^{m} \left(\vec{a}_i^{\mathrm{T}} \vec{x} - b_i \right)^4 \right)^{1/4}$$

First, we know that the original problem is equivalent to

$$\min_{\vec{x}} \quad \sum_{i=1}^{m} \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^4$$

Using change of variable, let $t_i = \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$. Thus, we have

$$\begin{aligned} \min_{\vec{x}, t_i} \quad & \sum_{i=1}^{m} t_i^2 \\ s.t. \quad & t_i = \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2, \ i = 1, 2, \ldots, m \end{aligned}$$

Since QCQP cannot have non-linear equality constraints, so we need to transform equality to inequality constraints. Suppose $t_i > \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$, then to minimize the sum of square of t_i, we can decrease t_i until it is equal to $\left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$, thus we can reformulate it into

$$\begin{aligned} \min_{\vec{x}, t_i} \quad & \sum_{i=1}^{m} t_i^2 \\ s.t. \quad & t_i \geq \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2, \ i = 1, 2, \ldots, m \end{aligned}$$

Question 28. The so-called *Chebyshev center* of a polyhedron is the deepest point inside the polyhedron. Suppose that the polyhedron is given by $P = \{\vec{x} \mid \vec{a_i}^{\mathrm{T}} \vec{x} \leq b_i, i = 1, 2, \ldots, m\}$. Formulate the problem of finding the Chebyshev center of P by a convex optimization model.

Suppose the Chebyshev center is at point $\vec{p}$, and the radius of the Euclidean ball is $r \geq 0$. The only constrain is that the whole ball should lie in the polyhedron (we only need the sphere to be in the polyhedron). Therefore,

$$\vec{a_i}^{\mathrm{T}} (\vec{p} + r\vec{u}) \leq b_i, \quad \forall \|\vec{u}\|_2 = 1, \ \forall i = 1, \ldots, m$$

However, this is the case when uncountable constraints are involved, so we need to change it into finite many constraints. Consider the supremum of all constraints, we have

$$\sup_{\|\vec{u}\|_2 = 1} \ \vec{a_i}^{\mathrm{T}} (\vec{p} + r\vec{u}) = \vec{a_i}^{\mathrm{T}} \vec{p} + r\|\vec{a_i}\|_2 \leq b_i, \quad \forall i = 1, \ldots, m$$

Therefore, we can obtain the formulation

$$\begin{aligned} \max_{\vec{p}, r} \quad & r \\ s.t. \quad & \vec{a_i}^{\mathrm{T}} \vec{p} + r\|\vec{a_i}\|_2 \leq b_i, \ i = 1, 2, \ldots, m \end{aligned}$$

Since the objective function and constraints are linear with respect to $\vec{p}$ and r, it is a convex problem.

Question 29. An ellipsoid may be given by the image of a ball under some linear transformation, e.g. $E = \{Bu + b \,|\, \|u\|_2 \leq 1\}$. Without losing generality we can also assume $B \succ 0$. Then the volume of E is proportional to $\det B$.

Consider again the polyhedron $P = \{\vec{x} \mid a_i^{\mathrm{T}} \vec{x} \leq b_i,\ i = 1, 2, \ldots, m\}$. Now the problem is to find the maximum volume ellipsoid inscribed inside P. Formulate the problem by convex optimization.

The constraint can be dealt with in a similar manner as that in the Question 28, but we need to be careful about the objective function here. We tend to maximize the volume, i..e, maximize the determinant of B. However, it is easy to show that $\det(B)$ is nonconvex and nonconcave function. Hence, we need to maximize $\log(\det(B))$ instead, because it is a concave function on $\mathcal{S}_{++}^n$. Thus, we have the formulation as follows

$$
\begin{aligned}
\max_{B,\,\vec{b}} \quad & \log(\det(B)) \\
s.t. \quad & \vec{a_i}^{\mathrm{T}} \vec{b} + \|B\vec{a_i}\|_2 \leq b_i,\ i = 1, 2, \ldots, m \\
& B \succ 0
\end{aligned}
$$

To prove the log-determinant function is concave on $\mathcal{S}_{++}^n$, it suffices to show $f(X)$ is concave in any direction. Define $g(t) = \log(\det(X + tV))$, where X and $X + tV$ are both positive definite. Then, there exists $X = X^{1/2}X^{1/2}$, such that

$$
\begin{aligned}
g(t) &= \log(\det(X^{1/2}X^{1/2} + tX^{1/2}X^{-1/2}VX^{-1/2}X^{1/2})) \\
&= \log(\det(X^{1/2}(I + tX^{-1/2}VX^{-1/2})X^{1/2})) \\
&= \log(\det(X)\det(I + tX^{-1/2}VX^{-1/2})) \\
&= \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2}))
\end{aligned}
$$

Note that $X^{1/2}$ and $I + tX^{-1/2}VX^{-1/2}$ are also positive definite, and assume the eigenvalues of $X^{-1/2}VX^{-1/2}$ are $\lambda_1, \ldots, \lambda_n$, then

$$
g(t) = \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2})) = \log(\det(X)) + \sum_{i=1}^{n} \log(1 + t\lambda_i)
$$

Thus, we have

$$
g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}, \quad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \leq 0
$$

Hence $g(t)$ is concave, meaning that $f(X)$ is concave in V-direction, but V is arbitrary, so $f(X)$ is concave in general.

Question 30. Let $A_i \in \mathcal{S}^{n\times n}$, $i = 1, 2, \ldots, m$. Therefore, $A_0 + x_1A_1 + \cdots + x_mA_m$ is a symmetric matrix. We wish to find the values of $x_1, \ldots, x_m$ so as to minimize the gap between the largest and the smallest eigenvalues of $A_0 + x_1A_1 + \cdots + x_mA_m$. Formulate this problem by SDP.

This question is trivial, the formulation is

$$
\begin{aligned}
\min_{\vec{x},L,U} \quad & U - L \\
s.t. \quad & L \cdot I_n \preceq A_0 + x_1A_1 + \cdots + x_mA_m \preceq U \cdot I_n
\end{aligned}
$$

where $\vec{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $L, U \in \mathbb{R}$, and I_n is $n \times n$ identity matrix.

Question 31. Let

$$
\mathcal{K} = \{\vec{x} \in \mathbb{R} \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}
$$

Show that $\mathcal{K}$ is a proper cone.

First, we show that $\mathcal{K}$ is closed. Take any convergent subsequence $\{\vec{v}_n\}_{n=1}^{\infty} \in \mathcal{K}$, for any $\vec{v}_n$, we have $\vec{v}_n^{(i)} \geq 0$ for $i = 1, \ldots, n$. Suppose the limit of this sequence is $\vec{v}$, then we have

$$\vec{v}^{(i)} = \lim_{n\to\infty} \vec{v}_n^{(i)} \geq 0$$

which means $\vec{v}$ is also in $\mathcal{K}$. This means any limit point of $\mathcal{K}$ is in itself, hence it is closed.

Second, we need to show that $\mathcal{K}$ is solid. For unit ball B, we can see that $[2n\ 2n-2\ \cdots\ 2]^{\mathrm{T}} + B$ is a ball in $\mathcal{K}$. This is because $B = \{\vec{v} \mid \|\vec{v}\|_2 = 1\}$, so any point in $[2n\ 2n-2\ \cdots\ 2]^{\mathrm{T}} + B$ can be expressed as $[v_1 + 2n, v_2 + 2n - 2, \cdots, v_n + 2]^{\mathrm{T}}$. Consider any two consecutive entries, W.O.L.G., we take the first two entries, $v_1 + 2n - v_2 - 2n + 2 = v_1 - v_2 + 2$, since $v_1^2 + v_2^2 \leq 1$, $|v_1 - v_2| < \sqrt{2}$, so $v_1 - v_2 + 2 > 0$ and this point is in $\mathcal{K}$. Hence, $\mathcal{K}$ cantains a ball and thus is solid.

Finally, we prove $\mathcal{K}$ is pointed. If $\vec{x} \in \mathcal{K}$, and $-\vec{x} \in \mathcal{K}$, then we will have $x_i \geq x_{i+1}$ and $x_i \leq x_{i+1}$ for $i = 1, \ldots, n-1$. Thus, $x_i = x_{i+1}$ for $i = 1, \ldots, n-1$, but $x_n \geq 0$ and $x_n \leq 0$, so $\vec{x} = \vec{0}$.

It's easy to check this is a convex cone by definition. For any $\vec{x} \in \mathcal{K}$, $\alpha\vec{x}$ is also in $\mathcal{K}$ for any $\alpha \geq 0$. For $\lambda \in [0, 1]$, it is trivial that $\lambda\vec{x} + (1-\lambda)\vec{y}$ is also in $\mathcal{K}$, if $\vec{x}$ and $\vec{y}$ are both in $\mathcal{K}$. Hence, $\mathcal{K}$ is a proper cone.

Question 32. Find $A \in \mathbb{R}^{n\times n}$ such that $\mathcal{K} = A\mathbb{R}_+^n$.

Take A as

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then for any $\vec{x} \in \mathbb{R}_+^n$, we have

$$A\vec{x} = [x_1 + \cdots + x_n, x_2 + \cdots + x_n, \cdots, x_1]^{\mathrm{T}}$$

which shows that $A\vec{x} \in \mathcal{K}$, because all x_i are nonnegative.

Also, for any $\vec{x} \in \mathcal{K}$, $A^{-1}\vec{x}$ is in $\mathbb{R}_+^n$, because

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A^{-1}\vec{x} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ \vdots \\ x_{n-1} - x_n \end{bmatrix} \geq \vec{0}$$

Therefore, $\mathcal{K} = A\mathbb{R}_+^n$.

Question 33. In general, if $\mathcal{K} \subset \mathbb{R}^n$ is a proper cone, and $M \in \mathbb{R}^{n\times n}$ is a non-singular matrix, then $M\mathcal{K}$ is also a proper cone.

First, we prove that $M\mathcal{K}$ is a convex cone. Since by definition, $M\mathcal{K} = \{M\vec{x} \mid \vec{x} \in \mathcal{K}\}$, for any element $\vec{y} \in M\mathcal{K}$, we have $\vec{y} = M\vec{x}$. Consider any $\alpha \geq 0$, $\alpha\vec{y} = M(\alpha\vec{x})$, since $\vec{x}$ is in cone $\mathcal{K}$, so is $\alpha\vec{x}$, and thus $\alpha\vec{y} \in M\mathcal{K}$. The convexity of $M\mathcal{K}$ also follows from the convexity of $\mathcal{K}$, similar arguments can be applied.

Then, we prove that $M\mathcal{K}$ is closed. This is trivial, since M is a linear transformation hence continuous. Continuous function maps a closed set to closed set, thus $M\mathcal{K}$ is closed because $\mathcal{K}$ is closed.

Next, we prove that $M\mathcal{K}$ is solid. Since there exists a unit ball in $\mathcal{K}$, take its interior, it is an open set, and will be mapped to an open set by M. Therefore, there is an open set in $M\mathcal{K}$, and there is a open ball contained in this open set, and of course in $M\mathcal{K}$.

Finally, we prove that $M\mathcal{K}$ is pointed. This is trivial, since $\vec{x} \in M\mathcal{K}$ means $M^{-1}\vec{x} \in \mathcal{K}$, and $-\vec{x} \in M\mathcal{K}$ means $-M^{-1}\vec{x} \in \mathcal{K}$. We know $\mathcal{K}$ is pointed, so $M^{-1}\vec{x} = \vec{0}$, which is equivalent to say $\vec{x} = \vec{0}$. Therefore, $M\mathcal{K}$ is pointed, and hence it is a proper cone.

Question 34. Compute $(M\mathcal{K})^*$.

By definition, we have

$$\begin{aligned}
(M\mathcal{K})^* &= \{\vec{y} \mid \vec{x}^{\mathrm{T}} M^{\mathrm{T}} \vec{y} \geq 0, \ \forall\, \vec{x} \in \mathcal{K}\} \\
&= \{\vec{y} \mid M^{\mathrm{T}}\vec{y} \in \mathcal{K}^*\} \\
&= \left(M^{\mathrm{T}}\right)^{-1} \mathcal{K}^*
\end{aligned}$$

Question 35. Derive the dual of the following *non-standard* conic optimization problem:

$$\begin{aligned}
\min_{\vec{x}} \quad & \vec{c}^{\mathrm{T}}\vec{x} \\
s.t. \quad & A_i\vec{x} + \vec{b}_i \in \mathcal{K}_i, \ i = 1, 2, \ldots, m
\end{aligned}$$

where $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_m$ are all closed convex cones.

Consider the Lagrangian function

$$L(\vec{x}, \vec{y}_i) = \vec{c}^{\mathrm{T}}\vec{x} + \sum_{i=1}^{m} \vec{y}_i^{\mathrm{T}}(A_i\vec{x} + \vec{b})$$

where $\vec{y}_i \in \mathcal{K}_i^*$. Then the dual function is

$$d(\vec{y}_i) = \min_{\vec{x}} L(\vec{x}, \vec{y}_i) = \begin{cases} \sum_{i=1}^{m} \vec{b}^{\mathrm{T}}\vec{y}_i & \text{when } \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}}\vec{y}_i = \vec{0} \\ -\infty & \text{when } \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}}\vec{y}_i \neq \vec{0} \end{cases}$$

Hence, the Lagrange dual problem is

$$\begin{aligned} \max_{\vec{y}_i} \quad & \sum_{i=1}^{m} \vec{b}^{\mathrm{T}} \vec{y}_i \\ s.t. \quad & \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}} \vec{y}_i = \vec{0} \\ & \vec{y}_i \in \mathcal{K}_i^*, \ i = 1, 2, \ldots, m \end{aligned}$$

Question 36. Suppose that $f(\vec{x})$ is a convex function, and its conjugate function is known to be $f^*(\vec{s})$. Consider the following optimization model

$$\begin{aligned} \min_{\vec{x}} \quad & f(\vec{x}) \\ s.t. \quad & A\vec{x} \leq \vec{b} \end{aligned}$$

Derive the Lagrangian dual of the above problem.

Recall the conjugate function $f^*(\vec{s})$ is given by

$$f^*(\vec{s}) = \sup_{\vec{x}} (\vec{s}^{\mathrm{T}} \vec{x} - f(\vec{x}))$$

The Lagrangian function is given by

$$L(\vec{x}, \vec{y}) = f(\vec{x}) + \vec{y}^{\mathrm{T}} (A\vec{x} - \vec{b})$$

Hence, the dual function $d(\vec{y})$ is given by

$$d(\vec{y}) = \min_{\vec{x}} L(\vec{x}, \vec{y}) = -\max_{\vec{x}} \left((-A^{\mathrm{T}} \vec{y})^{\mathrm{T}} - f(\vec{x}) \right) - \vec{b}^{\mathrm{T}} \vec{y} = -f^*(-A^{\mathrm{T}} \vec{y}) - \vec{b}^{\mathrm{T}} \vec{y}$$

where $\vec{y} \geq 0$. Therefore, the Lagrange dual problem is

$$\begin{aligned} \max_{\vec{y}} \quad & -f^*(-A^{\mathrm{T}} \vec{y}) - \vec{b}^{\mathrm{T}} \vec{y} \\ s.t. \quad & \vec{y} \geq \vec{0} \end{aligned}$$

Question 37. The channel capacity optimization problem is:

$$\begin{aligned} \min_{\vec{x}, \vec{y}} \quad & -\vec{c}^{\mathrm{T}} \vec{x} + \sum_{i=1}^{m} y_i \ln y_i \\ s.t. \quad & P\vec{x} = \vec{y} \\ & \vec{x} \geq \vec{0}, \ \vec{e}^{\mathrm{T}} \vec{x} = 1 \end{aligned}$$

What is the dual of the above problem?

The Lagrangian function is

$$L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda}) = -\vec{c}^{\mathrm{T}} \vec{x} + \sum_{i=1}^{m} y_i \ln y_i + \vec{u}^{\mathrm{T}} (P\vec{x} - \vec{y}) + u_0(\vec{e}^{\mathrm{T}} \vec{x} - 1) + \vec{\lambda}^{\mathrm{T}} (-\vec{x})$$

where $\vec{\lambda} \geq \vec{0}$ and $\vec{u} = (u_1, \cdots, u_m)^{\mathrm{T}}$. The dual function

$$d(\vec{u}, u_0, \vec{\lambda}) = \min_{\vec{x}, \vec{y}} L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda})$$

We can rewrite the Lagrangian function into separated form (separate $\vec{x}$, $\vec{y}$), which is

$$L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda}) = (-\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda})^{\mathrm{T}}\vec{x} + \sum_{i=1}^{m} y_i \ln y_i - \vec{u}^{\mathrm{T}}\vec{y} - u_0$$

Since for $\vec{x}$ part, it is an linear function, the coefficient must be zero, otherwise it will be unbounded (because in Lagrangian function, $\vec{x}$ is free variable). Thus,

$$-\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda} = \vec{0}$$

For $\vec{y}$ part, it is a convex function, hence the minimum is attained at the point where the gradient is zero, i.e.,

$$\ln y_i + 1 - u_i = 0 \Longrightarrow y_i = e^{u_i - 1}, \quad \forall\, i = 1, 2, \ldots, m$$

Hence, we can obtain the dual function

$$d(\vec{u}, u_0, \vec{\lambda}) = -\sum_{i=1}^{m} e^{u_i - 1} - u_0$$

And the Lagrange dual problem is given by

$$\begin{aligned} \max_{\vec{u}, u_0, \vec{\lambda}} \quad & -\sum_{i=1}^{m} e^{u_i - 1} - u_0 \\ s.t. \quad & -\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda} = \vec{0} \\ & \vec{\lambda} \geq \vec{0} \end{aligned}$$

Eliminate $\vec{\lambda}$, we have

$$\begin{aligned} \max_{\vec{u}, u_0} \quad & -\sum_{i=1}^{m} e^{u_i - 1} - u_0 \\ s.t. \quad & -\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} \geq \vec{0} \end{aligned}$$

Question 38. The sum of first k largest components of vector $\vec{x} \in \mathbb{R}^n$ $(k < n)$ is known to be a convex function (Why?). Denote this function to be $f(\vec{x})$. Formulate the following portfolio selection problem using $f(\vec{x})$: We wish to select from a total of n assets to form a portfolio (no short-selling is allowed). Asset i has an expected rate of return $\mu_i > 0$, and the covariance matrix is Σ. We wish to minimize the variance of the portfolio while requiring that the expected rate of return to the portfolio is at least μ. Moreover, the weight of the first k largest components of investment should not exceed half of the total investment.

To see why $f(\vec{x})$ is convex, we can see that

$$f(\vec{x}) = \sum_{i=1}^{k} x_{n_i} = \max\{x_{n_1} + \cdots + x_{n_k} \,|\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n\}$$

f is the maximum of C_n^k linear functions, so it must be convex.

Now let us formulate the portfolio problem. Since $\vec{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ means the percentage of different portfolio, so the sum of all entries must be one. Not short-selling means $x_i \geq 0$ for all i. If we denote $\vec{u} = (\mu_1, \cdots, \mu_n)^{\mathrm{T}}$, then since the expected rate of return is at least μ, we have $\vec{u}^{\mathrm{T}}\vec{x} \geq \mu$. The requirement on first k largest components yields $f(\vec{x}) \leq 0.5$. Finally, we need to minimize the variance of portfolio, so the objective function is $\vec{x}^{\mathrm{T}}\Sigma\vec{x}$. Therefore,

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{x}^{\mathrm{T}}\Sigma\vec{x} \\ s.t. \quad & \vec{e}^{\mathrm{T}}\vec{x} = 1 \\ & \vec{u}^{\mathrm{T}}\vec{x} \geq \mu \\ & f(\vec{x}) \leq 0.5 \\ & \vec{x} \geq \vec{0} \end{aligned}$$

Question 39. The condition that $f(x) \leq 0.5$ in Question 38 can be formulated by linear programming. How?

This is trivial if you use definition of $f(\vec{x})$,

$$f(\vec{x}) = \max\{x_{n_1} + \cdots + x_{n_k} \,|\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n\} \leq \frac{1}{2}$$

The above constraint is equivalent to

$$x_{n_1} + \cdots + x_{n_k} \leq \frac{1}{2}, \quad \forall\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n$$

Notice that there are C_n^k different choices of $\{n_1, \ldots, n_k\}$, so the original one non-linear constraint will be reformulated into C_n^k linear constraints.