# MAT3220 Additional Exercises: Convexity

李肖鹏 (116010114)

**Due date**: Tomorrow

**Question 1.** Why a real symmetric matrix will always have real (as opposed to complex) eigenvalues?

Suppose $\lambda_0$ is an eigenvalue of real symmetric matrix $A$, then there exists a nonzero eigenvector $\vec{a}$ such that

$$A\vec{a} = \lambda_0\vec{a} \tag{1}$$

Take complex conjugate on both sides of (1), we have

$$\overline{A\vec{a}} = \overline{\lambda_0\vec{a}} \Longrightarrow \overline{A}\cdot\overline{\vec{a}} = \overline{\lambda_0}\cdot\overline{\vec{a}} \Longrightarrow A\cdot\overline{\vec{a}} = \overline{\lambda_0}\cdot\overline{\vec{a}} \tag{2}$$

Also, take transpose on both sides of (1), we have

$$(A\vec{a})^{\mathrm{T}} = (\lambda_0\vec{a})^{\mathrm{T}} \Longrightarrow \vec{a}^{\mathrm{T}}A^{\mathrm{T}} = \lambda_0\vec{a}^{\mathrm{T}} \Longrightarrow \vec{a}^{\mathrm{T}}A = \lambda_0\vec{a}^{\mathrm{T}} \tag{3}$$

Multiply $\vec{a}^{\mathrm{T}}$ on the left on both sides of (2), and multiply $\overline{\vec{a}}$ on the right on both sides of (3), we have

$$\vec{a}^{\mathrm{T}}A\overline{\vec{a}} = \overline{\lambda_0}\vec{a}^{\mathrm{T}}\overline{\vec{a}} \quad \text{and} \quad \vec{a}^{\mathrm{T}}A\overline{\vec{a}} = \lambda_0\vec{a}^{\mathrm{T}}\overline{\vec{a}}$$

Hence, we conclude that

$$(\lambda_0 - \overline{\lambda_0})\|\vec{a}\|_2^2 = 0$$

Since $\vec{a} \neq \vec{0}$, $\|\vec{a}\|_2 \neq 0$, then $\lambda_0 = \overline{\lambda_0}$, meaning that $\lambda_0 \in \mathbb{R}$.

**Question 2.** Prove the following Cauchy-Schwarz inequality, i.e., for any $\vec{u}, \vec{v} \in \mathbb{R}^n$, we have

$$\vec{u}^{\mathrm{T}}\vec{v} \leq \|\vec{u}\|_2 \cdot \|\vec{v}\|_2$$

Consider the following inequality,

$$\begin{aligned} 0 \leq \|\vec{u} - \lambda\vec{v}\|_2^2 &= (\vec{u} - \lambda\vec{v})^{\mathrm{T}}(\vec{u} - \lambda\vec{v}) \\ &= (\vec{u}^{\mathrm{T}} - \lambda\vec{v}^{\mathrm{T}})(\vec{u} - \lambda\vec{v}) \\ &= \|\vec{u}\|_2^2 - 2\lambda\vec{u}^{\mathrm{T}}\vec{v} + \lambda^2\|\vec{v}\|_2^2 \end{aligned}$$

Since for any $\lambda$,

$$f(\lambda) = \|\vec{u}\|_2^2 - 2\lambda\vec{u}^{\mathrm{T}}\vec{v} + \lambda^2\|\vec{v}\|_2^2 \geq 0$$

We have

$$\Delta = 4(\vec{u}^{\mathrm{T}}\vec{v})^2 - 4\|\vec{u}\|_2^2\|\vec{v}\|_2^2 \leq 0$$

We will finally conclude that

$$\vec{u}^{\mathrm{T}}\vec{v} \leq \|\vec{u}\|_2 \cdot \|\vec{v}\|_2$$

**Question 3.** Use the Cauchy-Schwarz inequality to prove the so-called triangle inequality for the Euclidean norm,

$$\|\vec{x} + \vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

To prove $\|\vec{x} + \vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$, we only need to prove

$$(\vec{x} + \vec{y})^{\mathrm{T}}(\vec{x} + \vec{y}) \leq \vec{x}^{\mathrm{T}}\vec{x} + 2\|\vec{x}\|_2\|\vec{y}\|_2 + \vec{y}^{\mathrm{T}}\vec{y}$$

But the left hand side is just

$$\vec{x}^{\mathrm{T}}\vec{x} + 2\vec{x}^{\mathrm{T}}\vec{y} + \vec{y}^{\mathrm{T}}\vec{y}$$

By Cauchy-Schwarz inequality, $2\vec{x}^{\mathrm{T}}\vec{y} \leq 2\|\vec{x}\|_2\|\vec{y}\|_2$, hence, we finish the proof.

**Question 4.** For a square matrix, $A \in \mathbb{R}^{n\times n}$, its *trace* is $\mathrm{tr}\,(A) = \sum_{i=1}^{n} a_{ii}$. Prove that for any $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{m\times n}$, we have $\mathrm{tr}\,(XY^{\mathrm{T}}) = \mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$.

Consider the $(i,i)$-th entry of $XY^{\mathrm{T}}$, if we denote $X_i$ as the $i$-th row of $X$, and $Y_i$ as the $i$-th row of $Y$, then we have

$$(XY^{\mathrm{T}})_{i,i} = X_iY_i^{\mathrm{T}} = \sum_{j=1}^{n} X_{ij}Y_{ij}$$

Hence, the trace of $XY^{\mathrm{T}}$ can be computed by

$$\mathrm{tr}\,(XY^{\mathrm{T}}) = \sum_{i=1}^{m} X_iY_i^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$$

Similarly, consider the $(i,i)$-th entry of $YX^{\mathrm{T}}$, we have

$$(YX^{\mathrm{T}})_{i,i} = Y_iX_i^{\mathrm{T}} = \sum_{j=1}^{n} Y_{ij}X_{ij}$$

Hence, the trace of $YX^{\mathrm{T}}$ can be computed by

$$\mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m} Y_iX_i^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}X_{ij}$$

In conclusion,

$$\mathrm{tr}\,(XY^{\mathrm{T}}) = \mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}Y_{ij}$$

**Question 5.** Let $X \in \mathbb{R}^{m\times n}$ be a real matrix. The so-called Frobenius norm of $X$ is defined as

$$\|X\|_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2\right)^{1/2}$$

and its spectrum norm is defined as $\|X\|_2 := \big(\lambda_{\max}(X^\mathrm{T}X)\big)^{1/2}$. Prove that both $\|\cdot\|_F$ and $\|\cdot\|_2$ are indeed matrix norms.

We first prove $\|\cdot\|_F$ is matrix norms, by checking whether it satisfies the five defining properties. For property (1), it is obvious that $\|\cdot\|_F \geq 0$. For property (2), if $\|X\|_F = 0$, we can derive that all $X_{ij}^2$ are equal to zero, meaning that $X$ is zero matrix. For property (3),

$$\begin{aligned}
\|\alpha X\|_F &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(\alpha X_{ij})^2\right)^{1/2} \\
&= \left(\alpha^2\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} \\
&= |\alpha|\left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} = |\alpha|\|X\|_F
\end{aligned}$$

For property (4), to prove $\|X+Y\|_F \leq \|X\|_F + \|Y\|_F$, we only need to prove

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}+Y_{ij})^2 \leq \left[\left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2} + \left(\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij}^2\right)^{1/2}\right]^2$$

which is equivalent to say

$$\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}Y_{ij} \leq \left(\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2\right)^{1/2}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij}^2\right)^{1/2}$$

However, this is exactly Cauchy-Schwarz inequality, so the proof of property (4) is finished. For property (5),

$$\begin{aligned}
\|XY\|_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}X_{ik}Y_{kj}\right)^2 \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}X_{ik}^2\sum_{k=1}^{n}Y_{kj}^2\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}^2\left(\sum_{j=1}^{n}\sum_{k=1}^{n}Y_{kj}^2\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}^2\|Y\|_F^2 \\
&= \|X\|_F^2\|Y\|_F^2
\end{aligned}$$

Hence, $\|\cdot\|_F$ is matrix norm.

Then we prove $\|\cdot\|_2$ is matrix norm. For property (1), since $X^\mathrm{T}X$ is always positive semi-definite, so all of its eigenvalues are non-negative, hence $\|X\|_2 := \big(\lambda_{\max}(X^\mathrm{T}X)\big)^{1/2} \geq 0$. For property (2), if $\|X\|_2 = 0$, we can derive that all eigenvalues of $X^\mathrm{T}X$ are zero, but since it is symmetric, so it must be zero matrix. If $X^\mathrm{T}X$ is zero matrix, consider its $(i,i)$-th entry,

$$(X^\mathrm{T}X)_{ii} = X_i^\mathrm{T}X_i = 0 \Longrightarrow X_i = \overrightarrow{0}$$

where $X_i$ denote the $i$-th column of $X$. It is obvious that $X$ is zero matrix, and we finish the proof of property (2). For property (3), we have

$$\|\alpha X\|_2 := \left(\lambda_{\max}(\alpha^2 X^{\mathrm{T}} X)\right)^{1/2} = \left(\alpha^2 \lambda_{\max}(X^{\mathrm{T}} X)\right)^{1/2} = |\alpha| \|X\|_2$$

For property (4), we only need to prove,

$$\left(\lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y))\right)^{1/2} \leq \left(\lambda_{\max}(X^{\mathrm{T}} X)\right)^{1/2} + \left(\lambda_{\max}(Y^{\mathrm{T}} Y)\right)^{1/2}$$

Let $\mu = \lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y))$, then we can take a unit eigenvetor $\vec{v}$ corresponding to $\mu$, i.e.,

$$(X+Y)^{\mathrm{T}}(X+Y)\vec{v} = \mu \vec{v}, \quad \|\vec{v}\|_2 = 1$$

Then, we know

$$\begin{aligned}
\mu &= \vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} + \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} + 2(X\vec{v})^{\mathrm{T}}(Y\vec{v}) \\
&\leq \vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} + \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} + 2\|X\vec{v}\|_2 \|Y\vec{v}\|_2 \\
&= (\|X\vec{v}\|_2 + \|Y\vec{v}\|_2)^2 = \left(\sqrt{\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v}} + \sqrt{\vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v}}\right)^2
\end{aligned}$$

Since $X^{\mathrm{T}} X$ is a real symmetric matrix, according to spectral decomposition, there exists orthogonal matrix $T$, such that $T^{-1} X^{\mathrm{T}} X T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n$ is the eigenvalues of $X^{\mathrm{T}} X$. For any vector $\vec{v}$, suppose $(T^{\mathrm{T}} \vec{v})^{\mathrm{T}} = (w_1, \ldots, w_n)$, then

$$\begin{aligned}
\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} &= \vec{v}^{\mathrm{T}} T \operatorname{diag}(\lambda_1, \ldots, \lambda_n) T^{-1} \vec{v} = (T^{\mathrm{T}} \vec{v})^{\mathrm{T}} \operatorname{diag}(\lambda_1, \ldots, \lambda_n)(T^{\mathrm{T}} \vec{v}) \\
&= (w_1, \ldots, w_n) \operatorname{diag}(\lambda_1, \ldots, \lambda_n)(w_1, \ldots, w_n)^{\mathrm{T}} = \lambda_1 w_1^2 + \ldots + \lambda_n w_n^2 \\
&\leq \lambda_1 (w_1^2 + \ldots + w_n^2) = \lambda_1 \|T^{\mathrm{T}} \vec{v}\|_2^2 = \lambda_1 \|\vec{v}\|_2^2 = \lambda_1
\end{aligned}$$

Hence, $\vec{v}^{\mathrm{T}} X^{\mathrm{T}} X \vec{v} \leq \lambda_{\max}(X^{\mathrm{T}} X)$. Similarly, we have $\vec{v}^{\mathrm{T}} Y^{\mathrm{T}} Y \vec{v} \leq \lambda_{\max}(Y^{\mathrm{T}} Y)$. Therefore, we have

$$\lambda_{\max}((X+Y)^{\mathrm{T}}(X+Y)) \leq \left((\lambda_{\max}(X^{\mathrm{T}} X))^{1/2} + (\lambda_{\max}(Y^{\mathrm{T}} Y))^{1/2}\right)^2$$

which proves property (4). For property (5), we need to prove

$$\mu = \lambda_{\max}((XY)^{\mathrm{T}}(XY)) \leq \lambda_{\max}(X^{\mathrm{T}} X) \lambda_{\max}(Y^{\mathrm{T}} Y)$$

Similar to property (4), we will obtain

$$\begin{aligned}
\mu = \vec{v}^{\mathrm{T}} Y^{\mathrm{T}} X^{\mathrm{T}} X Y \vec{v} &\leq \lambda_{\max}(X^{\mathrm{T}} X) \|Y\vec{v}\|_2^2 \\
&\leq \lambda_{\max}(X^{\mathrm{T}} X) \lambda_{\max}(Y^{\mathrm{T}} Y) \|\vec{v}\|_2^2 = \lambda_{\max}(X^{\mathrm{T}} X) \lambda_{\max}(Y^{\mathrm{T}} Y)
\end{aligned}$$

Hence, $\|\cdot\|_2$ is matrix norm.

**Question 6.** Prove that for any $X \in \mathbb{R}^{m \times n}$ and $\vec{y} \in \mathbb{R}^m$,

$$\|X\vec{y}\|_2 \leq \|X\|_2 \cdot \|\vec{y}\|_2$$

Actually, we have already prove this during the proof of Question 5. Since $X^{\mathrm{T}} X$ is a real symmetric matrix, according to spectral decomposition, there exists orthogonal matrix $T$, such that

$T^{-1}X^{\mathrm{T}}XT = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n$ is the eigenvalues of $X^{\mathrm{T}}X$. For any vector $\vec{y}$, suppose $(T^{\mathrm{T}}\vec{y})^{\mathrm{T}} = (w_1, \ldots, w_n)$, then

$$\begin{aligned}\vec{y}^{\mathrm{T}}X^{\mathrm{T}}X\vec{y} &= \vec{y}^{\mathrm{T}}T\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)T^{-1}\vec{y} = (T^{\mathrm{T}}\vec{y})^{\mathrm{T}}\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(T^{\mathrm{T}}\vec{y}) \\ &= (w_1, \ldots, w_n)\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)(w_1, \ldots, w_n)^{\mathrm{T}} = \lambda_1 w_1^2 + \ldots + \lambda_n w_n^2 \\ &\leq \lambda_1(w_1^2 + \ldots + w_n^2) = \lambda_1\|T^{\mathrm{T}}\vec{y}\|_2^2 = \lambda_1\|\vec{y}\|_2^2\end{aligned}$$

However, by definition, $\lambda_1 = \lambda_{\max}(X^{\mathrm{T}}X) = \|X\|_2^2$, we then conclude that

$$\|X\vec{y}\|_2 \leq \|X\|_2 \cdot \|\vec{y}\|_2$$

**Question 7.** Prove that for any $X$, it holds that $\|X\|_2 \leq \|X\|_F$.

Use the same method as we did in Question 4, we can obtain

$$\mathrm{tr}\,(X^{\mathrm{T}}X) = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2$$

Since $X^{\mathrm{T}}X$ is positive semi-definite matrix, all of its eigenvalue is nonegative, so the trace of it is larger than or equal to the largest eigenvalue of it, i.e.,

$$\mathrm{tr}\,(X^{\mathrm{T}}X) \geq \lambda_{\max}\,(X^{\mathrm{T}}X)$$

Therefore,

$$\|X\|_2 = \left(\lambda_{\max}\,(X^{\mathrm{T}}X)\right)^{1/2} \leq \left(\mathrm{tr}\,(X^{\mathrm{T}}X)\right)^{1/2} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2\right)^{1/2} = \|X\|_F$$

**Question 8.** Compute the gradient of the quartic function

$$f(x) = (\vec{x}^{\mathrm{T}}A\vec{x})^2$$

where $A \in \mathcal{S}^n$.

First, we know that the derivative of the quadratic form with respect to vector $\vec{x}$ is given by (assuming that $A$ is symmetric)

$$\nabla_{\vec{x}}\left(\vec{x}^{\mathrm{T}}A\vec{x}\right) = 2A\vec{x}$$

Hence, by chain rule, we have

$$\nabla_{\vec{x}}f(\vec{x}) = 2\vec{x}^{\mathrm{T}}A\vec{x}(2A\vec{x}) = 4(\vec{x}^{\mathrm{T}}A\vec{x})A\vec{x}$$

**Question 9.** Compute the Hessian matrix of the quartic function

$$f(x) = (\vec{x}^{\mathrm{T}}A\vec{x})^2$$

where $A \in \mathcal{S}^n$.

We can see that the hessian matrix is given by

$$\nabla^2_{\vec{x}} f(\vec{x}) = \nabla_{\vec{x}}(4(\vec{x}^{\mathrm{T}} A\vec{x})A\vec{x})$$

Therefore, we have

$$\nabla^2_{\vec{x}} f(\vec{x}) = 4(A\vec{x})(A\vec{x})^{\mathrm{T}} + 8(\vec{x}^{\mathrm{T}} A\vec{x})A$$

**Question 10.** Prove that if $h(\vec{x})$ is twice continuously differentiable, then that $h(\vec{x})$ is convex in $\mathbb{R}^n$ is equivalent to $\nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$.

We first claim that $h(\vec{x})$ is convex in $\mathbb{R}^n$ if and only if for any $\vec{x}, \vec{y}$, we have

$$h(\vec{y}) \geq h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x})$$

If so, suppose $H_h(\vec{z}) = \nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$, by Taylor expansion, we have

$$h(\vec{y}) = h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x}) + \frac{1}{2}\left[(\vec{y} - \vec{x})^{\mathrm{T}} H_h(\vec{z})(\vec{y} - \vec{x})\right]$$

for some $\vec{z} \in [\vec{x}, \vec{y}]$. Therefore, we obtain

$$h(\vec{y}) \geq h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y} - \vec{x})$$

By our claim, we can conclude that $h(\vec{x})$ is convex.

If we suppose $h(\vec{x})$ is convex, then for any $\vec{x}$ and $\vec{d}$, some $\lambda > 0$ will yield $\vec{x} + \lambda\vec{d}$. By Taylor expansion, we have

$$h(\vec{x} + \lambda\vec{d}) = h(\vec{x}) + \lambda\nabla h(\vec{x})^{\mathrm{T}}\vec{d} + \frac{\lambda^2}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + o(\|\lambda\vec{d}\|^2)$$

From our claim, we have

$$h(\vec{x} + \lambda\vec{d}) \geq h(\vec{x}) + \lambda\nabla h(\vec{x})^{\mathrm{T}}\vec{d}$$

Hence, we have

$$\frac{\lambda^2}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + o(\|\lambda\vec{d}\|^2) \geq 0$$

which implies

$$\frac{1}{2}\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} + \|\vec{d}\|^2 o(1) \geq 0$$

Take $\lambda \to 0$, we conclude that $\vec{d}^{\mathrm{T}} H_h(\vec{x})\vec{d} \geq 0$, which means $H_h(\vec{x})$ is positive semi-definite for all $\vec{x}$. Thus, that $h(\vec{x})$ is convex in $\mathbb{R}^n$ is equivalent to $\nabla^2 h(\vec{x}) \succeq 0$ for all $\vec{x} \in \mathbb{R}^n$.

Now we prove our claim. First assume $h$ is convex, and let $\vec{z} = \lambda\vec{y} + (1 - \lambda)\vec{x}$ for some $\vec{x}, \vec{y}$ and $\lambda \in [0, 1]$. Since $h$ is convex, we have

$$h(\vec{z}) = h(\lambda\vec{y} + (1 - \lambda)\vec{x}) \leq \lambda h(\vec{y}) + (1 - \lambda)h(\vec{x})$$

and therefore,

$$h(\vec{z}) - h(\vec{x}) \leq \lambda h(\vec{y}) + (1 - \lambda)h(\vec{x}) - h(\vec{x}) = \lambda h(\vec{y}) - \lambda h(\vec{x})$$

Since we know

$$\nabla h(\vec{x})^{\mathrm{T}}\vec{d} = \lim_{\lambda\to 0+}\frac{h(\vec{x}+\lambda\vec{d})-h(\vec{x})}{\lambda}$$

and therefore,

$$\nabla h(\vec{x})^{\mathrm{T}}(\vec{y}-\vec{x}) = \lim_{\lambda\to 0+}\frac{h(\vec{x}+\lambda(\vec{y}-\vec{x}))-h(\vec{x})}{\lambda} = \lim_{\lambda\to 0+}\frac{h(\vec{z})-h(\vec{x})}{\lambda} \le h(\vec{y})-h(\vec{x})$$

Now we assume $h(\vec{y}) \ge h(\vec{x}) + \nabla h(\vec{x})^{\mathrm{T}}(\vec{y}-\vec{x})$ for any $\vec{x},\vec{y}$. Let $\vec{z} = \lambda\vec{y} + (1-\lambda)\vec{x}$, we have

$$h(\vec{y}) \ge h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\vec{y}-\vec{z}) \tag{1}$$

$$h(\vec{x}) \ge h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\vec{x}-\vec{z}) \tag{2}$$

Therefore, we have

$$\begin{aligned}
\lambda h(\vec{y}) + (1-\lambda)h(\vec{x}) &\ge \lambda h(\vec{z}) + \lambda\nabla h(\vec{z})^{\mathrm{T}}(\vec{y}-\vec{z}) + (1-\lambda)h(\vec{z}) + (1-\lambda)\nabla h(\vec{z})^{\mathrm{T}}(\vec{x}-\vec{z})\\
&= h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\lambda\vec{y}-\lambda\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}((1-\lambda)\vec{x}-(1-\lambda)\vec{z})\\
&= h(\vec{z}) + \nabla h(\vec{z})^{\mathrm{T}}(\lambda\vec{y}+(1-\lambda)\vec{x}-\vec{z})\\
&= h(\vec{z}) = h(\lambda\vec{y}+(1-\lambda)\vec{x})
\end{aligned}$$

Hence, we conclude that $h$ is convex. Therefore, we finish the proof of our claim.

**Question 11.** Prove that $\left(\prod_{i=1}^n x_i\right)^{1/n}$ is a concave function in $\mathbb{R}^n_{++}$.

Let $f(\vec{x}) = \left(\prod_{i=1}^n x_i\right)^{1/n}$, and we need to compute the hessian matrix of $f(\vec{x})$. First we have

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{f(\vec{x})}{nx_i} \quad \text{for all} \quad i = 1,\dots,n$$

Then we compute the second-order partial derivative, we have

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(\vec{x}) = \frac{f(\vec{x})}{n^2x_ix_j},\ \text{for } i\ne j; \qquad \frac{\partial^2 f}{\partial x_i^2}(\vec{x}) = \frac{f(\vec{x})}{n^2x_i^2}(1-n)$$

Therefore, we check the quadratic form of arbitrary vector $\vec{u} = (u_1,u_2,\dots,u_n)^{\mathrm{T}}$.

$$\begin{aligned}
\vec{u}^{\mathrm{T}}H_f(\vec{x})\vec{u} = \sum_{i=1}^n\sum_{j=1}^n H_{ij}u_iu_j &= \frac{f(\vec{x})}{n^2}\left(\sum_{i=1}^n\frac{1-n}{x_i^2}u_i^2 + \sum_{i=1}^n\sum_{\substack{j=1\\ j\ne i}}^n\frac{1}{x_ix_j}u_iu_j\right)\\
&= \frac{f(\vec{x})}{n^2}\left(\sum_{i=1}^n\sum_{j=1}^n\frac{u_iu_j}{x_ix_j} - n\sum_{i=1}^n\frac{u_i^2}{x_i^2}\right)\\
&= \frac{f(\vec{x})}{n^2}\left[\left(\sum_{i=1}^n\frac{u_i}{x_i}\cdot 1\right)^2 - \left(\sum_{i=1}^n 1^2\right)\left(\sum_{i=1}^n\left(\frac{u_i}{x_i}\right)^2\right)\right]\\
&\le \frac{f(\vec{x})}{n^2}\cdot 0 = 0
\end{aligned}$$

By what we proved previously, if the hessian matrix $H_f(\vec{x})$ is negative semi-definite, then $f$ is concave function in $\mathbb{R}^n_{++}$.

**Question 12.** Prove that

$$\frac{x_1^n}{x_2 x_3 \cdots x_n}$$

is a convex function in $\mathbb{R}^n_{++}$.

Let

$$f(\vec{x}) = \frac{x_1^n}{x_2 x_3 \cdots x_n}, \quad g(\vec{x}) = \ln f(\vec{x}) = n \ln x_1 - \sum_{i=2}^{n} \ln x_i$$

Then, we can compute

$$\nabla f(\vec{x}) = f(\vec{x}) \nabla g(\vec{x}), \text{ where } \nabla g(\vec{x}) = \begin{bmatrix} \frac{n}{x_1} & -\frac{1}{x_2} & \cdots & -\frac{1}{x_n} \end{bmatrix}^{\mathrm{T}}$$

Also, by chain rule, we have

$$\nabla^2 f(\vec{x}) = f(\vec{x}) \left( \nabla g(\vec{x}) \nabla g(\vec{x})^{\mathrm{T}} + \nabla^2 g(\vec{x}) \right), \text{ where } \nabla^2 g(\vec{x}) = \begin{bmatrix} -\frac{n}{x_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n^2} \end{bmatrix}$$

For any vector $\vec{u} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\vec{u}^{\mathrm{T}} \nabla^2 f(\vec{x}) \vec{u} &= f(\vec{x}) \left[ -n \left( \frac{u_1}{x_1} \right)^2 + \sum_{i=2}^{n} \left( \frac{u_i}{x_i} \right)^2 + \left( n \frac{u_1}{x_1} - \sum_{i=2}^{n} \frac{u_i}{x_i} \right)^2 \right] \\
&= f(\vec{x}) \left[ -n \left( \frac{u_1}{x_1} \right)^2 + \sum_{i=2}^{n} \left( \frac{u_i}{x_i} \right)^2 + \left( (n-1) \frac{u_1}{x_1} - \sum_{i=2}^{n} \frac{u_i}{x_i} \right)^2 + (2n-1) \frac{u_1^2}{x_1^2} - 2 \frac{u_1}{x_1} \sum_{i=2}^{n} \frac{u_i}{x_i} \right] \\
&= f(\vec{x}) \left[ \sum_{i=2}^{n} \left( \frac{u_1}{x_1} \right)^2 - \sum_{i=2}^{n} 2 \frac{u_1}{x_1} \frac{u_i}{x_i} + \sum_{i=2}^{n} \left( \frac{u_i}{x_i} \right)^2 + \left( (n-1) \frac{u_1}{x_1} - \sum_{i=2}^{n} \frac{u_i}{x_i} \right)^2 \right] \\
&= f(\vec{x}) \left[ \sum_{i=2}^{n} \left( \frac{u_1}{x_1} - \frac{u_i}{x_i} \right)^2 + \left( (n-1) \frac{u_1}{x_1} - \sum_{i=2}^{n} \frac{u_i}{x_i} \right)^2 \right] \geq 0
\end{aligned}$$

Hence, the Hessian of $f(\vec{x})$ is always positive semi-definite, which implies that $f(\vec{x})$ is a convex function on $\mathbb{R}^n_{++}$.

**Question 13.** Consider $X \in S^{n \times n}$, and so $X$ has $n$ real eigenvalues as we discussed before. Let them be

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$$

Prove that $\lambda_1(X)$ is a convex function.

First we prove a lemma. Suppose $f_\gamma : X \to \mathbb{R}$ is a family of convex functions, with $\gamma \in A$, some index set, and let $f(x) = \sup_{\gamma \in A} f_\gamma(x)$. Then, for any fixed $\alpha \in A$, $\lambda \in [0, 1]$,

$$\begin{aligned}
f_\alpha(\lambda x + (1-\lambda) y) &\leq \lambda f_\alpha(x) + (1-\lambda) f_\alpha(y) \\
&\leq \sup_{\gamma \in A} (\lambda f_\gamma(x) + (1-\lambda) f_\gamma(y)) \\
&\leq \lambda \sup_{\gamma \in A} f_\gamma(x) + (1-\lambda) \sup_{\gamma \in A} f_\gamma(y) \\
&= \lambda f(x) + (1-\lambda) f(y)
\end{aligned}$$

By taking the supremum of the left hand side, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Hence, $f(x)$ is also convex.

From Question 5, we know that for any unit vector $\vec{v}$, if $X$ is symmetric matrix, then $\vec{v}^{\mathrm{T}} X \vec{v} \leq \lambda_1$, and when $\vec{v}$ is the unit eigenvector corresponding to $\lambda_1$, the maximum value $\lambda_1$ can be obtained. Thus, we could consider

$$\lambda_1(X) = \sup_{\|\vec{v}\|_2=1} g_{\vec{v}}(X), \quad \text{where } \; g_{\vec{v}}(X) = \vec{v}^{\mathrm{T}} X \vec{v}$$

For any fixed $\vec{v}$, $g_{\vec{v}}(X)$ is linear with respect to $X$, hence convex. By the lemma we proved just now, the supreme of it, that is, $\lambda(X)$, must be convex.

**Question 14.** Prove that

$$\ln\left(\sum_{i=1}^{n} e^{x_i}\right)$$

is a convex function.

Let $f(\vec{x})$ denote the original function, then we can compute

$$\nabla f(\vec{x}) = \frac{1}{\sum_{i=1}^{n} e^{x_i}} \begin{bmatrix} e^{x_1} & \cdots & e^{x_n} \end{bmatrix}^{\mathrm{T}}$$

and denote $H = \nabla^2 f(\vec{x})$, we have

$$\hat{H} = \left(\sum_{k=1}^{n} e^{x_k}\right)^2 [H]_{ij} = \begin{cases} e^{x_i} \displaystyle\sum_{k=1}^{n} e^{x_k} - e^{x_i+x_j} & \text{when } i = j \\ -\, e^{x_i+x_j} & \text{when } i \neq j \end{cases}$$

We only need to prove $\hat{H}$ is positive semi-definite matrix. For any $\vec{u} \in \mathbb{R}$, we have

$$\begin{aligned} \vec{u}^{\mathrm{T}} \hat{H} \vec{u} &= \sum_{i=1}^{n} \sum_{j=1}^{n} [\hat{H}]_{ij} u_i u_j \\ &= \left(\sum_{i=1}^{n} e^{x_i} u_i^2\right) \cdot \left(\sum_{i=1}^{n} e^{x_i}\right) - \sum_{i,j=1}^{n} e^{x_i} e^{x_j} u_i u_j \\ &= \left(\sum_{i=1}^{n} e^{x_i} u_i^2\right) \cdot \left(\sum_{i=1}^{n} e^{x_i}\right) - \left(\sum_{i=1}^{n} e^{x_i} u_i\right)^2 \geq 0 \end{aligned}$$

where the last line holds by Cauchy-Schwarz inequality. Hence, $\hat{H}$ is positive semi-definite, which means $H$ is PSD, and $f$ is a convex function.

**Question 15.** Suppose that $f(\vec{x}) \geq 0$ is convex for $\vec{x} \in S$, and $g(\vec{x}) > 0$ is concave for $\vec{x} \in S$. Prove that

$$\frac{f(\vec{x})}{g(\vec{x})}$$

is a quasi-convex function.

We only need to prove that for all $a$, the level set (when $g(\vec{x}) > 0$)

$$L_a = \left\{ \vec{x} \in S \,\middle|\, \frac{f(\vec{x})}{g(\vec{x})} < a \right\} = \{\vec{x} \in S \,|\, f(\vec{x}) < ag(\vec{x})\}$$

is a convex set. Take any two elements $\vec{x}, \vec{y}$ in $L_a$, we have

$$f(\vec{x}) < ag(\vec{x}), \quad f(\vec{y}) < ag(\vec{y})$$

Therefore, since $f$ is convex, $g$ is concave, we have for $\lambda \in (0, 1)$,

$$\begin{aligned} f(\lambda\vec{x} + (1-\lambda)\vec{y}) &\le \lambda f(\vec{x}) + (1-\lambda)f(\vec{y}) \\ &< \lambda ag(\vec{x}) + (1-\lambda)ag(\vec{y}) \\ &\le ag(\lambda\vec{x} + (1-\lambda)\vec{y}) \end{aligned}$$

Hence, $\lambda\vec{x} + (1-\lambda)\vec{y} \in L_a$, which means $L_a$ is a convex set, and

$$\frac{f(\vec{x})}{g(\vec{x})}$$

is quasi-convex.

**Question 16.** Show that

$$\frac{\vec{a}^{\mathrm{T}}\vec{x} + b}{\vec{c}^{\mathrm{T}}\vec{x} + d}$$

is quasi-linear in $\{\vec{x} \,|\, \vec{c}^{\mathrm{T}}\vec{x} + d > 0\}$.

Let $f(\vec{x})$ denote the original function, we tend to prove both $f(\vec{x})$ and $-f(\vec{x})$ are quasi-convex. Consider the level set of $f(\vec{x})$,

$$\begin{aligned} S_\alpha &= \left\{ \vec{x} \,\middle|\, \vec{c}^{\mathrm{T}}\vec{x} + d > 0,\ \frac{\vec{a}^{\mathrm{T}}\vec{x} + b}{\vec{c}^{\mathrm{T}}\vec{x} + d} \le \alpha \right\} \\ &= \left\{ \vec{x} \,\middle|\, \vec{c}^{\mathrm{T}}\vec{x} + d > 0 \right\} \cap \left\{ \vec{x} \,\middle|\, \vec{a}^{\mathrm{T}}\vec{x} + b \le \alpha(\vec{c}^{\mathrm{T}}\vec{x} + d) \right\} \\ &= \left\{ \vec{x} \,\middle|\, \vec{c}^{\mathrm{T}}\vec{x} + d > 0 \right\} \cap \left\{ \vec{x} \,\middle|\, (\vec{a} - \alpha\vec{c})^{\mathrm{T}}\vec{x} \le \alpha d - b \right\} \\ &= S_\alpha^{(1)} \cap S_\alpha^{(2)} \end{aligned}$$

Since $S_\alpha^{(1)}$ and $S_\alpha^{(2)}$ are both half spaces, so they are both convex, and the intersection of two convex sets are convex, so $S_\alpha$ is convex, which shows $f(\vec{x})$ is quasi-convex.

Similarly, we can show that the level set of $-f(\vec{x})$ can also be written as the intersection of two half spaces, which are convex, so $-f(\vec{x})$ is also quasi-convex. Therefore, $f(\vec{x})$ is quasi-linear.

**Question 17.** Suppose that $f(\vec{x})$ is convex for $x \in S$, and $g(\vec{x}) > 0$ is concave for $\vec{x} \in S$. Prove that

$$\frac{[f(\vec{x})]^2}{g(\vec{x})}$$

is a convex function.

First we prove a lemma, for $a, b, c, d \in \mathbb{R}$ and $c, d > 0$,

$$\frac{(a+b)^2}{c+d} \le \frac{a^2}{c} + \frac{b^2}{d}$$

This is indeed true because

$$\begin{aligned}\frac{(a+b)^2}{c+d} - \left(\frac{a^2}{c} + \frac{b^2}{d}\right) &= \frac{(a+b)^2cd - (a^2d+b^2c)(c+d)}{(c+d)cd} \\ &= \frac{(a^2cd + 2abcd + b^2cd) - (a^2dc + b^2c^2 + a^2d^2 + b^2cd)}{(c+d)cd} \\ &= \frac{-(bc-ad)^2}{(c+d)cd} \leq 0\end{aligned}$$

Let $h(\vec{x}) = [f(\vec{x})]^2/g(\vec{x})$, then for $\lambda \in (0,1)$, we have

$$\begin{aligned}h(\lambda\vec{x} + (1-\lambda)\vec{y}) &= \frac{[f(\lambda\vec{x} + (1-\lambda)\vec{y})]^2}{g(\lambda\vec{x} + (1-\lambda)\vec{y})} \\ &\leq \frac{[\lambda f(\vec{x}) + (1-\lambda)f(\vec{y})]^2}{\lambda g(\vec{x}) + (1-\lambda)g(\vec{y})} \\ &\leq \frac{\lambda^2[f(\vec{x})]^2}{\lambda g(\vec{x})} + \frac{(1-\lambda)^2[f(\vec{y})]^2}{(1-\lambda)g(\vec{y})} \qquad \text{(By lemma)} \\ &= \lambda h(\vec{x}) + (1-\lambda)h(\vec{y})\end{aligned}$$

Hence, $h(\vec{x})$ is a convex function.

**Question 18.** Prove that $\prod_{i=1}^n x_i$ is quasi-concave in $\mathbb{R}^n_{++}$.

To prove $\prod_{i=1}^n x_i$ is quasi-concave, we only need to prove that the level set

$$S_\alpha = \left\{ \vec{x} \in \mathbb{R}^n_{++} \;\middle|\; \prod_{i=1}^n x_i \geq \alpha \right\}$$

is convex for any $\alpha$ (because the domain of the function is convex). If $\alpha \leq 0$, then the level set is reduced to be $S_\alpha = \mathbb{R}^n_{++}$, which is obviously convex. If $\alpha > 0$, then $S_\alpha$ is equivalent to

$$S_\alpha = \left\{ \vec{x} \in \mathbb{R}^n_{++} \;\middle|\; \sum_{i=1}^n \ln x_i \geq \ln\alpha \right\}$$

Consider any $\vec{x}, \vec{y} \in S_\alpha$, and $\lambda \in [0,1]$, it is easy to know $\lambda\vec{x} + (1-\lambda)\vec{y} \in \mathbb{R}^n_{++}$. Also, since $\sum_{i=1}^n \ln x_i \geq \ln\alpha$ and $\sum_{i=1}^n \ln y_i \geq \ln\alpha$, we have

$$\begin{aligned}\sum_{i=1}^n \ln(\lambda x_i + (1-\lambda)y_i) &\geq \sum_{i=1}^n (\lambda \ln x_i + (1-\lambda)\ln y_i) \\ &\geq \lambda\ln\alpha + (1-\lambda)\ln\alpha = \ln\alpha\end{aligned}$$

Therefore, $\lambda\vec{x} + (1-\lambda)\vec{y} \in S_\alpha$, which shows that $S_\alpha$ is convex.

**Question 19.** Show that $S := \{\vec{x} \mid \|\vec{x} - \vec{a}\|_2 \leq \|\vec{x} - \vec{b}\|_2\}$ is a convex region. Further prove that $\|\vec{x} - \vec{a}\|_2 / \|\vec{x} - \vec{b}\|_2$ is quasi-convex in $S$.

Consider the set $S$, we have

$$\begin{aligned}\{\vec{x} \mid \|\vec{x} - \vec{a}\|_2 \leq \|\vec{x} - \vec{b}\|_2\} &= \{\vec{x} \mid \vec{x}^{\mathrm{T}}\vec{x} - 2\vec{a}^{\mathrm{T}}\vec{x} + \vec{a}^{\mathrm{T}}\vec{a} \leq \vec{x}^{\mathrm{T}}\vec{x} - 2\vec{b}^{\mathrm{T}}\vec{x} + \vec{b}^{\mathrm{T}}\vec{b}\} \\ &= \{\vec{x} \mid 2(\vec{b} - \vec{a})^{\mathrm{T}}\vec{x} \leq \vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a}\}\end{aligned}$$

which shows that $S$ is a half-space. It is very easy to show by definition that a half-space is convex, and hence $S$ is convex.

Next we need to prove the level set of $\|\vec{x}-\vec{a}\|_2/\|\vec{x}-\vec{b}\|_2$, which is given by

$$S_\alpha = \{\vec{x} \in S \mid \|\vec{x}-\vec{a}\|_2/\|\vec{x}-\vec{b}\|_2 \le \alpha\}$$

is convex for all $\alpha$. If $\alpha < 0$, then $S_\alpha$ is empty set, hence trivially convex. If $\alpha \ge 1$, then $S_\alpha = S$, which we have proved is convex, so we only need to consider the case when $\alpha \in [0, 1)$. In this case, $S_\alpha$ is equivalent to

$$\{\vec{x} \in S \mid (1-\alpha^2)\vec{x}^{\mathrm{T}}\vec{x} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{x} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a}\}$$

Take $\vec{x}$ and $\vec{y}$ in $S_\alpha$, we have

$$(1-\alpha^2)\vec{x}^{\mathrm{T}}\vec{x} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{x} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{1}$$

$$(1-\alpha^2)\vec{y}^{\mathrm{T}}\vec{y} + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}\vec{y} \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{2}$$

Multiply (1) by $\lambda$ and (2) by $(1-\lambda)$, then consider the sum of them, for $\lambda \in [0.1]$, we have

$$(1-\alpha^2)\big[\lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y}\big] + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \le \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \tag{$*$}$$

Since

$$\begin{aligned} & \lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y} \ge (\lambda\vec{x}+(1-\lambda)\vec{y})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\ \iff\ & \lambda(1-\lambda)\vec{x}^{\mathrm{T}}\vec{x} + \lambda(1-\lambda)\vec{y}^{\mathrm{T}}\vec{y} \ge 2\lambda(1-\lambda)\vec{x}^{\mathrm{T}}\vec{y} \end{aligned}$$

which is obviously true, and since $1-\alpha^2 > 0$, we can obtain

$$\begin{aligned} & (1-\alpha^2)(\lambda\vec{x}+(1-\lambda)\vec{y})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\ \le\ & (1-\alpha^2)\big[\lambda\vec{x}^{\mathrm{T}}\vec{x} + (1-\lambda)\vec{y}^{\mathrm{T}}\vec{y}\big] + 2(\alpha^2\vec{b}-\vec{a})^{\mathrm{T}}(\lambda\vec{x}+(1-\lambda)\vec{y}) \\ \le\ & \alpha^2\vec{b}^{\mathrm{T}}\vec{b} - \vec{a}^{\mathrm{T}}\vec{a} \end{aligned}$$

which means $\lambda\vec{x}+(1-\lambda)\vec{y} \in S_\alpha$, and we conclude that $S_\alpha$ is convex, and the function is quasi-convex.

**Question 20.** Prove that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

is a log-concave function.

We need to prove that $g(x) = \ln\Phi(x)$ is concave function. Consider the first-order derivative of it, we have

$$g'(x) = \frac{e^{-x^2/2}}{\int_{-\infty}^{x} e^{-t^2/2}\,dt}$$

Then consider the second-order derivative of it, we have

$$g''(x) = e^{-x^2/2}\frac{-x\int_{-\infty}^{x} e^{-t^2/2}\,dt - e^{-x^2/2}}{\left(\int_{-\infty}^{x} e^{-t^2/2}\,dt\right)^2}$$

Let

$$h(x) = -x\int_{-\infty}^{x} e^{-t^2/2}\,dt - e^{-x^2/2}$$

we consider the monotonicity and limit of it. Compute

$$h'(x) = -x\int_{-\infty}^{x} e^{-t^2/2}\,dt - e^{-x^2/2} < 0$$

We know that $h(x)$ is strictly decreasing, the the supremum of it is its limit as $t \to -\infty$, however,

$$\lim_{x\to-\infty}\left[-x\int_{-\infty}^{x} e^{-t^2/2}\,dt - e^{-x^2/2}\right] = \lim_{x\to-\infty}\frac{\int_{-\infty}^{x} e^{-t^2/2}\,dt}{-x^{-1}} = \lim_{x\to-\infty}\frac{e^{-x^2/2}}{x^{-2}} = 0$$

Therefore, $h(x) < 0$ for all $x \in \mathbb{R}$, and we know that $g''(x) < 0$, which shows $g(x)$ is concave.

**Question 21.** Suppose $Q \in S_{++}^{n\times n}$. Prove that

$$2\vec{x}^{\mathrm{T}}\vec{y} \le \vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y}$$

for any $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Since $Q$ is positive definite matrix, there exists orthogonal matrix $P$ such that

$$\vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y} = \vec{x}^{\mathrm{T}}P^{\mathrm{T}}DP\vec{x} + \vec{y}^{\mathrm{T}}PD^{-1}P^{\mathrm{T}}\vec{y} = \vec{\bar{x}}^{\mathrm{T}}D\vec{\bar{x}} + \vec{\bar{y}}^{\mathrm{T}}D^{\mathrm{T}}\vec{\bar{y}}$$

If we suppose $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\vec{\bar{x}} = (\bar{x}_1, \ldots, \bar{x}_n)^{\mathrm{T}}$, and $\vec{\bar{y}} = (\bar{y}_1, \ldots, \bar{y}_n)^{\mathrm{T}}$, since all $\lambda_i > 0$, we have

$$\begin{aligned}
\vec{x}^{\mathrm{T}}Q\vec{x} + \vec{y}^{\mathrm{T}}Q^{-1}\vec{y} &= \lambda_1\bar{x}_1^2 + \ldots + \lambda_n\bar{x}_n^2 + \lambda^{-1}\bar{y}_1^2 + \ldots + \lambda^{-1}\bar{y}_n^2 \\
&\ge 2(\bar{x}_1\bar{y}_1 + \ldots + \bar{x}_n\bar{y}_n) \\
&= 2(P\vec{x})^{\mathrm{T}}P^{\mathrm{T}}\vec{y} = 2\vec{x}^{\mathrm{T}}PP^{\mathrm{T}}\vec{y} \\
&= 2\vec{x}^{\mathrm{T}}I_n\vec{y} = 2\vec{x}^{\mathrm{T}}\vec{y}
\end{aligned}$$

Hence, we finish the proof.

**Question 22.** Suppose $0 < p < 1$. Show that

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$$

is a concave function in $\mathbb{R}_{++}^n$.

Let $f(\vec{x})$ denote the original function, and $g(\vec{x}) = \ln f(\vec{x})$, we have

$$[\nabla g(\vec{x})]_i = \frac{1}{\sum_{k=1}^{n} x_k^p}x_i^{p-1}$$

and

$$[\nabla^2 g(\vec{x})]_{ij} = \begin{cases} \frac{1}{\left(\sum_{k=1}^{n} x_k^p\right)^2}\left[(p-1)x_i^{p-2}\sum_{k=1}^{n} x_k^p - px_i^{p-1}x_j^{p-1}\right] & \text{when } i = j \\ \frac{1}{\left(\sum_{k=1}^{n} x_k^p\right)^2}\left[-px_i^{p-1}x_j^{p-1}\right] & \text{when } i \ne j \end{cases}$$

Since we know
$$\nabla^2 f(\vec{x}) = f(\vec{x})\left[\nabla g(\vec{x})\nabla g(\vec{x}^{\mathrm{T}}) + \nabla^2 g(\vec{x})\right]$$
If we let $\bar{H} = f(\vec{x})^2\nabla^2 f(\vec{x})$, we only need to check $\bar{H}$ is negative semi-definite, then we can conclude that $f(\vec{x})$ is concave function. Take any vector $\vec{u}$, we consider for $1-p>0$,
$$\begin{aligned}
\vec{u}^{\mathrm{T}}\bar{H}\vec{u} &= (1-p)\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{p-1}x_j^{p-1}u_iu_j + (p-1)\left(\sum_{k=1}^{n} x_k^{p-2}u_k^2\right)\left(\sum_{k=1}^{n} x_k^{p}\right)\\
&= (1-p)\left[\left(\sum_{i=1}^{n} x_i^{p-1}u_i\right)^2 - \left(\sum_{k=1}^{n} x_k^{p-2}u_k^2\right)\left(\sum_{k=1}^{n} x_k^{p}\right)\right]\\
&\le 0
\end{aligned}$$
Therefore, $\bar{H}$ is negative semi-definite, and $f(\vec{x})$ is concave function in $\mathbb{R}^n_{++}$.

**Question 23.** If $f(\vec{x})$ is twice continuously differentiable and quasi-convex, then for any $\vec{x} \in \mathrm{dom}(f)$,
$$\vec{d}^{\mathrm{T}}\nabla f(\vec{x}) = 0 \Longrightarrow \vec{d}^{\mathrm{T}}\nabla^2 f(\vec{x})\vec{d} \ge 0$$

Suppose for some $\vec{x}$, $\vec{d}^{\mathrm{T}}\nabla^2 f(\vec{x})\vec{d} < 0$ under that condition. Let $h(t) = f(\vec{x} + t\vec{d})$, then $h'(0) = \vec{d}^{\mathrm{T}}\nabla f(\vec{x}) = 0$ and $h''(0) = \vec{d}^{\mathrm{T}}\nabla^2 f(\vec{x})\vec{d} < 0$. Then in a small neighborhood $(-\delta,\delta)$, 0 is a local maximum of $h(t)$. Then, we will have $h(0) > \max\{h(t_1), h(-t_1)\}$ for some $0 \ne t_1 \in (-\delta,\delta)$. Now we consider the level set $S_\alpha$ of $f(\vec{x})$, let $\alpha = \max\{h(t_1), h(-t_1)\}$, then $h(t_1) = f(\vec{x}+t_1\vec{d})$ and $h(-t_1) = f(\vec{x} - t_1\vec{d})$ are both in $S_\alpha$, but their convex combination $h(0) = f(\vec{x})$ is not in $S_\alpha$, so $f$ is not quasi-convex at least in that small neighborhood. Contradiction shows that our assumption is wrong, and $\vec{d}^{\mathrm{T}}\nabla^2 f(\vec{x})\vec{d} \ge 0$ for all $\vec{x}$.

**Question 24.** If the condition in Question 23 holds, then there must exist some real value $\alpha$ such that
$$\nabla^2 f(\vec{x}) + \alpha\nabla f(\vec{x})(\nabla f(\vec{x}))^{\mathrm{T}} \succeq 0$$
Also, the Hessian matrix of a quasi-convex function can have at most one negative eigenvalue

We first prove that the hessian matrix of quasi-convex function can never have two or more negative eigenvalues. If it does have, then take any two negative of them $\lambda_1$ and $\lambda_2$, with corresponding eigenvector $\vec{v}_1$ and $\vec{v}_2$. Since for symmetric matrix, it has orthogonal eigenbasis, we have $\vec{v}_1 \perp \vec{v}_2$. Let $\vec{u} = \nabla f(\vec{x})$, the orthogonal complement space of $\vec{u}$ has dimension $n-1$, but span $\{\vec{v}_1, \vec{v}_2\}$ has dimension 2, so the intersection of them always contains nontrivial vector $\vec{d}$. Therefore, $\vec{d}^{\mathrm{T}}\vec{u} = 0$, but if we consider $H = \nabla^2 f(\vec{x})$, we have
$$\begin{aligned}
\vec{d}^{\mathrm{T}}H\vec{d} &= \vec{d}^{\mathrm{T}}H(a\vec{v}_1 + b\vec{v}_2)\\
&= \vec{d}^{\mathrm{T}}(\lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2)\\
&= (a\vec{v}_1 + b\vec{v}_2)^{\mathrm{T}}(\lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2)\\
&= \lambda_1 a^2\|\vec{v}_1\|_2^2 + \lambda_2 b^2\|\vec{v}_2\|_2^2 < 0
\end{aligned}$$
which contradicts to what we proved in Question 23.

If $H$ is PSD, then we are done by choosing $\alpha = 0$. If $H$ has exactly one negative eigenvalue, $\lambda_1 < 0$, so $H$ is indefinite matrix. We now prove a more general theorem as follows

**Theorem [Finsler].** *For symmetric matrix $A, B \in \mathbb{R}^{n\times n}$ with $B$ indefinite, if $\vec{x}^{\mathrm{T}} B \vec{x} = 0 \implies \vec{x}^{\mathrm{T}} A \vec{x} \geq 0$, then $A + tB$ is positive semidefinite for some $t \in \mathbb{R}$.*

*Proof.* Define two sets as follows

$$F_1 = \{t \in \mathbb{R} \,|\, \vec{x} \in \mathbb{R}^n,\ \vec{x}^{\mathrm{T}}(-B)\vec{x} \geq 0 \implies \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0\}$$

$$F_2 = \{t \in \mathbb{R} \,|\, \vec{x} \in \mathbb{R}^n,\ \vec{x}^{\mathrm{T}} B\vec{x} \geq 0 \implies \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0\}$$

where $A(t) = A + tB$. If there exists real number $t_0 \in F_1 \cap F_2$, then $A(t_0)$ is positive semidefinite. Thus, we need to show $F_1 \cap F_2 \neq \varnothing$.

From our assumption, we have for $t \in \mathbb{R}$,

$$\vec{x}^{\mathrm{T}} B \vec{x} = 0 \implies \vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0$$

which implies $E(t) \subset C \cup D$, where

$$E(t) = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x}^{\mathrm{T}} A(t) \vec{x} < 0\},\ C = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x}^{\mathrm{T}} B \vec{x} > 0\},\ D = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x}^{\mathrm{T}} B \vec{x} < 0\}$$

The set $E(t)$ consists of at most two connected components (This is not trivial, you can consider the canonical form of quadratic form $\vec{x}^{\mathrm{T}} A(t) \vec{x} = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_r^2$, when there is only one negative term, $E(t)$ will be disconnected and has only two connected components; when the number of negative term is larger than or equal to 2, $E(t)$ will be connected), and these two components are symmetric (though each single component is not symmetric) with respect to the origin; the sets $C$ and $D$, whose union is disconnected, are also symmetric (here $C$ and $D$ itself is symmetric) with respect to origin. Since we can easily check that any connected component(s) of $E(t)$ must be contained in $C$ or $D$, the whole set $E(t)$ is contained in $C$ or $D$ for each fixed $t$. Therefore, for any $t \in \mathbb{R}$, $t \in F_1$ or $t \in F_2$, and this means $F_1 \cup F_2 = \mathbb{R}$.

Since $B$ is indefinite, It is easy to show that $F_1$ and $F_2$ are nonempty sets. Also, since quadratic function is always continuous, so $F_1$ and $F_2$ can be shown to be closed set easily. In this way, we can conclude that $F_1 \cap F_2 \neq \varnothing$. This just means there exists a $t$, no matter what the result of $\vec{x}^{\mathrm{T}} B \vec{x}$ is, we always have $\vec{x}^{\mathrm{T}} A(t) \vec{x} \geq 0$, meaning that $A(t) \succeq 0$.

$\square$

Then let $B = \vec{u}\vec{u}^{\mathrm{T}}$ and $A = H$ in the above theorem, we can directly obtain what we need to prove.

**Question 25.** For $X \in \mathcal{S}^{n\times n}$, its eigenvalues are denoted to be

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_{n-1}(X) \geq \lambda_n(X)$$

Let $1 \leq k \leq n$. Consider

$$f(X) := \sum_{i=1}^{k} \lambda_i(X)$$

Prove that $f(X)$ is a convex function. You could first show that

$$f(X) = \sup\{\mathrm{tr}\,(U^{\mathrm{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\mathrm{T}}U = I_k\}$$

If we prove that

$$f(X) = \sup\{\mathrm{tr}\,(U^{\mathrm{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\mathrm{T}}U = I_k\} \tag{$*$}$$

then $f(X)$ is obviously convex, because it can be regared as the supremum of $g(X) = \mathrm{tr}\,(U^{\mathrm{T}}XU)$, which is linear with respect to $X$ (trace function is linear, and $U^{\mathrm{T}}XU$ is also linear). Since linear function is convex, so the supremum of it must be convex. Thus, it suffices to prove $(*)$ is correct.

Take the eigen-decomposition of $X = Q^{\mathrm{T}}DQ$, where $Q$ is orthogonal matrix and $D$ is diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ of $X$ as its diagonal entries. If we let $\overline{U} = QU$ for all $U^{\mathrm{T}}U = I_k$, then

$$\overline{U}^{\mathrm{T}}\overline{U} = U^{\mathrm{T}}Q^{\mathrm{T}}QU = U^{\mathrm{T}}I_nU = U^{\mathrm{T}}U = I_k$$

Thus, we have

$$\left\{\mathrm{tr}\ \left(U^{\mathrm{T}}XU\right) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\mathrm{T}}U = I_k\right\} = \left\{\mathrm{tr}\ \left(\overline{U}^{\mathrm{T}}D\overline{U}\right) \,|\, \overline{U} \in \mathbb{R}^{n\times k},\ \overline{U}^{\mathrm{T}}\overline{U} = I_k\right\}$$

If we denote the $I$-th row of $\overline{U}$ to be $\vec{\overline{U}}_i$, and the $j$-th entry of $\vec{\overline{U}}_i$ to be $\overline{U}_{ij}$, then we have

$$\mathrm{tr}\ \left(\overline{U}^{\mathrm{T}}D\overline{U}\right) = \mathrm{tr}\ \left(D\overline{U}\,\overline{U}^{\mathrm{T}}\right) = \sum_{i=1}^{n} \lambda_i \left\|\vec{\overline{U}}_i\right\|_2^2$$

Since $\mathrm{tr}\ \left(\overline{U}^{\mathrm{T}}\overline{U}\right) = \mathrm{tr}\ (I_k) = k$, we have $\sum_{i=1}^{n} \left\|\vec{\overline{U}}_i\right\|_2^2 = k$. Also, notice that $\overline{U}$ is a $n \times k$ matrix whose $k \le n$ columns form an orthonormal set of vectors in $\mathbb{R}^n$, hence linearly independent. Thus, we can extend it to a basis of $\mathbb{R}^n$, and by applying Gram-Schmidt process, we can obtain an orthonormal basis of $\mathbb{R}^n$ including all $k$ columns of $\overline{U}$. In other words, we have extended the orginal $\overline{U}$ to a larger orthogonal matrix $\widetilde{U} = \left[\overline{U}, \overline{V}\right]$. Therefore, if we denote $\vec{\overline{V}}_i$ as the $i$-th row of $\overline{V}$

$$\left\|\vec{\overline{U}}_i\right\|_2^2 + \left\|\vec{\overline{V}}_i\right\|_2^2 = 1 \Longrightarrow \left\|\vec{\overline{U}}_i\right\|_2^2 \le 1$$

Therefore, if we consider the weighted average of $\left\|\vec{\overline{U}}_i\right\|_2^2$, i.e., $\sum_{i=1}^{n} \lambda_i \left\|\vec{\overline{U}}_i\right\|_2^2$, to maximize it, we should assign the maximum value to the maximum weight. However, each weight can be at most 1, and we have $k$ units in total, hence, the maximized case is that we allocate 1 to the largest $k$ weights, i.e.,

$$\sum_{i=1}^{n} \lambda_i \left\|\vec{\overline{U}}_i\right\|_2^2 \le \sum_{i=1}^{k} \lambda_i$$

If we choose the $k$ columns of $U$ to be $k$ eigenvectors of $X$, then we have $\mathrm{tr}\,(U^{\mathrm{T}}XU) = \lambda_1 + \cdots + \lambda_k$. Therefore,

$$\sup\{\mathrm{tr}\,(U^{\mathrm{T}}XU) \,|\, U \in \mathbb{R}^{n\times k},\ U^{\mathrm{T}}U = I_k\} = \lambda_1 + \cdots + \lambda_k = f(X)$$

and the proof is finished.

**Question 26.** A function $f : \mathbb{R}^n_{++} \mapsto \mathbb{R}$

$$h(\vec{x}) = cx_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$$

with $c > 0$ and $\lambda \in \mathbb{R}^n$ is called a *monomial*. Sum of monomials, $f(\vec{x}) = \sum_{i=1}^{k} h_i(\vec{x})$, is called a *posynomial*.

The so-called *geometric programming* problem is as follows,

$$
\begin{aligned}
(G) \quad \min_{\vec{x}} \quad & f_0(\vec{x}) \\
s.t. \quad & f_i(\vec{x}) \le 1, \ i = 1, 2, \dots, m \\
& h_j(\vec{x}) = 1, \ j = 1, 2, \dots, p
\end{aligned}
$$

where $f_i(\vec{x})$ are posynomials ($i = 1, 2, \dots, m$), and $h_j(\vec{x})$ are monomials ($j = 1, 2, \dots, p$). Show that $(G)$ can be formulated as convex optimization through a variable transformation.

First, we clarify some notations,

$$h_j(\vec{x}) = c_j x_1^{\lambda_{j,1}} x_2^{\lambda_{j,2}} \cdots x_n^{\lambda_{j,n}}, \ j = 1, \dots, p$$

Similarly,

$$f_i(\vec{x}) = \sum_{k=1}^{a_i} h_k^{(i)}(\vec{x}), \ i = 0, 1, \dots, m, \ h_k^{(i)}(\vec{x}) = c_k^{(i)} x_1^{\lambda_{k,1}^{(i)}} x_2^{\lambda_{k,2}^{(i)}} \cdots x_n^{\lambda_{k,n}^{(i)}}$$

Take $x_t = e^{y_t}$ for $t = 1, \dots, n$, the reformulation is

$$
\begin{aligned}
(G_1) \quad \min_{\vec{y}} \quad & \sum_{k=1}^{a_0} c_k^{(0)} \exp\left\{\sum_{t=1}^{n} \lambda_{k,t}^{(0)} y_t\right\} \\
s.t. \quad & \sum_{k=1}^{a_i} c_k^{(i)} \exp\left\{\sum_{t=1}^{n} \lambda_{k,t}^{(i)} y_t\right\} \le 1, \ i = 1, 2, \dots, m \\
& c_j \exp\left\{\sum_{t=1}^{n} \lambda_{j,t} y_t\right\} = 1, \ j = 1, 2, \dots, p
\end{aligned}
$$

To simplify it, we have

$$
\begin{aligned}
(G_2) \quad \min_{\vec{y}} \quad & \ln\left\{\sum_{k=1}^{a_0} c_k^{(0)} \exp\left\{\sum_{t=1}^{n} \lambda_{k,t}^{(0)} y_t\right\}\right\} \\
s.t. \quad & \ln\left\{\sum_{k=1}^{a_i} c_k^{(i)} \exp\left\{\sum_{t=1}^{n} \lambda_{k,t}^{(i)} y_t\right\}\right\} \le 0, \ i = 1, 2, \dots, m \\
& \sum_{t=1}^{n} \lambda_{j,t} y_t = -\ln c_j, \ j = 1, 2, \dots, p
\end{aligned}
$$

From Question 14, we have known that the log-sum-exponential function $\ln\left(\sum_{t=1}^{k} e^{y_t}\right)$ is convex, since all $c_t > 0$ are positive, this result can be easily generalized to the function $\ln\left(\sum_{t=1}^{k} c_t e^{y_t}\right)$. The objective function and inequality constraints of $(G_2)$ can be regarded as the composite of log-sum-exponential and affine function, so they are all convex. The equality constraints are all affine functions, so $(G_2)$ is a convex problem.

**Question 27.** Formulate the following $L_4$-norm approximation problem as QCQP,

$$\min_{\vec{x}} \quad \|A\vec{x} - b\|_4 = \left(\sum_{i=1}^{m} \left(\vec{a}_i^{\mathrm{T}} \vec{x} - b_i\right)^4\right)^{1/4}$$

First, we know that the original problem is equivalent to

$$\min_{\vec{x}} \sum_{i=1}^{m} \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^4$$

Using change of variable, let $t_i = \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$. Thus, we have

$$\begin{aligned} \min_{\vec{x}, t_i} \quad & \sum_{i=1}^{m} t_i^2 \\ s.t. \quad & t_i = \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2, \ i = 1, 2, \ldots, m \end{aligned}$$

Since QCQP cannot have non-linear equality constraints, so we need to transform equality to inequality constraints. Suppose $t_i > \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$, then to minimize the sum of square of $t_i$, we can decrease $t_i$ until it is equal to $\left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2$, thus we can reformulate it into

$$\begin{aligned} \min_{\vec{x}, t_i} \quad & \sum_{i=1}^{m} t_i^2 \\ s.t. \quad & t_i \geq \left(\vec{a_i}^{\mathrm{T}} \vec{x} - b_i\right)^2, \ i = 1, 2, \ldots, m \end{aligned}$$

**Question 28.** The so-called *Chebyshev center* of a polyhedron is the deepest point inside the polyhedron. Suppose that the polyhedron is given by $P = \{\vec{x} \mid \vec{a_i}^{\mathrm{T}} \vec{x} \leq b_i, i = 1, 2, \ldots, m\}$. Formulate the problem of finding the Chebyshev center of $P$ by a convex optimization model.

Suppose the Chebyshev center is at point $\vec{p}$, and the radius of the Euclidean ball is $r \geq 0$. The only constrain is that the whole ball should lie in the polyhedron (we only need the sphere to be in the polyhedron). Therefore,

$$\vec{a_i}^{\mathrm{T}}(\vec{p} + r\vec{u}) \leq b_i, \quad \forall \|\vec{u}\|_2 = 1, \ \forall i = 1, \ldots, m$$

However, this is the case when uncountable constraints are involved, so we need to change it into finite many constraints. Consider the supremum of all constraints, we have

$$\sup_{\|\vec{u}\|_2 = 1} \vec{a_i}^{\mathrm{T}}(\vec{p} + r\vec{u}) = \vec{a_i}^{\mathrm{T}} \vec{p} + r\|\vec{a_i}\|_2 \leq b_i, \quad \forall i = 1, \ldots, m$$

Therefore, we can obtain the formulation

$$\begin{aligned} \max_{\vec{p}, r} \quad & r \\ s.t. \quad & \vec{a_i}^{\mathrm{T}} \vec{p} + r\|\vec{a_i}\|_2 \leq b_i, \ i = 1, 2, \ldots, m \end{aligned}$$

Since the objective function and constraints are linear with respect to $\vec{p}$ and $r$, it is a convex problem.

**Question 29.** An ellipsoid may be given by the image of a ball under some linear transformation, e.g. $E = \{Bu + b \,|\, \|u\|_2 \leq 1\}$. Without losing generality we can also assume $B \succ 0$. Then the volume of $E$ is proportional to $\det B$.

Consider again the polyhedron $P = \{\vec{x} \,|\, a_i^{\mathrm{T}}\vec{x} \leq b_i,\ i = 1, 2, \dots, m\}$. Now the problem is to find the maximum volume ellipsoid inscribed inside $P$. Formulate the problem by convex optimization.

The constraint can be dealt with in a similar manner as that in the Question 28, but we need to be careful about the objective function here. We tend to maximize the volume, i..e, maximize the determinant of $B$. However, it is easy to show that $\det(B)$ is nonconvex and nonconcave function. Hence, we need to maximize $\log(\det(B))$ instead, because it is a concave function on $\mathcal{S}_{++}^n$. Thus, we have the formulation as follows

$$\begin{aligned}
\max_{B,\,\vec{b}} \quad & \log(\det(B)) \\
s.t. \quad & \vec{a_i}^{\mathrm{T}}\vec{b} + \|B\vec{a_i}\|_2 \leq b_i,\ i = 1, 2, \dots, m \\
& B \succ 0
\end{aligned}$$

To prove the log-determinant function is concave on $\mathcal{S}_{++}^n$, it suffices to show $f(X)$ is concave in any direction. Define $g(t) = \log(\det(X + tV))$, where $X$ and $X + tV$ are both positive definite. Then, there exists $X = X^{1/2}X^{1/2}$, such that

$$\begin{aligned}
g(t) &= \log(\det(X^{1/2}X^{1/2} + tX^{1/2}X^{-1/2}VX^{-1/2}X^{1/2})) \\
&= \log(\det(X^{1/2}(I + tX^{-1/2}VX^{-1/2})X^{1/2})) \\
&= \log(\det(X)\det(I + tX^{-1/2}VX^{-1/2})) \\
&= \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2}))
\end{aligned}$$

Note that $X^{1/2}$ and $I + tX^{-1/2}VX^{-1/2}$ are also positive definite, and assume the eigenvalues of $X^{-1/2}VX^{-1/2}$ are $\lambda_1, \dots, \lambda_n$, then

$$g(t) = \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2})) = \log(\det(X)) + \sum_{i=1}^{n} \log(1 + t\lambda_i)$$

Thus, we have

$$g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}, \quad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \leq 0$$

Hence $g(t)$ is concave, meaning that $f(X)$ is concave in $V$-direction, but $V$ is arbitrary, so $f(X)$ is concave in general.

**Question 30.** Let $A_i \in \mathcal{S}^{n \times n}$, $i = 1, 2, \dots, m$. Therefore, $A_0 + x_1A_1 + \cdots + x_mA_m$ is a symmetric matrix. We wish to find the values of $x_1, \dots, x_m$ so as to minimize the gap between the largest and the smallest eigenvalues of $A_0 + x_1A_1 + \cdots + x_mA_m$. Formulate this problem by SDP.

This question is trivial, the formulation is

$$\begin{aligned}
\min_{\vec{x}, L, U} \quad & U - L \\
s.t. \quad & L \cdot I_n \preceq A_0 + x_1A_1 + \cdots + x_mA_m \preceq U \cdot I_n
\end{aligned}$$

where $\vec{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $L, U \in \mathbb{R}$, and $I_n$ is $n \times n$ identity matrix.

**Question 31.** Let

$$\mathcal{K} = \{\vec{x} \in \mathbb{R} \,|\, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

Show that $\mathcal{K}$ is a proper cone.

First, we show that $\mathcal{K}$ is closed. Take any convergent subsequence $\{\vec{v}_n\}_{n=1}^{\infty} \in \mathcal{K}$, for any $\vec{v}_n$, we have $\vec{v}_n^{(i)} \geq 0$ for $i = 1, \dots, n$. Suppose the limit of this sequence is $\vec{v}$, then we have

$$\vec{v}^{(i)} = \lim_{n\to\infty} \vec{v}_n^{(i)} \geq 0$$

which means $\vec{v}$ is also in $\mathcal{K}$. This means any limit point of $\mathcal{K}$ is in itself, hence it is closed.

Second, we need to show that $\mathcal{K}$ is solid. For unit ball $B$, we can see that $[2n\ 2n-2\ \cdots\ 2]^{\mathrm{T}} + B$ is a ball in $\mathcal{K}$. This is because $B = \{\vec{v} \mid \|\vec{v}\|_2 = 1\}$, so any point in $[2n\ 2n-2\ \cdots\ 2]^{\mathrm{T}} + B$ can be expressed as $[v_1 + 2n, v_2 + 2n - 2, \cdots, v_n + 2]^{\mathrm{T}}$. Consider any two consecutive entries, W.O.L.G., we take the first two entries, $v_1 + 2n - v_2 - 2n + 2 = v_1 - v_2 + 2$, since $v_1^2 + v_2^2 \leq 1$, $|v_1 - v_2| < \sqrt{2}$, so $v_1 - v_2 + 2 > 0$ and this point is in $\mathcal{K}$. Hence, $\mathcal{K}$ cantains a ball and thus is solid.

Finally, we prove $\mathcal{K}$ is pointed. If $\vec{x} \in \mathcal{K}$, and $-\vec{x} \in \mathcal{K}$, then we will have $x_i \geq x_{i+1}$ and $x_i \leq x_{i+1}$ for $i = 1, \dots, n-1$. Thus, $x_i = x_{i+1}$ for $i = 1, \dots, n-1$, but $x_n \geq 0$ and $x_n \leq 0$, so $\vec{x} = \vec{0}$.

It's easy to check this is a convex cone by definition. For any $\vec{x} \in \mathcal{K}$, $\alpha\vec{x}$ is also in $\mathcal{K}$ for any $\alpha \geq 0$. For $\lambda \in [0, 1]$, it is trivial that $\lambda\vec{x} + (1-\lambda)\vec{y}$ is also in $\mathcal{K}$, if $\vec{x}$ and $\vec{y}$ are both in $\mathcal{K}$. Hence, $\mathcal{K}$ is a proper cone.

**Question 32.** Find $A \in \mathbb{R}^{n\times n}$ such that $\mathcal{K} = A\mathbb{R}_+^n$.

Take $A$ as

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then for any $\vec{x} \in \mathbb{R}_+^n$, we have

$$A\vec{x} = [x_1 + \cdots + x_n, x_2 + \cdots + x_n, \cdots, x_1]^{\mathrm{T}}$$

which shows that $A\vec{x} \in \mathcal{K}$, because all $x_i$ are nonnegative.

Also, for any $\vec{x} \in \mathcal{K}$, $A^{-1}\vec{x}$ is in $\mathbb{R}_+^n$, because

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A^{-1}\vec{x} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ \vdots \\ x_{n-1} - x_n \end{bmatrix} \geq \vec{0}$$

Therefore, $\mathcal{K} = A\mathbb{R}_+^n$.

**Question 33.** In general, if $\mathcal{K} \subset \mathbb{R}^n$ is a proper cone, and $M \in \mathbb{R}^{n\times n}$ is a non-singular matrix, then $M\mathcal{K}$ is also a proper cone.

First, we prove that $M\mathcal{K}$ is a convex cone. Since by definition, $M\mathcal{K} = \{M\vec{x} \mid \vec{x} \in \mathcal{K}\}$, for any element $\vec{y} \in M\mathcal{K}$, we have $\vec{y} = M\vec{x}$. Consider any $\alpha \geq 0$, $\alpha\vec{y} = M(\alpha\vec{x})$, since $\vec{x}$ is in cone $\mathcal{K}$, so is $\alpha\vec{x}$, and thus $\alpha\vec{y} \in M\mathcal{K}$. The convexity of $M\mathcal{K}$ also follows from the convexity of $\mathcal{K}$, similar arguments can be applied.

Then, we prove that $M\mathcal{K}$ is closed. This is trivial, since $M$ is a linear transformation hence continuous. Continuous function maps a closed set to closed set, thus $M\mathcal{K}$ is closed because $\mathcal{K}$ is closed.

Next, we prove that $M\mathcal{K}$ is solid. Since there exists a unit ball in $\mathcal{K}$, take its interior, it is an open set, and will be mapped to an open set by $M$. Therefore, there is an open set in $M\mathcal{K}$, and there is a open ball contained in this open set, and of course in $M\mathcal{K}$.

Finally, we prove that $M\mathcal{K}$ is pointed. This is trivial, since $\vec{x} \in M\mathcal{K}$ means $M^{-1}\vec{x} \in \mathcal{K}$, and $-\vec{x} \in M\mathcal{K}$ means $-M^{-1}\vec{x} \in \mathcal{K}$. We know $\mathcal{K}$ is pointed, so $M^{-1}\vec{x} = \vec{0}$, which is equivalent to say $\vec{x} = \vec{0}$. Therefore, $M\mathcal{K}$ is pointed, and hence it is a proper cone.

**Question 34.** Compute $(M\mathcal{K})^*$.

By definition, we have

$$\begin{aligned}
(M\mathcal{K})^* &= \{\vec{y} \mid \vec{x}^{\mathrm{T}} M^{\mathrm{T}} \vec{y} \geq 0,\ \forall\, \vec{x} \in \mathcal{K}\} \\
&= \{\vec{y} \mid M^{\mathrm{T}}\vec{y} \in \mathcal{K}^*\} \\
&= \left(M^{\mathrm{T}}\right)^{-1} \mathcal{K}^*
\end{aligned}$$

**Question 35.** Derive the dual of the following *non-standard* conic optimization problem:

$$\begin{aligned}
\min_{\vec{x}} \quad & \vec{c}^{\,\mathrm{T}}\vec{x} \\
s.t. \quad & A_i\vec{x} + \vec{b}_i \in \mathcal{K}_i,\ i = 1, 2, \ldots, m
\end{aligned}$$

where $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_m$ are all closed convex cones.

Consider the Lagrangian function

$$L(\vec{x}, \vec{y}_i) = \vec{c}^{\,\mathrm{T}}\vec{x} + \sum_{i=1}^{m} \vec{y}_i^{\,\mathrm{T}}(A_i\vec{x} + \vec{b})$$

where $\vec{y}_i \in \mathcal{K}_i^*$. Then the dual function is

$$d(\vec{y}_i) = \min_{\vec{x}} L(\vec{x}, \vec{y}_i) = \begin{cases} \sum_{i=1}^{m} \vec{b}^{\,\mathrm{T}}\vec{y}_i & \text{when } \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}}\vec{y}_i = \vec{0} \\ -\infty & \text{when } \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}}\vec{y}_i \neq \vec{0} \end{cases}$$

Hence, the Lagrange dual problem is

$$\begin{aligned}
\max_{\vec{y}_i} \quad & \sum_{i=1}^{m} \vec{b}^{\mathrm{T}} \vec{y}_i \\
s.t. \quad & \vec{c} + \sum_{i=1}^{m} A_i^{\mathrm{T}} \vec{y}_i = \vec{0} \\
& \vec{y}_i \in \mathcal{K}_i^*, \ i = 1, 2, \ldots, m
\end{aligned}$$

**Question 36.** Suppose that $f(\vec{x})$ is a convex function, and its conjugate function is known to be $f^*(\vec{s})$. Consider the following optimization model

$$\begin{aligned}
\min_{\vec{x}} \quad & f(\vec{x}) \\
s.t. \quad & A\vec{x} \le \vec{b}
\end{aligned}$$

Derive the Lagrangian dual of the above problem.

Recall the conjugate function $f^*(\vec{s})$ is given by

$$f^*(\vec{s}) = \sup_{\vec{x}} (\vec{s}^{\mathrm{T}} \vec{x} - f(\vec{x}))$$

The Lagrangian function is given by

$$L(\vec{x}, \vec{y}) = f(\vec{x}) + \vec{y}^{\mathrm{T}} (A\vec{x} - \vec{b})$$

Hence, the dual function $d(\vec{y})$ is given by

$$d(\vec{y}) = \min_{\vec{x}} L(\vec{x}, \vec{y}) = -\max_{\vec{x}} \left((-A^{\mathrm{T}} \vec{y})^{\mathrm{T}} \boldsymbol{x} - f(\vec{x})\right) - \vec{b}^{\mathrm{T}} \vec{y} = -f^*(-A^{\mathrm{T}} \vec{y}) - \vec{b}^{\mathrm{T}} \vec{y}$$

where $\vec{y} \ge 0$. Therefore, the Lagrange dual problem is

$$\begin{aligned}
\max_{\vec{y}} \quad & -f^*(-A^{\mathrm{T}} \vec{y}) - \vec{b}^{\mathrm{T}} \vec{y} \\
s.t. \quad & \vec{y} \ge \vec{0}
\end{aligned}$$

**Question 37.** The channel capacity optimization problem is:

$$\begin{aligned}
\min_{\vec{x}, \vec{y}} \quad & -\vec{c}^{\mathrm{T}} \vec{x} + \sum_{i=1}^{m} y_i \ln y_i \\
s.t. \quad & P\vec{x} = \vec{y} \\
& \vec{x} \ge \vec{0}, \ \vec{e}^{\mathrm{T}} \vec{x} = 1
\end{aligned}$$

What is the dual of the above problem?

The Lagrangian function is

$$L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda}) = -\vec{c}^{\mathrm{T}} \vec{x} + \sum_{i=1}^{m} y_i \ln y_i + \vec{u}^{\mathrm{T}} (P\vec{x} - \vec{y}) + u_0(\vec{e}^{\mathrm{T}} \vec{x} - 1) + \vec{\lambda}^{\mathrm{T}} (-\vec{x})$$

where $\vec{\lambda} \geq \vec{0}$ and $\vec{u} = (u_1, \cdots, u_m)^{\mathrm{T}}$. The dual function

$$d(\vec{u}, u_0, \vec{\lambda}) = \min_{\vec{x}, \vec{y}} L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda})$$

We can rewrite the Lagrangian function into separated form (separate $\vec{x}$, $\vec{y}$), which is

$$L(\vec{x}, \vec{y}, \vec{u}, u_0, \vec{\lambda}) = (-\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda})^{\mathrm{T}}\vec{x} + \sum_{i=1}^{m} y_i \ln y_i - \vec{u}^{\mathrm{T}}\vec{y} - u_0$$

Since for $\vec{x}$ part, it is an linear function, the coefficient must be zero, otherwise it will be unbounded (because in Lagrangian function, $\vec{x}$ is free variable). Thus,

$$-\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda} = \vec{0}$$

For $\vec{y}$ part, it is a convex function, hence the minimum is attained at the point where the gradient is zero, i.e.,

$$\ln y_i + 1 - u_i = 0 \Longrightarrow y_i = e^{u_i - 1}, \quad \forall\, i = 1, 2, \ldots, m$$

Hence, we can obtain the dual function

$$d(\vec{u}, u_0, \vec{\lambda}) = -\sum_{i=1}^{m} e^{u_i - 1} - u_0$$

And the Lagrange dual problem is given by

$$\begin{aligned} \max_{\vec{u}, u_0, \vec{\lambda}} \quad & -\sum_{i=1}^{m} e^{u_i - 1} - u_0 \\ s.t. \quad & -\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\,\vec{e} - \vec{\lambda} = \vec{0} \\ & \vec{\lambda} \geq \vec{0} \end{aligned}$$

Eliminate $\vec{\lambda}$, we have

$$\begin{aligned} \max_{\vec{u}, u_0} \quad & -\sum_{i=1}^{m} e^{u_i - 1} - u_0 \\ s.t. \quad & -\vec{c} + P^{\mathrm{T}}\vec{u} + u_0\vec{e} \geq \vec{0} \end{aligned}$$

**Question 38.** The sum of first $k$ largest components of vector $\vec{x} \in \mathbb{R}^n$ ($k < n$) is known to be a convex function (Why?). Denote this function to be $f(\vec{x})$. Formulate the following portfolio selection problem using $f(\vec{x})$: We wish to select from a total of $n$ assets to form a portfolio (no short-selling is allowed). Asset $i$ has an expected rate of return $\mu_i > 0$, and the covariance matrix is $\Sigma$. We wish to minimize the variance of the portfolio while requiring that the expected rate of return to the portfolio is at least $\mu$. Moreover, the weight of the first $k$ largest components of investment should not exceed half of the total investment.

To see why $f(\vec{x})$ is convex, we can see that

$$f(\vec{x}) = \sum_{i=1}^{k} x_{n_i} = \max\{x_{n_1} + \cdots + x_{n_k} \,|\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n\}$$

$f$ is the maximum of $C_n^k$ linear functions, so it must be convex.

Now let us formulate the portfolio problem. Since $\vec{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ means the percentage of different portfolio, so the sum of all entries must be one. Not short-selling means $x_i \geq 0$ for all $i$. If we denote $\vec{u} = (\mu_1, \cdots, \mu_n)^{\mathrm{T}}$, then since the expected rate of return is at least $\mu$, we have $\vec{u}^{\mathrm{T}}\vec{x} \geq \mu$. The requirement on first $k$ largest components yields $f(\vec{x}) \leq 0.5$. Finally, we need to minimize the variance of portfolio, so the objective function is $\vec{x}^{\mathrm{T}}\Sigma\vec{x}$. Therefore,

$$\begin{aligned}
\min_{\vec{x}} \quad & \vec{x}^{\mathrm{T}}\Sigma\vec{x} \\
s.t. \quad & \vec{e}^{\mathrm{T}}\vec{x} = 1 \\
& \vec{u}^{\mathrm{T}}\vec{x} \geq \mu \\
& f(\vec{x}) \leq 0.5 \\
& \vec{x} \geq \vec{0}
\end{aligned}$$

**Question 39.** The condition that $f(x) \leq 0.5$ in Question 38 can be formulated by linear programming. How?

This is trivial if you use definition of $f(\vec{x})$,

$$f(\vec{x}) = \max\{x_{n_1} + \cdots + x_{n_k} \,|\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n\} \leq \frac{1}{2}$$

The above constraint is equivalent to

$$x_{n_1} + \cdots + x_{n_k} \leq \frac{1}{2}, \quad \forall\, 1 \leq n_1 < n_2 < \cdots < n_k \leq n$$

Notice that there are $C_n^k$ different choices of $\{n_1, \ldots, n_k\}$, so the original one non-linear constraint will be reformulated into $C_n^k$ linear constraints.