MAT3006*: Real Analysis
Homework 11

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Extra Problem 1. Recall the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) & x \in \mathbb{R} \end{cases}$$

whose solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x-y,t)\phi(y)\,dy$$

where $\Gamma(x,t)$ is the fundamental solution of heat equation given by

$$\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0$$

which is the solution of heat equation with $\phi(x)$ equal to delta function $\delta(x)$.

(i) Prove for any fixed $y \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\Gamma(x-y,t) = \frac{\partial^2}{\partial x^2}\Gamma(x-y,t), \quad \forall\, x \in \mathbb{R},\ \forall\, t > 0$$

For each fixed $y \in \mathbb{R}$, we have

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma(x-y,t) &= -\frac{1}{2\sqrt{4\pi}} t^{-3/2} e^{-\frac{(x-y)^2}{4t}} + \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \frac{(x-y)^2}{4t^2} \\ &= \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \left[-\frac{1}{2t} + \frac{(x-y)^2}{4t^2} \right] \end{aligned}$$

Also, we have

$$\frac{\partial}{\partial x}\Gamma(x-y,t) = -\frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \frac{x-y}{2t}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\Gamma(x-y,t) &= \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \frac{(x-y)^2}{4t^2} - \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \frac{1}{2t} \\ &= \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-\frac{(x-y)^2}{4t}} \left[-\frac{1}{2t} + \frac{(x-y)^2}{4t^2} \right] \end{aligned}$$

Therefore, we can see the desired equality holds.

(ii) Suppose $\phi \in L^1(\mathbb{R})$ from now on, and prove $u(x,t)$ satisfies the equation $u_t(x,t) = u_{xx}(x,t)$ for $x \in \mathbb{R}$, $t > 0$.

By part (i), we have known

$$\int_{\mathbb{R}} \Gamma_t(x-y,t)\phi(y)\;dy = \int_{\mathbb{R}} \Gamma_{xx}(x-y,t)\phi(y)\;dy$$

Thus, it suffices to show that

$$u_t(x,t) = \frac{\partial}{\partial t}\int_{\mathbb{R}} \Gamma(x-y,t)\phi(y)\;dy = \int_{\mathbb{R}} \Gamma_t(x-y,t)\phi(y)\;dy \tag{1}$$

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{\mathbb{R}} \Gamma(x-y,t)\phi(y)\;dy = \int_{\mathbb{R}} \Gamma_{xx}(x-y,t)\phi(y)\;dy \tag{2}$$

For $u_t(x,t)$, we can fixed each x, then denote $f(y,t) = \Gamma(x-y,t)\phi(y)$, which is define on $\mathbb{R}\times\mathbb{R}^+$. First, for each fixed t, $f(y,t) = C_1 e^{-C_2(y-C_3)^2}\phi(y)$ is in $L^1(\mathbb{R})$, where $C_1 > 0$, $C_3 \in \mathbb{R}$ and $C_2 \geq 0$ are independent of y. This is because $|f(y,t)| \leq C_1|\phi(y)|$ and $\phi(y) \in L^1(\mathbb{R})$. Second, for each fixed y, $\frac{f}{\partial t}$ obviously exists for $t > 0$. Finally, for a fixed $t_0 > 0$, for all $t \in (t_0/2, 3t_0/2)$,

$$\left|\frac{\partial f}{\partial t}\right| \leq e^{-\frac{(x-y)^2}{6t_0}}\left[C_1 t_0^{-3/2} + C_2 t_0^{-5/2}(x-y)^2\right]|\phi(y)| = g(y)|\phi(y)|$$

for some constant $C_1, C_2 > 0$. Notice that $g(y) \in L^\infty(\mathbb{R})$ because $x^n e^{-x^2} \in L^\infty(\mathbb{R})$ for any $n \in \mathbb{N}$. Also, since $\phi(y) \in L^1(\mathbb{R})$, $g(y)|\phi(y)| \in L^1(\mathbb{R})$. Therefore, by differentiation across integral sign, (1) is proved.

For $u_{xx}(x,t)$, we need to use differentiation across integral sign twice. This time we fixed each t, then denote $f(y,x) = \Gamma(x-y,t)\phi(y)$, which is defined on $\mathbb{R}\times\mathbb{R}$. First, for each fixed x, $f(y,x) = C_1 e^{-C_2(y-C_3)^2}\phi(y)$ is in $L^1(\mathbb{R})$, which is exactly the same as the $u_t(x,t)$ case. Second, for each fixed y, $\frac{f}{\partial x}$ obviously exists for $x \in \mathbb{R}$. Finally, for a fixed $x_0 \in \mathbb{R}$, if $x_0 = y$, then for all $x \in (x_0-\delta, x_0+\delta)$,

$$\begin{aligned}
-(x-y)^2 &= -(x-x_0+x_0-y)^2 \\
&\leq -(x-x_0)^2 - (x_0-y)^2 + 2|x-x_0||x_0-y| \\
&\leq -(x_0-y)^2 + 2\delta|x_0-y|
\end{aligned}$$

Therefore, we can find a dominating function $(C_1, C_2 > 0)$,

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}\right| &\leq C_1 e^{-C_2(x-y)^2}|x-y||\phi(y)| \\
&\leq C_1 e^{-C_2(x-y)^2}(\delta + |x_0-y|)|\phi(y)| \\
&\leq C_1 e^{-C_2(x_0-y)^2 + 2C_2\delta|x_0-y|}(\delta + |x_0-y|)|\phi(y)| \\
&= h(y)|\phi(y)| \in L^1_y(\mathbb{R})
\end{aligned}$$

because $h(y) \in L^\infty(\mathbb{R})$. This is enough to show that

$$u_{xx}(x,t) = \frac{\partial}{\partial x}\int_{\mathbb{R}} \Gamma_x(x-y,t)\phi(y)\;dy$$

Again, denote $f(y,x) = \Gamma_x(x-y,t)\phi(y)$, then for fixed x, $f(y,x) = C_1(x-y)e^{-C_2(x-y)^2}\phi(y)$, so $f(y,x) \in L^1(\mathbb{R})$. Second, for each fixed y, $\frac{\partial f}{\partial x}$ obviously exists. Finally, for all $x \in (x_0-\delta, x_0+\delta)$,

$$\left|\frac{\partial f}{\partial x}\right| \leq e^{-C_1(x_0-y)^2 + 2C_1\delta|x_0-y|}[C_2 + C_3(|x_0-y|+\delta)^2]|\phi(y)| = h(y)|\phi(y)| \in L^1_y(\mathbb{R})$$

because $h(y) \in L^\infty(\mathbb{R})$. Therefore, we have shown that

$$u_{xx}(x,t) = \int_{\mathbb{R}} \Gamma_{xx}(x-y,t)\phi(y)\,dy$$

This finishes the proof of $u(x,t)$ satisfying heat equation (without initial condition).

(iii) Prove $\|u(\cdot,t) - \phi(\cdot)\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0+$.

Notice that we have already known $\Gamma(x-y,t)\phi(y)$ is in $L^1(\mathbb{R})$, by change of variable with $y = x + \sqrt{4t}z$ eliminating y, we obtain

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)\,dy = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-z^2} \phi(x+\sqrt{4t}z)\,dz$$

Therefore, by generalized Minkowski inequality, we have

$$\|u(x,t) - \phi(x)\|_{L^1_x(\mathbb{R})} \le \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-z^2} \|\phi(x+\sqrt{4t}z) - \phi(x)\|_{L^1_x(\mathbb{R})}\,dz$$

Notice that $\|\phi(x+\sqrt{4t}z) - \phi(x)\|_{L^1_x(\mathbb{R})} \le 2\|\phi(x)\|_{L^1_x(\mathbb{R})}$, so we can see

$$\|u(x,t) - \phi(x)\|_{L^1_x(\mathbb{R})} \le 2\|\phi(x)\|_{L^1_x(\mathbb{R})} \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-z^2}\,dz = 2\|\phi(x)\|_{L^1_x(\mathbb{R})} < \infty$$

By continuity of L^1-norm, $\|\phi(x+\sqrt{4t_k}z) - \phi(x))\|_{L^1(\mathbb{R})} \to 0$ as $k \to \infty$ for any sequence $t_k \to 0+$ as $k \to \infty$. Notice that the dominant function is given by

$$\frac{2}{\sqrt{\pi}} e^{-z^2} \|\phi(x)\|_{L^1(\mathbb{R})} \in L^1(\mathbb{R})$$

Therefore, by DCT, we have

$$\lim_{k\to\infty} \|u(\cdot,t_k) - \phi(\cdot)\|_{L^1(\mathbb{R})} \le \lim_{k\to\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-z^2} \|\phi(x+\sqrt{4t_k}z) - \phi(x))\|_{L^1(\mathbb{R})}\,dz = 0$$

This is enough to show $\|u(\cdot,t) - \phi(\cdot)\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0+$.

(iv) Prove that $|u(x,t)| \le \frac{1}{\sqrt{4\pi t}}\|\phi\|_{L^1(\mathbb{R})}$, for all $x \in \mathbb{R}$, all $t > 0$. Give physical intepretation of this result.

Since $e^{-\frac{(x-y)^2}{4t}} \le 1$ for any x, y and $t > 0$, we obtain

$$|u(x,t)| = \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)\,dy \right| \le \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} |\phi(y)|\,dy = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} |\phi(y)|\,dy$$

Thus, $|u(x,t)| \le \frac{1}{\sqrt{4\pi t}}\|\phi\|_{L^1(\mathbb{R})}$, for all $x \in \mathbb{R}$, all $t > 0$. The physical intepretation is that if the initial energy $\|\phi\|_{L^1(\mathbb{R})}$ is finite, then as time tends to infinity, the temperature will decrease to zero uniformly over different position with speed no slower than $O\left(\frac{1}{\sqrt{t}}\right)$.

Extra Problem 2. Prove that step functions are not dense in $L^\infty(0,1)$.

Consider $f(x) = \sum_{i=1}^{\infty} I_{(\frac{1}{2n}, \frac{1}{2n-1})}(x)$, where $I_{(\frac{1}{2n}, \frac{1}{2n-1})}(x)$ is the indicator function on interval $(\frac{1}{2n}, \frac{1}{2n-1})$. Then it is obvious that $f(x) \in L^\infty(0,1)$ because $0 \le f(x) \le 1$. Suppose there exists a sequence of step functions $f_k(x) = \sum_{i=1}^{N_k} c_i^{(k)} I_{\left(a_i^{(k)}, b_i^{(k)}\right)}(x)$ s.t. all $\left(a_i^{(k)}, b_i^{(k)}\right)$ are pairwise disjoint,

$c_i^{(k)} \neq 0$ and $f_k \to f$ in $L^\infty(0,1)$ as $k \to \infty$. However, consider for each fixed k, we can find $L^k = \min_{i=1}^{N_k} a_i^{(k)}$.

If $L^k = 0$, and WLOG, suppose the minimum is attained at $i = 1$, then $|f_k(x)| = |c_1^{(k)}| > 0$ on $\left(0, b_1^{(k)}\right)$ where $b_1^{(k)} > 0$. In this way we can find large enough n s.t. $\frac{1}{2n-1} < b_1^{(k)}$, then $|f_k(x) - f(x)| = |c_1^{(k)}|$ on $(\frac{1}{2n+1}, \frac{1}{2n})$. However, on interval $(\frac{1}{2n+2}, \frac{1}{2n+1})$, $|f_k(x) - f(x)| = |c_1^{(k)} - 1|$. Therefore, $\|f_k - f\|_\infty \geq \max\{|c_1^{(k)}|, |c_1^{(k)} - 1|\} \geq \frac{1}{2}$.

If $L^k > 0$, and WLOG, suppose the minimum is attained at $i = 1$, then $f_k(x) = 0$ on $(0, a_1^k)$. Similarly, we can find n large s.t. $\frac{1}{2n-1} < a_1^k$, then $f(x) = 1$ on $(\frac{1}{2n}, \frac{1}{2n-1})$. This implies that $\|f_k - f\|_\infty \geq 1$.

Thus, for all k, for whatever L^k, we always have $\|f_k - f\|_\infty \geq \frac{1}{2}$, then f_k cannot converge to f in $L^\infty(0,1)$.

Extra Problem 3. Let $f(x)$ be measurable and bounded on $\mathbb{R}$ and periodic with period $T > 0$. Let $g \in L^1(0, a)$, where $0 < a < \infty$. Prove that as $\epsilon \to 0+$,

$$\int_0^a f(x/\epsilon) g(x)\, dx \to \langle f \rangle \int_0^a g(x)\, dx, \quad \langle f \rangle = \frac{1}{T} \int_0^T f(y)\, dy$$

First consider $g = I_{(b,c)}$ where $0 \leq b < c \leq a$, we have

$$\begin{aligned}
\int_0^a f(x/\epsilon) g(x)\, dx &= \int_b^c f(x/\epsilon)\, dx \\
&= \epsilon \int_0^{(c-b)/\epsilon} f(z)\, dz && (z = (x-b)/\epsilon) \\
&= (c-b) \frac{1}{m} \int_0^m f(z)\, dz && (m = (c-b)/\epsilon)
\end{aligned}$$

Since $m = nT + r$ where $0 \leq r < T$, and $\left| \int_{nT}^{nT+r} f(z)\, dz \right| \leq M$ for some constant M and for all r, we have

$$\int_0^a f(x/\epsilon) g(x)\, dx = (c-b) \left[\frac{n}{m} \int_0^T f(z)\, dz + \frac{1}{m} \int_{nT}^{nT+r} f(z)\, dz \right]$$

Note that $\frac{n}{m} = \frac{m-r}{Tm} = \frac{1 - r/m}{T} \to \frac{1}{T}$ as $m \to \infty$. Thus, as $\epsilon \to 0+$, $m \to \infty$, and

$$\int_0^a f(x/\epsilon) g(x)\, dx \to (c-b) \frac{1}{T} \int_0^T f(z)\, dz = \langle f \rangle \int_0^a I_{(b,c)}(x)\, dx = \langle f \rangle \int_0^a g(x)\, dx$$

Thus, by linearity, it is easy to see if g is a step function on $(0, a)$, the desired result holds as well. Now consider general $g \in L^1(0, a)$, since step function is dense in $L^1(0, a)$, there exists $g_n(x) \to g(x)$ in $L^1(0, a)$. For arbitrary fixed δ, there exists N_1 s.t. for all $n \geq N_1$,

$$\left| \langle f \rangle \int_0^a g_n(x)\, dx - \langle f \rangle \int_0^a g(x)\, dx \right| < \frac{\delta}{3}$$

Since $f(x)$ is bounded on $(0, a)$, we can find N_2 s.t. for all $n \geq N_2$,

$$\left| \int_0^a f(x/\epsilon) g_n(x)\, dx - \int_0^a f(x/\epsilon) g(x)\, dx \right| < \frac{\delta}{3}$$

Take $N = \max\{N_1, N_2\}$, we have proved

$$\left|\int_0^a f(x/\epsilon)g_N(x)\,dx - \langle f\rangle \int_0^a g_N(x)\,dx\right| < \frac{\delta}{3}$$

By triangular inequality,

$$\left|\int_0^a f(x/\epsilon)g(x)\,dx - \langle f\rangle \int_0^a g(x)\,dx\right| < \delta$$

Since this is true for arbitrary $\delta > 0$, this is enough to prove the desired result.

Extra Problem 4.

(i) For all measurable subset $A \subset [0, 2\pi]$, prove that

$$\lim_{t\to\infty} \int_A \cos(tx)\,dx = 0$$

Consider any sequence t_k s.t. $t_k \to \infty$ as $k \to \infty$, then it suffices to show that

$$\lim_{k\to\infty} \int_A \cos(t_k x)\,dx = 0$$

Note that

$$\int_A \cos(t_k x)\,dx = \int_0^{2\pi} \cos(t_k x) I_A(x)\,dx$$

Since A is a bounded set, $I_A(x) \in L^1(0, 2\pi)$. Also, $|\cos(t_k x)| \le 1$ for all $x \in [0, 2\pi]$ and for any $c \in [0, 2\pi]$,

$$\int_0^c \cos(t_k x)\,dx = \frac{1}{t_k}\sin(t_k x)\Big|_0^c = \frac{\sin(ct_k)}{t_k} \to 0$$

as $k \to \infty$. Thus, by generalized Riemann-Lebesgue theorem, $\int_0^{2\pi} \cos(t_k x) I_A(x)\,dx \to 0$.

(ii) Let $t_k \to \infty$ as $k \to \infty$. Define $E = \{x \in [0, 2\pi] \,|\, \sin(t_k x) \text{ converges as } k \to \infty\}$. Prove $m(E) = 0$.

Similar to the proof of Egorov's theorem, let $f_k(x) = \sin(t_k x)$ and $f(x) = \lim_{k\to\infty} f_k(x)$. Denote

$$E^i_{k,l} = \{x \in [0, 2\pi] \,|\, |f_{k+l}(x) - f_k(x)| < 1/i\}$$

Then we can write $E = \bigcap_{i=1}^\infty \bigcap_{l=1}^\infty \bigcup_{k=1}^\infty E^i_{k,l}$. It is easy to see $E^i_{k,l}$ is measurable because f_k is continuous function. Thus, E is also measurable. Notice that

$$\int_E \sin^2(t_k x)\,dx = \int_E \frac{1 - \cos(2t_k x)}{2}\,dx$$

For LHS, since $|\sin^2(t_k x)| \le 1$ and $m(E) \le 2\pi$, we can use DCT to obtain

$$\lim_{k\to\infty} \int_E \sin^2(t_k x)\,dx = \int_E f^2(x)\,dx$$

Similarly, since $|f(x)\sin(t_k x)| \le 1$, by DCT again,

$$\lim_{k\to\infty} \int_E f(x)\sin(t_k x)\,dx = \int_E f^2(x)\,dx$$

Now we need to prove $\lim_{k\to\infty}\int_E f(x)\sin(t_k x)\,dx = 0$ by Riemann-Lebesgue theorem. We know $|f(x)| \in L^1(E)$ and $\sin(t_k x)$ is uniformly bounded by 1. Thus, it suffices to show $\lim_{k\to\infty}\int_0^c \sin(t_k x)\,dx = 0$ for all $c \in [0, 2\pi]$. This is trivial by using the same argument in part (i). Therefore, we obtain

$$\lim_{k\to\infty}\int_E \sin^2(t_k x)\,dx = 0$$

Now consider RHS, by part (i),

$$\lim_{k\to\infty}\int_E \frac{1-\cos(2t_k x)}{2}\,dx = \frac{m(E)}{2} - \lim_{k\to\infty}\frac{1}{2}\int_E \cos(2t_k x)\,dx = \frac{m(E)}{2}$$

This shows that $\frac{m(E)}{2} = 0$, i.e., $m(E) = 0$.

Extra Problem 5. Suppose $f \in L^1(0,1)$. Let $g(x) = \int_x^1 \frac{f(t)}{t}\,dt$, $0 < x \le 1$. Prove that $g \in L^1(0,1)$, $\lim_{x\to 0+} xg(x) = 0$ and $\int_0^1 g(x)\,dx = \int_0^1 f(t)\,dt$.

Notice that $g(x) = \int_0^1 \frac{f(t)}{t} I_E(t,x)\,dt$ for $E = \{(t,x) \in \mathbb{R}^2 \,|\, 0 \le x \le t \le 1\}$. Apply nonnegative version of Fubini's theorem to $\frac{|f(t)|}{t} I_E(t,x)$ on $(t,x) \in [0,1]\times[0,1]$, we obtain

$$\int_0^1 |g(x)|\,dx \le \int_0^1\int_0^1 \frac{|f(t)|}{t} I_E(t,x)\,dt\,dx = \int_0^1 \frac{|f(t)|}{t}\int_0^1 I_E(t,x)\,dx\,dt = \int_0^1 |f(t)|\,dt < \infty$$

This implies that $g \in L^1(0,1)$. The above result also implies that $\frac{|f(t)|}{t} I_E(t,x)$ is in $L^1([0,1]\times[0,1])$. Then, $\frac{f(t)}{t} I_E(t,x)$ is in $L^1([0,1]\times[0,1])$ and we can apply L^1-version of Fubini's theorem to it, i.e.,

$$\int_0^1 g(x)\,dx = \int_0^1\int_0^1 \frac{f(t)}{t} I_E(t,x)\,dt\,dx = \int_0^1 \frac{f(t)}{t}\int_0^1 I_E(t,x)\,dx\,dt = \int_0^1 f(t)\,dt$$

Now take arbitrary sequence $a_n > 0$ s.t. $a_n \to 0$ as $n \to \infty$. Also, let $g_n(t) = \frac{a_n}{t} I_E(a_n, t)$, then for each fixed $c \in [0,1]$, $|g_n(t)| \le 1$ for all $t \ge a_n$. Since $g_n(t) \to 0$ a.e. on $[0,1]$, by DCT, $\int_0^c g_n(t)\,dt \to 0$ as $n \to \infty$. Since $f(t) \in L^1(0,1)$, by generalized Riemann-Lebesgue theorem, we have $\int_0^1 f(t)g_n(t)\,dx \to 0$ as $n\to\infty$, i.e. $a_n g(a_n) \to 0$ as $n \to \infty$. This shows that $\lim\limits_{x\to 0+} xg(x) = 0$.

Extra Problem 6. Let $f \in L^1(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n)$. Prove that

(i) $(f*g)(x)$ is uniformly continuous in x on $\mathbb{R}^n$.

Let $F = (f*g)(x)$, consider

$$|F(x+h) - F(x)| = \left|\int_{\mathbb{R}^n} [f(x+h-y) - f(x-y)]g(y)\,dy\right| \le \|g\|_{L^\infty}\|f(u+h) - f(u)\|_{L^1_u} \to 0$$

as $|h| \to 0$ by continuity of L^1-norm and finiteness of $\|g\|_{L^\infty}$. Thus, for any fixed $\epsilon > 0$, there exists $\delta > 0$ s.t. when $|h| < \delta$, $\|f(u+h) - f(u)\|_{L^1_u} < \frac{\epsilon}{\|g\|_{L^\infty}}$, so $|F(x+h) - F(x)| < \epsilon$ and this proves the uniform continuity of F.

(ii) If $g \in L^1(\mathbb{R}^n)$, then $(f*g)(x) \to 0$ as $|x| \to \infty$.

Since simple function with bounded support is dense in $L^1(\mathbb{R})$, there exists $f_k \to f$ in L^1, where f_k is simple function with bounded support. This shows

$$|(f*g)(x)| \le \|f - f_k\|_{L^1}\|g\|_{L^\infty} + \left|\int_{\mathbb{R}^n} f_k(x-y)g(y)\,dy\right|$$

Similarly, we can find a sequence of simple function g_k with bounded support and $g_n \to g$ in L^1. Then,

$$\left|\int_{\mathbb{R}^n} f_k(x-y)g(y)\ dy\right| \le \|f_k\|_{L^\infty}\|g-g_n\|_{L^1} + \left|\int_{\mathbb{R}^n} f_k(x-y)g_n(y)\ dy\right|$$

Since f_k is simple function, it must be in L^∞ space, and $f_k(x-y)g_n(y) = 0$ for large enough $|x|$. This is because if the radius of the support of f_k is r_k and the radius of support of g_n is R_n, then if $|x| > r_k + R_n$, either $|y| > R_n$ or $|x-y| > r_k$, so either $f_k(x-y) = 0$ or $g_n(y) = 0$. This implies that

$$|(f*g)(x)| \le \|f-f_k\|_{L^1}\|g\|_{L^\infty} + \|f_k\|_{L^\infty}\|g-g_n\|_{L^1} + \left|\int_{\mathbb{R}^n} f_k(x-y)g_n(y)\ dy\right|$$

First take $|x| \to \infty$ on both sides, we have

$$\lim_{|x|\to\infty} |(f*g)(x)| \le \|f-f_k\|_{L^1}\|g\|_{L^\infty} + \|f_k\|_{L^\infty}\|g-g_n\|_{L^1}$$

Then take $n \to \infty$ on both sides, since LHS is independent of n, we have

$$\lim_{|x|\to\infty} |(f*g)(x)| \le \|f-f_k\|_{L^1}\|g\|_{L^\infty}$$

Finally, take $k \to \infty$ on both sides, since LHS is independent of k, we obtain $\lim_{|x|\to\infty} |(f*g)(x)| \le 0$, i.e., $(f*g)(x) \to 0$ as $|x| \to \infty$.

Extra Problem 7. Consider Fourier transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\ dx$$

Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Since step function is dense in $L^1(\mathbb{R})$, and a step function is a linear combination of characteristic functions of bounded intervals in $\mathbb{R}$, there exists $f_k(x) = \sum_{j=1}^{N_k} c_j^k I_{(a_j^k, b_j^k)}$ s.t. $f_k \to f$ in $L^1(\mathbb{R})$. Therefore, as $k \to \infty$,

$$\left|\hat{f}(\xi) - \hat{f}_k(\xi)\right| \le \int_{\mathbb{R}} |f(x) - f_k(x)|\ dx \to 0$$

Also notice that as $|\xi| \to \infty$,

$$\left|\hat{f}_k(\xi)\right| \le \sum_{j=1}^{N_k} |c_j^k| \left|\int_{a_j^k}^{b_j^k} e^{-2\pi i x\xi}\ dx\right| \le \sum_{j=1}^{N_k} |c_j^k| \frac{1}{\pi|\xi|} \to 0$$

Thus, for any fixed $\epsilon > 0$, we can find a large enough K s.t. $|\hat{f}(\xi) - \hat{f}_K(\xi)| < \frac{\epsilon}{2}$ and then find a large M, s.t. for all $|\xi| > M$, $|\hat{f}_K(\xi)| < \epsilon/2$. Then by triangular inequality, for all $|\xi| > M$, $|\hat{f}(\xi)| < \epsilon$. Since for each ϵ we can find such M, $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Extra Problem 8. Let $f(x)$ be nonnegative measurable on $[0,1]$. Prove that if there exists constant $A < \infty$ s.t. $\int_0^1 f^k(x)\ dx = A$ for all $k \ge 1$, then $f(x) = I_E(x)$ a.e. on $[0,1]$ for some $E \subset [0,1]$.

Let $g(x) = f(x)(1-f(x))$, then we have

$$\int_0^1 g^2(x)\ dx = \int_0^1 f^2\ dx - 2\int_0^1 f^3\ dx + \int_0^1 f^4\ dx = A - 2A + A = 0$$

Thus, $g(x) = 0$ a.e. on $[0,1]$. Denote $F = \{x \in [0,1] \,|\, g(x) = 0\}$, then $m(F) = 1$, and over the set F, $f(x) = 1$ or $f(x) = 0$. Thus, let $E = \{x \in [0,1] \,|\, f(x) = 1\}$, and we can see that $f(x) = I_E(x)$ on set F. Thus, $f(x) = I_E(x)$ a.e. on $[0,1]$.

Extra Problem 9. Suppose $f \in L^1(\mathbb{R})$, $f(0) = 0$, $f'(0)$ exists. Prove that $\frac{f(x)}{x} \in L^1(\mathbb{R})$.

By definition of derivative and assumption,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{x \to 0} \frac{f(x)}{x} = c$$

for some finite constant c. Thus, there exists $\delta > 0$ s.t. for all $|x| < \delta$, $|f(x)/x - c| < 1$, so $|f(x)/x| < 1 + |c|$. This shows

$$\begin{aligned}
\int_{\mathbb{R}} \left| \frac{f(x)}{x} \right| dx &= \int_{-\delta}^{\delta} \left| \frac{f(x)}{x} \right| dx + \int_{-\infty}^{-\delta} \left| \frac{f(x)}{x} \right| dx + \int_{\delta}^{\infty} \left| \frac{f(x)}{x} \right| dx \\
&\le 2\delta(1 + |c|) + \frac{1}{\delta} \int_{-\infty}^{-\delta} |f(x)|\, dx + \frac{1}{\delta} \int_{\delta}^{\infty} |f(x)|\, dx \\
&\le 2\delta(1 + |c|) + \frac{2}{\delta} \int_{\mathbb{R}} |f(x)|\, dx < \infty
\end{aligned}$$

Therefore, $\frac{f(x)}{x} \in L^1(\mathbb{R})$.

Extra Problem 10. Let $f \in L^1(\mathbb{R})$, and $a > 0$. Define $F(x) = \sum_{n=-\infty}^{\infty} f(x/a + n)$. Prove the series converges absolutely for almost all $x \in \mathbb{R}$, $F \in L^1([0,a])$ and F is periodic with period a.

Let $G(x) = \sum_{n=-\infty}^{\infty} |f(x/a + n)|$, then consider

$$\int_0^a G(x)\, dx = \int_0^a \sum_{n=-\infty}^{\infty} |f(x/a + n)|\, dx = \sum_{n=-\infty}^{\infty} \int_0^a |f(x/a + n)|\, dx$$

where the last equality is due to integration term by term for nonnegative function. Since $f \in L^1(\mathbb{R})$, by change of variable, let $u = x/a + n$,

$$\sum_{n=-\infty}^{\infty} \int_0^a |f(x/a + n)|\, dx = a \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f(u)|\, du = a \int_{\mathbb{R}} |f(u)|\, du < \infty$$

This implies that $G(x) \in L^1(0,a)$, and since $|F(x)| \le G(x)$, so $F \in L^1(0,a)$. Notice that

$$F(x + a) = \sum_{n=-\infty}^{\infty} f(x/a + n + 1) = \sum_{n=-\infty}^{\infty} f(x/a + n) = F(x)$$

so $F(x)$ is periodic with period a. Similarly $G(x)$ is also periodic with period a. Since $G \in L^1(0,a)$, G is a.e. finite on $(0,a)$. By periodicity and countable subadditivity, G is a.e. finite on $\mathbb{R}$. This implies that the series $F(x)$ is convergent absolutely for almost all $x \in \mathbb{R}$.