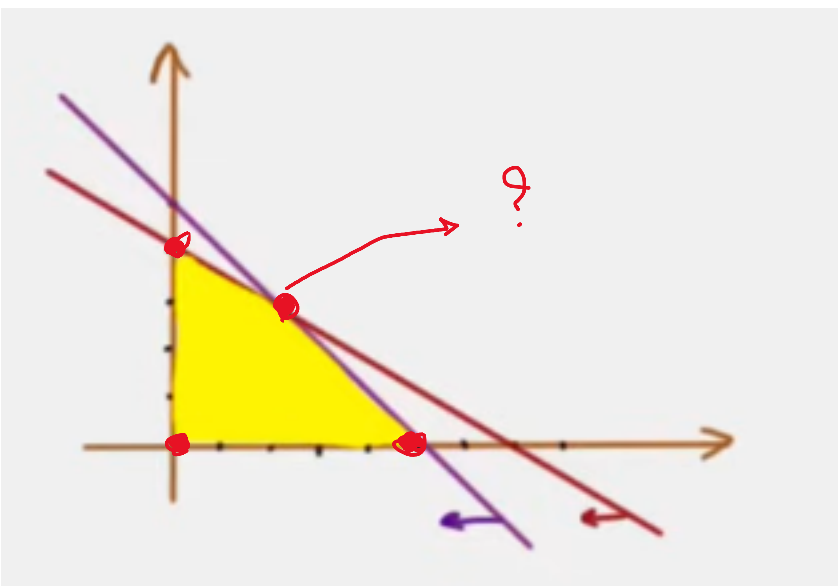
Branch and Bound Method to Solve Integer Linear Programming Problems

Given the following maximization, solve minding that x1 and x2 are integers.

$$\max \quad Z = 5x_1 + 6x_2$$
$$x_1 + x_2 \leq 5$$
$$4x_1 + 7x_2 \leq 28$$
$$x_1, x_2 \geq 0,\ x_1, x_2 \text{ are integer}$$

First let's "Relax" the constraints and find points that help us to draw the feasible region.

$$x_1 + x_2 \leq 5 \quad \begin{bmatrix} 0 \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$
$$4x_1 + 7x_2 \leq 28 \quad \begin{bmatrix} 0 \\ 4 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$



We know that, in this case the optimal answer is one of the vertices. Let's list them5

(0,0), (5, 0), (0,4), but how do we calculate the 3rd point?

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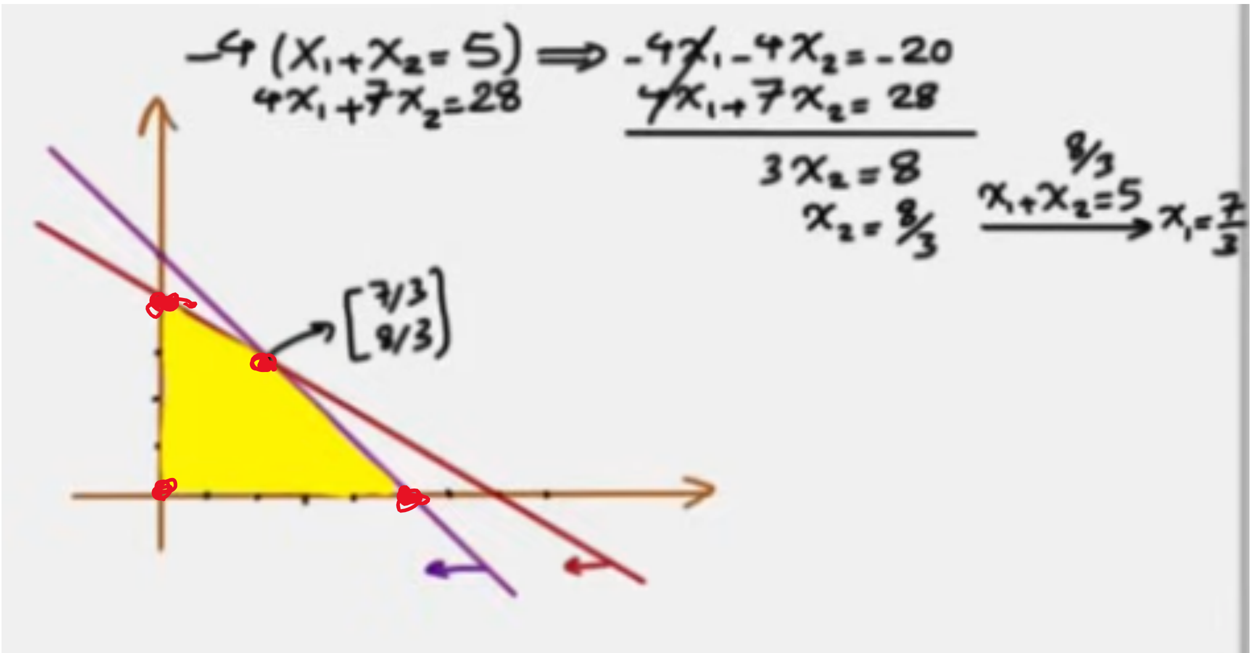
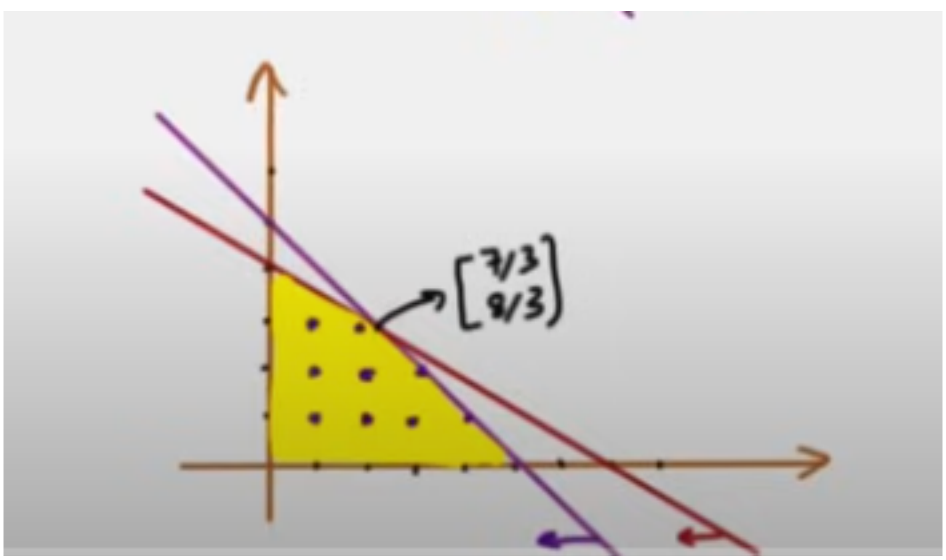


(0,0), (5, 0), (0,4), $(\frac{7}{3},\frac{8}{3})$, so the optimal solution is the point $(\frac{7}{3},\frac{8}{3})$, but values are not integers!

Let's bring back the main integer constraints, so our feasible area is not really the yellow part, but the dots in it.



So how do we solve this issue? check that rounding down or up may not be the solution too!

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We plug in the $(\frac{7}{3},\frac{8}{3})$ to the objective function and we start the branching from there as a baseline.

$$x_1 = \frac{7}{3} \approx 2.33$$

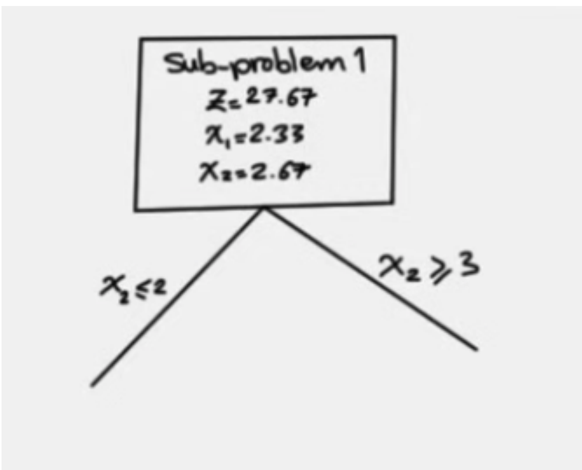
$$x_2 = \frac{8}{3} \approx 2.67$$

$$Z = 5x_1 + 6x_2$$

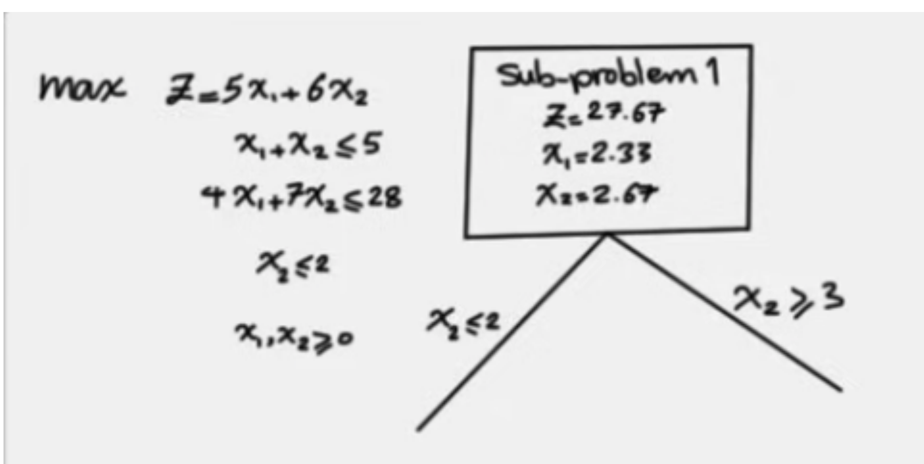
$$Z = 5\,(2.33) + 6\,(2.67) = 27.67$$

Now, we can use Branch and Bound method:

We need to find the variable with largest decimal value, here x_2 .



Now this gives us a new constraint and then let's solve the original optimization problem using this new constraints and find the feasible region:



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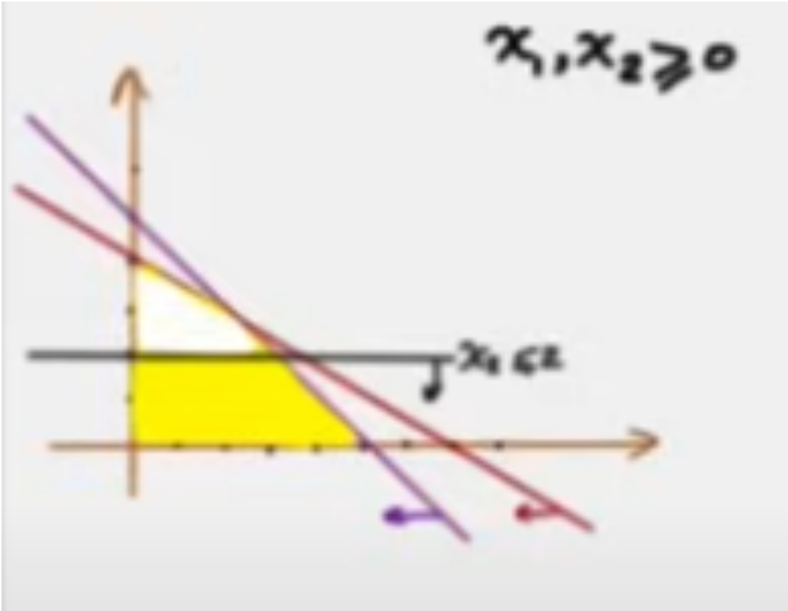


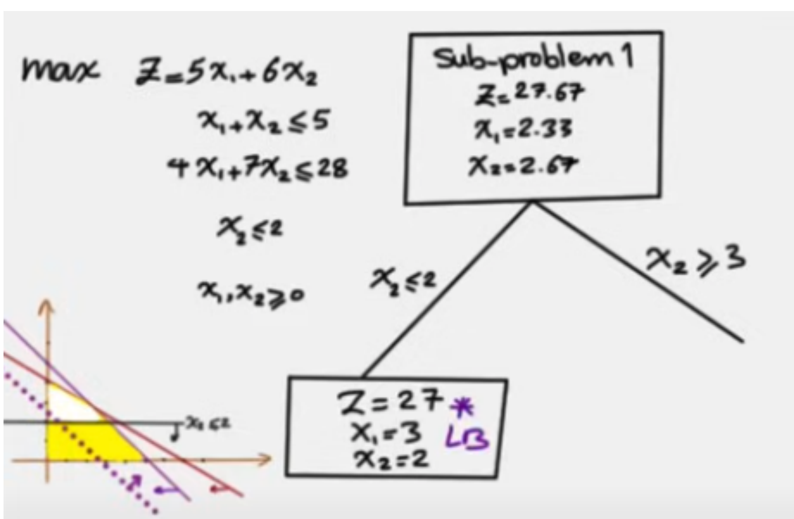
We can see that $x_1 = 3$ and $x_2 = 2$. We plug it in to the original function to find the value for Z.

$Z = 5x_1 + 6x_2$

$x_1 = 3$ and $x_2 = 2$.

Z = 27

We will add the details to the branch as follows:



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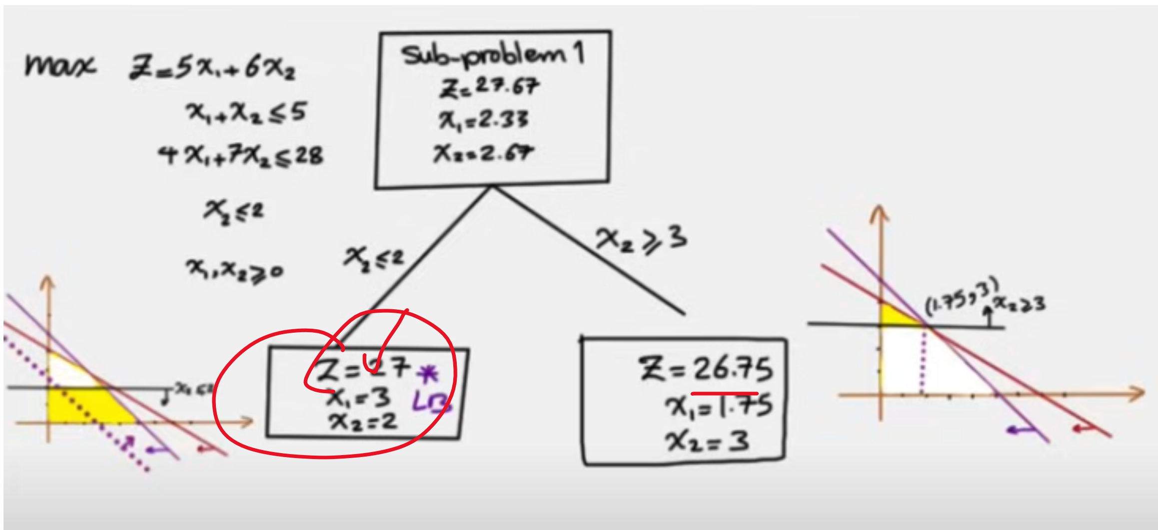
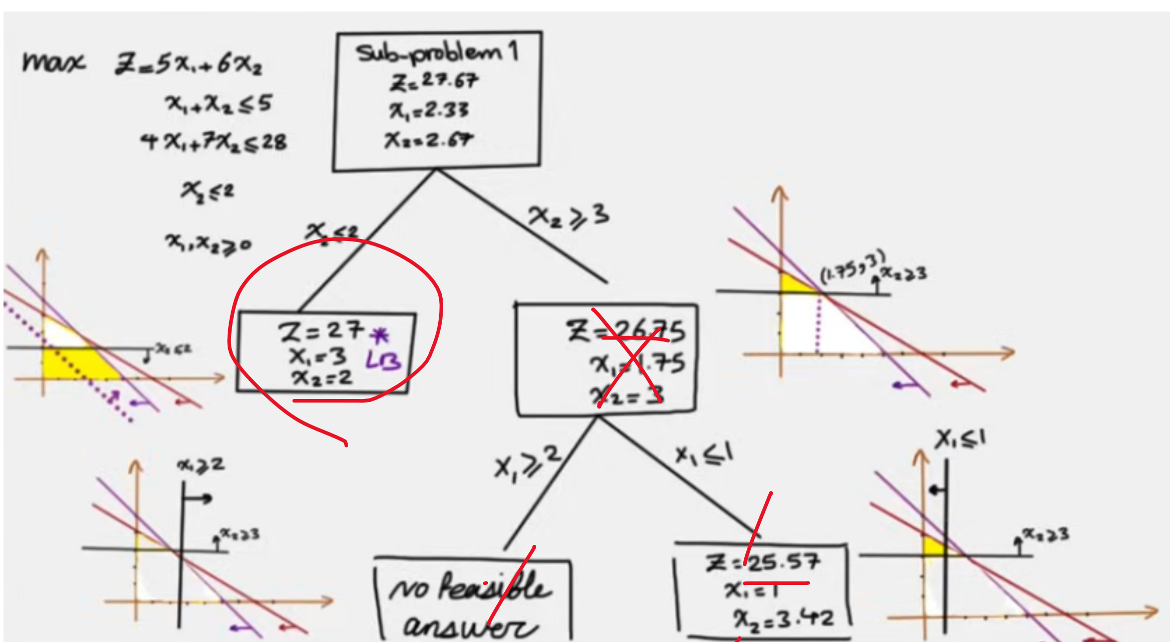
Note that we did not round the variables but rather find an integer solution to the problem and 27 is too close to our original base line.

For now, we call this our Lower Bound (LB) but let’s move to the right branch and do the same thing.   Maybe we will find a better one, let’s see!

Let’s move to the right side, doing the same process that we did for the left , we get the right hand sinde of our branch.



Unfortunately, $x_1 = 1.75$ which is not an integer, so we need to branch again!



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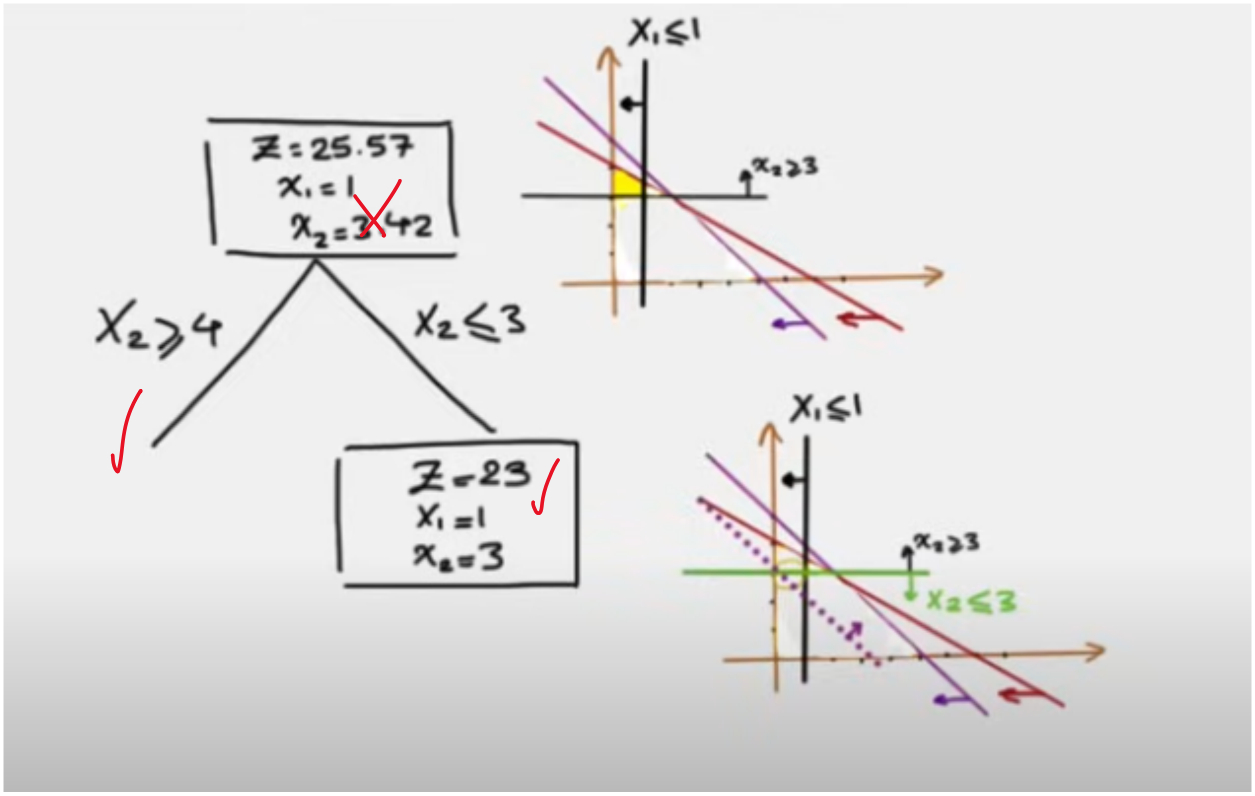
We can see that right side of the second branch has a feasible answer but $x_2 = 3.42$, this means we need to branch out again:



Remember that we need to include the whole path constraints so $x_2 \geq 3$ and $x_2 \leq 3$, together will change our feasible solution to a line. Z becomes 23, smaller than our lower bound solution which was 27, so we call this branch "fathomed" branch. Now let's see the left branch:

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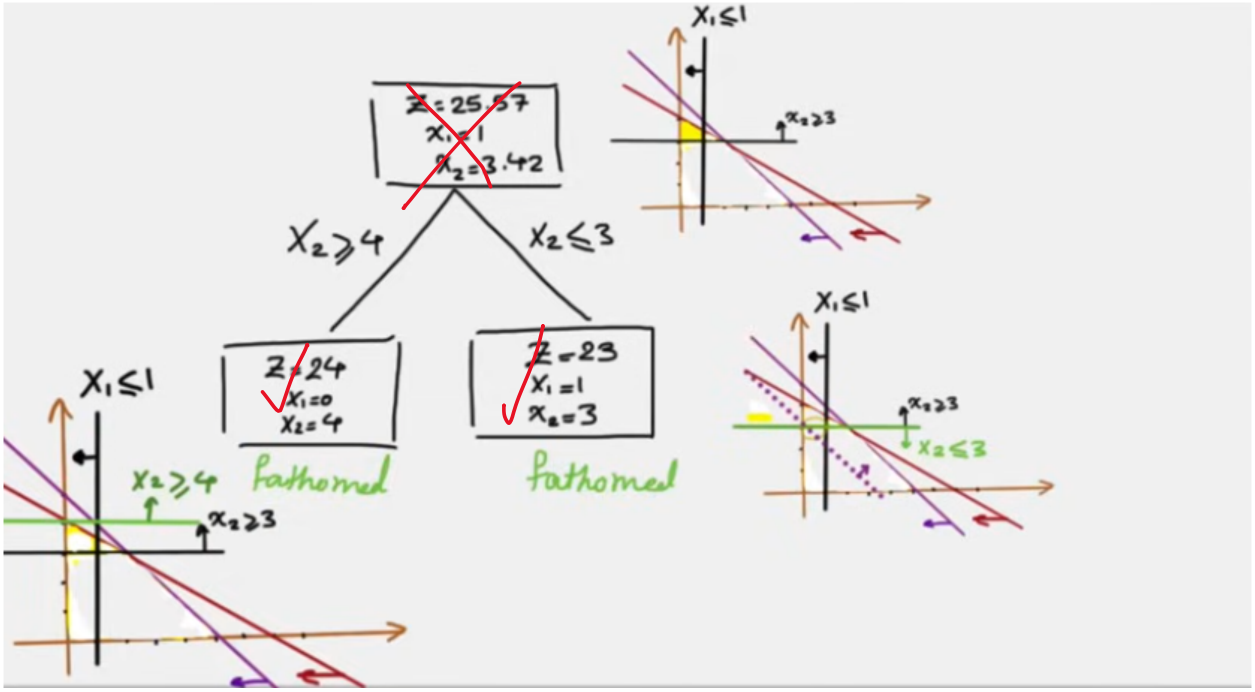
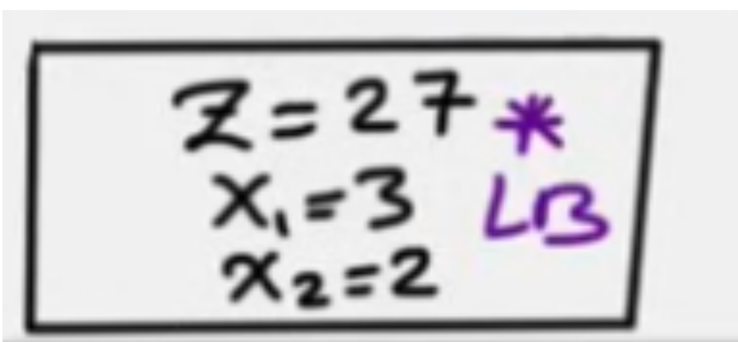


We got to the end of our branching and all variables are integer, however, we could not come up with a better solution that our original lower band.



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