Part II Astrophysical Fluid Dynamics

William Royce

wr286@cam.ac.uk github

February 23, 2025

Part II Physics, The University of Cambridge

Preface

Fluid dynamical forces drive most of the fundamental processes in the Universe and so play a crucial role in our understanding of astrophysics. Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is "frozen in" to a highly conducting plasma.

Take anything in the universe, throw a bunch of it in a box, and turn up the heat. Then it doesn't matter what you started with, the motion of this substance will be governed by the equations of fluid dynamics.

This is a remarkable statement. There are lots of different things in the universe and we go to great lengths to understand their properties. Yet if you heat them, most of the differences disappear. When things get hot, everything looks the same.

Here are some examples. Take any element in the periodic table and heat it until it melts, so that it is either a liquid or a gas. The motion of every element is governed by the same set of equations. The only reminder of what you started with is to be found in a handful of parameters of these equations which describe, among other things, the density and viscosity of the fluid. These will differ from element to element. But the basic set of equations are the same, regardless of whether you started with an alkaline earth metal or an inert gas.

This same story holds if we turn our attention to more exotic substances. For example, inside every proton and neutron sit three quarks. They have been trapped there since the Big Bang, held in place by the grip of the strong nuclear force. However, earlier this century, experimenters succeeded in colliding nuclei together with energies that were so high that the protons and neutrons themselves melted, freeing their imprisoned quarks and forming a novel state of matter known as a the quark-gluon plasma. This plasma only lasts for a fraction of a second before it cools and once again forms protons and neutrons. But during that fraction of a second it moves. And the movement is described by the laws of fluid mechanics.

Here is an even more extreme example. Take spacetime itself. It is possible for spacetime to collapse in on itself to form a black hole and, due to the work of Hawking, we know that these black holes are hot objects. So a black hole can be viewed as a way to heat spacetime. Surprisingly, if you look at the equations that govern the event horizon of a black hole, you will once again find the laws of fluid mechanics.

All of which is to say that there is a wonderful universality to the laws that govern fluids. In certain circumstances, these laws describe literally everything. And this makes them interesting.

The reasons underlying this universality are well understood. At the microscopic level, fluids are ridiculously complicated objects, consisting of, say, 10^{23} atoms, each following its own path, while acting through various forces on the atoms around it. But much of this motion is fleeting and we lose little if we ignore it. Instead, we care only about patterns in the collective motion of the atoms that survive over long time scales. It turns out that these long-lived modes are all related to familiar conservation laws – conservation of mass, momentum and energy – and these conservation laws are universal and obeyed by all substances. This, ultimately, is why all fluids look the same: the equations of fluid dynamics are essentially the equations that govern how conserved quantities evolve in time.

In addition to the universal aspect of fluid mechanics, the subject also has enormous practical applications. It explains, for example, why planes fly. Fluid mechanics explains how oil flows through pipes and how the motion of the atmosphere manifests itself in the climate, and how many decades of focussing on the former has resulted in an urgent and desperate need to better understand the latter.

In this course we explore the basics of fluid mechanics. Our focus will not be on quarks and black holes, but nor will it be any particular application of fluid mechanics. Instead our goal is simply to understand the different things that fluids can do. Fluids are everywhere and they have a tendency to move. The purpose of these lectures is simply to construct and explore the equation governing this motion.

As we've stressed above, the motion of all fluids is described by the same basic set of equations. Prominent among these is the Navier-Stokes equation, accompanied by one or two of further equations describing the conservation of mass and, in some cases, the flow of heat. One of the themes of fluid mechanics is that a wonderful diversity of different behaviour emerges from these equations. As these lectures progress, we will find ourselves falling into a routine. Like Monet and his haystacks, we will return to these same theme over and over again, not because we did anything wrong the first time but because there is always something new to see. Attacking the same set of equations, but with slight change to the boundary condition, or a novel approximation scheme, will often yield something new and surprising. One of the delights of the subject lies in finding such riches sitting inside such simple equations.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs.

Contents

List of Figures

Chapter 1

Basic Principles

1.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g., density ρ, pressure p), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples: (Fluids in the Universe)

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

Real gases are not perfectly continuous, since they are made of individual particles: atoms, molecules, ions, and/or electrons. In our discussion, we shall use the concept of a *fluid element*.

Definition 1.1 Fluid Element
This is a region of fluid (size ℓ_{region}) that is

1. Small enough that there are no significant variations of any property q

that interests us

$$l_{\text{region}} \ll l_{\text{scale}} \sim \frac{q}{|\nabla q|}. \tag{1.1}$$

2. Large enough to contain sufficient particles to be considered in the continuum limit

$$n l_{\text{region}}^3 \gg 1, \tag{1.2}$$

where n is the number density of particles.

If such fluid elements can be defined, continuum description is valid, otherwise, we must describe the system at the particle-level.

1.2 Collisional and Collisionless Fluids

Definition 1.2 Collisional Fluid
Any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.

$$l_{\text{region}} \gg \lambda, \tag{1.3}$$

where λ is the mean free path, the typical distance travelled by a particle before its direction of travel is significantly changed due to particle collisions.

The mean free path in a gas of neutral particles is

$$\lambda = \frac{1}{n\sigma} \tag{1.4}$$

where n is the number density of particles, and σ is the cross section for collisions.

Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p. We can relate ρ, T and p with an *equation of state*:

$$p = p(\rho, T). \tag{1.5}$$

Almost all fluids considered in this course are collisional.

A typical cross section for atoms or small molecules is $\sigma \sim 10^{-15}$ cm^2, or about 1 gigabarn. In air at room temperature, $n \sim 10^{19}$ cm^{-3}, and hence $\lambda \sim 10^{-4}$ cm $\sim 1\,\mu$m. Thus, a volume of air that is larger than several microns on a side can be treated as a continuous fluid. In the interstellar medium (ISM), particle number densities are much lower than in the Earth's atmosphere. In a molecular cloud, $n \sim 1000$ cm^{-3}, and hence $\lambda \sim 10^{12}$ cm ~ 0.07 AU. In the warm neutral medium, $n \sim 0.50.5$ cm^{-3}, and hence $n \sim 2 \times 10^{15}$ cm ~ 100 AU $\sim 6 \times 10^{-4}$ pc.

Definition 1.3 Collisionless Fluid
Particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$.

So, distribution of particle speeds locally does not correspond to the maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples: (Collisionless Fluids)

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless).

1.2.1 Example of ICM

The situation is more complicated in a plasma. Consider a gas of fully ionised hydrogen. The effective radius of interaction r_{e} for a free electron can be found by setting the magnitude of its potential energy at a distance r_{e} from an electron or proton equal to its thermal kinetic energy. We first suggest

$$e^2/r_{\text{e}} \sim m_{\text{e}}v_{\text{e}}^2. \tag{1.6}$$

Since $m_{\text{e}}v_{\text{e}}^2 \sim k_{\text{B}}T$, where T is the kinetic temperature of the free electrons, we can write

$$r_{\text{e}} \sim \frac{e^2}{m_{\text{e}}v_{\text{e}}^2} \sim \frac{e^2}{k_{\text{B}}T}. \tag{1.7}$$

The cross section is thus

$$\sigma \sim \pi r_{\text{e}}^2 \sim \frac{\pi e^4}{k_{\text{B}}^2 T^2}, \tag{1.8}$$

and the mean free path for an electron is[1]

$$\lambda \sim \frac{k_{\text{B}}^2 T^2}{\pi e^4 n}. \tag{1.9}$$

In the hot ionised interstellar medium, $T \sim 10^6$ K and $n \sim 3 \times 10^{-3}$ cm^{-3}. The mean free path is thus $\lambda \sim 4 \times 10^{19}$ cm ~ 10 pc.

Treating as fully ionised plasma of electrons and ions, the mean free path in the intracluster medium is set by Coulomb collisions and a full analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln \Lambda}, \tag{1.10}$$

[1]Note: a more accurate calculation would contain the Coulomb logarithm $\ln \Lambda$, but this is good enough for an order-of-magnitude estimate.

where n_e is the electron number density, and Λ is the ratio of largest to smallest impact parameter. For $T \gtrsim 4 \times 10^5\,\mathrm{K}$ we have $\ln \Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \simeq 23\,\mathrm{kpc}\left(\frac{T_e}{10^8\,\mathrm{K}}\right)^2 \left(\frac{n_e}{10^{-3}\,\mathrm{cm}^{-3}}\right)^{-1}. \tag{1.11}$$

So we have

$$\overbrace{R_{\mathrm{galaxy}} \sim \lambda_e}^{\text{collisionless}} \!\!\!\!\!\!\!\! \underbrace{ \ll R_{\mathrm{cluster}}}_{\text{collisional}} \sim 1\,\mathrm{Mpc}. \tag{1.12}$$

Although the intracluster medium is thousands of times higher in pressure than the Warm/Hot Intergalactic Medium (WHIM), it is kept in hydrostatic equilibrium by the gravitational pull of the dark matter in clusters.

Chapter 2

Formulation of the Fluid Equations

2.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow. The partial time derivative of a function Q, which we have written as $\partial Q/\partial t$, is the rate of change as viewed by an observer at a fixed coordinate position, who is watching the gas flow by. Such an observer is called an **Eulerian** observer. An alternative point of view is that of an observer who is moving along with the bulk flow of the gas. Such an observer is called a **Lagrangian** observer.

Definition 2.1 Eulerian Description
One considers the properties of the fluid measured in a frame of reference that is fixed in space.

So we consider quantities like

$$\rho(\mathbf{r}, t), \quad p(\mathbf{r}, t), \quad T(\mathbf{r}, t), \quad \mathbf{v}(\mathbf{r}, t). \tag{2.1}$$

Definition 2.2 Lagrangian Description
One considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest. The Lagranian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals). These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

These frameworks form the foundation of the two principal methods in computational fluid dynamics. The Eulerian approach employs grid-based codes, where space is divided into a fixed grid, and fluid flows through the grid. In contrast, the Lagrangian approach uses smoothed particle codes, treating fluid elements as smoothed particles that move through continuous space. An important application of Lagrangian methods includes grid codes designed for modeling stellar collapse and supernova explosions.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position $\mathbf{r}$ and time t. At time $t + \delta t$ the element will be at position $\mathbf{r} + \delta\mathbf{r}$. The change in quantity Q of the fluid element is

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \left[\frac{Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right] \tag{2.2}$$

but

$$Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = Q(\mathbf{r}, t) + \frac{\partial Q}{\partial t}\delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q + \mathcal{O}\Big(\delta t^2, |\delta\mathbf{r}|^2, \delta t |\delta\mathbf{r}|\Big), \tag{2.3}$$

so

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \left[\frac{\partial Q}{\partial t} + \frac{\delta\mathbf{r}}{\delta t} \cdot \boldsymbol{\nabla} Q + \mathcal{O}(\delta t, |\delta\mathbf{r}|) \right], \tag{2.4}$$

which gives us

Definition 2.3 Lagrangian Derivative
The Lagrangian (or material) derivative describes the time rate of change of some physical quantity (like heat or momentum) of a fluid element that is subjected to a space-and-time-dependent macroscopic velocity field.

$$\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian}\\ \text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian}\\ \text{time derivative}}} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} Q}_{\substack{\text{“convective”}\\ \text{time derivative}}} . \tag{2.5}$$

For example, in fluid dynamics, the velocity field is the flow velocity, and the quantity of interest might be the temperature of the fluid. In this case, the material derivative then describes the temperature change of a certain fluid parcel with time, as it flows along its pathline (trajectory).

2.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

Definition 2.4 Streamline
Families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element.

Parametrising the streamline by a coordinate s such that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s} \right), \tag{2.6}$$

and demanding $\mathrm{d}\mathbf{r}/\mathrm{d}s \parallel \mathbf{u}$, we get

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}. \tag{2.7}$$

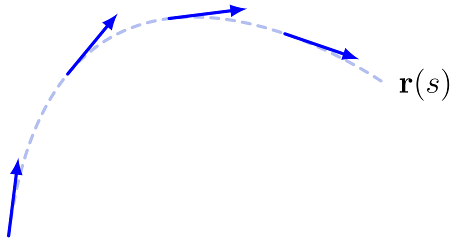

Fig. 2.1: Streamline

Definition 2.5 Particle Paths
Trajectories of individual fluid elements given by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t). \tag{2.8}$$

For small time intervals, particle paths follow streamlines since $\mathbf{u}$ can be treated as approximately steady.

Definition 2.6 Streaklines
Locus of points of all fluid that have passed through a given spatial point in the past.

$$\mathbf{r}(t) = \mathbf{r}_0 \tag{2.9}$$

for some given t in the past. Imagine the point as a source of “dye” or “smoke”.

Streamlines, particle paths, and streaklines all coincide if the flow is steady (i.e. $\partial\mathbf{u}/\partial t = 0$).

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

2.3 Conservation of Mass

Consider a fixed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. If there are no sources or sinks of mass within the volume, we can say

$$\text{rate of change of mass in } \mathcal{V} = -\text{rate that mass is flowing out across } \mathcal{S} \tag{2.10}$$

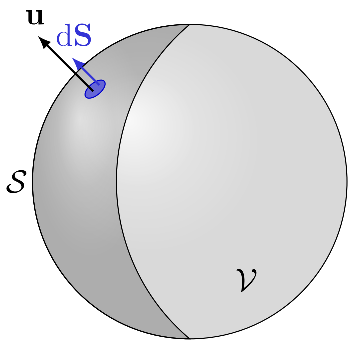

Fig. 2.2: Mass flow of a fluid element

this gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \,\mathrm{d}V = -\int_{\mathcal{S}} \rho\,\mathbf{u}\cdot \mathrm{d}\mathbf{S} \\
\implies &\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\,\mathrm{d}V \\
\implies &\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right)\mathrm{d}V = 0. \qquad (2.11)
\end{aligned}$$

This is true for all volumes $\mathcal{V}$. So we must have the *Eulerian continuity equation*,

Equation 2.1 Eulerian Continuity Equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad (2.12)$$

It states that mass is conserved. Moreover, the flow of matter is continuous; mass does not disappear at one point and simultaneously appear at another point some distance away.

The Lagrangian expression of mass conservation is easily found:

$$\begin{aligned}
\frac{\mathrm{D}\rho}{\mathrm{D}t} &= \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho \\
&= -\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) + \mathbf{u}\cdot\boldsymbol{\nabla}\rho \\
&= -\rho\boldsymbol{\nabla}\cdot\mathbf{u}, \qquad (2.13)
\end{aligned}$$

where we have used the vector identity $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = \rho\boldsymbol{\nabla}\cdot\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho$ for the final equality. Thus we have the *Lagrangian continuity equation*,

Equation 2.2 Lagrangian Continuity Equation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad (2.14)$$

The equations of compressible hydrodynamics are very complicated. They are difficult to integrate numerically – in 3D computing time scales like (grid size)^4. But there are useful *approximations* that give insight.

Definition 2.7 Incompressible Flow
Fluid elements maintain a constant density, i.e.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{2.15}$$

We can now see that incompressible flows must be divergence free, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. This a good model for liquids, but also a surprisingly good approximation for gases provided the flow is subsonic.

Definition 2.8 Irrotational Flow
In an Irrotational flow, there is no vorticity, i.e. $\boldsymbol{\nabla} \times \mathbf{u} = 0$.

The vorticity field $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{u}$ moves with the fluid, and vorticity is usually generated at boundaries, so often the bulk of a flow is irrotational. If $\boldsymbol{\nabla} \times \mathbf{u} = 0$ the velocity field can be generated from a scalar potential $\mathbf{u} = \boldsymbol{\nabla}\Phi$.[1] If the fluid is both incompressible and irrotational, the velocity potential satisfies Laplace's equation $\nabla^2\Phi = 0$. In this important case we can use potential theory to find $\mathbf{u}$ and Bernoulli's equation to find the pressure.

2.3.1 Bernoulli Equation for Compressible Fluids

Theorem 2.1 Bernoulli's Principle
The Bernoulli's principle states that the quantity

$$\frac{1}{2}u^2 + \Psi + \mathcal{E} + \frac{p}{\rho} \tag{2.16}$$

must be conserved along a streamline if some conditions are matched, namely: steady and irrotational flow of an inviscid fluid, subject to conservative forces.

There is an important theorem which is often useful for cases involving steady flow, expressing the conservation of energy as it transported along a streamline: Bernoulli's equation.

The derivation for compressible fluids depends upon conservation of mass, and conservation of energy, ignoring viscosity, and thermal effects. For *steady flow*, consider *streamlines* connecting areas A_1 and A_2.

Conservation of mass implies that in Fig. 2.3, in the interval of time Δt, the amount of mass passing through the boundary defined by the area A_1 is equal to

[1] Note: in fluid dynamics, this conventionally has a plus sign.

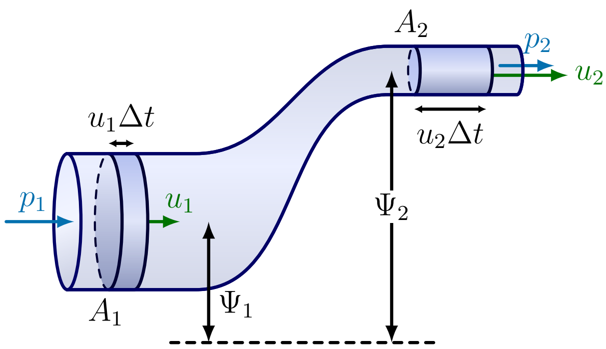

Fig. 2.3: A streamtube of fluid moving to the right. Indicated are pressure $p_{1,2}$, potential $\Psi_{1,2}$, flow speed $u_{1,2}$, distance $u_{1,2}\Delta t$, and cross-sectional area $A_{1,2}$.

the amount of mass passing outwards through the boundary defined by the area A_2:

$$0 = \Delta M_1 - \Delta M_2 = \rho_1 A_1 u_1 \Delta t - \rho_2 A_2 u_2 \Delta t. \tag{2.17}$$

Conservation of energy is applied in a similar manner: It is assumed that the change in energy of the volume of the streamtube bounded by A_1 and A_2 is due entirely to energy entering or leaving through one or the other of these two boundaries. Clearly, in a more complicated situation such as a fluid flow coupled with radiation, such conditions are not met. Nevertheless, assuming this to be the case and assuming the flow is steady so that the net change in the energy is zero,

$$\Delta E_1 - \Delta E_2 = 0, \tag{2.18}$$

where ΔE_1 and ΔE_2 are the energy entering through A_1 and leaving through A_2, respectively. The energy entering through A_1 is the sum of the kinetic energy entering, the energy entering in the form of potential gravitational energy of the fluid, the fluid thermodynamic internal energy per unit of mass ($\mathcal{E}_1$) entering, and the energy entering in the form of mechanical $p\,\mathrm{d}V$ work:

$$\Delta E_1 = \left(\frac{1}{2}\rho u_1^2 + \Psi_1\rho_1 + \mathcal{E}_1\rho_1 + p_1\right) A_1 u_1 \Delta t, \tag{2.19}$$

where Ψ is the potential due to gravity. A similar expression for ΔE_2 may easily be constructed. So now setting $0 = \Delta E_1 - \Delta E_2$:

$$0 = \left(\frac{1}{2}u_1^2 + \Psi_1 + \mathcal{E}_1 + \frac{p_1}{\rho_1}\right)\rho_1 A_1 u_1 \Delta t + \left(\frac{1}{2}u_2^2 + \Psi_2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}\right)\rho_2 A_2 u_2 \Delta t. \tag{2.20}$$

Now, using the previously-obtained result from conservation of mass, this may be simplified to obtain

$$\boxed{\frac{1}{2}u^2 + \Psi + \mathcal{E} + \frac{p}{\rho} = \text{const.},} \tag{2.21}$$

which is the **Bernoulli equation** for compressible flow.

For an ideal gas, the internal energy is (6.80) $\mathcal{E} = \frac{1}{\gamma-1}\frac{p}{\rho}$,

$$\frac{1}{2}u^2 + \Psi + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \text{const.} \tag{2.22}$$

So for **incompressible flow**, when ρ is constant (2.15) (i.e. $\gamma = \infty$, $\mathcal{E} = 0$), then

$$\frac{1}{2}u^2 + \Psi + \frac{p}{\rho} = \text{const.} \tag{2.23}$$

along a streamline.

Note: if a streamline is/is not curved, then a pressure gradient perpendicular to the streamline is/is not needed to provide a centripetal force. (Vertically there may be a pressure gradient due to gravity, but that is not related to the curvature of the flow).

Example 2.1 Streamline Equations
Determine the equation of a general streamline of the flow $u_\phi = a$, $u_r = b$, $u_Z = 0$ in cylindrical polar coordinates, and sketch the flow. Include more than one streamline in your sketches. Repeat for the flow $u_\phi = ar^2$, $u_r = br^2$, $u_Z = 0$. If the flows are steady, and the density at a given radius is independent of ϕ, find the radial dependence of the density in both cases.

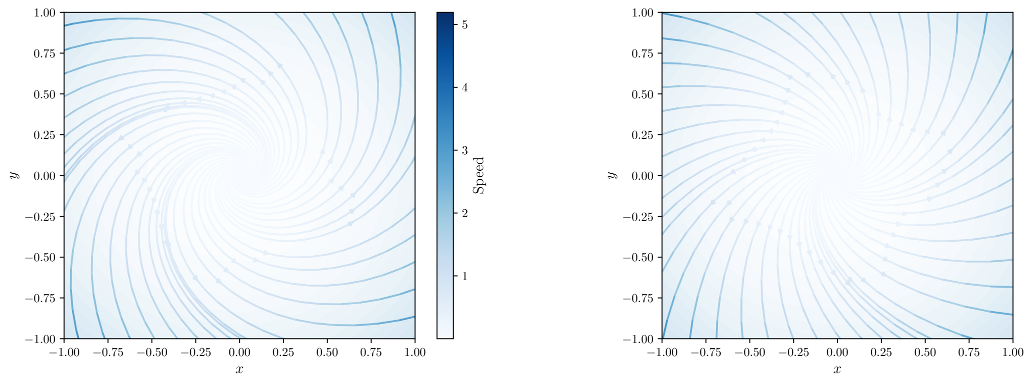

Fig. 2.4: Streamlines of the flow $u_\phi = a$, $u_r = b$ in the plane. Left: $a/b = 1.5$. Right: $b/a = 1.5$.

Solution In cylindrical coordinates, the position vector $\mathbf{r} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$ can be differentiated to give the relationship between the coordinate components and the velocities,

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_r + \dot{z}\hat{\mathbf{e}}_z \\ &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\phi}\hat{\mathbf{e}}_\phi + \dot{z}\hat{\mathbf{e}}_z \quad \text{since} \quad \mathrm{d}\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\phi\,\mathrm{d}\phi\,, \end{aligned} \tag{2.24}$$

letting us make the identifications $u_r = \dot{r}$ and $u_\phi = r\dot{\phi}$. From the velocity components given in the question, we can write

$$\dot{R} = b \quad \text{and} \quad R\dot{\phi} = a, \tag{2.25}$$

thus by application of the chain rule

$$\frac{\mathrm{d}R}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = b \quad \Longrightarrow \quad a\frac{\mathrm{d}R}{R} = b\,\mathrm{d}\phi\,. \tag{2.26}$$

With the differential $\mathrm{d}(\ln R) = \frac{\mathrm{d}R}{R}$, the equation above simplifies to $a\,\mathrm{d}(\ln R) = b\,\mathrm{d}\phi$, from which integrating gives

$$R = R_0 e^{b\phi/a}, \tag{2.27}$$

for a constant R_0.

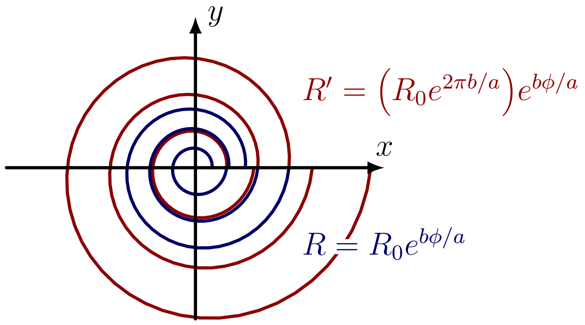

Fig. 2.5: Streamlines for $R = R_0 e^{b\phi/a}$. Different values of the constant R_0 allow different streamlines in the medium to be described, and additionally allows the streamlines to be continuous despite the domain restriction on ϕ.

Steady flow is characterised by $\dot{\mathbf{u}} = \mathbf{0}$, let us consider the time derivative of the Eulerian continuity equation (2.12) to see what condition this implies on $\dot{\rho}$,

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right) &= 0 \\ \Longrightarrow \quad \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\mathbf{u}\right)\frac{\partial \rho}{\partial t} &= -\boldsymbol{\nabla}\cdot\left(\rho\frac{\partial \mathbf{u}}{\partial t}\right). \end{aligned} \tag{2.28}$$

The right hand side of the last equality vanishes for steady flow, and since in general the bracketed operator on the left hand side does *not* vanish for an arbitrary flow, then steady flow also implies $\partial\rho/\partial t = 0$, as expected.

Thus, the continuity equation reduces to $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$, or equivalently in cylindrical coordinates using the appropriate expression for $\boldsymbol{\nabla}\cdot\mathbf{u}$[2]

$$\underbrace{\left(u_r\frac{\partial}{\partial r} + u_\phi\frac{\partial}{\partial \phi}\right)}_{\mathbf{u}\cdot\boldsymbol{\nabla}}\rho + \rho\underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)}_{\boldsymbol{\nabla}\cdot\mathbf{u}} = 0. \tag{2.30}$$

and following substitution of the given components gives the equation

$$\mathrm{d}(\ln\rho) = -\,\mathrm{d}(\ln r)\,, \tag{2.31}$$

and thus the radial dependence of the density is

$$\rho \propto r^{-1}. \tag{2.32}$$

[2]In cylindrical coordinates, the divergence of a vector field $\mathbf{A} = (A_r, A_\phi, A_z)$ is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \tag{2.29}$$

Following the exact same procedure for the second set of conditions, we find the equation of the streamlines is the same as for the first flow (2.27), but now Eq. (2.31) instead becomes

$$\mathrm{d}(\ln \rho) = -3\,\mathrm{d}(\ln r)\,, \tag{2.33}$$

and this time we get for the density dependence

$$\rho \propto r^{-3}. \tag{2.34}$$

Although the particle paths are the same, we note that the velocity increases with the radial distance and the fluid spirals around faster ◀

Example 2.2 Density in Steady Flow
Show that for a steady flow with $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, the density ρ is constant along the streamlines. Need ρ be constant throughout the medium?

Solution For *steady* flow, streamlines coincide with the particle paths, and so we consider the Lagrangian description. From the Lagrangian continuity equation (2.14), we see that if $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, then $\mathrm{D}\rho/\mathrm{D}t = 0$, i.e. density is constant *along* a streamline. This does not imply density is constant across different streamlines throughout the fluid.

If we instead consider the Eulerian description, the continuity equation (2.12), it reduces to $\partial\rho/\partial t + \cancel{\rho\boldsymbol{\nabla}\cdot\mathbf{u}} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = 0$. If we take $\mathbf{u}\perp\boldsymbol{\nabla}\rho$, then $\mathbf{u}$ is constant (i.e. a streamline) and so ρ is constant. However, for an arbitrary $\mathbf{u}$ the density gradient causes a force to change the velocity, so we do not remain along a streamline, and the so in the bulk fluid the density does not remain constant. ◀

Example 2.3 Cylindrical Flow
If $\mathbf{r} = (x, y, 0)$ and $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$ and a flow velocity is given for $r \geq a$ by $\mathbf{u} = U\left(1 + \frac{a^2}{r^2}\right)\hat{\mathbf{e}}_x - 2Ua^2xr^{-3}\hat{\mathbf{e}}_r$, (where $\hat{\mathbf{e}}_x = \mathbf{x}/|\mathbf{x}|$) show that the streamlines obey $U\left(r - \frac{a^2}{r}\right)\sin\phi = \text{const.}$, where $\phi = \tan^{-1} y/x$. Sketch the streamlines and explain what the flow represents physically. (Hint: consider which coordinate system is better suited for this problem.)

Solution To rewrite the equation of flow in terms of cylindrical polars, we need $\hat{\mathbf{e}}_x$ in terms of the new basis vectors,

$$\begin{aligned}\hat{\mathbf{e}}_x &= \boldsymbol{\nabla}x = \boldsymbol{\nabla}(r\cos\phi)\\ &= \frac{\partial(r\cos\phi)}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(r\cos\phi)}{\partial\phi}\hat{\mathbf{e}}_\phi\\ &= \cos\phi\,\hat{\mathbf{e}}_r - \sin\phi\,\hat{\mathbf{e}}_\phi.\end{aligned} \tag{2.35}$$

Thus, we can rewrite the flow as

$$\mathbf{u} = U\left(1 - \frac{a^2}{r^2}\right)\cos\phi\,\hat{\mathbf{e}}_r - U\left(1 + \frac{a^2}{r^2}\right)\sin\phi\,\hat{\mathbf{e}}_\phi. \tag{2.36}$$

From the components $u_r = \dot{r}$ and $u_\phi = r\dot{\phi}$, we can use the chain rule to work towards finding the equation of the streamlines,

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\mathrm{d}r}{\mathrm{d}t} \Bigg/ \underbrace{\frac{\mathrm{d}\phi}{\mathrm{d}t}}_{u_\phi/r} = \frac{ru_r}{u_\phi}$$

$$\implies \quad \frac{\mathrm{d}r}{\mathrm{d}\phi} = r\frac{U\left(1-\frac{a^2}{r^2}\right)\cos\phi}{-U\left(1+\frac{a^2}{r^2}\right)\sin\phi}$$

$$\implies \frac{\mathrm{d}r}{r}\frac{1+a^2/r^2}{1-a^2/r^2} = -\,\mathrm{d}(\ln\sin\phi)\,. \tag{2.37}$$

Integrating this we quickly see that $\ln(r - a^2/r) = -\ln(\sin\phi)+\text{const.}$, so the streamlines obey

$$\left(r - \frac{a^2}{r}\right)\sin\phi = \text{const.}, \tag{2.38}$$

representing the flow around a cylinder, as illustrated in Fig. 2.6.

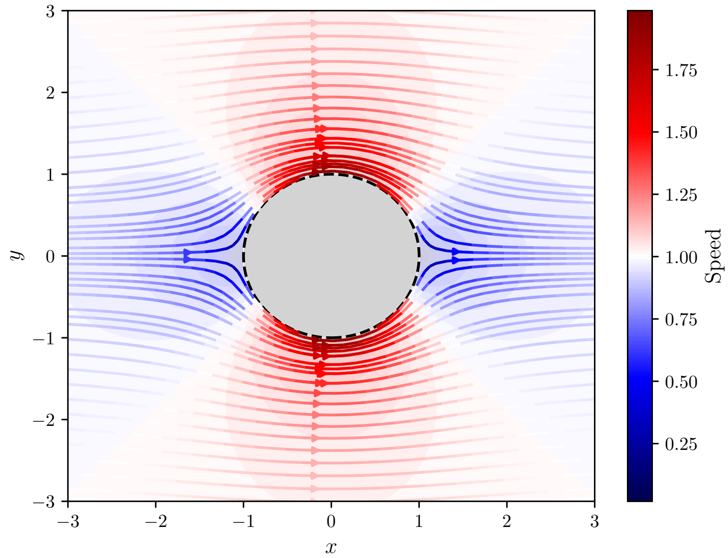

Fig. 2.6: Streamline plots of Eq. (2.36), for $U = R = 1$ for a cross-section of the flow at constant z. Colourmap chosen to highlight the areas of relative low and high speeds.

If we take the limit of large r in Eq. (2.38), we see it reduces to $r\sin\phi = \text{const.}$, which is the equation for lines of constant y, so far away from whatever disturbance may be close to the origin, there is a uniform flow in the $\hat{\mathbf{x}}$ direction.

Now, considering the limit of $r \gtrsim a$, we can see that at $r = a$, Eq. (2.38) reduces to $(r - r)\sin\phi = 0$, which is satisfied for all ϕ. Since a streamline is defined by a constant value of (2.38), the fact that (2.38) vanishes along $r = a$ means that the cylinder $r = a$ itself is a streamline. This is consistent with the no-penetration condition: the fluid does not cross the cylinder's surface.

Consider writing Eq. (2.38) as $\Psi_u = U(r - \frac{a^2}{r})\sin\phi$, such that Fig. 2.6 is now a plot of constant Ψ_u (i.e. the streamlines). To understand the behaviour of the streamlines near $r = a$, we may take the radial component of $\nabla\Psi_u$. The radial velocity (perpendicular to the surface) in cylindrical coordinates is $u_r = -\frac{\partial\Psi_u}{\partial r}$, which vanishes at $r = a$, meaning that there is no flow in the radial (normal) direction at the surface, ensuring that the fluid cannot penetrate the cylinder. Consequently, the velocity on the surface is entirely tangential, so the fluid moves only along the cylinder.

Note that this equation represents the flow around an infinitely long cylinder – not merely a disc – since there is no dependence on the z-coordinate, ensuring symmetry along the cylinder's axis. ◀

Example 2.4 2D Flow and Streaklines
A steady 2D flow is described by $u_x = 2/x$, $u_y = 1$. Find and sketch the streamlines. Find also a general expression for the surface density of the flow $\Sigma(x, y)$ assuming it can be written as a separable function of x and y.
Radioactive nuclei are introduced in a small patch at (x_0, y_0) so as to maintain a fixed concentration there. These nuclei decay such that their number per unit mass is given by $Q = Q_0 e^{-t}$ where t is the time since introduction into the flow. Show that the surface density of radioactive nuclei (i.e. number per unit area) attains a maximum along the radioactive streakline if x_0 is less than a critical value, and determine the coordinates of this maximum.

Solution From $\dot{x} = 2/x$ and $\dot{y} = 1$, we can write $\frac{1}{2}x\,\mathrm{d}x = \mathrm{d}y$, and integrating gives

$$y = \frac{1}{4}\left(x^2 - x_0^2\right) + y_0, \tag{2.39}$$

where x_0 and y_0 are just our chosen integration constants.

Assuming a solution separable in x and y gives us $\Sigma(x, y) = X(x)Y(y)$, and we can substitute this into the continuity equation (2.12) with the steady flow condition to give

$$\nabla \cdot (\Sigma\mathbf{u}) = 0 \quad \Longrightarrow \quad \left(\frac{2}{x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)X(x)Y(y) + X(x)Y(y)\frac{\partial}{\partial x}\left(\frac{2}{x}\right) = 0, \tag{2.40}$$

which we can evaluate and rearrange as

$$\frac{2}{x}\frac{X'}{X} - \frac{2}{x^2} = -\frac{Y'}{Y} = \text{const.}, \tag{2.41}$$

where the prime denotes differentiation with respect to the single argument of each function. We set this equal to a constant, since if a function only of x is equal to a function only of y for all allowed values of (x, y), then its partial derivative with respect to either must vanish, i.e. it is constant.

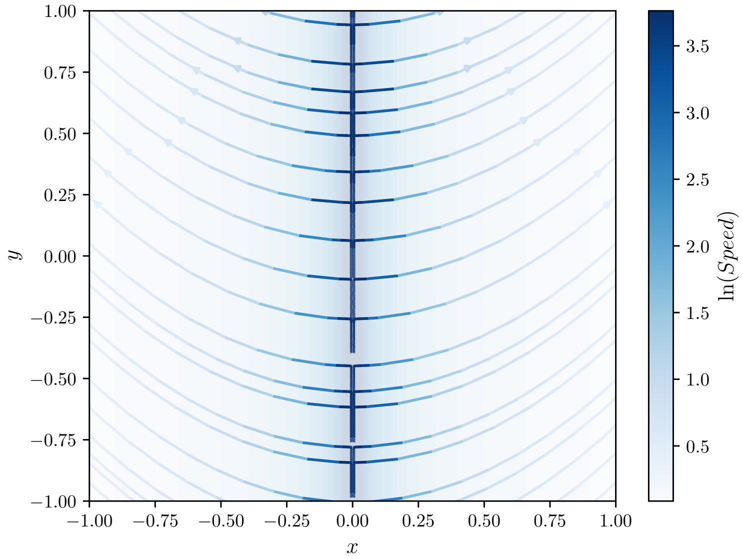

Fig. 2.7: Streamlines for $u_x = 1/x$, $u_y = 1$, with a logarithmic scale due to the diverging nature of the flow at $x = 0$.

Solving Eq. (2.41) for $Y(y)$,

$$\frac{Y'}{Y} = -c \quad \Longrightarrow \quad \frac{\mathrm{d}(\ln Y)}{\mathrm{d}y} = -c$$
$$\Longrightarrow \quad Y \propto e^{-cy}, \tag{2.42}$$

where c is a constant. We find for $X(x)$ that

$$\frac{\mathrm{d}(\ln X)}{\mathrm{d}x} = \frac{1}{2}cx + \frac{1}{x} \quad \Longrightarrow \quad X \propto |x|e^{cx^2/4}, \tag{2.43}$$

where the absolute value of x arises bearing in mind both signs are allowed, and $\int^x 1/x \, \mathrm{d}x = \ln|x|$.

Substituting the relationship for y along the streamline (2.39),

$$\Sigma(x, y) \propto |x|e^{\frac{1}{4}cx^2}e^{-cy} \quad \Longrightarrow \quad \Sigma = \Sigma_0 \frac{|x|}{x_0} \quad \text{given} \quad (2.39) \tag{2.44}$$

where the x^2 terms have cancelled in the exponential and the constant ones absorbed into Σ_0 and x_0.

We can write the surface density n of radioactive nuclei as

$$n = Q\Sigma = Q_0\Sigma_0 \frac{|x|}{x_0} e^{-t}. \tag{2.45}$$

We need to find the time t since introduction of the radioactive particles along the streakline,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{x} \quad \Longrightarrow \quad \frac{1}{4}(x^2 - x_0^2) = t, \tag{2.46}$$

since at $t = 0$, the particles are at $x = x_0$. Substituting into (2.45) gives

$$n = n_0 |x| e^{-\frac{1}{4}x^2}. \tag{2.47}$$

This function is symmetric under the discrete transformation $x \to -x$, so let us find the maxima as

$$\frac{\mathrm{d}n}{\mathrm{d}|x|} = n_0 e^{-\frac{1}{4}|x|^2} - \frac{1}{2} n_0 |x|^2 e^{-\frac{1}{4}|x|^2} \quad \Longrightarrow \quad |x| = \sqrt{2}, \tag{2.48}$$

thus, if $|x_0| < \sqrt{2}$, then the surface concentration will reach this maximum before decaying again.

Physically, this scenario involves a balance between two competing effects. On one hand, as illustrated in Fig. 2.7, the streamlines converge and become more densely packed as time increases and the flow moves away from the y-axis, leading to a greater concentration of particles. On the other hand, nuclear decay continuously reduces the particle count over time. Therefore, we are interested in determining the position where the particles are most concentrated – before decay significantly depletes their numbers. ◀

2.4 Conservation of Momentum

2.4.1 Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of a surface element is

$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}\,. \tag{2.49}$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$\mathrm{d}F_i = \sum_j \sigma_{ij}\,\mathrm{d}S_j\,. \tag{2.50}$$

where σ_{ij} is the stress tensor – the force in direction i acting on a surface with normal along j.

Isotropic[3] pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}. \tag{2.52}$$

2.4.2 Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field $\mathbf{g}$ and internal pressure forces. Let the fluid element have volume $\mathcal{V}$ and surface $\mathcal{S}$.

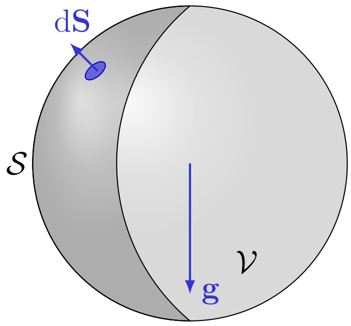

Fig. 2.8: A fluid element subject to gravity

Pressure acting on the surface element gives a force $-p\,\mathrm{d}\mathbf{S}$. The pressure force on an element projected in direction $\hat{\mathbf{n}}$ is $-p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$. So, the net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_{\mathcal{S}} p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V\,. \tag{2.53}$$

The rate of change of momentum of a fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = \underbrace{-\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V}_{(2.53)} + \int_{\mathcal{V}} \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V\,. \tag{2.54}$$

[3] An invariant tensor or invariant pseudo-tensor is one which has the same components in all frames,

$$T'_{ijk\ldots} = T_{ijk\ldots}. \tag{2.51}$$

Invariant tensors and invariant pseudo-tensors are both called isotropic tensors. All scalars are isotropic (from the transformation law for an order-zero tensor). There are no non-zero isotropic vectors or axial-vectors. The most general second-order isotropic tensor is $\lambda\delta_{ij}$ where λ is a scalar. Isotropic tensors don't have any "preferred" direction.

In the limit that $\int \mathrm{d}V \to \delta V$ we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho \mathbf{u} \delta V) \cdot \hat{\mathbf{u}} = -\delta V \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} p + \delta V \rho \mathbf{g} \cdot \hat{\mathbf{n}}$$

$$\implies \quad \hat{\mathbf{n}} \cdot \mathbf{u} \underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho \delta V)}_{\substack{=0 \text{ by mass} \\ \text{conservation}}} + \rho \delta V \hat{\mathbf{n}} \cdot \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} p + \delta V \rho \mathbf{g} \cdot \hat{\mathbf{n}}$$

$$\implies \quad \delta V \hat{\mathbf{n}} \cdot \left(\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p - \rho \mathbf{g} \right) = 0. \tag{2.55}$$

This must be true for all $\hat{\mathbf{n}}$ and all δV. So we arrive at the Lagrangian momentum equation,

Equation 2.3 Lagrangian Momentum Equation

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \tag{2.56}$$

Or instead from Eq. (2.5) we have the Eulerian momentum equation,

Equation 2.4 Eulerian Momentum Equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \tag{2.57}$$

These are the equivalent of "$F = ma$" for the fluid element. Note the importance of the pressure *gradients*.

Now consider the Eulerian rate of change of momentum density $\rho \mathbf{u}$ and introduce a more compact notation

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t (\rho u_i) \\ &= \rho \partial_t u_i + u_i \partial_t \rho \\ &= \underbrace{-\rho u_j \partial_j u_i - \partial_j p \delta_{ij} + \rho g_i}_{\text{given} \quad (2.57)} - u_i \underbrace{\partial_j (\rho u_j)}_{\text{given } (2.12)}, \end{aligned} \tag{2.58}$$

where we have used notation

$$\partial_j \equiv \frac{\partial}{\partial x_j} \tag{2.59}$$

and employed summation convention (summation over the repeated indices).

This gives

$$\partial_t (\rho u_i) = -\partial_j \Big(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor} \\ \text{due to bulk flow} \\ \text{“Ram Presure”}}} + \underbrace{p \delta_{ij}}_{\substack{\text{stress tensor} \\ \text{due to random} \\ \text{thermal motions}}} \Big) + \rho g_i = -\partial_j \sigma_{ij} + \rho g_i \tag{2.60}$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$\sigma_{ij} = p\delta_{ij} + \rho u_i u_j. \tag{2.61}$$

In component free language we write[4]

$$\partial_t(\rho\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underbrace{\left(\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\underline{\mathbf{I}}}\right)}_{\substack{\text{flux of}\\ \text{momentum}\\ \text{density}}} + \rho\mathbf{g}. \tag{2.63}$$

The tensor form of the momentum conservation equation tells us that the time derivative of a conserved quantity plus the divergence of a flux is equal to a source term; this is the standard form of a conservation equation.

2.4.2.1 Bernoulli's Equation Revisited

Consider the gradient of Eq. (2.23), $\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right)$. Using the identity (2.70), $\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$. So,

$$\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = \boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Psi + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \rho\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}). \tag{2.64}$$

We also have the Eulerian momentum equation (2.57),

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi, \tag{2.65}$$

and combining this with the right hand side of the above Eq. (2.64) gives

$$\boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Psi + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \rho\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) = -\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}), \tag{2.66}$$

and thus

$$\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = -\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}). \tag{2.67}$$

If the flow is *steady*, i.e. $\rho\,\partial\mathbf{u}/\partial t = 0$, then taking the dot product of each side of the previous equation with $\mathbf{u}$ gives

$$\mathbf{u}\cdot\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = 0. \tag{2.68}$$

[4]The tensor product of the velocities is

$$\mathbf{u}\otimes\mathbf{u} = \begin{pmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{pmatrix}. \tag{2.62}$$

This again proves that $p + \rho\Psi + \frac{1}{2}\rho u^2$ *is constant on a streamline* (Bernoulli's equation). If the flow is *steady and irrotational* (i.e. $\boldsymbol{\nabla} \times \mathbf{u} = 0$), then $p + \rho\Psi + \frac{1}{2}\rho u^2$ is a *constant everywhere*.

Note: this can be generalised to time-dependent flows. If the flow is irrotational and derived from a velocity potential, i.e. $\mathbf{u} = \boldsymbol{\nabla}\Phi$, then there is an important generalisation: $p + \rho\Psi + \frac{1}{2}\rho u^2 + \rho\, \partial\Phi/\partial t$ is constant everywhere, and at all times.

2.4.2.2 Example: Flow in a Pipe in the y-direction

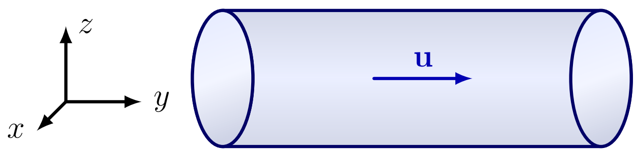

Fig. 2.9: Flow in a pipe

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure. Following from equation (2.61),

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{2.69}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

Example 2.5 Vector Identities and Time Evolution of Vorticity
Use the summation convention to prove:

$$\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) \equiv \boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{A} \cdot \mathbf{A}\right) - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{A} \tag{2.70}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) \equiv 0 \tag{2.71}$$

$$\boldsymbol{\nabla} \times (\psi\mathbf{A}) \equiv \psi\boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \times \boldsymbol{\nabla}\psi. \tag{2.72}$$

Using the above identities and the curl of the momentum equation, show that if $\boldsymbol{\nabla} \times \mathbf{u} = 0$ everywhere at time $t = t_0$, then it remains so provided that the pressure is a function of the density only.

Solution For the first identity,

$$\begin{aligned} \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \epsilon_{ijk} A_j (\boldsymbol{\nabla} \times \mathbf{A})_k \hat{\mathbf{e}}_i \\ &= \epsilon_{ijk}\epsilon_{klm} A_j \partial_l A_m \hat{\mathbf{e}}_i \\ &= \epsilon_{kij}\epsilon_{klm} A_j \partial_l A_m \hat{\mathbf{e}}_i \quad \text{since} \quad \epsilon_{ijk} = \epsilon_{kij} \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l A_m \hat{\mathbf{e}}_i \\ &= A_j \partial_i A_j \hat{\mathbf{e}}_i - A_j \partial_j A_i \hat{\mathbf{e}}_i \end{aligned} \tag{2.73}$$

where we have applied the very useful identity $\epsilon_{kij}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$.[5] Recognising the first term as half of a total derivative, we find

$$\begin{aligned}\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \tfrac{1}{2}\partial_i(\mathbf{A} \cdot \mathbf{A})\hat{\mathbf{e}}_i - (\mathbf{A} \cdot \boldsymbol{\nabla})b_i\hat{\mathbf{e}}_i \\ &= \boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{A} \cdot \mathbf{A}\right) - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{A}. \qquad (2.77)\end{aligned}$$

For the second identity, the proof focuses on the antisymmetry of the levi-civita symbol,

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) &= \epsilon_{ijk}\partial_j(\boldsymbol{\nabla}\psi)_k\hat{\mathbf{e}}_i \\ &= \epsilon_{ijk}\partial_j\partial_k\psi\hat{\mathbf{e}}_i \qquad (2.78) \\ &= \epsilon_{ikj}\partial_k\partial_j\psi\hat{\mathbf{e}}_i, \qquad (2.79)\end{aligned}$$

where we have relabelled the dummy indices $j \leftrightarrow k$. Using the commutativity of the partial derivative, we can write

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) = \epsilon_{ikj}\partial_j\partial_k\psi\hat{\mathbf{e}}_i, \qquad (2.80)$$

and now using the total antisymmetry of ϵ_{ijk},

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) = -\epsilon_{ijk}\partial_j\partial_k\psi\hat{\mathbf{e}}_i, \qquad (2.81)$$

which is also equal to the negative of itself (2.78), hence $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi$ must vanish for all scalar fields ψ.

The final identity is the easiest, for it follows almost directly from application of the chain rule,

$$\begin{aligned}\boldsymbol{\nabla} \times (\psi\mathbf{A}) &= \epsilon_{ijk}\partial_j(\psi A_k)\hat{\mathbf{e}}_i \\ &= \epsilon_{ijk}\psi\partial_j A_k\hat{\mathbf{e}}_i + \epsilon_{ijk}A_k\partial_j\psi\hat{\mathbf{e}}_i \\ &= \psi(\boldsymbol{\nabla} \times \mathbf{A})_i\hat{\mathbf{e}}_i - \epsilon_{ikj}A_k\partial_j\psi\hat{\mathbf{e}}_i \quad \text{since} \quad \epsilon_{ijk} = -\epsilon_{ikj} \\ &= \psi(\boldsymbol{\nabla} \times \mathbf{A}) - (\mathbf{A} \times \boldsymbol{\nabla})\psi. \qquad (2.82)\end{aligned}$$

[5] The Levi-Civita symbol is related to the Kronecker delta. In three dimensions, the relationship is given by the following equations (vertical lines denote the determinant):[3]

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix} \qquad (2.74) \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). \qquad (2.75)\end{aligned}$$

A special case of this result occurs when one of the indices is repeated and summed over:

$$\sum_{i=1}^{3}\epsilon_{ijk}\epsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \qquad (2.76)$$

In Einstein notation, the duplication of the i index implies the sum on i. The previous is then denoted $\epsilon_{ijk}\epsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

We can write the Eulerian momentum equation (2.57) as

$$\dot{\mathbf{u}} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi, \tag{2.83}$$

where we used (3.1) to express $\mathbf{g}$ as the gradient of the potential $\mathbf{g} = -\boldsymbol{\nabla}\Psi$. Using the proven identity (2.70),

$$\dot{\mathbf{u}} = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) - \boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi. \tag{2.84}$$

Taking the curl of this equation, using Eq. (2.71) we find that multiple terms vanish, leaving us with

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla} \times \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) + \boldsymbol{\nabla} p \times \boldsymbol{\nabla}\left(\frac{1}{\rho}\right). \tag{2.85}$$

Given that the fluid is barotropic ($p = p(\rho)$), the pressure gradient can be expressed as $\boldsymbol{\nabla} p = p'(\rho)\boldsymbol{\nabla}\rho$, thus $\boldsymbol{\nabla} p \parallel \boldsymbol{\nabla}\rho$ and the cross product of these terms vanishes, since $\boldsymbol{\nabla}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\boldsymbol{\nabla}\rho$. This leaves

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla} \times \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}), \tag{2.86}$$

so, if $\boldsymbol{\nabla} \times \mathbf{u}$ initially, then the right hand side vanishes giving that $\boldsymbol{\nabla} \times \mathbf{u}$ is a conserved quantity of the flow. ◀

Chapter 3

Gravitation

3.1 Basics

Define the gravitational potential Ψ such that the gravitational acceleration $\mathbf{g}$ is

Equation 3.1 Gravitational Acceleration

$$\mathbf{g} = -\boldsymbol{\nabla}\Psi \tag{3.1}$$

If ℓ is some closed loop, we have (using the curl theorem)

$$\oint_\ell \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{g}) \cdot \mathrm{d}\mathbf{S} = -\int_{\mathcal{S}} [\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Psi)] \cdot \mathrm{d}\mathbf{S} = 0, \tag{3.2}$$

as curl of any gradient is zero. So gravity is a *conservative force* – the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point $\mathbf{r}$ to ∞ is thus

$$-\int_{\mathbf{r}}^{\infty} \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \boldsymbol{\nabla}\Psi \cdot \mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r}), \tag{3.3}$$

which is *independent of path*.

A particular important case is the gravity of a point mass, which has

$$\Psi = -\frac{GM}{r} \qquad \text{if mass at origin} \tag{3.4}$$

$$\Psi = -\frac{GM}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{if mass at location } \mathbf{r}'. \tag{3.5}$$

For a system of point masses we have

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}'_i|} \tag{3.6}$$

$$\implies \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3} \tag{3.7}$$

Replacing $M_i \to \rho_i \delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G \int_{\mathcal{V}} \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V' \,. \tag{3.8}$$

Take divergence of both sides

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{g} &= -G \int_{\mathcal{V}} \rho(\mathbf{r}')\boldsymbol{\nabla}_{\mathbf{r}} \cdot \left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right] \mathrm{d}V' \\
&= -G \int_{\mathcal{V}} \rho(\mathbf{r}')\boldsymbol{\nabla}_{\mathbf{r}} \cdot \left[\boldsymbol{\nabla}_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \mathrm{d}V' \\
&= -G \int_{\mathcal{V}} \rho(\mathbf{r}') \underbrace{\nabla^2_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)}_{4\pi\delta(\mathbf{r}-\mathbf{r}')} \mathrm{d}V' \\
&= -4\pi G \int_{\mathcal{V}} \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' \\
&= -4\pi G\rho(\mathbf{r}). \qquad (3.9)
\end{aligned}
$$

where we have made use of our knowledge of the Green's function for the 3D laplacian operator.[1] Thus we arrive at *Poisson's equation for gravitation*,

Equation 3.2 Poisson's Equation for Gravitation

$$
\boldsymbol{\nabla} \cdot \mathbf{g} = -\nabla^2\Psi = -4\pi G\rho \qquad (3.12)
$$

We can also express Poisson's equation in integral form: for some volume $\mathcal{V}$ bounded by surface $\mathcal{S}$ we have

$$
\begin{aligned}
\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{g}\,\mathrm{d}V &= -4\pi G \int_{\mathcal{V}} \rho\,\mathrm{d}V \\
\implies \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G M. \qquad (3.13)
\end{aligned}
$$

This is useful for calculating $\mathbf{g}$ when the mass distribution obeys some symmetry.

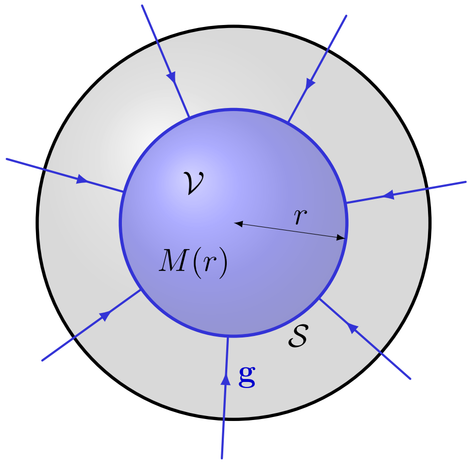

Fig. 3.1: Spherical distribution of mass

3.1.1 Spherical Distribution of Mass

By symmetry $\mathbf{g}$ is radial and $|\mathbf{g}|$ is constant over a $r=$ const. shell. So

$$
\begin{aligned}
\int_{\mathcal{S}} \mathbf{g}\cdot \mathrm{d}\mathbf{S} &= -4\pi G \underbrace{M(r)}_{\substack{\text{mass}\\ \text{enclosed}}} \\
\implies \quad -4\pi r^2|\mathbf{g}| &= -4\pi G M(r) \\
\implies \quad |\mathbf{g}| &= \frac{GM(r)}{r^2} \\
\therefore \mathbf{g} &= -\frac{GM(r)}{r^2}\hat{\mathbf{r}}.
\end{aligned}
\tag{3.14}
$$

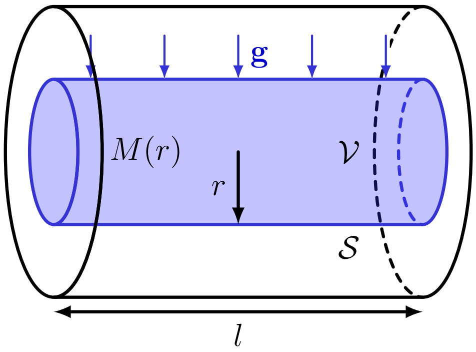

Fig. 3.2: Cylindrical distribution of mass

3.1.2 Infinite Cylindrically Symmetric Mass

By symmetry, $\mathbf{g}$ is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \Longrightarrow \qquad -2\pi r l |\mathbf{g}| &= -4\pi G l \underbrace{M(r)}_{\substack{\text{enclosed mass} \\ \text{per unit length}}} \\ \therefore \mathbf{g} &= -\frac{2GM(r)}{r}\hat{\mathbf{r}}. \end{aligned} \tag{3.15}$$

3.1.3 Infinite Planar Distribution of Mass

Assume infinite and homogeneous in x and y, $\rho = \rho(z)$.

By symmetry, $\mathbf{g}$ is in the $-\hat{\mathbf{z}}$ direction and is constant on a $z = \text{const.}$ surface.

[1]The Green's function (or fundamental solution) for the Laplacian (or Laplace operator) in three variables is used to describe the response of a particular type of physical system to a point source.

The free-space Green's function for the Laplace operator in three variables is given in terms of the reciprocal distance between two points and is known as the "Newton kernel" or "Newtonian potential". That is to say, the solution of the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{3.10}$$

is

$$G(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{3.11}$$

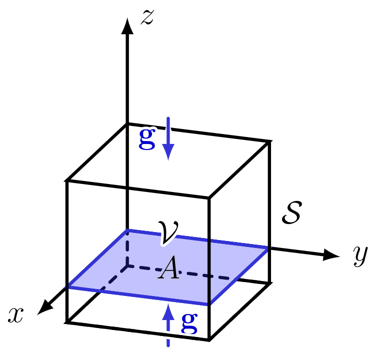

Fig. 3.3: Planar distribution of mass

So, if we also have reflection symmetry about $z = 0$,

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \implies \quad -2|\mathbf{g}|A &= -4GA \int_{-z}^{z} \rho(z) \, \mathrm{d}z \\ \therefore \mathbf{g} &= -4\pi G \hat{\mathbf{z}} \int_0^z \rho(z) \, \mathrm{d}z \quad \forall z \geq 0. \end{aligned} \tag{3.16}$$

(For planar distribution of finite height $z_{\max}$, $\mathbf{g}$ is constant for $z \geq z_{\max}$.)

Example 3.1 Infalling Stars
A particle is released at rest at radius r_0 from the centre of a body mass M. Compute

(a) its initial acceleration,

(b) the time it takes to reach the centre of the body,

for the two cases

(i) that the body is a point mass,

(ii) that the body is a uniform sphere radius r_0.

A cluster consists initially of stars at rest, distributed in a uniform sphere. Find how long it takes a star to reach the centre as a function of its initial radius in the cluster and comment on your results.

Solution For the body that is a point mass (i), the gravitational potential is given by (3.4) $\Psi = -\frac{GM}{r}$, so considering the conservation of energy *per unit mass* at the initial radius r_0, and at an arbitrary radius r gives

$$-\frac{GM}{r_0} = \frac{1}{2}u^2 - \frac{GM}{r}, \tag{3.17}$$

where u is the speed of the particle at r. Rearranging this for $\dot{r}$ gives

$$2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \quad \implies \quad \frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}, \tag{3.18}$$

where we have chosen the negative sign to reflect the particles inwards trajectory, i.e. the radial distance is decreasing with time.

We can solve this integral by making the substitution $r = r_0 \cos^2\theta$, such that

$$\mathrm{d}r = -2r_0 \cos\theta \sin\theta \, \mathrm{d}\theta \,. \tag{3.19}$$

Substituting this into the right of Eq. (3.18) we see the simplification that motivated our chosen substitution,

$$\begin{aligned} 2r_0 \cos\theta \sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \sqrt{\frac{2GM}{r_0}(\sec^2\theta - 1)} \\ &= \sqrt{\frac{2GM}{r_0}} \tan\theta \end{aligned} \tag{3.20}$$

and thus we obtain the nicer looking *separable* differential equation

$$2r_0 \cos^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2GM}{r_0}}. \tag{3.21}$$

Finally, we can do the integral with the appropriate bounds,

$$\underbrace{\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta}_{\pi/4} = \sqrt{\frac{GM}{2r_0^3}} \int_0^t \mathrm{d}t' \,, \tag{3.22}$$

to obtain the time to the centre $t = \sqrt{\pi^2 r_0^3/8GM}$.

The acceleration for this case is given differentiation of Eq. (3.18), or by simply recalling the gravitational acceleration (3.14),

$$\ddot{r} = -\frac{GM}{r^2}, \tag{3.23}$$

so initially has a value $\ddot{r} = -GM/r_0^2$.

For the second case (ii), we again use the gravitational potential equation (3.4), $\Psi = -\frac{GM(r)}{r}$. However, since the shells outside the particle do not contribute to the potential, only the mass enclosed within radius r matters. This enclosed mass scales as $M(r) = Mr^3/r_0^3$, and the acceleration is thus

$$\ddot{r} = -\frac{G}{r^2}\frac{r^3}{r_0^3}M = -\frac{GM}{r_0^3}r. \tag{3.24}$$

Notice that this is simple harmonic motion with angular frequency $\omega^2 = GM/r_0^3$, and so the appropriate solution is $r = r_0\cos(\omega t)$ giving the time to reach the centre of the body as $t = \sqrt{\pi^2 r_0^3/4GM}$. Initially, the situation is identical to (i), and thus the acceleration must again be given by (3.23).

Let us discuss the final part of the question. We could, for example, assume that all the stars begin the collapse at the same time, and so we would in fact use the result as if the stars at radius less than that of the infalling particles behaved like a point mass, scaled appropriately to the amount of mass confined in this region, and then the outside of this shell would fall under the influence of a point mass with an time to reach the centre similar to $t = \sqrt{\pi^2 r_0^3/GM}$, but now taking $M \to Mr^3/r_0^3$ and $r_0 \to r$, giving the time as $t = \sqrt{\pi^2 r_0^6/GMr^3}$.

Instead, if we assume that the background average mass distribution remains constant, one might derive the free-fall time as $t = \sqrt{\pi^2 r_0^3/4GM}$. However, this result raises a conceptual issue: how can an individual star collapse while the surrounding mass distribution in the cluster remains unchanged? In reality, stars within a cluster are not stationary but instead possess random, individual motions – referred to as peculiar motions[2] – superimposed on any overall organised motion of the cluster.

These peculiar motions are quantified by the velocity dispersion, which measures the spread in stellar velocities about the cluster's mean velocity. This dispersion imparts a kinetic pressure analogous to thermal pressure in a gas cloud, counteracting gravitational attraction. For the cluster to remain in equilibrium, the gravitational forces that tend to pull the stars together must be balanced by the kinetic energy associated with these random motions. If the velocity dispersion is too low, gravity will dominate, leading to collapse; if it is too high, the cluster may disperse as stars overcome the gravitational binding energy.

In addition to random motions, any organised rotational motion within the cluster provides centrifugal support, which further counterbalances gravity. Although random motions are typically the dominant factor in maintaining cluster stability, rotation can significantly contribute to the overall structural integrity and long-term evolution of the cluster by inhibiting gravitational collapse. ◀

3.2 Potential of a Spherical Mass Distribution

We found in Eq. (3.14), for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \quad |\mathbf{g}| = \frac{G}{r^2}\underbrace{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'}_{M(r)} = \frac{\mathrm{d}\Psi}{\mathrm{d}r}, \tag{3.25}$$

so,

$$\Psi(r_0) - \Psi(\infty) = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r\,. \tag{3.26}$$

[2]**Peculiar motion** or peculiar velocity refers to the velocity of an object relative to a rest frame – usually a frame in which the average velocity of some objects is zero.

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\begin{aligned} \Psi(r_0) &= -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\bigg|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r \\ \implies \Psi(r_0) &= -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r\,, \end{aligned} \tag{3.27}$$

where we have made an assumption that $M(r) \to 0$ as $r \to 0$, and $M(r)/r \to 0$ as $r \to \infty$.

We find that $\Psi(r_0)$ is affected by matter outside of r_0 through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r.

3.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}, \tag{3.28}$$

and the energy required to take a unit mass to ∞ is $-\Psi$. The energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j, \tag{3.29}$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution, the natural extension is

$$\Omega = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V\,. \tag{3.30}$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r)r^2\Psi(r)\,\mathrm{d}r \tag{3.31}$$

Integrate by parts, choosing parts $u \equiv \Psi$, $\mathrm{d}v \equiv 4\pi\rho r^2\,\mathrm{d}r$ so that $v = \int_0^r 4\pi\rho'^2\,\mathrm{d}r' = M(r)$, then

$$\Omega = \frac{1}{2}\left[M(r)\Psi(r)\bigg|_0^\infty - \int_0^\infty M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,\mathrm{d}r\right]. \tag{3.32}$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (the boundary term) is zero. Noting further that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}, \tag{3.33}$$

we conclude

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r\,. \tag{3.34}$$

Integrate again by parts, choosing $u \equiv GM(r)^2$, $\mathrm{d}v \equiv \mathrm{d}r\,/r^2$,

$$\Omega = \underbrace{\frac{1}{2}GM(r)^2\frac{1}{r}\bigg|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r$$

$$\implies \quad \Omega = -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M\,. \tag{3.35}$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r}\,\mathrm{d}M(r)\,. \tag{3.36}$$

3.4 The Virial Theorem

Theorem 3.1 Virial Theorem
For a system of N particles bound by conservative forces with potential energy Ω, the time-averaged total kinetic energy $\langle T\rangle$ and the potential energy $\langle\Omega\rangle$ satisfy the relation

$$2\langle T\rangle + \langle\Omega\rangle = 0. \tag{3.37}$$

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems. Here, we will examine the **scalar virial theorem** ($\exists$ general tensor virial theorem).

Consider the motion of a cloud of particles (atoms, stars, galaxies, ...). A particle with mass m_i at $\mathbf{r}_i$ is acted upon by a force

$$\mathbf{F}_i = m_i\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2}. \tag{3.38}$$

Consider the 2nd derivative of the scalar moment of inertia, $I_i = m_i r_i^2$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i r_i^2\right) &= m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_i\cdot\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)\\ &= m_i\mathbf{r}_i\cdot\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2\\ &= \mathbf{r}_i\cdot\mathbf{F}_i + \underbrace{m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2}_{2\times\text{Kinetic Energy } T_i}\,.\end{aligned} \tag{3.39}$$

If $I \equiv \sum_i m_i r_i^2$ then we can sum the previous equation over all particles to give

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{\substack{V,\text{ the }\textbf{Virial}\\ \text{(R. Clausius)}}} + 2T. \tag{3.40}$$

The word **virial** for the first term on the right-hand side of the equation derives from *vis*, the Latin word for “force” or “energy”, and was given its technical definition by Rudolf Clausius in 1870. [1]

In the absence of external forces (i.e. an isolated system), we have that $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ where $\mathbf{F}_{ij}$ is the force exerted on the i^{th} particle by the j^{th} particle. Consider any two particles with m_i and m_j at $\mathbf{r}_i$ and $\mathbf{r}_j$, Newton’s 3$^{\text{rd}}$ Law says

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \tag{3.41}$$

and so their contribution to the virial is $\mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. We then have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j). \tag{3.42}$$

If there are no non-gravitational interactions except for possibly when $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \quad \text{where} \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \tag{3.43}$$

Thus we have

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}, \tag{3.44}$$

where each term is the work done to separate each pair of particles to infinity against gravity.

And so, $V = \Omega$ and we can use the above to write

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + \Omega. \tag{3.45}$$

If the system is in a steady state (“relaxed”), then $I =$ const. and we can state the *Virial theorem*

$$\boxed{2T + \Omega = 0.} \tag{3.46}$$

Here, the kinetic energy T has contributions from local flows and random/thermal motions. The Virial theorem implies *gravitational potential sets the “temperature” or velocity dispersion of the system.*

3.4.1 Implications of the Virial Theorem

Connects mass, velocity and size of a gravitating system

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M \left\langle v^2 \right\rangle \quad \text{and} \quad \Omega = -\frac{1}{2} \int_0^\infty \frac{GM(r)^2}{r^2} \, \mathrm{d}r = -\int_0^\infty \frac{GM(r)}{r} \, \mathrm{d}M = -\frac{GM^2}{\bar{r}}. \tag{3.47}$$

So, invoking the virial theorem $2T = -\Omega$,

$$M \left\langle c^2 \right\rangle = \frac{GM^2}{\bar{r}} \quad \Longrightarrow \quad \left\langle v^2 \right\rangle = \frac{GM}{\bar{r}}. \tag{3.48}$$

Gravitating systems have a negative specific "heat" capacity,

$$\begin{aligned} E_{\text{total}} &= T + \Omega \\ &= -T = -\frac{1}{2} M \left\langle v^2 \right\rangle = -\frac{GM^2}{\bar{r}}. \end{aligned} \tag{3.49}$$

As a general rule, gravitationally bound systems have negative heat capacities. This is because in equilibrium (and remember we can't do classical thermodynamics without equilibrium anyway), some form of the virial theorem will apply. If the system has only kinetic energy T and potential energy Ω, then the total energy is of course $E_{\text{total}} = T + \Omega$, where $E_{\text{total}} < 0$ for bound systems. In virial equilibrium where the potential energy is purely gravitational, then we also have $T = -\Omega$. As a result, $E_{\text{total}} = -T$, and so adding more energy results in a decrease in temperature.

Broadly, this is why gravitation creates structure from initially smooth conditions. A dramatic manifestation is the gravothermal collapse (e.g. globular clusters). Note that, for a gravitating gas ball, T is directly related to the gas temperature.

Examples include stars and globular clusters. Imagine adding energy to such systems by heating up the particles in the star or giving the stars in a cluster more kinetic energy. The extra motion will work toward slightly unbinding the system, and everything will spread out. But since (negative) potential energy counts twice as much as kinetic energy in the energy budget, everything will be moving even slower in this new configuration once equilibrium is reattained.

Chapter 4

Equations of State and the Energy Equation

4.1 The Equation of State

In three dimensions, the (scalar) equation of mass conservation and the (vector) equation of momentum conservation can be written as four independent scalar equations. Given appropriate boundary conditions, these must be solved in order to find the density (scalar field), pressure (scalar field), gravitational potential (scalar field), and velocity components (3D vector field); a total of six degrees of freedom.

To close the system of equations, we need additional information. Specifically, we need to find relations between Ψ, p and the other fluid variables such as ρ and $\mathbf{u}$.

$\Psi(r)$ and ρ are related via Poisson's equation (and/or we sometimes consider an externally imposed gravitational potential).

p and the other thermodynamic properties of the system are related by the *equation of state* (EoS), depending upon microphysics of the fluid. This is only valid for collisional fluids.

Most astrophysical fluids are quite dilute (particle separation much larger than effective particle size) and can be well approximated as ideal gases. The corresponding EoS is

$$p = p(\rho, T) = nk_B T = \frac{k_B}{\mu m_p}\rho T, \tag{4.1}$$

where n is the number density of particles, and μ is the mean particle mass in units of the proton mass m_p. (Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars and white dwarfs.)

The ideal gas EoS introduces another scalar field into the description of the fluid, the temperature $T(\mathbf{r}, t)$. In general, we need to solve another PDE that describes the conservation of energy through heating and cooling processes in order to close the set of equations. We shall move on to this in Section 4.2.

4.1.1 Barotropic Fluids

However, for special cases, we can relate T and ρ without the need to solve a separate energy equation.

Definition 4.1 Barotropic Fluid
Fluids for which p is *only* a function of ρ are known as barotropic fluids.

4.1.1.1 Electron Degeneracy Pressure

Important in systems with free electrons that are (relatively) cold and dense.

$$p = \frac{\pi^2 \hbar^2}{5 m_{\mathrm{e}} m_{\mathrm{ion}}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}, \quad \text{(non-relativistic)} \tag{4.2}$$

e.g., interiors of white dwarfs, iron core in massive stars, deep interior of Jupiter.

4.1.1.2 Isothermal Ideal Gas

$$p = A\rho, \quad A = \frac{k_B T}{\mu m_p} = \text{const.} \tag{4.3}$$

T is constant so that $p \propto \rho$. Valid most commonly when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance at some well-defined temperature.

4.1.1.3 Adiabatic Ideal Gas

Ideal gas undergoes *reversible* thermodynamic changes such that

$$p = K\rho^{\gamma} \tag{4.4}$$

where K, γ are constants.

The first law of thermodynamics is

$$\underbrace{đQ}_{\substack{\text{heat absorbed by}\\ \text{unit mass of fluid}\\ \text{from surrounding}}} = \underbrace{\mathrm{d}\mathcal{E}}_{\substack{\text{change in internal}\\ \text{energy of unit}\\ \text{mass of fluid}}} + \underbrace{p\,\mathrm{d}V}_{\substack{\text{work done by}\\ \text{unit mass of fluid}}} . \tag{4.5}$$

Here đ is a Pfaffian operator – change in quantity depends on the path taken through the thermodynamic phase space. For an ideal gas, we can write

$$p = \frac{\mathcal{R}_*}{\mu}\rho T, \quad \mathcal{E} = \mathcal{E}(T), \tag{4.6}$$

where $\mathcal{R}_*$ is a modified gas constant, $\mathcal{R}_* = k_\mathrm{B}/m_\mathrm{p}$.

So, the first law of thermodynamics reads

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V\,, \end{aligned} \tag{4.7}$$

where we define specific heat capacity at constant volume as $C_V \equiv \mathrm{d}\mathcal{E}/\mathrm{d}T$ and have noted that for unit mass we have $\rho = 1/V$.

For a *reversible* change we have $đQ = 0$, so

$$\begin{aligned} & C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V = 0 \\ \implies \quad & C_V\,\mathrm{d}(\ln T) + + \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}(\ln V) = 0 \\ & \implies V \propto T^{-C_V\mu/\mathcal{R}_*} \qquad (4.8) \\ \implies & p \propto T^{1+C_V\mu/\mathcal{R}_*} \quad \text{given} \quad (4.6). \qquad (4.9) \end{aligned}$$

C_V depends on the number of degrees of freedom with which the gas can store kinetic energy, f such that

$$C_V = f\frac{k_B}{2\mu m_p} = f\frac{\mathcal{R}_*}{2\mu}. \tag{4.10}$$

i.e. using here $\mathcal{R}_* = k_B/m_p$. From equipartition, there is an internal energy contribution of $\frac{1}{2}k_B T$ per particle per degree of freedom. Monatomic gas has $f = 3 \implies C_V = 3\mathcal{R}_*/2\mu$; diatomic gas at a few $\times$ 100 K (if two rotational modes excited) has $f = 5 \implies C_V = 5\mathcal{R}_*/2\mu$.

Returning to the ideal gas law,

$$\begin{aligned} p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \text{with} \quad & \rho = 1/V \quad \text{for a unit mass of fluid} \\ \implies \quad & pV = \frac{\mathcal{R}_* T}{\mu} \\ \implies \quad & p\,\mathrm{d}V + V\,\mathrm{d}p = \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}T\,, \qquad (4.11) \end{aligned}$$

but,

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= \underbrace{\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} + \frac{\mathcal{R}_*}{\mu}\right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}}\,\mathrm{d}T - V\,\mathrm{d}p\,, \end{aligned} \tag{4.12}$$

so,

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu} = \frac{k_B}{\mu m_p}. \tag{4.13}$$

Let us define the ratio of specific heat capacities

$$\gamma \equiv \frac{C_p}{C_v} = \frac{f+2}{f}, \tag{4.14}$$

so that, for the reversible/adiabatic processes discussed above, we have

$$p \propto T^{1+C_V\mu/\mathcal{R}_*} \implies p \propto T^{\gamma/(\gamma+1)} \tag{4.15}$$

$$V \propto T^{-C_V\mu/\mathcal{R}_*} \implies V \propto T^{-1/(\gamma-1)} \tag{4.16}$$

which we can combine to give

$$p \propto \rho^\gamma \tag{4.17}$$

We say that a fluid element behaves *adiabatically* if $p = K\rho^\gamma$ with $K =$ constant. A fluid is **isentropic** if all fluid elements behave adiabatically with the same value of K. $\ln K$ is proportional to the entropy per unit mass.

Example 4.1 Stellar Wind
A stellar wind behaves as a steady adiabatic spherical outflow of a perfect monatomic gas (so $\gamma = c_p/c_V$) from the surface of the star, so at radius a the density is ρ_0, temperature T_0, and outflow velocity u_0. If the fluid motions are dominated by the star's gravitational potential, determine the temperature as a function of the radius from the star centre.
If the flow velocity u_0 at radius a is just the gravitational escape velocity from that point do pressure effects ever become significant?

Solution The flow is steady, so the particles follow the streamlines, along which we can apply the Bernoulli principle (see Subsection 2.3.1) to know that $\frac{1}{2}u^2 + \Psi + \frac{\gamma}{\gamma-1}\frac{p}{\rho}$ is constant (2.22), given the gas is ideal. Outside the surface of the star, the gravitational acceleration is $\Psi = -\frac{GM}{r}$ (3.4), thus we can write

$$\frac{1}{2}u^2 - \frac{GM}{r} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \text{const.} \tag{4.18}$$

In regions where gravitational effects dominate over pressure contributions, this simplifies to

$$\frac{1}{2}u^2 - \frac{GM}{r} = \text{const.} \tag{4.19}$$

Applying this equation at both the reference radius r_0 at the surface of the star and an arbitrary radial distance r, we obtain

$$u^2 - \frac{2GM}{r} = u_0^2 - \frac{2GM}{r_0}. \tag{4.20}$$

We could also instead use the momentum equation (2.57), and the gravitational acceleration for a point mass $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$ to write

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{4.21}$$

where we have neglected the pressure effects relative to the gravitational term. Noticing the radial dependence reduces the grad operator to $\boldsymbol{\nabla} \equiv \partial/\partial r\,\hat{\mathbf{r}}$, it follows along the $\hat{\mathbf{r}}$ direction with the appropriate integration bounds that

$$u\,\mathrm{d}u = -\frac{GM}{r^2}\,\mathrm{d}r \quad\implies\quad u^2 - u_0^2 = 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right), \tag{4.22}$$

as before.

The conservation of mass (Eulerian continuity equation (2.12)), when integrated over spherical shells, implies a constant outwards mass flux,

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathcal{V}} \rho\,\mathrm{d}V &= -\rho\int_{\mathcal{S}} \mathbf{u}\cdot\mathrm{d}\mathbf{S} \\ \implies \quad \text{const.} &= -\rho\int_{\mathcal{S}} \mathbf{u}\cdot\mathrm{d}\mathbf{S} \\ \implies \quad \rho_0 u_0 r_0^2 &= \rho u r^2. \end{aligned} \tag{4.23}$$

By eliminating $u(r)$, an explicit expression for the density profile is derived:

$$\begin{aligned} \left(\frac{\rho_0 u_0 r_0^2}{\rho r^2}\right)^2 - \frac{2GM}{r} &= u_0^2 - \frac{2GM}{r_0} \\ \implies \quad \rho &= \frac{\rho_0 u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}. \end{aligned} \tag{4.24}$$

Since gas flow is adiabatic, then we have the additional relation $p \propto \rho^\gamma$ (4.17) and the ideal gas law $p = \frac{\mathcal{R}_*}{\mu}\rho T$ (4.6), which combine to give the relation $T \propto \rho^{\gamma-1}$, so

$$T = T_0\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}. \tag{4.25}$$

Substituting ρ using (4.24) gives

$$T = T_0\left(\frac{u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\right)^{\gamma-1} = T_0\left(\frac{u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\right)^{2/3}, \tag{4.26}$$

i.e. $T \propto r^{-4/3}$ for an ideal monatomic gas with $\gamma = 5/3$. If the gas is very hot at r_0 then thermal effects are important at first, but the rapid cooling as the gas expands means that the flow becomes very supersonic.

The escape velocity from the surface is given by

$$\frac{1}{2}u_0^2 - \frac{GM}{r_0} = 0 \quad\implies\quad u_0^2 = \frac{2GM}{r_0}, \tag{4.27}$$

so if the initial outflow velocity is equal to the escape velocity, then the density is

$$\rho = \rho_0 \left(\frac{r_0}{r}\right)^{3/2}, \tag{4.28}$$

and so T is given by $T = T_0\left(\frac{r_0}{r}\right)^{\frac{3}{2}(\gamma-1)} = T_0\left(\frac{r_0}{r}\right)$, so $T \propto r^{-1}$. Using the adiabatic relation again, $p \propto \left(r^{-3/2}\right)^{5/3} \propto r^{-5/2}$ and $\boldsymbol{\nabla} p \propto r^{-7/2}$, indicating that pressure effects rapidly diminish with increasing radius.

To check whether pressure effects ever become significant, we can compare the magnitude of the pressure and gravitational terms in the momentum equation (2.57), $|\boldsymbol{\nabla} p/\rho\mathbf{g}|$.

$$\left|\frac{\boldsymbol{\nabla} p}{\rho\mathbf{g}}\right| \propto \frac{r^{-7/2}}{r^{-3/2}r^{-2}} = \text{const.} \tag{4.29}$$

Thus, since this ratio remains constant, there is never a point at which the pressure effects become significant. ◀

This analysis provides a physically motivated model for stellar winds, including the solar wind, where thermal expansion and gravitational forces dictate the structure of the outflow. The results demonstrate that while thermal pressure initially contributes to driving the wind, its role becomes negligible at large distances, where the flow becomes highly supersonic.

4.2 The Energy Equation

In general, the equation of state will not be barotropic and we will need to solve a separate partial differential equation which follows the energy conservation in the flow through heating and cooling processes in the gas, the *energy equation*.

From the first law of thermodynamics we have

$$đQ = \mathrm{d}\mathcal{E} + \underbrace{p\,\mathrm{d}V}_{\mathrm{d}W=-p\mathrm{d}V} \quad \text{in absence of dissipative processes,} \tag{4.30}$$

so, applying this to a given fluid element in the Lagrangian framework

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{\mathrm{D}W}{\mathrm{D}t} + \frac{đQ}{\mathrm{d}t}, \tag{4.31}$$

with, for a unit mass $V = 1/\rho$, **??**

$$\frac{\mathrm{D}W}{\mathrm{D}t} = -p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{4.32}$$

and

$$\frac{\text{đ}Q}{\mathrm{d}t} \equiv -\dot{Q}_{\text{cool}} \quad \text{rate of cooling per unit mass,} \tag{4.33}$$

therefore,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\text{cool}}. \tag{4.34}$$

The total energy per unit volume is

$$E = \rho\left(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}}\right), \tag{4.35}$$

so from a simple application of the product rule,

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\mathrm{D}\rho}{\mathrm{D}t}\frac{E}{\rho} + \rho\left(\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\text{cool}}\right), \tag{4.36}$$

where,

$$\frac{\mathrm{D}E}{\mathrm{D}t} \equiv \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E \qquad (2.5) \tag{4.37}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad (2.14) \tag{4.38}$$

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi \qquad (2.56) \tag{4.39}$$

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} \equiv \frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Psi. \qquad (2.5) \tag{4.40}$$

Substituting Eqs. (4.37-4.40) into Eq. (4.36) it follows that

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{E}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p - \cancel{\rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi} + \rho\frac{\partial\Psi}{\partial t} + \cancel{\rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi} - \frac{p}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \rho\dot{Q}_{\text{cool}}$$

$$\implies \quad \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E = -(E+p)\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p + \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\text{cool}}, \tag{4.41}$$

which, upon using the vector identity $\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) \equiv \phi\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot(\boldsymbol{\nabla}\phi)$, gives the *Energy equation*,

Equation 4.1 Energy Equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \rho\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho\dot{Q}_{\text{cool}} \tag{4.42}$$

In many settings, $\partial\Psi/\partial t = 0$, i.e. Ψ depends on position only. If, further, we have no cooling ($\dot{Q}_{\text{cool}} = 0$), then this equation expresses the conservation of energy in which the Eulerian change in total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$.

4.3 Heating and Cooling Processes

The $\dot{Q}_{\rm cool}$ term in the energy equation describes processes that locally cool ($\dot{Q}_{\rm cool} > 0$) or locally heat ($\dot{Q}_{\rm cool} < 0$) the fluid. There are many such processes and a full discussion of them would be lengthy as they depend upon the detailed microphysics of the system under consideration. Here, we discuss just a small number of important cases relevant to the energy-budget of a *thermal gas* (i.e. one in which the bulk of the particles are in thermodynamic equilibrium) in an astrophysical setting (diffuse and dominant element is hydrogen).

1. *Cooling by radiation*: energy carried away from fluid by photons.
   - Energy loss by recombination of an ionised gas. Free electron captured by ion puts ion in excited state. Electron cascades down energy levels, eventually forming ground state, so ion de-excites via line emission. Number of recombinations per ion per unit time $\propto n_{\rm e}$, and the number of recombinations per unit volume per unit time $\propto n_{\rm e} \times n_{\rm ion}$ and $\dot{Q} = \rho f(T)$;
   - Energy loss by free-free emission (free electrons accelerated in electric fields of ions)
     $$L_{\rm ff} \propto n_{\rm e} n_{\rm p} T^{1/2}. \tag{4.43}$$
   - Collisionally-excited atomic line radiation. Electron-ion collisions lead to excited electronic states (inelastic collision with ground state atom). Excited state decays back to ground state via the emission of photons at well-defined energy χ. Number of collisions per ion per unit time $\propto n_{\rm e}$, and the number of collisions per unit volume per unit time $\propto n_{\rm e} \times n_{\rm ion}$.
     $$L_{\rm e} \propto n_{\rm e} n_{\rm ion} e^{-\chi/kT} \chi/\sqrt{T} \tag{4.44}$$
     In cold gas clouds with $T \sim 10^4\,$K, H cannot be excited so cooling occurs through trace species (O^+, O^{++}, N^+).

   These are all two-body interactions $\implies$ cooling rate per unit volume proportional to ρ^2 . Recalling that $\dot{Q}_{\rm cool}$ is defined per unit mass, such processes give $\dot{Q}_{\rm cool} = \rho f(T)$.

2. *Heating by cosmic rays*: Heating can occur through the dissipation of kinetic energy via internal processes within the fluid, e.g. shocks (Chapter 6) and viscosity (Chapter 9). Heating can also occur from an external agent, heating and energy transport via high-energy (often highly relativistic) particles that are diffusing/streaming through the thermal fluid.
   - High energy particles ionise atoms in fluid, excess energy put into freed e^-. High-energy electrons proceed to collide with atoms/ions, which ends up thermalising the energy as heat in fluid.
     $$\begin{aligned} &\text{ionisation rate per unit volume} \propto \text{CR flux} \times \rho \\ \implies \quad &\dot{Q}_{\rm cool} \propto \text{CR flux. (independent of } \rho) \end{aligned} \tag{4.45}$$

Combining these cases, we can parametrise $\dot{Q}_{\rm cool}$ as:

$$\dot{Q}_{\rm cool} = \underbrace{A\rho T^{\alpha}}_{\substack{\text{radiative}\\ \text{cooling}}} - \underbrace{H}_{\text{CR heating}}, \tag{4.46}$$

where α depends upon the physics of the dominant radiative cooling process.

4.4 Energy Transport Processes

Transport processes move energy through the fluid. Important examples are:

1. *Thermal conduction*: transport of thermal energy down temperature gradients due to diffusion of the hot e^- into cooler regions. Relevant in, for example
   - Interiors of white dwarfs;
   - Supernova shock fronts;
   - ICM plasma.

   There is also thermal conduction associated with ions, but it is smaller than the electron thermal conduction by a factor of $\sqrt{m_{\rm ion}/m_e} \sim 43$.

   The energy flux per unit area is

   $$\mathbf{F}_{\rm cond} = -\kappa \boldsymbol{\nabla} T, \tag{4.47}$$

   where κ is thermal conductivity (computed from kinetic theory).

   The local rate of change of E per unit volume is

   $$-\boldsymbol{\nabla} \cdot \mathbf{F}_{\rm cond} = \kappa \nabla^2 T. \tag{4.48}$$

2. *Convection*: transport of energy due to fluctuating or circulating fluid flows in presence of entropy gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.

3. *Radiation transport*: Transport of energy through system due to radiation, relevant in optically-thick systems (mean free path of photon much shorter than size of system).

   If scattering opacity dominates, then we have radiative diffusion. If $\epsilon_{\rm rad}$ is the energy density of the radiation field, the radiative flux through the fluid is

   $$\mathbf{F}_{\rm rad} \propto -\boldsymbol{\nabla} \epsilon_{\rm rad}. \tag{4.49}$$

   The general topic of radiation transport through a fluid flow is very complex and beyond the scope of this course. Important in stellar interiors, supernova explosions, and black hole accretion disks.

Chapter 5

Hydrostatic Equilibrium, Atmospheres and Stars

We now have the full set of equations describing the dynamics of an ideal (inviscid, dilute, unmagnetised) non-relativistic fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \qquad \text{Continuity equation (2.12)} \tag{5.1}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \qquad \text{Momentum equation (2.57)} \tag{5.2}$$

$$\nabla^2 \Psi = 4\pi G \rho \qquad \text{Poisson's equation (3.12)} \tag{5.3}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho \frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho \dot{Q}_{\text{cool}} \qquad \text{Energy equation (4.42)} \tag{5.4}$$

$$E = \rho \left(\frac{1}{2} u^2 + \Psi + \mathcal{E} \right) \qquad \text{Definition of total energy (4.35)} \tag{5.5}$$

$$p = \frac{k_B}{\mu m_p} \rho T \qquad \text{EoS for ideal gas (4.1)} \tag{5.6}$$

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho} \qquad \text{Internal energy for ideal gas (6.80)} \tag{5.7}$$

We proceed to use those equations to explore astrophysically relevant situations.

This chapter starts with the simplest, but important, case – fluid systems that are in static equilibrium with pressure forces balancing gravity.

5.1 Hydrostatic Equilibrium

Definition 5.1 Hydrostatic Equilibrium
A fluid system is in a state of hydrostatic equilibrium if

$$\mathbf{u} = \mathbf{0}, \qquad \frac{\partial}{\partial t} = 0. \tag{5.8}$$

Then, the continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{5.9}$$

The momentum equation gives

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi = 0, \tag{5.10}$$

resulting in the *equation of hydrostatic equilibrium*

Equation 5.1 Equation of Hydrostatic Equilibrium

$$\frac{1}{\rho}\boldsymbol{\nabla}p = -\boldsymbol{\nabla}\Psi \tag{5.11}$$

Assuming a barotropic equation of state $p = p(\rho)$, this system of equations can be solved.

Example 5.1 Maximum Mass of Incompressible Planet
A planet is composed of a material that is incompressible, density ρ, at pressures $\leq p_0$. Show that the maximum mass of such a planet that is incompressible throughout its interior is given by

$$M_{\max} = \frac{2}{3\rho^2}\sqrt{\frac{1}{2\pi}\left(\frac{3p_0}{G}\right)^3} \tag{5.12}$$

Solution The gravitational acceleration inside the planet, assuming spherical symmetry, is given by $\mathbf{g} = -\frac{GM(r)}{r^2}\hat{\mathbf{r}}$ (3.14)where $M(r)$ is the mass enclosed within a radius r. Under the assumption of constant density ρ, the enclosed mass is $M(r) = \frac{4}{3}\pi r^3\rho$. Substituting this into the expression for gravitational acceleration, and using the equation of hydrostatic equilibrium (5.11), we arrive at the differential equation governing the pressure distribution within the planet:

$$\mathbf{g} = -\frac{4}{3}\pi G\rho r\hat{\mathbf{r}} \quad \Longrightarrow \quad \boldsymbol{\nabla}p = -\frac{4}{3}\pi G\rho^2 r\hat{\mathbf{r}}. \tag{5.13}$$

Since this expression is negative, it confirms that the pressure decreases monotonically with increasing radius. Therefore, the pressure must reach its maximum value at the centre, which we denote as p_0. The outer boundary of the planet, $r_{\max}$, is defined as the point where the pressure vanishes. To determine $r_{\max}$, we integrate the hydrostatic equation from $r = 0$ (where $p = p_0$) to $r = r_{\max}$ (where $p = 0$):

$$\begin{aligned}\int_{p_0}^{0} \mathrm{d}p &= -\frac{4}{3}\pi G\rho^2 \int_0^{r_{\max}} r\,\mathrm{d}r \\ \Longrightarrow \qquad p_0 &= \frac{2}{3}\pi G\rho^2 r_{\max}^2. \end{aligned} \tag{5.14}$$

Thus the maximum radius of such a planet is given by $r_{\max} = \sqrt{3p_0/(2\pi G\rho^2)}$. To find the maximum mass $M_{\max}$, we substitute $r_{\max}$ into the spherical mass formula:

$$\begin{aligned} M_{\max} &= \frac{4}{3}\pi\rho r_{\max}^3 \\ &= \frac{4}{3}\pi\rho\left(\frac{1}{\rho^2}\frac{1}{2\pi}\frac{3p_0}{G}\right)^{3/2} \\ &= \frac{2}{3\rho^2}\sqrt{\frac{1}{2\pi}\left(\frac{3p_0}{G}\right)^3}. \end{aligned} \tag{5.15}$$

This result provides an upper bound on the mass of a self-gravitating sphere of constant density in hydrostatic equilibrium. ◀

5.1.1 Isothermal Atmosphere with Constant (Externally Imposed) g

Suppose $\mathbf{g} = -g\hat{\mathbf{z}}$. Then the equation of hydrostatic equilibrium with an isothermal equation of state reads

$$\text{Isothermal} \quad \implies \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \implies \quad p = A\rho, \quad A = \text{const.}$$

$$A\frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\Psi = -g\hat{\mathbf{z}}$$

$$\implies \quad \ln\rho = -\frac{gz}{A} + \text{const.}$$

$$\therefore \rho = \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T}z\right), \tag{5.16}$$

i.e. exponential atmosphere.

Examples of this is the Earth's atmosphere: $T \sim 300\,\text{K}$ and $\mu \sim 28 \implies$ e-folding $\sim 9\,\text{km}$. The highest astronomical observatories are at $z \sim 4\,\text{km}$, so have ρ and $p \sim 60\%$ of sea level.

Example 5.2 Atmospheric Density Variation and Disturbances

(a) If the Earth's atmosphere can be approximated by a perfect static gas at constant temperature of 300 K subject to a uniform gravitational field, find the variation in number density (molecules per cubic metre) with height above the Earth's surface. If the number density of molecules at the Earth's surface is $3 \times 10^{25}\ \text{m}^{-3}$, estimate the height above the Earth where the fluid approximation breaks down, and compare it with the height at which the assumption of constant gravity breaks down.

(b) The Earth runs in to a cloud (made of hydrogen) which is stationary with respect to the Sun. Estimate the number density the cloud would have to have in order that it seriously disturbed the Earth's atmosphere. (*Hint: You might take "seriously disturb" to mean that the ram pressure is comparable with the atmosphere pressure.*)

(Some data possibly relevant to this question: $T \sim 300$ K, $\mu \sim 30$, $k_\text{B} = 1.381 \times 10^{-23}\ \text{JK}^{-1}$, $m_\text{p} = 1.673 \times 10^{-27}$ kg, $g = 10\ \text{ms}^{-2}$, $M_\odot = 2 \times 10^{30}$ kg, collisional cross-section $\sigma \sim 2 \times 10^{-19}\ \text{m}^2$.)

Solution For hydrostatic equilibrium, the equation of state derived in Subsection

5.1.1 gives

$$\rho = \rho e^{-z/H} \quad \text{where} \quad H = \frac{\mathcal{R}_* T}{\mu g}. \tag{5.17}$$

For an ideal gas, $p = \frac{\mathcal{R}_*}{\mu}\rho T$ (4.6), and the number density is related to the density as $n\mu m_\mathrm{p} = \rho$, so

$$n = \frac{\rho_0}{\mu m_\mathrm{p}} e^{-z/H} = n_0 e^{-z/H}, \tag{5.18}$$

for a constant n_0. At the surface of the Earth, $n(z = R) = 3 \times 10^{25}\ \mathrm{m}^{-3}$, and so the constant becomes $n_0 = 3 \times 10^{25} e^{R/H}\ \mathrm{m}^{-3}$.

Consider the number of particles within a region of the natural length scale H defined by Eq. (5.17), $nH^3 = n_0 H^3 e^{-z/H}$. The fluid approximation breaks down when it can no longer be considered in the continuum limit with $nl^3_\mathrm{region} \sim 1$ (1.2), so

$$\begin{aligned} n_0 H^3 e^{-z/H} \sim 1 \quad &\Longrightarrow \quad z = H \ln\left(n_0 H^3\right) \\ \Longrightarrow \quad z \sim H\ln 3 &+ 25H \ln 10 + R + 3H \ln H, \end{aligned} \tag{5.19}$$

which gives the height above the surface at which the breakdown occurs as $z - R = 700$ km, using the data given to us in the example and assuming the gravitational acceleration remains constant.

But is that last assumption justified? We shall now look for the height at which the constant gravity assumption breaks down and compare the two. g is given as $10\ \mathrm{ms}^{-1}$, so one way to look at the breakdown could be the height at which to 1 sf. it changes to $9\ \mathrm{ms}^{-1}$. From the value at the surface,

$$g_R = 10\ \mathrm{ms}^{-1} = \frac{GM}{R^2} \quad \Longrightarrow \quad GM = R^2 \cdot 10\ \mathrm{ms}^{-1}, \tag{5.20}$$

and hence for the height at which it changes to $9\ \mathrm{ms}^{-1}$,

$$9\ \mathrm{ms}^{-1} = \frac{R^2}{r^2} 10\ \mathrm{ms}^{-1} \quad \Longrightarrow \quad r - R = \left(\frac{\sqrt{10}}{3} - 1\right) R, \tag{5.21}$$

giving $r - R \sim 350$ km. Although these are estimates, it is clear that the breakdown of the constant gravity assumption occurs before the fluid approximation is no longer valid.

From Eq. (2.60), the contribution to the stress tensor due to bulk flow – the *ram pressure* – has magnitude $\sim \rho u^2$. If the Earth has an orbital period T, then its orbital velocity is $v = 2\pi R/T$, where R is the orbital radius. Equating the ram pressure of the H_2 gas cloud to the Earth's atmospheric pressure,

$$\begin{aligned} P = \rho u^2 &= \rho \frac{4\pi^2}{T^2} R^2 \\ &= n m_\mathrm{p} \frac{8\pi^2}{T^2} R^2, \end{aligned} \tag{5.22}$$

and so we estimate the critical number density as

$$n = \frac{PT^2}{8\pi^2 R^2 m_\mathrm{p}} \approx 1.7 \times 10^{21}\ \mathrm{m}^{-3}. \tag{5.23}$$

◀

5.1.2 Vertical Density Structure of an Isothermal, Rotationally-Supported, Geometrically-Thin Gas Disk Orbiting a Central Mass

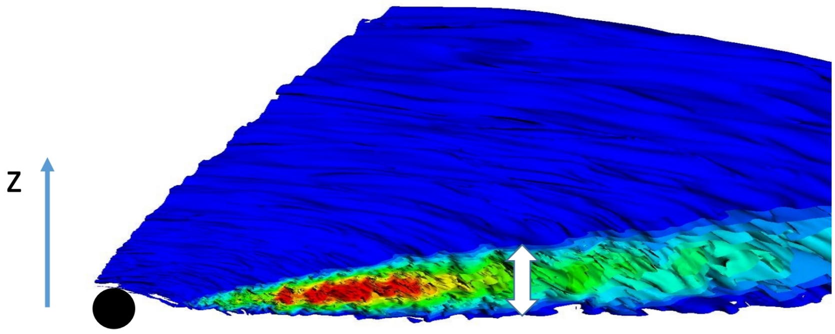

Fig. 5.1: Vertical density structure of an isothermal, rotationally- supported, geometrically-thin gas accretion disk orbiting a central mass.

At a given patch of the disk, transform into a locally co-moving and co-rotating frame. In z-direction, pressure forces balance z-component of gravity,

$$g_z \approx -\frac{GM}{r^2}\frac{z}{r} \approx -\frac{GMz}{R^3}. \tag{5.24}$$

So, hydrostatic equilibrium gives,

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial z} &= g_z \\ \implies \quad A\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= -\frac{GM}{R^3}z \\ \implies \quad A\ln\rho &= -\frac{GM}{2R^3}z^2 + \text{const.} \\ \implies \quad \rho &= \rho_0 \exp\left(-\frac{GMz^2}{2R^3A}\right) \\ \therefore \rho = \rho_0 \exp\left(-\frac{\Omega^2}{2A}z^2\right) &\quad \text{where} \quad \Omega^2 = \frac{GM}{R^3} \end{aligned} \tag{5.25}$$

Example 5.3 Ring of Isothermal Fluid Orbiting a Star
An equilibrium ring of isothermal fluid orbits a star with mass M_* at radius

R. In the plane of the ring, mechanical equilibrium results from a balance of centrifugal force and the gravitational force of the central object; normal to the ring (ie. vertically) equilibrium is between the vertical component of the gravitational force of the central object and vertical pressure gradients in the ring gas.
Show that in the limit that the ring thickness $H \ll R$, the vertical density stratification in the ring is a Gaussian and determine its e-folding length in terms of the gas temperature and the angular velocity at the ring, Ω. Hence determine an upper limit to the temperature such that the ring is thin ($H \ll R$) and calculate this temperature if the ring's radius is that of the Earth's orbit around the Sun.
[*Hint: In the limit of $H \ll R$ you can find an approximate expression for gravitational acceleration.*]

Solution In the rest frame of a unit mass fluid element in the ring, mechanical equilibrium in radial direction requires a balance between the centrifugal acceleration $\Omega^2 R$ and the gravitational acceleration GM/R^2, leading to the condition

$$\Omega^2 R = \frac{GM}{R^2} \qquad \Longrightarrow \qquad \Omega^2 = \frac{GM}{R^3}. \tag{5.26}$$

For vertical equilibrium, we consider the gravitational tidal force acting on an element at height z above the midplane. In the limit where the ring is thin ($H \ll R$), the vertical component of gravity can be approximated as

$$\mathbf{g}_{\mathrm{T}} = \mathbf{g}(z)\sin\theta \quad \text{where} \quad \sin\theta \approx \frac{z}{R}. \tag{5.27}$$

Using the gravitational acceleration at displacement $R\hat{\mathbf{r}} + z\hat{\mathbf{z}}$ from the central mass,

$$g(z) = \frac{GM}{R^2 + z^2}, \tag{5.28}$$

expanding to first order in z/R,

$$\mathbf{g}_{\mathrm{T}} \approx -\frac{GMz}{R^3}\hat{\mathbf{z}}. \tag{5.29}$$

From the equation of hydrostatic equilibrium (5.11) $\boldsymbol{\nabla} p = \rho\mathbf{g}$, and the chain rule,

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{GM}{R^3}\rho z, \tag{5.30}$$

and using the equation of state for an isothermal ideal gas (4.1) $p = \frac{\mathcal{R}_*}{\mu}\rho T$, we can rewrite the equilibrium equation in terms of just ρ and z

$$\frac{\mathcal{R}_*}{\mu}T\,\mathrm{d}(\ln\rho) = -\frac{GM}{R^3}\,\mathrm{d}\!\left(\tfrac{1}{2}z^2\right) \implies \quad \rho = \rho_0 \exp\left(-\frac{GM\mu z^2}{2\mathcal{R}_* T R^3}\right),$$

the density profile we obtained in Subsection 5.1.2 Eq. (5.25), which can be expressed as

$$\rho = \rho_0 \exp\left(-\frac{\Omega^2\mu}{2\mathcal{R}_* T}z^2\right), \tag{5.31}$$

comparing to the standard gaussian formula distribution $\propto \exp\left(-x^2/2\sigma^2\right)$, we obtain the characteristic vertical scale height $\sigma = \sqrt{\mathcal{R}_* T/\Omega^2\mu}$, and so the constraint $H \ll R$ in terms of the temeprature becomes

$$T \ll \frac{\Omega^2 R^2 \mu}{\mathcal{R}_*}. \tag{5.32}$$

For a ring at Earth's orbital radius ($R \approx 1.5 \times 10^{11}$ m) about the Sun ($M_\odot \approx 2.0 \times 10^{30}$ kg), we obtain that for the thin-ring approximation to be valid, the temperature must be significantly lower than $\sim 10^5$ K. ◀

If a system is self-gravitating (rather than having an externally imposed gravitational field), we also have

$$\nabla^2 \Psi = 4\pi G \rho. \tag{5.33}$$

This must be solved together with the equation of hydrostatic equilibrium (5.11) $\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi$.

Example 5.4 Self-Gravitating Fluid Slab
A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z| < a$. Find the gravitational field everywhere and the pressure distribution within the slab. [Hint: check the limits when integrating the pressure gradient.]
If a galactic disk is approximated by a uniform density slab with density 10^{-18} kg m^{-3} and $a = 10^{18}$ m, determine the velocity of a star at the midplane if it starts from rest at $z = a$, and the period of its oscillation.

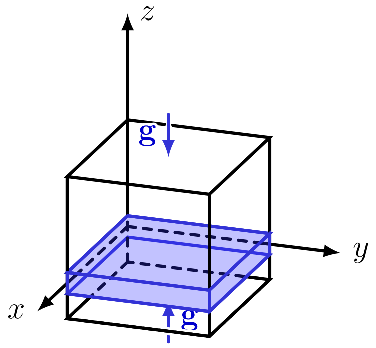

Fig. 5.2: A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z| < a$.

Solution Using Gauss' theorem and Eq. (3.12), we proceed very similarly to

Subsection 3.1.3,

$$\begin{aligned}\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{g}\,\mathrm{d}V &= -4\pi G\rho\int_{\mathcal{V}}\mathrm{d}V\\ \int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} &= -4\pi G\rho\int_{\mathcal{S}}\mathrm{d}S\int_{-z}^{z}\mathrm{d}z'\\ -2|\mathbf{g}|\int_{\mathcal{S}}\mathrm{d}S &= -4\pi G\rho\int_{\mathcal{S}}\mathrm{d}S\int_{-z}^{z}\mathrm{d}z'\\ |\mathbf{g}| &= 4\pi G\rho\int_{0}^{z}\mathrm{d}z'\,. \end{aligned}\tag{5.34}$$

We know $\mathbf{g}$ must lie parallel to $\hat{\mathbf{z}}$, and so there are only contributions from the constant z planes as $\mathbf{g}$ lies tangential to the other planes in Fig. 5.2. For $|z|\leq a$,

$$\begin{aligned}|\mathbf{g}| &= 4\pi G\rho z\\ \mathbf{g} &= -4\pi G\rho z\hat{\mathbf{z}}.\end{aligned}\tag{5.35}$$

For $|z|\geq a$,

$$|\mathbf{g}| = 4\pi G\rho a \qquad\Longrightarrow\qquad \mathbf{g} = \begin{cases}-4\pi G\rho a\hat{\mathbf{z}} & z\geq a\\ 4\pi G\rho a\hat{\mathbf{z}} & z\leq -a\end{cases}.\tag{5.36}$$

In hydrostatic equilibrium we can use Eqs. (5.11) and (3.1) to write

$$\boldsymbol{\nabla}p = -\rho\boldsymbol{\nabla}\Psi = \rho\mathbf{g}.\tag{5.37}$$

From the apparent symmetry of the situation in Fig. 5.2, $\boldsymbol{\nabla} = \partial/\partial z\,\hat{\mathbf{z}}$, and so we can find the pressure distribution as

$$\begin{aligned}\frac{\partial p}{\partial z} &= \rho\left(-4\pi G\rho z\right)\\ p &= -4\pi G\rho^2\int_{a}^{z} z'\,\mathrm{d}z' &&(5.38)\\ &= 2\pi G\rho^2\left(a^2-z^2\right) &&(5.39)\end{aligned}$$

Within the galactic disc, the motion can be approximated as simple harmonic oscillations with angular frequency Ω, since the gravitational acceleration is proportional to z,

$$\mathbf{g} = -4\pi G\rho z\hat{\mathbf{z}} \qquad\Longrightarrow\qquad \ddot{z} = -\underbrace{4\pi G\rho}_{\Omega^2}\,z,\tag{5.40}$$

which gives the time period of oscillation as $\sqrt{\pi/G\rho} = 6.9\times 10^6$ yr. To find the velocity at the midpoint, we notice that $\ddot{z} = \dot{z}\frac{\mathrm{d}\dot{z}}{\mathrm{d}z}$, so it follows

$$\begin{aligned}\dot{z}\,\mathrm{d}\dot{z} &= -4\pi G\rho z\,\mathrm{d}z\\ \int_0^v \dot{z}\,\mathrm{d}\dot{z} &= -4\pi G\rho\int_a^0 z\,\mathrm{d}z\\ \Longrightarrow\qquad v &= 2a\sqrt{\pi G\rho} = 29\ \mathrm{km\ s^{-1}}.\end{aligned}\tag{5.41}$$

◀

5.1.3 Isothermal Self-Gravitating Slab

Consider a static, isothermal slab in x and y which is symmetric about $z = 0$ (e.g. two clouds collide and generate a shocked slab of gas between them as in Example 6.3).

$$\text{Isothermal} \quad \Longrightarrow \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \Longrightarrow \quad p = A\rho, \quad A = \text{const.}$$

also, $\boldsymbol{\nabla} = \partial/\partial z$ due to symmetry, $p = p(z)$, $\Psi = \Psi(z)$.

Then the equation of hydrostatic equilibrium (5.11) becomes

$$\begin{aligned} A\frac{1}{\rho}\boldsymbol{\nabla}\rho &= -\boldsymbol{\nabla}\Psi \\ \Longrightarrow \quad A\frac{\mathrm{d}}{\mathrm{d}z}(\ln\rho) &= -\frac{\mathrm{d}\Psi}{\mathrm{d}z} \\ \Longrightarrow \quad \Psi = -A\ln(\rho/\rho_0) + \Psi_0 \quad &\text{where} \quad \rho_0 = \rho(z=0) \\ \therefore \rho &= \rho_0 e^{-(\Psi-\Psi_0)/A}. \end{aligned} \tag{5.42}$$

Since $A \propto T$, we note that this last equation has the form of a Boltzmann distribution.

Poisson's equation is

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}. \tag{5.43}$$

Let's change variables to $\chi = -(\Psi - \Psi_0)/A$, $Z = z\sqrt{2\pi G\rho_0/A}$ so that Poisson's equation becomes

$$\begin{aligned} \frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} = -2e^{\chi} \quad \text{with} \quad \chi = \frac{\mathrm{d}\chi}{\mathrm{d}Z} &= 0 \quad \text{at} \quad Z = 0 \\ \Longrightarrow \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z}\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} &= -2\frac{\mathrm{d}\chi}{\mathrm{d}Z}e^{\chi} \\ \Longrightarrow \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2\right] &= -2\frac{\mathrm{d}}{\mathrm{d}Z}(e^{\chi}) \\ \Longrightarrow \quad \left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2 &= C_1 - 4e^{\chi}. \end{aligned} \tag{5.44}$$

But we have boundary condition $\mathrm{d}\chi/\mathrm{d}Z = 0$ when $\chi = 0 \implies C_1 = 4$.

$$\therefore \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 2\sqrt{1-e^{\chi}} \quad \Longrightarrow \quad \int \frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} = 2\int \mathrm{d}Z\,. \tag{5.45}$$

Change variables $e^{\chi} = \sin^2\theta$

$$\Longrightarrow \quad e^{\chi}\,\mathrm{d}\chi = 2\sin\theta\cos\theta\,\mathrm{d}\theta \quad \text{or} \quad \mathrm{d}\chi = \frac{2\cos\theta}{\sin\theta}\,\mathrm{d}\theta\,. \tag{5.46}$$

So, we can evaluate the χ integral

$$\begin{aligned}\int \frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} &= \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \\ &= \int \frac{2\,\mathrm{d}\theta}{\sin\theta} \\ &= \int 2\frac{1}{2}\frac{1+t^2}{t}\,\mathrm{d}\theta \\ &= 2\int\frac{\mathrm{d}t}{t} \\ &= 2\ln t + C_2, \end{aligned} \tag{5.47}$$

by setting

$$t = \tan\tfrac{\theta}{2} \quad\Longrightarrow\quad \mathrm{d}t = \tfrac{1}{2}(1+t^2)\,\mathrm{d}\theta\,, \tag{5.48}$$

and by noting

$$\sin\theta \equiv 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = \frac{2t}{1+t^2} = e^{\chi/2}. \tag{5.49}$$

So, Poisson's equation becomes

$$2\ln t = 2Z + C_2. \tag{5.50}$$

Now, $\chi = 0$ at $Z = 0 \implies \theta = \pi/2,\ t = 1 \implies C_2 = 0$, so $t = e^Z$

$$\Longrightarrow \quad \sin\theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}. \tag{5.51}$$

This gives

$$\boxed{\begin{aligned}\Psi - \Psi_0 &= 2A\ln\left[\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)\right] \quad (5.52)\\ \rho &= \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)} \quad (5.53)\end{aligned}}$$

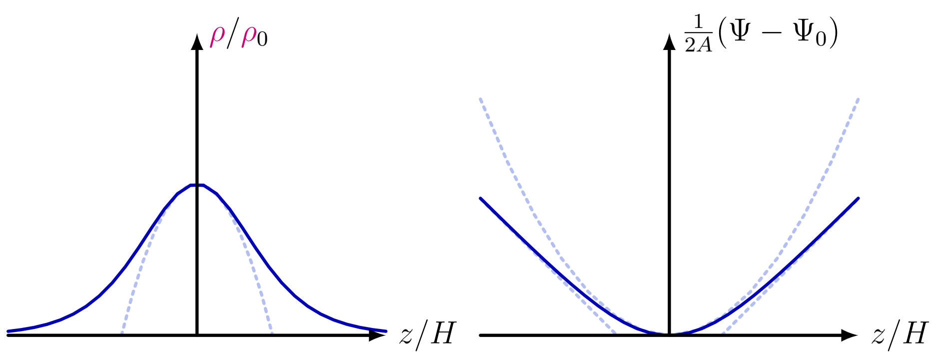

Fig. 5.3: Density distribution (5.52) and gravitational potential (5.53) for isothermal self-gravitating slab. Dashed lines show the limiting parabolic and linear behaviours.

Example 5.5 Density Distribution for Isothermal Slab
Sketch the density distribution for an isothermal slab and discuss the asymptotic limits $z \to 0$, $z \to \infty$.
A galactic disc can be well approximated in its vertical structure by an isothermal slab of gas, temperature T, central density ρ_0. If a star falls from rest from a height z_0, show that its vertical velocity at height z is given by

$$\dot{z}^2 = \frac{4k_{\mathrm{B}}T}{\mu m_{\mathrm{p}}} \ln\left[\frac{\cosh(z_0/H)}{\cosh(z/H)}\right]. \tag{5.54}$$

Solution The density distribution for an isothermal self-gravitating slab was derived above in Subsection 5.1.3 as

$$\rho = \rho_0 \operatorname{sech}^2(z/H) \quad \text{where} \quad \frac{1}{H} = \sqrt{\frac{2\pi G\rho_0}{A}}. \tag{5.55}$$

The density is parabolic about the origin as $\rho \to 1 - (z/H)^2$, and tends to zero as $z \to \infty$, which are both most easily seen from considering the behaviour of $\cosh x$ in the respective limits.

Let us briefly consider an additional derivation, we start by writing the equation of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\mathrm{d}\Psi}{\mathrm{d}z}. \tag{5.56}$$

Now set $\mathrm{d}m = \rho\,\mathrm{d}z$ and use $p = c_s^2\rho$ (see Eq. (6.17) in Subsection 6.1.1.1 later) to get

$$c_s^2\frac{\mathrm{d}\rho}{\mathrm{d}m} = -\frac{\mathrm{d}\Psi}{\mathrm{d}z} \quad\Longrightarrow\quad c_s^2\frac{\mathrm{d}^2\rho}{\mathrm{d}m^2} = -\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2}\frac{\mathrm{d}z}{\mathrm{d}m} = -4\pi G, \tag{5.57}$$

using on the last equality Poisson's equation for gravitation (3.12). This allows us to integrate to find the form of $\rho(m)$. Applying the boundary condition $m = 0$ at $z = 0$ we get

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}z} = \frac{2\pi G}{c_s^2}\left(M^2 - m^2\right), \tag{5.58}$$

where M is the total mass of the (half) slab. This has solution, through a standard integral, $m = M\tanh\left(2\pi GMz/v_s^2\right)$, and we're there, since $M = \sqrt{c_s^2\rho_0/2\pi G}$.

From conservation of the energy per unit mass, $\Psi_0 = \Psi + \frac{1}{2}\dot{z}^2$, it follows that

$$\begin{aligned}\dot{z}^2 &= 2(\Psi_0 - \Psi)\\ &= 4A\Big(\ln[\cosh(z_0/H)] - \ln[\cosh(z/H)]\Big)\\ &= \frac{4k_{\mathrm{B}}T}{\mu m_{\mathrm{p}}}\ln\left[\frac{\cosh(z_0/H)}{\cosh(z/H)}\right].\end{aligned} \tag{5.59}$$

◀

5.2 Stars as Self-Gravitating Polytropes

Polytropes are useful as they provide simple solutions (albeit in some cases via numerical integration) for the internal structure of a star that can be tabulated and used for estimates of various quantities. They are much simpler to manipulate than the full rigorous solutions of *all* the equations of stellar structure. But the price of this simplicity is assuming a power law relationship between pressure and density which must hold (including a fixed constant) throughout the star.

Consider a spherically-symmetric self-gravitating system in hydrostatic equilibrium; from now on we will refer to this as a "star". We have

$$\begin{aligned} \boldsymbol{\nabla} p &= -\rho \boldsymbol{\nabla} \Psi \\ \implies \quad \frac{\mathrm{d}p}{\mathrm{d}r} &= -\rho \frac{\mathrm{d}\Psi}{\mathrm{d}r}. \quad \text{(spherical polar)} \end{aligned} \tag{5.60}$$

Now, $\rho > 0$ within a star which implies p is a monotonic function of Ψ. Also

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad \implies \quad \rho = -\frac{\mathrm{d}p}{\mathrm{d}\Psi}. \tag{5.61}$$

So ρ is a monotonic function of Ψ,

$$\therefore p = p(\Psi) \quad \text{and} \quad \rho = \rho(\Psi) \quad \implies \quad p = p(\rho), \tag{5.62}$$

i.e. non-rotating stars are barotropes!

Definition 5.2 Polytrope
A barotropic equation of state can be written as

$$p = K\rho^{1+1/n}, \tag{5.63}$$

where in general $n = n(\rho)$. When the parameter $n =$ constant, known as the polytropic index, we say that we have a *polytropic* EoS and the structure is called a **polytrope**.

This is purely for convenience, and in the knowledge that over a limited range one can always fit the p–ρ relation as a power law. In fact it turns out to be a resonably good approximation to use a single power law (i.e. single n) throughout the entire interior of some stars, so that a polytropic approach is a reasonable one. It is important to note that in general the adiabatic constant γ wis not equal to the polytropic power law $1 + \frac{1}{n}$ in general,

$$1 + \frac{1}{n} \neq \gamma. \tag{5.64}$$

We only have $1 + 1/n = \gamma$ (i.e. $p \propto \rho^{\gamma}$ (4.17)) if the star is isentropic (constant entropy throughout) due to, for example, mixing by convective motions throughout. In general a polytropic star is not isentropic but just has a barotropic equation of state that can be approximated as a power law.

Example 5.6 Internal Energy Scaling of a Polytrope
Show that if

$$\psi = -\frac{GM_s}{(r^2 + b^2)^{1/2}} \tag{5.65}$$

is the gravitational potential for a spherical distribution of matter then its density $\rho \propto \psi^5$.
Deduce the pressure and hence show that the equation of state is polytropic with $n = 5$.
Find the total internal energy U as a function of K, M_s and b where $K = P\rho^{-6/5}$ thus showing that it scales as $KM_s^{6/5}b^{-3/5}$.

Solution Poisson's law for gravitation (3.12) states $\nabla^2\Psi = 4\pi G\rho$, and for a spherically symmetric potential, the laplacian reduces to $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$, we can solve for the density as

$$4\pi G\rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right). \tag{5.66}$$

We can evaluate the first derivative as

$$\frac{\partial\Psi}{\partial r} = \frac{GM_s r}{\left(r^2 + b^2\right)^{3/2}}, \tag{5.67}$$

and evaluate

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{GM_s r^3}{\left(r^2 + b^2\right)^{3/2}}\right) &= \frac{3GM_s r^2}{\left(r^2 + b^2\right)^{3/2}} - \frac{3GM_s r^4}{\left(r^2 + b^2\right)^{5/2}} \\ &= \frac{3GM_s b^2 r^2}{\left(r^2 + b^2\right)^{5/2}}.\end{aligned} \tag{5.68}$$

Substituting into Poisson's equation,

$$\begin{aligned}4\pi G\rho &= \frac{3GM_s b^2}{\left(r^2 + b^2\right)^{5/2}} \\ &= -\frac{3b^2}{G^4M_s^4}\left(-\frac{G^5M_s^5}{\left(r^2 + b^2\right)^{5/2}}\right),\end{aligned} \tag{5.69}$$

thus the density distribution is

$$\rho = -\frac{3}{4\pi}\frac{b^2}{G^5M_s^4}\Psi^5 \qquad \left(= \frac{3}{4\pi}b^2\frac{M_s}{\left(r^2 + b^2\right)^{5/2}}\right), \tag{5.70}$$

where we used the form of Ψ to express the density in terms of the potential to see the relation $\rho \propto \psi^5$. We have already found $\boldsymbol{\nabla}\Psi$ in the hydrostatic equilibrium, so we can write the pressure gradient as

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}r} &= \frac{3}{4\pi}b^2\frac{M_s}{\left(r^2 + b^2\right)^{5/2}} \cdot \frac{GM_s r}{\left(r^2 + b^2\right)^{3/2}} \\ &= -\frac{3GM_s^2b^2}{4\pi}\frac{r^2}{\left(r^2 + b^2\right)^4}.\end{aligned} \tag{5.71}$$

Integrating this can get the pressure itself, using the boundary condition that $p = 0$ as $r \to \infty$,

$$\int_0^p \mathrm{d}p' = -\frac{3GM_s^2b^2}{4\pi}\int_\infty^r \frac{r'}{(r'^2+b^2)^4}\,\mathrm{d}r'$$

$$p = \frac{GM_s^2b^2}{8\pi}\frac{1}{(r^2+b^2)^3} \tag{5.72}$$

$$= \underbrace{\frac{G}{8\pi}b^{-2/5}M_s^{4/5}\left(\frac{4\pi}{3}\right)^{6/5}}_{K}\rho^{6/5} \quad \text{given} \quad (5.70) \tag{5.73}$$

So we can see that this is a polytropic equation of state ($p = K\rho^{1+1/n}$), and comparing we find $n = 5$ and the entropy dependent constant K.

We have the internal energy per unit mass as (6.80) $\mathcal{E} = \frac{1}{\gamma-1}\frac{p}{\rho}$, so the internal energy for an infinitesimal volume is

$$\mathrm{d}U = \frac{1}{\gamma-1}p\,\mathrm{d}V\,, \tag{5.74}$$

using the spherical symmetry gives the internal energy contribution from a spherical shell at radius r to $r + \mathrm{d}r$ as

$$\mathrm{d}U = \frac{1}{\gamma-1}4\pi r^2 p\,\mathrm{d}r\,, \tag{5.75}$$

and thus for the total internal energy, we integrate over all space

$$U = \frac{1}{2}\frac{1}{\gamma-1}GM_s^2b^2\int_0^\infty \frac{r^2}{(r^2+b^2)^3}\,\mathrm{d}r\,. \tag{5.76}$$

The integral on the right evaluates as $\pi/16b^3$,[1] therefore

$$U = \frac{\pi}{32}\frac{1}{\gamma-1}\frac{GM_s^2}{b}. \tag{5.79}$$

[1] We are motivated to begin by integration by parts as we can tell the boundary term will vanish, giving

$$\int_0^\infty \frac{r^2}{(r^2+b^2)^3}\,\mathrm{d}r = \frac{1}{4}\int_0^\infty \frac{1}{(r^2+b^2)^2}\,\mathrm{d}r\,, \tag{5.77}$$

this may not look like progress but this leaves the integrand ripe for another substitution, $r = b\tan u$,

$$\int_0^\infty \frac{r^2}{(r^2+b^2)^3}\,\mathrm{d}r = \frac{1}{4}\int_0^{\pi/2}\frac{1}{b^4}\frac{1}{(1+\tan^2 u)^2}b\sec^2 u\,\mathrm{d}u$$

$$= \frac{1}{4b^3}\underbrace{\int_0^{\pi/2}\cos^2 u\,\mathrm{d}u}_{\pi/4} = \frac{\pi}{16b^3}, \tag{5.78}$$

where we quoted the result of a simple standard integral.

If we wish, we can rearrange the expression for K to show

$$\frac{GM_s^2}{b} = 8\pi\left(\frac{3}{4\pi}\right)^{6/5} KM_s^{6/5}b^{-3/5}, \tag{5.80}$$

such to rewrite the internal energy as

$$U = \frac{1}{\gamma-1}\frac{\pi^2}{4}\left(\frac{3}{4\pi}\right)^{6/5} KM_s^{6/5}b^{-3/5}. \tag{5.81}$$

Note the index $n = 5$ is no indication of the internal energy content $u = p/(\gamma-1)$. Indeed, if $\gamma = 6/5$, a polytrope is gravitationally unstable, the critical value being $4/3$. As a model for a radiative star, $n = 5$ is quite acceptable, and convectively stable (see later **??**). ◀

Assuming a polytropic EoS, the equation of hydrostatic equilibrium gives

$$\begin{aligned} -\boldsymbol{\nabla}\Psi = \frac{1}{\rho}\boldsymbol{\nabla}\left(K\rho^{1+1/n}\right) &= \frac{n+1}{n}K\rho^{1/n-1}\boldsymbol{\nabla}\rho \\ &= (n+1)\boldsymbol{\nabla}\left(K\rho^{1/n}\right) \\ \implies \qquad \rho &= \left(\frac{\Psi_T-\Psi}{(n+1)K}\right)^n, \end{aligned} \tag{5.82}$$

defining $\Psi_T \equiv \Psi$, where $\rho = 0$ on the surface. If the central density is ρ_c and central potential is Ψ_c, we have

$$\rho_c = \left(\frac{\Psi_T-\Psi_c}{(n+1)K}\right)^n, \tag{5.83}$$

so we can write,

$$\rho = \rho_c\left(\frac{\Psi_T-\Psi}{\Psi_T-\Psi_c}\right)^n. \tag{5.84}$$

Feeding this into Poisson's equation gives

$$\nabla^2\Psi = 4\pi G\rho_c\Bigg(\underbrace{\frac{\Psi_T-\Psi}{\Psi_T-\Psi_c}}_{\theta}\Bigg)^n, \tag{5.85}$$

defining a remapped potential coordinate $\theta = (\Psi_T-\Psi)/(\Psi_T-\Psi_c)$, and we then get

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T-\Psi_c}\theta^n. \tag{5.86}$$

Imposing spherical symmetry and writing in spherical polars, this becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\frac{4\pi G\rho_c}{\Psi_T-\Psi_c}\theta^n. \tag{5.87}$$

Defining a scaled radial coordinate $\xi = r\sqrt{4\pi G\rho_c/(\Psi_T-\Psi_c)}$, we finally get the *Lane-Emden equation of index n*,

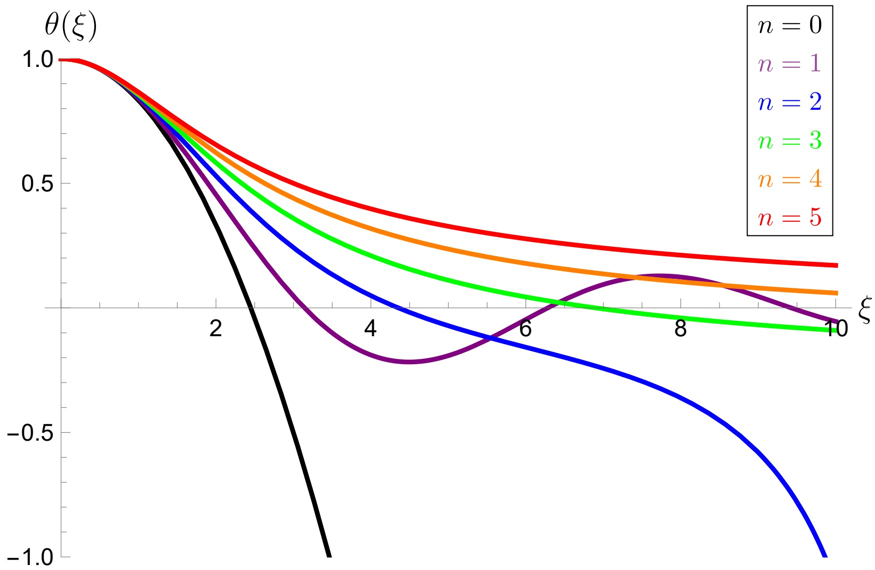

Fig. 5.4: Solutions $\theta(\xi)$ to the Lane-Emden equation for various values of n.

Equation 5.2 Lane-Emden Equation of Index n

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \tag{5.88}$$

The appropriate boundary conditions for the Lane-Emden equation require that at the centre of the star where $\xi = 0$, $\theta(0)$ must be one. Furthermore, since $\mathrm{d}p/\mathrm{d}r$ approaches 0 as $r \to 0$, we need $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. (Zero force at $\xi = 0$, enclosed mass $\to 0$ as $\xi \to 0$. It is not true if there is a point mass in the middle of the sphere) The outer boundary (the surface) is the first location where $\rho = 0$, or equivalently $\theta(\xi) = 0$. That location is called ξ_1. The formal solution may have additional zeros at larger values of ξ, but $\xi > \xi_1$ is not relevant for stellar models.

The Lane-Emden equation can be solved analytically for $n = 0$, 1 and 5; otherwise solve numerically. For $n = 5$, the first zero of $\theta(\xi)$, which is proportional to the radius of the polytrope, occurs at infinity. For $n > 5$, the binding energy is positive, and hence such a polytrope cannot represent a real star.

For all other polytrope indices n, a numerical solution to the Lane-Emden equation must be calculated. A display of solutions for several values of n between 0 to 6 is given in Fig. 5.4. Note that the radius of the star is defined by the first zero in the solution, and the solution at larger values of ξ is not relevant for computing stellar models.

5.2.1 Solution for $n = 0$

This is a somewhat singular case, physically corresponding to a fluid that is at constant uniform density and incompressible.

$$\begin{aligned} \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\theta^n = -1 \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\xi^2 \\ \implies \quad \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} &= -\frac{1}{3}\xi^3 - C \\ \therefore \theta &= -\frac{\xi^2}{6} + \frac{C}{\xi} + D. \end{aligned} \tag{5.89}$$

We need $\theta = 1$ at $\xi = 0 \implies C = 0, D = 1$ and hence

$$\theta = 1 - \frac{\xi^2}{6}. \tag{5.90}$$

5.2.2 Isothermal Spheres (Case $n \to \infty$)

The isothermal case $p = K\rho$ corresponds to $n \to \infty$). Let's combine Eqs. (5.60) and (5.63),

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad &\text{and} \quad p = K\rho \\ \implies \quad \frac{\mathrm{d}\Psi}{\mathrm{d}r} &= -\frac{K}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} \\ \implies \quad \Psi - \Psi_c &= -K\ln(\rho/\rho_c). \end{aligned} \tag{5.91}$$

From Poisson's equation

$$\begin{aligned} \nabla^2\Psi &= 4\pi G\rho \\ \implies \quad \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= 4\pi G\rho \\ \implies \quad \frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) &= -4\pi G\rho. \end{aligned} \tag{5.92}$$

Let $\rho = \rho_c e^{-\psi}$ (defining $\psi = \Psi/K$, and $\Psi_c = 0$), we set

$$r = a\xi, \quad a = \sqrt{\frac{K}{4\pi G\rho_c}}, \tag{5.93}$$

then,

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) = e^{-\psi}, \tag{5.94}$$

with $\psi = \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0$ at $\xi = 0$.

This replaces the Lane-Emden equation in the case where the system is isothermal.

At large radii, this has solutions of the form $\rho \propto r^{-2}$ (see Fig. 5.5), so the enclosed mass $\propto r$. Thus, the mass of an isothermal sphere of self-gravitating gas tends to ∞ as the radius tends to ∞. This is why we cannot adopt our usual convention of defining $\Psi = 0$ at ∞.

So, to be physical (finite total mass), isothermal spheres need to be truncated at some finite radius. There needs to be some confining pressure by an external medium. These are called **Bonnor-Ebert spheres**, whose density profile depends on ξ_{cut}. E.g. dense gas cores in molecular clouds are well fitted by such Bonnor-Ebert spheres. Stability requires $\rho_c/\rho_{\mathrm{ext}} < 14$.

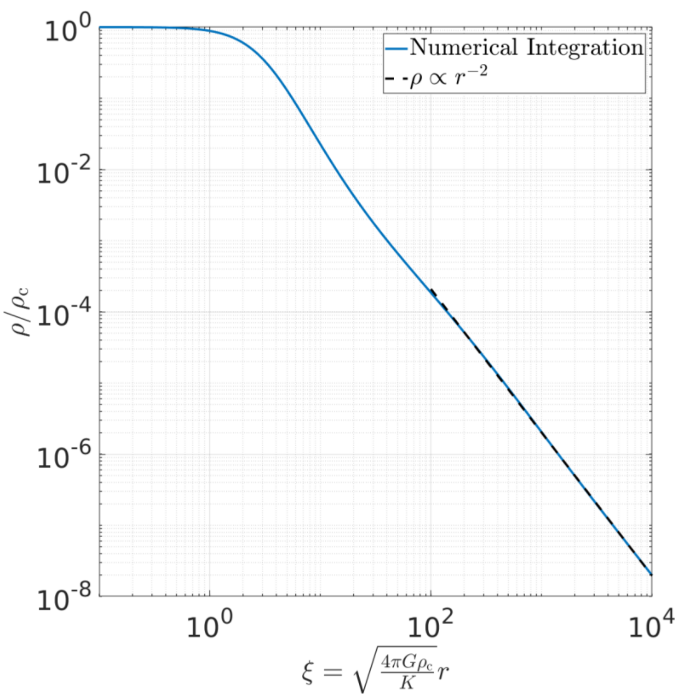

Fig. 5.5: Numerical integration solution to $1/\xi^2\, \mathrm{d}/\mathrm{d}\xi\, \left(\xi^2\, \mathrm{d}\psi/\mathrm{d}\xi\right) = e^{-\psi}$, at large radii, this has solutions of the form $\rho \propto r^{-2}$.

5.3 Scaling Relations

In many circumstances, stars behave as polytropes, amongst which is the case of fully convective stars with $p(\rho)$ where the pressure–density relation is very close to the adiabatic relation. In such a star ideal gas pressure dominates, assuming the gas is monatomic with $\gamma = 5/3$, we have $p = K\rho^{5/3} \implies n = 3/2$. White dwarfs well below the Chandrasekhar mass also correspond to $n = 3/2$.

For such cases we have to solve the equations numerically, but we can make some progress with the general properties of such stars through so-called scaling relations. The basis principle is that one treats all the stars characterised by a given polytropic index n as belonging to a family, distinguished from each other by the single parameter of the value of the central density, ρ_c (we assume for now that all stars in a particular family also share the same value of the polytropic constant K). If we can find how quantities such as the mass of the star and its radius vary as functions of ρ_c, we can eliminate ρ_c and discover the relationship between masses and radii for such stars.

The reason that we can do this is that *all* stars with a given value of n share the same $\theta(\xi)$.[2] The *shape* of the density distribution within each star in the family is identical, given by the appropriately scaled solution of the Lane-Emden equation. The value of ρ_c does however determine the mapping between ξ, r, θ, and ρ.

Thus one can find how mass and radius vary as a function of ρ_c and, eliminating ρ_c, obtain *scaling relations* relating the mass and radius.

Example 5.7 Mass-Radius Relation for Polytropes
Derive the mass-radius relation for polytropic stars [equation of state $p = K\rho^{1+1/n}$] on the assumption that K varies with stellar mass in such a way as to maintain a constant central temperature independent of mass.

Solution To be concrete, for now, consider a family of stars with $p = K\rho^{1+1/n}$, where both n and K are fixed across the family. All stars with given n have the same $\theta(\xi)$ since the Lane-Emden equation does not depend on ρ_c. Recall the relations

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n \quad \Longrightarrow \quad \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n} \tag{5.95}$$

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}\, r \quad \Longrightarrow \quad \xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{(1+n)K}}\, r \tag{5.96}$$

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n = \rho_c\theta^n. \tag{5.97}$$

The surface of the polytropic star is at $\xi = \xi_{\max}$ defined as location where we have the first zero of the solution of the Lane-Emden equation, $\theta(\xi_{\max}) = 0$. Let

[2] To see why this is the case, trace the derivation of (5.88) and see that this equation, which does not involve ρ_c, is true for all ρ_c.

$r_{\max}$ be the corresponding physical radius. Then the total mass of the polytrope is

$$\begin{aligned} M &= \int_0^{r_{\max}} 4\pi r^2 \rho \, \mathrm{d}r \\ &= 4\pi \rho_c \left[\frac{4\pi G \rho_c^{1-1/n}}{(1+n)K} \right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n \xi^2 \, \mathrm{d}\xi}_{\substack{\text{same for all} \\ \text{polytrope of index } n}} \\ \therefore M &\propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)}. \end{aligned} \tag{5.98}$$

From the definition of ξ above in Eq. (5.96), we also know that

$$r_{\max} \propto \rho_c^{\frac{1}{2}\left(\frac{1}{n}-1\right)}. \tag{5.99}$$

Eliminating density ρ_c thus gives the **mass-radius relation for polytropic stars**

$$\boxed{M \propto R^{\frac{3-n}{1-n}}.} \tag{5.100}$$

For $\gamma = 5/3$, $n = 3/2$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EoS is due to e^- degeneracy pressure rather than gas pressure). As we consider progressively more massive white dwarfs, the bulk of the electrons need to be in high energy levels (Fermi surface is higher energy). At some point, the electrons become relativistic, standard kinetic theory shows that the equation of state "softens" from $p = K\rho^{5/3}$ to $p = K'\rho^{4/3}$ (corresponding to $n = 3$). The scaling relation is Eq. (5.100). So, for $n = 3$, the mass is independent of radius, i.e. there is only one permitted mass for the configuration. This is the Chandrasekar mass, about $1.4\,M_\odot$, plays a special part in Type-Ia supernovae.

But for most main-sequence stars we do *not* observe $M \propto R^{-3}$, instead across much of the mass sequence we see $M \propto R$. The reason is that stars do not share the same polytropic constant K. Let's write the temperature at the core in terms of the central density and K

$$\left.\begin{aligned} p &= K\rho^{1+1/n} \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \quad \Longrightarrow \quad T_c = \frac{\mu K}{\mathcal{R}_*}\rho_c^{1/n}. \tag{5.101}$$

Thermonuclear reactions in the core that power stars are extremely temperature sensitive, so across the main sequence the stars will adjust as to approximately keep T_c similar in the cores. So we can say that

$$K \propto \rho_c^{-1/n}. \tag{5.102}$$

Substituting this into the above expression for mass gives

$$M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \quad \Longrightarrow \quad M \propto R. \tag{5.103}$$

◀

We can also use these techniques to examine the behaviour of an individual star that is gaining or losing mass. In this case, when can the $K = \text{const.}$ or $T_c = \text{const.}$ relations be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust. The time to adjust to new hydrostatic equilibrium is roughly the time for a sound wave (Chapter 6) to propagate across the star, which is less than a day for the Sun,

$$t_{\rm hd} \sim R/c_s < 1\,\text{day}. \tag{5.104}$$

The thermal timescale, on which a star can lose a significant amount of energy, is

$$t_{\rm th} \sim \frac{\text{energy content of the star}}{\text{luminosity}} \sim \frac{GM^2}{RL}, \tag{5.105}$$

which is $\sim 30\,$Myr for the Sun. So, mass loss/gain is followed by rapid readjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time.

Example 5.8 Internal Energy of Polytropes
Explain (for the case of a polytrope, index n) why the internal energy per kg, $\mathcal{E}$, is equal to $\int_0^\rho \frac{p}{\rho'^2}\,\mathrm{d}\rho'$, if and only if $\gamma = 1 + \frac{1}{n}$.
Calculate how the total internal energy of a polytropic star varies with stellar mass, assuming all stars share the same polytropic constant K. [*Hint: Determine how the total mass* $M = \int_0^{R_0} 4\pi r^2 \rho\,\mathrm{d}r$ *and the total internal energy* $U = \int_0^{R_0} 4\pi r^2 \rho\epsilon\,\mathrm{d}r$ *scale with the central density* ρ_c.]

Solution Let us consider the work done per unit mass, $đW = -p\,\mathrm{d}V$, with the substitution $V = 1/\rho$ to express this as

$$\Delta W = \int_0^\rho \frac{p}{\rho'^2}\,\mathrm{d}\rho'\,. \tag{5.106}$$

From the polytropic equation of state $p = K\rho^{1+1/n}$ (5.63), we can evaluate this integral as

$$\Delta W = n\frac{p}{\rho}. \tag{5.107}$$

Since the internal energy per unit mass is $\mathcal{E} = \frac{1}{\gamma-1}\frac{p}{\rho}$, for the work done to equal the internal energy per unit mass, we require $\gamma = 1 + \frac{1}{n}$. Thus, the given integral expression for $\mathcal{E}$ holds if and only if $\gamma = 1 + \frac{1}{n}$.

From Eq. (5.107), we now can use the polytropic equation to write the internal energy per unit mass as $\mathcal{E} = nK\rho^{1/n}$, so the total internal energy becomes

$$U = 4\pi nK \int_0^{R_0} r^2 \rho^{1+1/n}\,\mathrm{d}r\,. \tag{5.108}$$

To determine the scaling relations, we use $\rho = \rho_c \theta^n$ (5.84) and $\xi = r\sqrt{4\pi G \rho_c^{1-1/n}/(1+n)K}$ (5.96)

$$\begin{aligned} U &= 4\pi n K \rho_c^{1+1/n} \left(\frac{4\pi G \rho_c^{1-1/n}}{(1+n)K}\right)^{-3/2} \int_0^{\xi_0} \xi^2 \theta^{1+n}\, \mathrm{d}\xi \\ &= 4\pi n K \rho_c^{(5/n-1)/2} \left(\frac{4\pi G}{1+n}\right)^{-3/2} \underbrace{\int_0^{\xi_0} \xi^2 \theta^{1+n}\, \mathrm{d}\xi}_{\substack{\text{same for all} \\ \text{polytrope of index } n}} . \end{aligned} \tag{5.109}$$

The scaling is thus

$$U \propto \rho_c^{(5/n-1)/2}, \tag{5.110}$$

assuming all stars share the same polytropic constant K.

We already derived in Eq. (5.98) that the total mass scales as

$$M \propto \rho_c^{(3/n-1)/2}, \tag{5.111}$$

and thus the scaling of the internal energy with the mass is

$$U \propto M^{\frac{5-n}{3-n}}. \tag{5.112}$$

If you use $K \propto \rho_c^{-1/n}$ (5.102) you get $U \propto M$ of course, as this is the assumption that the central temperature is constant. ◀

5.3.1 Spherical rotating star

A spherical rotating polytropic star with angular velocity Ω gains non-rotating mass on less than the thermal timescale. How does Ω evolve?

Conservation of angular momentum gives $MR^2\Omega = \text{const}$. So, if $\Omega \to \Omega + \Delta\Omega$ then $MR^2 \to MR^2 + \Delta(MR^2)$, and to first order in small quantities,

$$\begin{aligned} MR^2\Delta\Omega + \Omega\Delta(MR^2) = 0 \\ \implies \quad \frac{\Delta\Omega}{\Omega} = -\frac{\Delta(MR^2)}{MR^2}. \end{aligned} \tag{5.113}$$

But we can use

$$R \propto M^{\frac{1-n}{3-n}} \tag{5.114}$$

to say

$$\begin{aligned} \frac{\Delta\Omega}{\Omega} &\propto -\Delta\left(M^{\frac{5-3n}{3-n}}\right) \\ \implies \quad \frac{\Delta\Omega}{\Omega} &\propto -\left(\frac{5-3n}{3-n}\right)\Delta M, \end{aligned} \tag{5.115}$$

so,

$$\Delta M > 0 \quad \implies \quad \begin{cases} \Delta\Omega < 0 & \text{if} \quad \dfrac{5-3n}{3-n} > 0 \quad (\text{e.g. } n = \frac{3}{2}) \text{ Spin down} \\ \Delta\Omega > 0 & \text{if} \quad \dfrac{5-3n}{3-n} < 0 \quad (\text{e.g. } n = 2) \text{ Spin up} \end{cases} . \tag{5.116}$$

5.3.2 Star in a binary system

A star in a binary system loses mass to its companion. The donor star loses mass, $\Delta M < 0$. So since $R \propto M^{(1-n)/(3-n)}$, the radius will increase if $1 < n < 3$.

So there is the potential for unstable (runaway) mass transfer (but need to look at evolution of the size of the Roche lobe to conclusively decide whether the process is unstable).

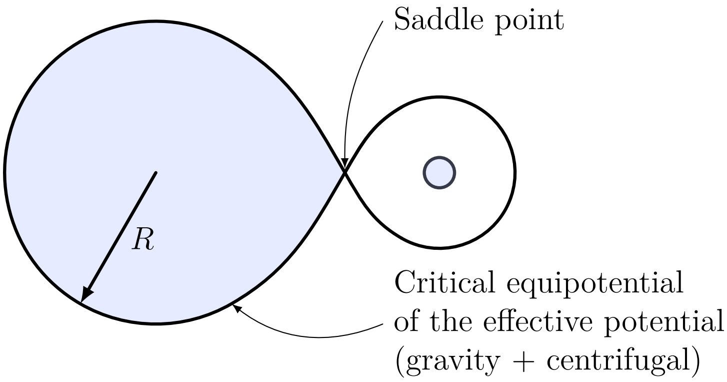

Fig. 5.6: Roche lobe overflow. Schematic of a binary star system in a semidetached configuration, the filled regions represent the two stars. The black line represents the inner critical Roche equipotential, made up of two Roche lobes that meet at the Lagrangian point L_1. In a semidetached configuration one star fills its Roche lobe.

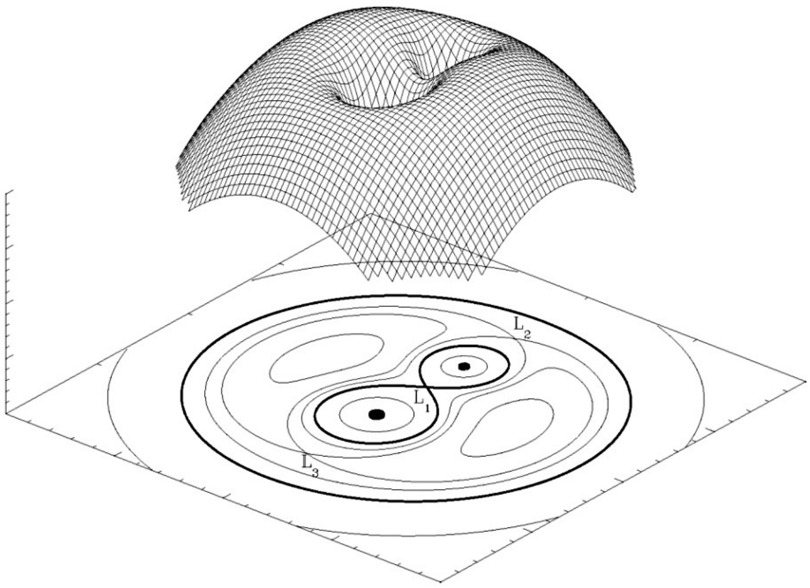

Fig. 5.7: A three-dimensional representation of the Roche potential in a binary star with a mass ratio of 2, in the co-rotating frame. The droplet-shaped figures in the equipotential plot at the bottom of the figure are called the Roche lobes of each star. L_1, L_2 and L_3 are the points of Lagrange where forces cancel out. Mass can flow through the saddle point L_1 from one star to its companion, if the donor star fills its Roche lobe. [7]

Chapter 6

Sound Waves, Supersonic Flows and Shock Waves

6.1 Sound Waves

We now start discussion of how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{6.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{6.2}$$

The equilibrium around which we will perturb is

$$\begin{aligned} \rho &= \rho_0 \quad \text{(uniform and constant)} \\ p &= p_0 \quad \text{(uniform and constant)} \\ \mathbf{u} &= \mathbf{0}. \end{aligned} \tag{6.3}$$

We consider small perturbations and write in Lagrangian terms (Lagrangian meaning the change of quantities are for a *given fluid element*)

$$\begin{aligned} p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u}. \end{aligned} \tag{6.4}$$

The relation between Lagrangian and Eulerian perturbations is:

$$\underbrace{\delta \rho}_{\substack{\text{Eulerian} \\ \text{perturbation}}} = \underbrace{\Delta \rho}_{\substack{\text{Lagrangian} \\ \text{perturbation}}} - \underbrace{\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \rho_0}_{\substack{\text{Element displacement dot} \\ \text{Gradient of unperturbed state}}} \tag{6.5}$$

In the present example, $\boldsymbol{\nabla} \rho_0 = 0$ and so $\delta \rho = \Delta \rho$, but the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we will address later.

Substitute the perturbations into the fluid equations and ignore terms that are 2$^{\text{nd}}$ order (or higher) in the perturbed quantities:

Start with continuity equation (6.1):

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla}\cdot[(\rho_0+\Delta\rho)\Delta\mathbf{u}] = 0$$

$$\implies \underbrace{\frac{\partial\rho_0}{\partial t}}_{=0} + \frac{\partial\Delta\rho}{\partial t} + \underbrace{\boldsymbol{\nabla}\rho_0\cdot\Delta\mathbf{u}}_{=0} + \underbrace{\boldsymbol{\nabla}(\Delta\rho)\cdot\Delta\mathbf{u}}_{2^{\text{nd}}\text{ order}} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) + \underbrace{\Delta\rho\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})}_{2^{\text{nd}}\text{ order}} = 0$$

$$\therefore \frac{\partial}{\partial t}(\Delta\rho) + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{6.6}$$

And similarly, the momentum equation (6.2):

$$\frac{\partial}{\partial t}(\Delta\mathbf{u}) + \underbrace{(\Delta\mathbf{u}\cdot\boldsymbol{\nabla})\Delta\mathbf{u}}_{2^{\text{nd}}\text{ order}} = -\frac{1}{\rho_0+\Delta\rho}\boldsymbol{\nabla}(p_0+\Delta p)$$

$$\begin{aligned}
\implies \quad \frac{\partial}{\partial t}(\Delta\mathbf{u}) &= -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2}\right)\Big(\underbrace{\boldsymbol{\nabla}p_0}_{=0} + \boldsymbol{\nabla}(\Delta p)\Big) \\
&= -\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta p) \qquad (6.7)\\
\therefore \frac{\partial}{\partial t}(\Delta\mathbf{u}) &= -\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0}, \quad \text{assuming barotropic EoS,} \qquad (6.8)
\end{aligned}$$

where we assumed a barotropic equation of state $p = p(\rho)$ to write the Taylor expansion about ρ_0 as $\Delta p = \partial p/\partial\rho\,|_{\rho=\rho_0}\Delta\rho$ to first order.

Now, taking the partial derivative of Eq. (6.6) with respect to time, and following from the commutativity of partial derivatives

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Delta\rho) &= -\rho_0\frac{\partial}{\partial t}[\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})] \\
&= -\rho_0\boldsymbol{\nabla}\cdot\left[\frac{\partial}{\partial t}(\Delta\mathbf{u})\right] \\
&= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho) \quad \text{given} \quad (6.8). \qquad (6.9)
\end{aligned}$$

Thus we arrive at the wave equation

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho).} \tag{6.10}$$

This admits solutions of the form $\Delta\rho = \Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Substituting into the wave equation we get

$$\begin{aligned}
(-i\omega)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}(ik)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
\therefore \omega^2 &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}k^2. \qquad (6.11)
\end{aligned}$$

The (phase) speed of the wave is $v_p = \omega/k$, so the sound wave travels at speed determined by the derivative of $p(\rho)$

$$c_\mathrm{s} = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}}. \tag{6.12}$$

Consider a 1D wave and substitute

$$\begin{aligned}\Delta\rho &= \Delta\rho_0 e^{i(kx-\omega t)}\\ \Delta u &= \Delta u_0 e^{i(kx-\omega t)}\end{aligned} \tag{6.13}$$

into Eq. (6.6). We get

$$\begin{aligned}&-i\omega\Delta\rho + \rho_0 ik\Delta u = 0\\ \Longrightarrow \qquad &\Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_\mathrm{s}\frac{\Delta\rho}{\rho_0}.\end{aligned} \tag{6.14}$$

So we learn that

- Fluid velocity and density perturbations are in phase (since $\Delta u/\Delta\rho \in \mathbb{R}$);
- A disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small, since

$$\Delta u_0 = c_\mathrm{s}\frac{\Delta\rho_0}{\rho_0} \ll c_\mathrm{s}. \tag{6.15}$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes acceleration of the fluid elements, this induces further density perturbations, making disturbances propagate.

Sound speed depends on how the pressure forces react to density changes. If the EoS is “stiff” (i.e. high $\mathrm{d}p/\mathrm{d}\rho$), then restoring force is large and propagation is rapid.

6.1.1 Examples of $\mathrm{d}p/\mathrm{d}\rho$

Notes about these two examples:

- We see that $c_\mathrm{s,I}$ and $c_\mathrm{s,A}$ differ by only $\sqrt{\gamma}$;
- Thermal behaviour of the perturbations does *not* have to be the same as that of the unperturbed structure! E.g. in the Earth’s atmosphere, the background is approximately isothermal but sound waves are adiabatic.
- Waves for which c_s is not a function of ω are called non-dispersive. The shape of a wave packet is preserved.

6.1.1.1 Isothermal Case

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_T \tag{6.16}$$

In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant T. Then

$$\begin{aligned} p &= \frac{\mathcal{R}_*}{\mu}\rho T \\ \therefore c_{s,\mathrm{I}} &= \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \sqrt{\frac{p}{\rho}}. \end{aligned} \tag{6.17}$$

6.1.1.2 Adiabatic Case

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S \tag{6.18}$$

No heat exchange between fluid elements; compressions heat up and rarefactions cool down from $p\,\mathrm{d}V$ work. So

$$p = K\rho^\gamma$$

$$\implies \quad \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S = \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho} \tag{6.19}$$

$$\therefore c_{s,\mathrm{A}} = \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}} = \sqrt{\frac{\gamma p}{\rho}}. \tag{6.20}$$

Example 6.1 Fluid Velocities in Sound Waves
Derive that for a linear sound wave (i.e. one in which $\Delta\rho/\rho$ is small) the velocity of fluid motion is $\ll c_s$. Estimate the maximum longitudinal fluid velocity in the case of a sound wave in air at s.t.p. in the case of a disturbance which sets up pressure fluctuations of order 0.1%.

Solution We consider a small perturbation in the general form of a linear 3D sound wave. These perturbations are assumed to be of the form

$$\begin{aligned} \Delta p &= \Delta p_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ \Delta \rho &= \Delta \rho_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ \Delta \mathbf{u} &= \Delta \mathbf{u}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \end{aligned} \tag{6.21}$$

Substituting these into the continuity equation (6.6), assuming small perturbations, we get to first order

$$-i\omega\Delta\rho + \rho_0 i\mathbf{k}\cdot\Delta\mathbf{u} = 0, \tag{6.22}$$

and solving for the velocity perturbation

$$|\Delta \mathbf{u}| = \frac{\omega}{k\cos\theta}\frac{\Delta\rho}{\rho_0} = c_\mathrm{s}\frac{\Delta\rho}{\rho_0}. \tag{6.23}$$

It may at first seem that we could, in principle, have $\cos\theta = 0$, corresponding to transverse oscillations. However, in a non-viscous liquid, transverse waves cannot be sustained. This is because such a medium lacks shear forces, which are required to restore transverse displacements and propagate shear waves.

In solids, transverse waves can exist because atomic or molecular interactions generate restoring forces perpendicular to the direction of wave propagation. However, in a perfect (non-viscous) fluid, the stress tensor contains only isotropic pressure forces, and there are no internal mechanisms to oppose or restore transverse deformations. Consequently, a transverse wave in a non-viscous fluid would not experience any restoring force, leading to zero frequency oscillations ($\omega = 0$), meaning that such waves do not propagate at all.

Thus, in a non-viscous fluid, only longitudinal sound waves ($\cos\theta = 1$) are physically allowed, where density and pressure perturbations propagate parallel to the wave vector. This confirms that the speed of sound is given by $c_s = \omega/k$. For small density perturbations in the linear regime, it is clear from our derivation that:

$$|\Delta \mathbf{u}| \ll c_s. \tag{6.24}$$

This ensures that fluid motion in a sound wave remains subsonic, validating the assumption of small perturbations.

In the Earth's atmosphere, the background is approximately isothermal but sound waves are adiabatic, so we have from Eq. (6.19)

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} = \gamma\frac{p_0}{\rho_0}. \tag{6.25}$$

Substitute now our wave solutions into Eq. (6.7),

$$-i\omega\Delta\mathbf{u} = -i\frac{1}{\rho_0}\mathbf{k}\Delta p \quad |\Delta\mathbf{u}| = \frac{1}{c_s}\frac{1}{\rho_0}\Delta p, \tag{6.26}$$

for longitudinal waves, and using the above Eq. (6.25)

$$|\Delta\mathbf{u}| = \sqrt{\frac{p_0}{\gamma\rho_0}}\frac{\Delta p}{p_0}. \tag{6.27}$$

STP is a can of worms, and not very unique! So for $\Delta p/p_0 \sim 0.001$, let's take $p_0 = 10^5$ Pa, $\rho = 1.3$ kg m^{-3}, and assuming a diatomic atmosphere $\gamma = 7/5$, we thus have $|\Delta\mathbf{u}| = 23$ cm s^{-1}. ◀

6.2 Sound Waves in a Stratified Atmosphere

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$.

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on z-dependent terms, taking $\mathbf{u} = u\hat{\mathbf{z}}$. The continuity (2.12) and momentum equations (2.57) are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{6.28}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{6.29}$$

and the equilibrium is (see Subsection 5.1.1)

$$\begin{aligned} u_0 &= 0 \\ \rho_0(z) &= \tilde{\rho}e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu} \\ p_0(z) &= \frac{\mathcal{R}_* T}{\mu}\rho_0(z) = \tilde{p}e^{-z/H}. \end{aligned} \tag{6.30}$$

Consider a Lagrangian perturbation:

$$\begin{aligned} u &\to \Delta u \\ \rho_0 &\to \rho_0 + \Delta\rho \\ p_0 &\to p_0 + \Delta p. \end{aligned} \tag{6.31}$$

Remember from Eq. (6.5) that $\delta\rho = \Delta\rho - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho$. So we have

$$\left.\begin{aligned} \delta\rho &= \Delta\rho - \xi_z\frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z\frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u \end{aligned}\right\} \quad \text{Eulerian to Lagrangian perturbation relation,} \tag{6.32}$$

and

$$\Delta\mathbf{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \underbrace{\mathbf{u}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}}_{2^{\text{nd}}\text{ order}} \approx \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{6.33}$$

Substituting perturbed eulerian quantities into the Eulerian continuity equation

(6.28)

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho_0+\delta\rho)+\frac{\partial}{\partial z}[(\rho_0+\delta\rho)\delta u_z]=0\\
\implies\quad &\frac{\partial}{\partial t}\left(\rho_0+\Delta\rho-\xi_z\frac{\partial\rho_0}{\partial z}\right)+\frac{\partial}{\partial z}(\rho_0\Delta u_z)=0\quad(\text{ignoring }2^{\text{nd}}\text{ order terms})\\
\implies\quad &\underbrace{\frac{\partial\rho_0}{\partial t}}_{=0}+\frac{\partial\Delta\rho}{\partial t}-\frac{\partial\xi_z}{\partial t}\frac{\partial\rho_0}{\partial z}-\underbrace{\xi_z\frac{\partial}{\partial t}\frac{\partial\rho_0}{\partial z}}_{=0}+\frac{\partial\rho_0}{\partial z}\Delta u_z+\rho_0\frac{\partial\Delta u_z}{\partial z}=0\\
\implies\quad &\frac{\partial\Delta\rho}{\partial t}-\underbrace{\Delta u_z}_{\partial\xi_z/\partial t}\frac{\partial\rho_0}{\partial z}+\frac{\partial\rho_0}{\partial z}\Delta u_z+\rho_0\frac{\partial\Delta u_z}{\partial z}=0\\
&\therefore\ \frac{\partial\Delta\rho}{\partial t}+\rho_0\frac{\partial\Delta u_z}{\partial z}=0.
\end{aligned}\tag{6.34}
$$

To perform this next calculation, we need a relation that is obtained from the Lagrangian continuity equation:

$$
\begin{aligned}
&\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\boldsymbol{\nabla}\cdot\mathbf{u}=0\\
\implies\quad &\Delta\rho+\left(\rho_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}\right)\Delta t=0\quad(\text{integrating over a short time interval }\Delta t)\\
&\therefore\ \Delta\rho+\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}=0.
\end{aligned}\tag{6.35}
$$

Following a similar method for the Eulerian momentum equation (6.29), we write in terms of the perturbed quantities

$$
\frac{\partial(\delta u)}{\partial t}+\underbrace{\delta u\frac{\partial(\delta u)}{\partial z}}_{2^{\text{nd}}\text{ order}}=-\frac{1}{\rho_0+\delta\rho}\frac{\partial}{\partial z}(p_0+\delta p)-g,\tag{6.36}
$$

discarding the second term, and using the perturbation relation (6.32),

$$
\frac{\partial\Delta u}{\partial t}=-\left(\frac{1}{\rho_0}-\frac{\Delta\rho}{\rho_0^2}+\frac{1}{\rho_0^2}\xi_z\frac{\partial\rho_0}{\partial z}\right)\frac{\partial}{\partial z}\left(p_0+\Delta p-\xi_z\frac{\partial p_0}{\partial z}\right).\tag{6.37}
$$

Given the exponential equilibrium state equations (6.30), we can evaluate the derivatives easily to give

$$
\frac{\partial\Delta u}{\partial t}=-\left(\frac{1}{\rho_0}-\frac{\Delta\rho}{\rho_0^2}-\frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-\frac{1}{H}p_0+\frac{\partial\Delta p}{\partial z}+\frac{1}{H}\frac{\partial\xi_z}{\partial z}p-\frac{\xi_z}{H}\frac{1}{H}p_0\right)-g,\tag{6.38}
$$

and using the isothermal equation of state $p_0=gH\rho_0$,

$$
\frac{\partial\Delta u}{\partial t}=-\left(\frac{1}{\rho_0}-\frac{\Delta\rho}{\rho_0^2}-\frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-g\rho_0+\frac{\partial\Delta p}{\partial z}+g\frac{\partial\xi_z}{\partial z}\rho_0-g\frac{\xi_z}{H}p_0\right)-g.\tag{6.39}
$$

Now, using Eq. (6.35) to write

$$\frac{\partial \Delta u}{\partial t} = -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2} - \frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-g\rho_0 + \frac{\partial \Delta p}{\partial z} - g\Delta\rho - g\frac{\xi_z}{H}\rho_0 - g\right), \tag{6.40}$$

it follows to first order in small quantities (both $\Delta\rho$ and ξ_z are small here) that

$$\frac{\partial \Delta u}{\partial t} = g - \frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z} + g\frac{\Delta\rho}{\rho_0} + g\frac{\xi_z}{H} - g\frac{\Delta\rho}{\rho_0} - g\frac{\xi_z}{H} - g, \tag{6.41}$$

and a large amount of these terms cancel to give

$$\begin{aligned} \frac{\partial \Delta u_z}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z} \\ \Longrightarrow \quad \frac{\partial \Delta u_z}{\partial t} &= -\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}, \quad c_u^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}. \end{aligned} \tag{6.42}$$

Let's now derive the wave equation and dispersion relation. Take the partial derivative of Eq. (6.34) with respect to time

$$\begin{aligned} \frac{\partial^2 \Delta\rho}{\partial t^2} + \rho_0\frac{\partial}{\partial z}\left(\frac{\partial \Delta u_z}{\partial t}\right) &= 0 \\ \Longrightarrow \quad \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0\frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}\right) &= 0, \end{aligned} \tag{6.43}$$

where the last step involved substitution from Eq. (6.42). If the medium is isothermal, then c_u is independent of z. So,

$$\begin{aligned} \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0\frac{c_u^2}{\rho_0}\frac{\partial^2 \Delta\rho}{\partial z^2} + \rho_0\frac{c_u^2}{\rho_0^2}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z} &= 0 \\ \therefore \quad \underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z}}_{\substack{\text{extra piece associated} \\ \text{with stratification}}} &= 0. \end{aligned} \tag{6.44}$$

Now,

$$\begin{aligned} \frac{\partial \rho_0}{\partial z} &= \frac{\partial}{\partial z}\left(\tilde{\rho} e^{-z/H}\right) \\ &= -\frac{1}{H}\tilde{\rho} e^{-z/H} \\ &= -\frac{\rho_0}{H}. \end{aligned} \tag{6.45}$$

So,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_u^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0. \tag{6.46}$$

Look for solutions of the form $\Delta\rho \propto e^{i(kz-\omega t)}$,

$$\implies \quad -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{ik}{H}, \tag{6.47}$$

which reveals the *dispersion relation*

$$\boxed{\omega^2 = c_u^2\left(k^2 - \frac{ik}{H}\right).} \tag{6.48}$$

We can also write this as

$$k^2 - \frac{ik}{H} - \frac{\omega^2}{c_u^2} = 0, \tag{6.49}$$

and solve the quadratic for $k(\omega)$ to give

$$k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}. \tag{6.50}$$

Let's take $\omega \in \mathbb{R}$. We have two cases to examine if we wish to understand the implications of this dispersion relation.

6.2.1 Case I: $\omega > c_u/2H$

Examine the real and imaginary parts of k:

$$\mathrm{Im}\{k\} = \frac{1}{2H} \tag{6.51}$$

$$\mathrm{Re}\{k\} = \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.52}$$

So the density perturbation is

$$\Delta\rho \propto \underbrace{e^{-z/2H}}_{1}\,\underbrace{e^{i\left(\pm\sqrt{(\omega/c_u)^2-(1/2H)^2}z-\omega t\right)}}_{2} \tag{6.53}$$

corresponding to

1. Exponentially decaying amplitude with increasing height;
2. Wave with phase velocity

$$v_{\mathrm{ph}} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.54}$$

where v_{ph} is function of ω, meaning that the wave is dispersive. A wave packet consisting of different ω's will change shape as it propagates.

As before, we can relate Δu to $\Delta \rho$:

$$\Delta u_z = \frac{\Delta \rho}{\rho_0} \frac{\omega}{k}, \tag{6.55}$$

with

$$\Delta \rho \propto e^{-z/2H} \tag{6.56}$$

$$\rho_0 \propto e^{-z/H}, \tag{6.57}$$

giving

$$\Delta u_z \propto e^{+z/2H}, \quad \frac{\Delta \rho}{\rho_0} \propto e^{+z/2H}. \tag{6.58}$$

Thus the perturbed velocity and the fractional density variation both *increase* with height. In the absence of dissipation (e.g. viscosity), the kinetic energy flux ($\propto \Delta\rho\Delta u$) is conserved and the amplitude of the wave increases until

$$\Delta u \sim c_{\mathrm{s}}, \quad \frac{\Delta \rho}{\rho_0} \sim 1, \tag{6.59}$$

where the linear treatment breaks down and the sound wave "steepens" into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere!

6.2.2 Case II: $\omega < c_u/2H$

In this case, we find that k is purely imaginary. So,

$$\Delta \rho \propto e^{-|k|z} e^{i\omega t}. \tag{6.60}$$

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength, giving rise to reflections.

6.3 Transmission of Sound Waves at Interfaces

Consider two non-dispersive media with a boundary at $x = 0$. Suppose we have a sound wave travelling from $x < 0$ to $x > 0$. Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by r and t the amplitude of the reflected and transmitted waves, respectively.

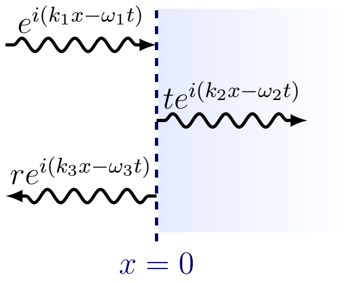

Fig. 6.1: Waves at boundary $x = 0$.

At the boundary $x = 0$, variables must be single valued and the accelerations are finite, thus oscillations in the second medium must have the same frequency,

$$\omega_1 = \omega_2 = \omega_3 = \omega. \tag{6.61}$$

The reflected wave is in the same medium as the incident so

$$k_3 = -k_1. \quad \text{(phase speed reversed)} \tag{6.62}$$

The amplitude of a sound wave is continuous at $x = 0$ hence

$$1 + r = t, \tag{6.63}$$

and the derivative of the amplitude is continuous at $x = 0$ thus

$$k_1(1 - r) = k_2 t. \tag{6.64}$$

We can combine these relations to get

$$t = \frac{2k_1}{k_1 + k_2}, \quad r = \frac{k_1 - k_2}{k_1 + k_2}. \tag{6.65}$$

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

1. If $c_{s,2} > c_{s,1}$, then $k_2 < k_1 \implies r > 0$, i.e reflected wave in phase with incident;
2. If $c_{s,2} < c_{s,1}$, then $r < 0 \implies$ reflected wave is π out of phase with incident wave;
3. If $c_{s,2} \ll c_{s,1}$, then $k_2 \gg k_1 \implies t \ll 1$, i.e. wave almost completely reflected.

6.4 Supersonic Fluids and Shocks

Consider some source of disturbance where signals propagate outwards at the speed of sound. Sound waves are characteristics of the set of hyperbolic equations. Wavefronts are circular since the speed of sound is isotropic.

If the source is moving, then the centre of each subsequent wavefront is displaced. What happens if the source moves faster than the sound speed?

Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding c_s. The linear theory applied to sound waves breaks down.

When thinking about the sound speed, recall that the chemical composition of the fluid matters, $c_\mathrm{s} \propto \mu^{-1/2}$

$$c_\mathrm{s} \text{ in } \underbrace{\text{atomic Hydrogen}}_{\substack{\mu\approx 1 \\ \text{e.g. ISM}}} \gg c_\mathrm{s} \text{ in } \underbrace{\text{diatomic Nitrogen}}_{\substack{\mu=28 \\ \text{e.g. Earth's} \\ \text{atmosphere}}}, \quad \text{for a given } T. \tag{6.66}$$

Disturbances in a fluid always propagate at the sound speed relative to the fluid itself. Consider an observer at the centre of a spherical disturbance, watching the fluid flow past at speed v.

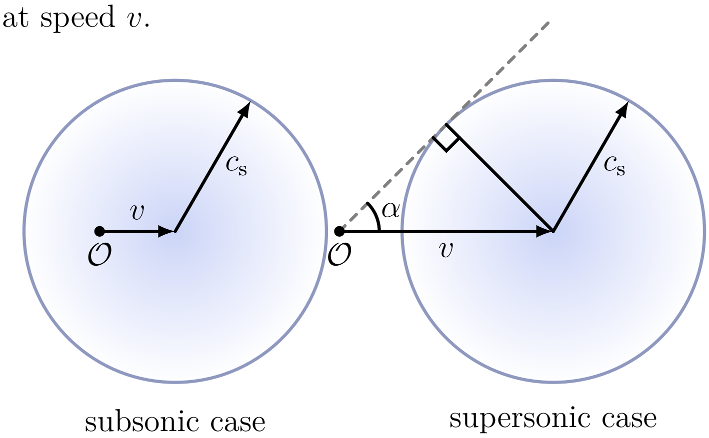

Fig. 6.2: Subsonic flow vs supersonic flow.

The velocity of the disturbance relative to the observer, v', is the vector sum of the fluid velocity and the disturbance velocity relative to the fluid.

- Subsonic case: v' sweeps 4π steradians;
- Supersonic case: disturbance always to the right. If we continuously produce a disturbance, the envelope of the disturbances will define a cone, named the *Mach cone*, with opening angle α given by

Definition 6.1 Mach Cone
A Mach wave propagates across the flow at the Mach angle α, which is the angle formed between the Mach wave wavefront and a vector that

points opposite to the vector of motion. It is given by

$$\sin\alpha = \frac{c_s}{v}. \tag{6.67}$$

We are interested in determining the Mach angle because small disturbances in a supersonic flow are confined to the cone formed by the Mach angle. There is *no upstream influence in a supersonic flow*; disturbances are only transmitted downstream within the cone.

Definition 6.2 Mach Number
The ratio of the flow speed to the sound speed is called the Mach number

$$\mathcal{M} \equiv \frac{v}{c_s} \tag{6.68}$$

$$\sin\alpha = \frac{1}{\mathcal{M}}. \tag{6.69}$$

Imagine an obstacle in a supersonic flow – disturbances cannot propagate upstream from the obstacle so the flow cannot adjust to the presence of the obstacle. The flow properties must change discontinuously once the obstacle is reached, giving a shock!

Let's analyse the properties of shocks.

6.5 The Rankine-Hugoniot Relations

A shock is a discontinuous flow. But we don't allow any old discontinuity. Instead, the discontinuity itself has certain properties. And these are derived from the conservation laws.

The discontinuity splits the flow into two, as shown in Figure 6.3. On the left, the flow has values u_1, ρ_1 and p_1; on the right values u_2, ρ_2 and p_2. To make life particularly simple, we'll assume that each of these flows is constant in space and time. All of the physics arises from the discontinuity.

We analyse a shock by applying conservation of mass, momentum and energy across the shock front. Let's specialise to the case of fluid entering a plane-parallel shock normally. On each side of the shock, the properties are uniform, but density, pressure, and velocity are discontinuous across the shock.

We'll consider the rest frame of the shock itself, so it is fixed at some position, say $x = 0$. We model the discontinuity as in infinitely thin surface and, as such, it can't carry any conserved charge density. Any mass that enters from one side

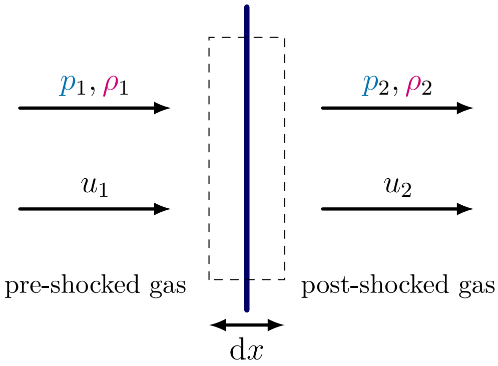

Fig. 6.3: The geometry of a plane parallel shock separating two flows; we are in the shock's frame of reference, with pre-shocked gas flowing in from the left.

must exit through the other. The same holds for momentum and energy. This means that each of the conserved currents must coincide on the left and right. Mass conservation (2.12) gives

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \\ \Longrightarrow \quad &\frac{\partial}{\partial t}\left(\int_{-\mathrm{d}x/2}^{\mathrm{d}x/2} \rho \,\mathrm{d}x\right) + \rho u_x\Big|_{x=\mathrm{d}x/2} - \rho u_x\Big|_{x=-\mathrm{d}x/2} = 0, \end{aligned} \tag{6.70}$$

where we have integrated over a small region dx around the shock.

Let's take d$x \to 0$ and assume a steady state such that mass does not continually accumulate at $x = 0$. Then

$$\frac{\partial}{\partial t}\left(\int \rho \,\mathrm{d}x\right) = 0, \tag{6.71}$$

which implies the 1st *Rankine-Hugoniot relation*,

Equation 6.1 1st Rankine-Hugoniot Relation

$$\rho_1 u_1 = \rho_2 u_2 \tag{6.72}$$

Apply similar analysis to the momentum equation (2.63):

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + p) - \rho\frac{\partial \Psi}{\partial x} \\ \Longrightarrow \quad &\frac{\partial}{\partial t}\left(\int \rho u_x \,\mathrm{d}x\right) = -\left(\rho u_x u_x + p\right)\Big|_{x=\mathrm{d}x/2} + \left(\rho u_x u_x + p\right)\Big|_{x=-\mathrm{d}x/2}, \end{aligned} \tag{6.73}$$

which gives the 2nd *Rankine-Hugoniot relation*,

Equation 6.2 2nd Rankine-Hugoniot Relation

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{6.74}$$

We note that u_y and u_z do not change across the shock front (can be immediately seen by looking at the y- and z-components of the momentum equation).

Now for the energy equation (4.42). Start with the adiabatic case so that the gas cannot cool and hence we have $\dot{Q}_{\text{cool}}$. Also take gravitational potential to have no time-dependence. Then

$$\begin{aligned}
\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] &= \underbrace{-\rho \dot{Q}_{\text{cool}}}_{=0} + \underbrace{\rho \frac{\partial \Psi}{\partial t}}_{=0} \\
\implies \quad \frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] &= 0 \\
\implies \quad \frac{\partial}{\partial t}\left(\int E \,\mathrm{d}x \right) + (E+p)u_x \Big|_{x=\mathrm{d}x/2} - (E+p)u_x \Big|_{x=-\mathrm{d}x/2} &= 0 \\
\implies \quad (E_1 + p_1)u_1 &= (E_2 + p_2)u_2. \qquad (6.75)
\end{aligned}$$

Since we know from equation (4.35) that $E = \rho\left(\frac{1}{2}u^2 + \mathcal{E} + \Psi\right)$, this becomes

$$\frac{1}{2}\rho_1 u_1^3 + \rho_1 \mathcal{E}_1 u_1 + \rho_1 \Psi_1 u_1 + p_1 u_1 = \frac{1}{2}\rho_2 u_2^3 + \rho_2 \mathcal{E}_2 u_2 + \rho_2 \Psi_2 u_2 + p_2 u_2. \qquad (6.76)$$

But $\Psi_1 = \Psi_2$ and $\rho_1 u_1 = \rho_2 u_2$ (6.72), so terms involving Ψ cancel out. We are left with the 3$^{\text{rd}}$ *Rankine-Hugoniot relation*,

Equation 6.3 3$^{\text{rd}}$ Rankine-Hugoniot Relation (Adiabatic)

$$\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2} \qquad (6.77)$$

This is Bernoulli's theorem 7.1 applied to the shock.

For an ideal gas, we have

$$\left.\begin{aligned} \mathcal{E} &= C_V T \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \implies \mathcal{E} = \frac{C_V \mu}{\mathcal{R}_*}\frac{p}{\rho} \qquad (6.78)$$

$$\left.\begin{aligned} \gamma &= \frac{C_p}{C_V} \\ C_p - C_V &= \frac{\mathcal{R}_*}{\mu} \end{aligned}\right\} \implies C_V(\gamma - 1) = \frac{\mathcal{R}_*}{\mu}, \qquad (6.79)$$

which combine to give the internal energy per unit mass

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}.} \qquad (6.80)$$

If we assume that γ does not change across the shock (e.g. there is no disassociation of molecules), the 3rd R-H relation becomes

$$\begin{aligned} &\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} \\ \implies \quad &\frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma-1} = \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma-1}, \end{aligned} \tag{6.81}$$

since, for the adiabatic case, recall from (6.19) the sound speed is

$$c_\mathrm{s}^2 = \left.\frac{\partial p}{\partial \rho}\right|_S = \frac{\gamma p}{\rho}. \tag{6.82}$$

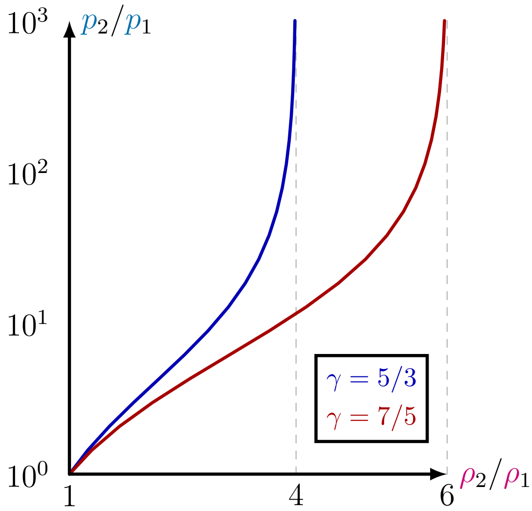

Fig. 6.4: Plot of Eq. (6.89) for various values of γ.

We have three conditions which fix u_2, ρ_2, and p_2 uniquely in terms of the initial flow data. To see this in more detail, we first use (6.72) and (6.74) to derive the relation

$$\rho_1 u_1^2 = \left(1 - \frac{\rho_1}{\rho_2}\right)^{-1}(p_2 - p_1). \tag{6.83}$$

Since the left-hand-side is positive, the right-hand side must also be positive. That gives us two possibilities: either pressure and density both increase across the shock

$$p_2 > p_1 \quad \text{and} \quad \rho_2 > \rho_1 \tag{6.84}$$

or the opposite happens. Clearly these two options are related by a parity flip, so we'll assume that the above occurs and the pressure is greater on the right of the shock. Then, from (6.72), we have

$$u_2 = \frac{\rho_1}{\rho_2}u_1. \tag{6.85}$$

This tells us that $|u_2| < |u_1|$, so the speed of the flow is smaller on the right of the shock. Note, however, that we haven't yet said anything about the sign of u_1 and u_2, i.e. is the flow left-to-right or right-to-left? We'll come to this shortly.

6.5.1 The Size of the Shock

The *shock compression ratio* is defined to be

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}. \tag{6.86}$$

Obviously it's a measure of how big the discontinuity is. We can also get an expression for r in terms of the pressure difference on each side but, for this, we need our final R-H matching condition associated to conservation of energy (6.77). For an ideal gas, this reads

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}. \tag{6.87}$$

We can use (6.83) and (6.85) to write this as

$$\frac{1}{2}\frac{\rho_1+\rho_2}{\rho_1\rho_2}(p_2-p_1) = \frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right). \tag{6.88}$$

Now substituting $\rho_2 = r\rho_1$, we find

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\rho_1 u_1^2}{\rho_2 u_2^2} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}. \tag{6.89}$$

This form of r puts some bounds on the strength of the shock.

Example 6.2 Post-Shock Energy in the Strong Shock Limit
In the reference frame where the unshocked gas is static, show that the post-shock kinetic energy is the same as the post-shock internal energy in the strong shock limit.

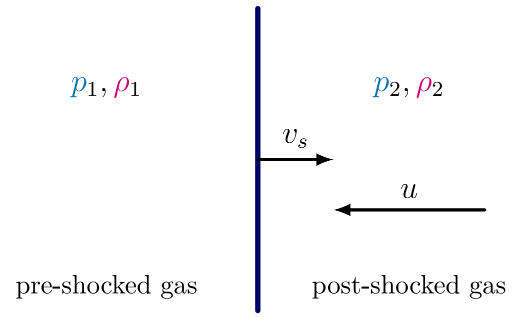

Fig. 6.5: The geometry of a plane parallel shock separating two flows; we are in the pre-shock fluid's frame of reference, with post-shocked gas flowing in from the right.

Solution We analyse the shock in the rest frame of the unshocked gas, meaning the shock front propagates with velocity v_s. Transforming to the conventions in Fig. 6.3, the pre- and post-shock velocities in this frame are:

$$\begin{aligned} u_1 &= v_s \\ u_2 &= v_s - u. \end{aligned} \tag{6.90}$$

In the strong shock limit ($p_2 \gg p_1$), the Rankine-Hugonoit relations simplify to

$$\rho_1 u_1 = \rho_2 u_2 \implies \rho_1 v_s = \rho_2 (v_s - u) \tag{6.91}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \implies \rho_1 v_s^2 = \rho_2 (v_s - u)^2 + p_2 \tag{6.92}$$

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \implies \frac{1}{2}v_s^2 = \frac{1}{2}(v_s - u)^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \tag{6.93}$$

These three equations provide sufficient constraints to determine the post-shock parameters in terms of the pre-shock parameters through their ratios.

We can define the shock compression ratio r as in (6.86) and aim to solve for this. We can rearrange the mass conservation equation (6.91) as

$$v_s - u = \frac{\rho_1}{\rho_2} v_s, \tag{6.94}$$

and substituting into the momentum equation (6.92),

$$\begin{aligned} \rho_1 v_s^2 &= \rho_2 \left(\frac{\rho_1}{\rho_2} v_s\right)^2 + p_2 \\ &= \left(1 - \frac{1}{r}\right)^{-1} p_2. \end{aligned} \tag{6.95}$$

Now, substituting this and (6.91) into the energy equation (6.93),

$$\frac{1}{2}v_s^2 = \frac{1}{2}v_s^2 \frac{1}{r^2} + \frac{\gamma}{\gamma - 1} v_s^2 \left(1 - \frac{1}{r}\right)\frac{\rho_1}{\rho_2} \tag{6.96}$$

$$\frac{1}{2} = \frac{1}{2r^2} + \frac{\gamma}{\gamma - 1}\left(1 - \frac{1}{r}\right)\frac{1}{r} \tag{6.97}$$

dividing through by v_s^2 which is non-zero. Multiplying through by $2r^2$, we can write this as

$$r^2 - 1 = \frac{2\gamma(r - 1)}{\gamma - 1}, \tag{6.98}$$

and we can cancel the common factors, as $r \neq 1$, to obtain

$$r = \frac{\gamma + 1}{\gamma - 1}, \tag{6.99}$$

as we shall show later in Eq. (6.112) again from the limit of the general solution.

Using this with $u_2 = \frac{1}{r} u_1$, we find

$$u = v_s - u_2 \implies u = \left(1 - \frac{\gamma - 1}{\gamma + 1}\right) v_s = \frac{2}{\gamma + 1} v_s. \tag{6.100}$$

This result shows that the post-shock gas moves slower than the shock front, as expected. The post-shock kinetic energy per unit volume is $e_{\mathrm{K}} = \frac{1}{2}\rho_2 u^2$, and so with these results this becomes

$$
\begin{aligned}
e_{\mathrm{K}} &= \frac{1}{2}(r\rho_1)\left(\frac{2}{\gamma+1}\right)^2 v_s^2 \\
&= \frac{1}{2}\frac{\gamma+1}{\gamma-1}\frac{4}{(\gamma+1)^2}\rho_1 v_s^2 \\
&= \frac{2}{\gamma^2-1}\rho_1 v_s^2,
\end{aligned}
\tag{6.101}
$$

which is all in terms of the unshocked gas parameters. Finally, we wish to look at the internal energy density per unit volume $e_{\mathrm{I}} = \rho\mathcal{E} = \frac{1}{\gamma-1}p$ (6.80), but first, we can substitute r into (6.95) using (6.99),

$$
p_1 = \left(1 - \frac{1}{r}\right)\rho_1 v_s^2 = \left(1 - \frac{\gamma-1}{\gamma+1}\right)\rho_1 v_s^2 = \frac{2}{\gamma+1}\rho_1 v_s^2,
\tag{6.102}
$$

and so the internal energy becomes

$$
\begin{aligned}
e_{\mathrm{I}} &= \frac{1}{\gamma-1}\frac{2}{\gamma+1}\rho_1 v_s^2 \\
&= \frac{2}{\gamma^2-1}\rho_1 v_s^2,
\end{aligned}
\tag{6.103}
$$

showing as required that the post-shock kinetic energy is the same as the post-shock internal energy in the strong shock limit. ◀

There is more physics to extract from our expressions for the compression ratio. We again define the Mach number (6.68) of the incoming flow as $\mathcal{M}_1 \equiv u_1/c_{s,1}$. Each side of the flow has a normal Mach number, $\mathcal{M}_1$ and $\mathcal{M}_2$. Note, in particular, the speed of sound also differs on either side of the shock. We'll now show that we can express the normal Mach numbers $\mathcal{M}_1$ and $\mathcal{M}_2$ on either side of the flow directly in terms of the compression factor r.

The algebra is a little fiddly so we'll tread slowly. Following from (6.74) which reads $\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$, if we divide through by $\rho_1 u_1^2$ then, after a little rearranging, we find

$$
\begin{aligned}
1 + \frac{p_1}{\rho_1 u_1^2} = \underbrace{\frac{\rho_2 u_2^2}{\rho_1 u_1^2}}_{1/r} + \frac{p_2}{\rho_1 u_1^2} \quad &\Longrightarrow \quad 1 - \frac{1}{r} = \frac{1}{\rho_1 u_1^2}(p_2 - p_1) \\
&\Longrightarrow \quad \rho_1 u_1^2 = \frac{r}{r-1}(p_2 - p_1).
\end{aligned}
\tag{6.104}
$$

Now compute $r/(r-1)$ using the expression in (6.89) involving pressure. This will give the result that we want,

$$
\begin{aligned}
\rho_1 u_1^2 &= \frac{1}{2}[(\gamma-1)p_1 + (\gamma+1)p_2] \\
\rho_2 u_2^2 &= \frac{1}{2}[(\gamma+1)p_1 + (\gamma-1)p_2]
\end{aligned}
\tag{6.105}
$$

Note that if the first equation above in Eq. (6.105) is true, then (6.89) immediately implies that the second is also true.

Now we've done the hard work. We use the expression for the speed of sound $c_s^2 = \gamma p/\rho$ to write the two equations in (6.105) as

$$\gamma\mathcal{M}_1^2 = \frac{1}{2}\left[(\gamma-1)+(\gamma+1)\frac{p_2}{p_1}\right]$$
$$\gamma\mathcal{M}_2^2 = \frac{1}{2}\left[(\gamma+1)\frac{p_1}{p_2}+(\gamma-1)\right] \tag{6.106}$$

To finish, we just need an expression for the pressure ratios. We can easily get this from (6.89). It is

$$\frac{p_2}{p_1} = \frac{r(\gamma+1)-(\gamma-1)}{(\gamma+1)-r(\gamma-1)}. \tag{6.107}$$

Finally we get the results that we wanted: the Mach numbers before and after the shock are

$$\mathcal{M}_1^2 = \frac{2r}{(\gamma+1)-r(\gamma-1)}$$
$$\mathcal{M}_2^2 = \frac{2}{r(\gamma+1)-(\gamma-1)}. \tag{6.108}$$

The key takeaway from these equations is that, for $r < r \leq r_{\max}$, we always have $\mathcal{M}_1$ and $\mathcal{M}_2 < 1$, as shown in Figure 6.6. This means that shocks only form in supersonic flows, where the speed of the fluid exceeds the speed of sound. After the shock, the speed of the fluid is reduced below the sound speed. (Although, as the reduction of the fluid speed is limited by a factor of $r \leq r_{\max}$, for very fast flows this is achieved by increasing the pressure, and hence increasing the sound speed, rather than by reducing the flow speed.)

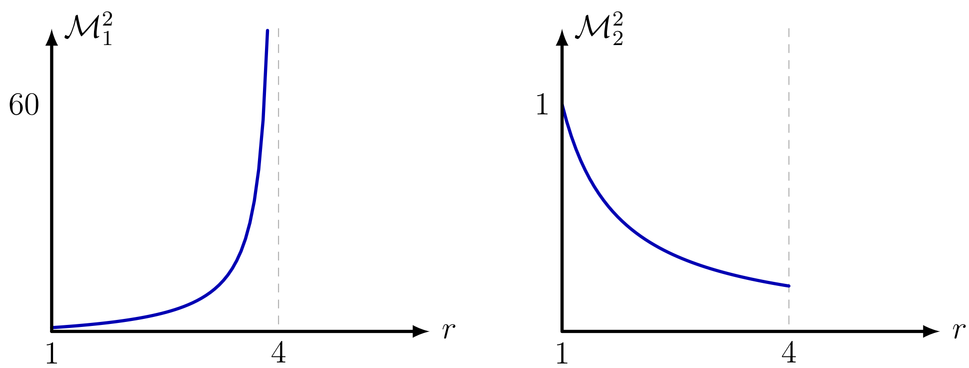

Fig. 6.6: The initial and final Mach numbers (6.108) as a function of the compression ratio r, plotted for $\gamma = 5/3$. We have $\mathcal{M}_1 \geq 1$ and $\mathcal{M}_2 \leq 1$ for all values of r.

From a physical perspective, the equations (6.108) are kind of backwards: the compression factor r doesn't determine the initial speed $\mathcal{M}_1$. It's the other way round! We can easily invert these equations to get the compression factor in terms of the initial Mach number,

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}. \tag{6.109}$$

Similarly, the jump in pressure, given in (6.107), is also determined by the initial speed

$$\frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}. \tag{6.110}$$

Then from the ideal gas equation of state (4.1) we can also now show

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{((\gamma-1)\mathcal{M}_1^2+2)(2\gamma\mathcal{M}_1^2-(\gamma-1))}{(\gamma+1)^2\mathcal{M}_1^2}. \tag{6.111}$$

In the limit of a *strong shock*, $p_2 \gg p_1$, we get directly from Eq. (6.89)

$$r = \frac{\rho_2}{\rho_1} \to r_{\max} \to \frac{\gamma+1}{\gamma-1}. \tag{6.112}$$

This is the largest compression factor that we can have. For $\gamma = 5/3$, this gives $r_{\max} = 4$. Note that the discontinuity is very different for pressure and speed: if the pressure changes by an infinite amount, the speed changes only by a factor of 4. So there is a maximum possible density contrast across an adiabatic shock - with stronger and stronger shocks, the thermal pressure of the shocked gas increases and prevents further compression. Also, it follows from Eq. (6.111) that for strong shocks

$$\begin{aligned} T_2 &= \frac{2\gamma(\gamma-1)\mathcal{M}_1^2}{(\gamma+1)^2}T_1 \\ \Longrightarrow \quad k_BT_2 &= \frac{4(\gamma-1)}{(\gamma+1)^2}\left(\frac{1}{2}\mu m_p u_1^2\right) \quad \text{since} \quad c_{s,1}^2 = \frac{\gamma k_B T_1}{\mu m_p} && (6.113) \\ \Longrightarrow \quad k_BT_2 &= \frac{3}{8}\left(\frac{1}{2}\mu m_p u_1^2\right) \quad \text{for} \quad \gamma = \frac{5}{3}. && (6.114) \end{aligned}$$

This makes explicit the notion that the kinetic energy of the pre-shock fluid is being converted into random motion of the post-shock flow.

There's a very basic question that we haven't yet addressed. Which way is the flow is going? Is the fluid moving left-to-right, so u_1, $u_2 > 0$ as shown in Figure 6.3? Or is it moving right-to-left, with u_1, $u_2 < 0$. In other words, does the pressure increase in the direction of the flow or decrease in the direction of the flow? It turns out that the answer to this question lies in the second law of thermodynamics.

Note that, since $p_2 \gg p_1$, and $\rho_2 \leq 4\rho_1$, we have

$$\frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma} \quad \text{i.e. } K_1 \neq K_2. \tag{6.115}$$

The gas has jumped adiabats during its passage through the shock. Shocking the gas produces a *non-reversible* change, due to viscous processes operating within the shock. We always find that flow decelerates from supersonic to subsonic; bulk kinetic energy converted into disorganised motions (heat). We say that the shock is *compressive.*

While the R-H conditions are symmetric in the up- and down-stream quantities, the thermodynamic requirement that entropy increases dictates the direction of the jump (i.e. a fast/cold upstream flow shocking to produce a slow/fast downstream flow).

It is interesting that we can derive R-H conditions using the inviscid equations that do not explicitly include dissipation/entropy-generating terms.

The fact that the entropy is not constant across the discontinuity means that shocks are necessarily dissipative. There's something a little surprising about this. We've worked with the Euler equation which, as mentioned previously, enjoys the symmetry of time reversal. Moreover, we've also used the adiabatic condition $p\rho^{-\gamma} =$ constant for an ideal gas. Nonetheless, the discontinuity is a violent event and allows dissipative behaviour to be hidden in the singularity, even though the underlying equations did not themselves have dissipation. See Appendix A.2 for a continued discussion of these ideas on singularities.

Physically, we've captured the dissipation by allowing the internal, heat energy $\mathcal{E}$ to increase downstream. A fuller understanding of the dissipation mechanism would need us to look more closely at the shock wave by understanding the role that viscosity plays in thickening the discontinuity. But the results above tell us that, ultimately, fact these microscopic details don't affect the amount of dissipation: that's fully determined by the properties of the initial flow and some basic conservation laws.

6.5.1.1 Weak-Shock Limit

There's also something familiar hiding in this unfamiliar setting. Suppose that we have a weak shock, meaning $p_2 = p_1 + \Delta p$ with $\Delta p \ll p_1$. Then we have $r \approx 1 + \Delta p/\gamma p_1$. We can also write, $\rho_2 = \rho_1 + \Delta\rho$ and this gives $r \approx 1 + \Delta\rho/\rho_1$. Equating these, we have

$$\frac{\Delta p}{\Delta \rho} = \frac{\gamma p_1}{\rho_1}. \tag{6.116}$$

But this is the equation for the speed of sound in an ideal gas (see, for example, (6.20))

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = c_s^2 = \frac{\gamma p}{\rho}. \tag{6.117}$$

We previously derived this result for linearised (i.e. small) sound waves. Here we make contact with the shock waves. A very weak shock wave can be viewed as the limit of a very strong sound wave.

6.5.2 Isothermal Shocks

Not all shocks are adiabatic! To consider the other extreme, let's discuss *isothermal shocks*. Here we have $\dot{Q}_{\rm cool} \neq 0$ such that the shocked gas cools to produce $T_2 = T_1$. Whether a shock is isothermal or adiabatic depends on whether the "cooling length" is smaller or larger than the system size, respectively.

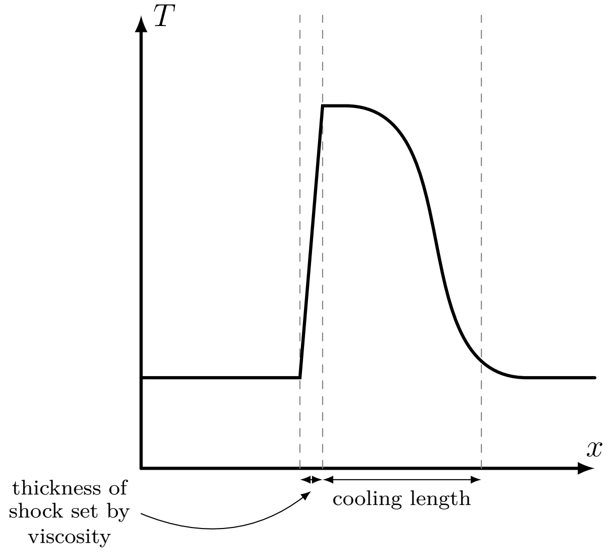

Fig. 6.7: Temperature profile through a shock.

For isothermal shocks, the first two R-H equations are unchanged:

$$\rho_1 u_1 = \rho_2 u_2 \tag{6.118}$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \tag{6.119}$$

but the 3rd R-H equation is replaced by

$$T_1 = T_2. \tag{6.120}$$

Now, following from Eq. (6.17),

$$c_{s,\rm I} = \sqrt{\frac{\mathcal{R}_* T}{\mu}} \quad \Longrightarrow \quad c_{s,1} = c_{s,2} \tag{6.121}$$
$$= \sqrt{\frac{p}{\rho}} \quad \Longrightarrow \quad p = c_{s,\rm I}^2 \rho. \tag{6.122}$$

So, the 2$^{\text{nd}}$ R-H equation becomes

$$\begin{aligned} \rho_1(u_1^2 + c_\mathrm{s}^2) &= \rho_2(u_2^2 + c_\mathrm{s}^2) \\ \implies \quad u_1 + \frac{c_\mathrm{s}^2}{u_1} &= u_2 + \frac{c_\mathrm{s}^2}{u_2} \quad (\text{since } \rho_1 u_1 = \rho_2 u_2) \\ \implies \quad c_\mathrm{s}^2\left(\frac{1}{u_1} - \frac{1}{u_2}\right) &= u_2 - u_1 \\ \implies \quad c_\mathrm{s}^2 \frac{u_2 - u_1}{u_1 u_2} &= u_2 - u_1 \end{aligned}$$

and thus,

Equation 6.4 3$^{\text{rd}}$ Rankine-Hugoniot Relation (Isothermal)

$$c_\mathrm{s}^2 = u_1 u_2 \tag{6.123}$$

Thus we see that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_\mathrm{s}}\right)^2 = \mathcal{M}_1^2, \tag{6.124}$$

where $\mathcal{M}_1$ is the Mach number of the upstream flow. So the density compression can be very large.

Note that since $c_s^2 = u_1 u_2$ and $u_1 > c_\mathrm{s}$ (condition for a shock), we must have $u_2 < c_\mathrm{s}$. So flow behind the shock is subsonic. In fact this is always true for any shock and is necessary to preserve causality (the post shock gas must know about the shock!).

Example 6.3 Collision Timescale and Shock Properties in Cloud Collisions

Two identical clouds, radius 3×10^{16} m, temperature 10 K, collide with each other with relative velocity 4 km s^{-1}. What is the time interval t_c, over which each cloud falls into the shock? If the cooling rate in the shocked gas is $Q^- = 10^{-4}$J s^{-1} kg^{-1} decide whether the shock is approximately adiabatic or isothermal.

If the clouds colliding produces an isothermal shock, what is the thickness of the shocked layer, x, at the moment that the entirety of each cloud has been shocked? At later times the layer relaxes into a structure that can be approximated by a hydrostatic isothermal slab, column density 0.1 kg m^{-2}. What fraction of the cloud masses remains within thickness x in this hydrostatic structure?

[*Hint: Ignore edge effects and variations of column density in the plane of the slab.*]

Solution For two identical clouds, assumed approximately spherical, colliding with a relative speed of 4 km s^{-1}, in the centre of mass frame (i.e. the shock frame) each cloud approaches the collision interface with a speed of $u_1 = 2 \times 10^3$ m s^{-1}.

Given the cloud radius, the time interval over which all the gas from each cloud falls into the shock is approximately

$$t_c \approx \frac{2r}{u_1} = 3 \times 10^{13}\ \mathrm{s} = 9.5 \times 10^5\ \mathrm{yr}. \tag{6.125}$$

To determine the nature of the shock, we compare the timescale of the shock and the time for the dissipation of almost all kinetic energy to heat.

$$t_{\mathrm{cool}} \approx \frac{\frac{1}{2}u_1^2}{Q^-} = 2 \times 10^{10}\ \mathrm{s} = 600\ \mathrm{yr}. \tag{6.126}$$

Since $t_{\mathrm{cool}} \ll t_c$, the shocked gas cools very rapidly compared to the time over which each cloud is processed by the shock. This means that the shock loses its thermal energy via radiation almost immediately, and the gas is kept at nearly constant (isothermal) temperature during the collision.

In an isothermal shock the gas remains at constant temperature so that the pressure is given by $p = \rho k_{\mathrm{B}}T/\mu m_{\mathrm{p}}$ (4.1). The jump condition across the shock derived from the conservation of momentum is $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$ (6.119). In the strong shock limit the pre-shock pressure p_1 is negligible compared to the post-shock pressure p_2, and additionally neglecting u_2 for an approximately static shock layer,

$$\rho_1 u_1^2 = p_2, \tag{6.127}$$

and immediately after the shock the gas is rapidly cooled back to T, so that the post-shock pressure is $p_2 = \rho_2 k_{\mathrm{B}}T/\mu m_{\mathrm{p}}$. Thus, we can approximate

$$\rho_1 u_1^2 = \frac{k_{\mathrm{B}}}{\mu m_{\mathrm{p}}}\rho_2 T, \tag{6.128}$$

and rearranging to obtain the density compression ratio (6.86)

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1^2 \mu m_{\mathrm{p}}}{k_{\mathrm{B}}T}. \tag{6.129}$$

For example, if the gas is molecular hydrogen (H_2) then using the given values one finds

$$\frac{\rho_2}{\rho_1} \approx 97. \tag{6.130}$$

When the clouds collide, the shocked layer forms at the interface. An estimate for the thickness of the shocked layer is the effective length over which the shock processes the gas is reduced by the compression ratio. For a head-on collision of two clouds, one may approximate the initial shocked thickness as

$$x \approx \frac{4R}{r} = \frac{4 \times 3 \times 10^{16}\ \mathrm{m}}{97} = 1.2 \times 10^{15}\ \mathrm{m}. \tag{6.131}$$

After the collision, the shocked gas relaxes into a configuration that can be approximated as an isothermal slab in vertical hydrostatic equilibrium. For such a slab, the vertical scale height H is given by the density distribution $\rho = \rho_0 \,\mathrm{sech}^2(z/H)$ (5.53). A standard expression for the scale height in an isothermal slab is

$$H^2 = \frac{A}{2\pi G\rho_0} = \frac{\mathcal{R}_* T}{2\pi\mu G\rho_0}. \tag{6.132}$$

For an infinite, planar (slab) geometry, the gravitational acceleration at a distance z from the midplane (where $z = 0$) is determined by the mass contained in the slab. Ignoring the edge effects as instructed, one may show that Eq. (3.16) holds for a thin, self-gravitating slab, as shown in Subsection 3.1.3:

$$\mathbf{g} = -2\pi G\hat{\mathbf{z}} \underbrace{\int_{-z}^{z} \rho(z)\,\mathrm{d}z}_{\Sigma(z)}, \tag{6.133}$$

where $\Sigma(z)$ is the column density from the midplane up to height z,

$$\Sigma(z) = 2\int_0^z \rho(z')\,\mathrm{d}z'\,, \tag{6.134}$$

where we used the symmetry of the density profile about $z = 0$. The total column density is then

$$\Sigma = 2\int_0^\infty \rho(z)\,\mathrm{d}z\,. \tag{6.135}$$

Using the known density profile (5.53), we find[1]

$$\Sigma = 2\rho_0 \int_0^\infty \mathrm{sech}^2(z/H)\,\mathrm{d}z = 2\rho_0 H. \tag{6.136}$$

Substituting this into Eq. (6.132) above gives

$$H = \frac{2\rho_0}{\Sigma}\frac{\mathcal{R}_* T}{2\pi\mu G\rho_0} = \frac{\mathcal{R}_* T}{\pi\mu G\Sigma}, \tag{6.137}$$

and for molecular Hydrogen with the given column density, $H \approx 2 \times 10^{15}$ m.

To find the fraction of mass remaining within a vertical distance $|z| < x$, we find the ratio $\Sigma(x)/\Sigma$,

$$\frac{\Sigma(x)}{\Sigma} = \frac{\displaystyle\int_0^x \mathrm{sech}^2(z/H)\,\mathrm{d}z}{\displaystyle\int_0^\infty \mathrm{sech}^2(z/H)\,\mathrm{d}z} = \frac{\tanh z|_0^{x/H}}{\tanh z|_0^\infty} = \tanh(x/H). \tag{6.138}$$

This indicates that about 54% of the cloud mass ends up being confined within the initial shocked layer thickness x. Note the column density is astrophysically reasonable and corresponds to a H_2 density of about 5×10^8 m^{-3}. ◀

[1] The integral is made easy by simply recalling $\mathrm{d}/\mathrm{d}x\,(\tanh x) = \mathrm{sech}^2\,x$.

Example 6.4 Downstream Mach Number from a Strong Shock
Show that for a strong shock (where the upstream Mach number $\mathcal{M}_1$ is large), the downstream Mach number $\mathcal{M}_2$ satisfies

$$\mathcal{M}_2^2 \simeq \frac{\gamma - 1}{2\gamma}. \tag{6.139}$$

Hence obtain an equation for the sound-speed ratio c_2/c_1.
A shock from a supernova travelling through the surrounding interstellar medium is observed to be travelling with speed 3000 km/s. What is the temperature immediately behind the shock?
[*Hint: You may assume, if you wish, the surrounding interstellar medium to have temperature* $100\,\mathrm{K}$ *and density* 10^7 *particles* m^{-3}.]

Solution ◀

6.6 Theory of Supernova Explosions

An important application of shock wave theory is to supernova explosions in the interstellar medium (ISM). A supernova (SN) deposits about 10^{51} erg ($= 10^{44}$ J) of energy into the surrounding medium, the shocked medium expands, sweeps up more gas, and creates large bubbles in the ISM.

Consider the following system:

- Initially uniform density interstellar medium (ISM) at rest, with density ρ_0;
- Instantaneous point-like explosion with energy E;
- Ignore temperature of the ambient ISM ($T_0 = 0$), thus no confinement of explosion by an external pressure.

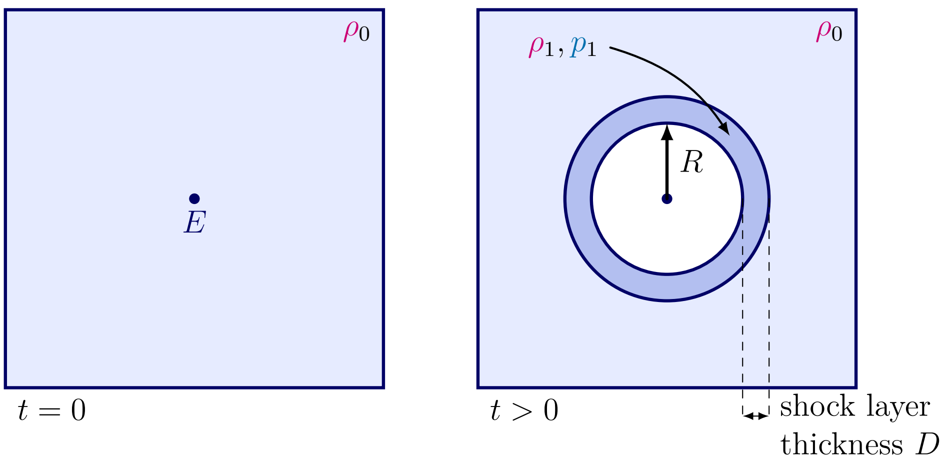

Fig. 6.8: Supernova explosion.

Given that $T_0 = 0$, the shock has $M \to \infty$. Assuming an adiabatic shock, we sweep mass into a shell with density ρ, given by

$$\rho_1 = \rho_0 \frac{\gamma+1}{\gamma-1}. \tag{6.140}$$

If all the mass is swept up into a shell then

$$\begin{aligned} \frac{4\pi}{3}\rho_0 R^3 &= 4\pi\rho_1 R^2 D \quad \text{(assuming } D \ll R) \\ \therefore D &= \frac{1}{3}\left(\frac{\gamma-1}{\gamma+1}\right)R. \end{aligned} \tag{6.141}$$

For $\gamma = 5/3$, we have $D \approx 0.08R$ which justifies the assumption $D \ll R$.

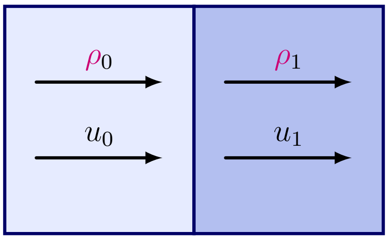

Fig. 6.9: Situation seen in shock frame.

Assume that all gas in the shell moves with a common velocity. Fig. 6.9 shows the frame of a local patch of the shock, and so

$$\begin{aligned} \rho_0 u_0 &= \rho_1 u_1 \\ \implies u_1 &= \frac{\rho_0}{\rho_1}u_0 = \frac{\gamma-1}{\gamma+1}u_0. \end{aligned} \tag{6.142}$$

Thus, relative to the unshocked gas, the velocity of the shocked gas U is

$$U = u_0 - u_1 = \frac{2u_0}{\gamma+1}. \tag{6.143}$$

Then, the rate of change of momentum of the shocked shell is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma+1}\right]. \tag{6.144}$$

This momentum gain is provided by pressure acting on the inside surface of the shell - call this p_{in}. Let's make the ansatz that this is related to the pressure within the shell by

$$p_{\text{in}} = \alpha p_1, \tag{6.145}$$

and we now relate p_1 and u_0 using the R-H jump condition: we have

$$
\begin{aligned}
p_0 + \rho_0 u_0^2 &= p_1 + \rho_1 u_1^2 \\
\implies \quad p_1 &= \rho_0 u_0^2 \left[1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2}\right] \quad \text{(since } p_0 = 0 \text{ by assumption)} \\
&= \rho_0 u_0^2 \left[1 - \frac{\gamma - 1}{\gamma + 1}\right] \quad \text{(assuming a strong shock)} \\
&= \frac{2}{\gamma + 1} \rho_0 u_0^2. \qquad (6.146)
\end{aligned}
$$

So, equating the rate of change of momentum in the shocked shell to the pressure acting on the inside surface of the shell, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma + 1}\right] &= 4\pi R^2 p_{\mathrm{in}} \\
&= 4\pi R^2 \alpha p_1 \\
&= 4\pi R^2 \alpha \frac{2}{\gamma + 1} \rho_0 u_0^2 \qquad (6.147)
\end{aligned}
$$

$$
\begin{aligned}
\implies \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 u_0\right] &= 3\alpha R^2 u_0^2 \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 \dot{R}\right] &= 3\alpha R^2 \dot{R}^2 \quad \text{since} \quad u_0 \equiv \dot{R} \qquad (6.148)
\end{aligned}
$$

This admits solutions of the form $R \propto t^b$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3b} b t^{b-1}\right) &= 3\alpha t^{2b}\left(b t^{b-1}\right)^2 \\
\implies \quad b(4b-1)t^{4b-2} &= 3\alpha b^2 t^{4b-2} \quad \text{(cancellation of } t^{4b-2} \text{ justifies assumed form of solution)} \\
\implies \quad b = 0 \quad &\text{(not physical)} \quad \text{or} \quad b = \frac{1}{4 - 3\alpha} \\
\implies \quad R \propto t^{1/(4-3\alpha)}, \quad &u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \propto R^{3\alpha-3}. \qquad (6.149)
\end{aligned}
$$

To determine α, we need to consider energy conservation. For an adiabatic shock, the explosion energy is conserved and transformed into kinetic and internal energy:

- Kinetic energy of the shell is

$$\frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2. \qquad (6.150)$$

- Internal energy per unit mass is

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho}, \qquad (6.151)$$

and so the internal energy per unit volume is

$$\rho\mathcal{E} = \frac{1}{\gamma - 1}p. \tag{6.152}$$

Since the shell is very thin, it has a small volume and so most of the internal energy is in the central cavity which contains little mass

$$\text{Internal energy of cavity} \approx \frac{4\pi}{3}R^3 \underbrace{\frac{p_{\text{in}}}{\gamma - 1}}_{\rho\mathcal{E}} = \frac{4\pi}{3}R^3\alpha\frac{p_1}{\gamma - 1}. \tag{6.153}$$

So, energy conservation says that

$$\begin{aligned} E &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 U^2 + \frac{4\pi}{3}R^3\alpha\frac{p_1}{\gamma-1} \\ &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 \underbrace{\left(\frac{2u_0}{\gamma+1}\right)^2}_{\text{Eq. (6.143)}} + \frac{4\pi}{3}R^3\alpha \underbrace{\frac{2}{\gamma+1}\rho_0 u_0^2}_{\text{Eq. (6.146)}} \frac{1}{\gamma-1} \qquad (6.154) \\ &= \frac{4\pi}{3}R^3 u_0^2\left[\frac{1}{2}\rho_0\frac{4}{(\gamma+1)^2} + \alpha\rho_0\frac{2}{(\gamma+1)(\gamma-1)}\right], \qquad (6.155) \end{aligned}$$

from which we conclude that

$$E \propto R^3 u_0^2 \propto t^{(6\alpha-3)/(4-3\alpha)}. \tag{6.156}$$

But E must be conserved. So we need $\alpha = 1/2$ to remove time dependence of E. Using $\alpha = 1/2$ we find

$$\boxed{R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}.} \tag{6.157}$$

6.6.1 Similarity Solutions

The above problem only has 2 parameters, E and ρ_0. Look at their dimensions

$$[E] = \frac{ML^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}. \tag{6.158}$$

These cannot be combined to give quantities with the dimension of length or time. So, there is no natural length scale or time scale in the problem!

Given some time t, the only way to combine E, ρ_0 and t to give a length scale is

$$\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}. \tag{6.159}$$

We can define a dimensionless distance parameter

$$\xi \equiv \frac{r}{\lambda} = r\left(\frac{\rho_0}{Et^2}\right)^{1/5}. \tag{6.160}$$

Then, for any variable in the problem $X(r,t)$, we will have

$$X = X_1(t)\tilde{X}(\xi), \tag{6.161}$$

i.e. X is a function of scaled distance ξ and always has the same shape scaled up/down by the time dependence factor $X_1(t)$.

So,

$$\frac{\partial X}{\partial r} = X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial r}\right|_t \tag{6.162}$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi)\frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial t}\right|_r . \tag{6.163}$$

ξ is neither a Lagrangian nor an Eulerian coordinate. It labels a particular feature in the flow (e.g. shock wave) that can move through the fluid. So we can write

$$R_{\text{shock}} \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \tag{6.164}$$

Let's put some numbers in for the case of supernova explosions,

$$R(t) = \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \quad \text{(we will assume } \xi_0 \sim 1\text{)} \tag{6.165}$$

$$u_0(t) = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\xi_0\left(\frac{E}{\rho_0 t^3}\right)^{1/5} = \frac{2}{5}\frac{R}{t}. \tag{6.166}$$

In a supernova we have

$$E \approx 10^{44}\,\mathrm{J} = 10^{51}\,\mathrm{erg} \tag{6.167}$$

$$\rho_0 = \rho_{\text{ISM}} \approx 10^{-21}\,\mathrm{kg\,m^{-3}}. \tag{6.168}$$

So the similarity solution gives

$$\left.\begin{array}{l} R \approx 0.3t^{2/5}\,\mathrm{pc} \\ u_0 \approx 10^5 t^{-3/5}\,\mathrm{km\,s^{-1}} \end{array}\right\} \text{where } t \text{ is measured in yrs.} \tag{6.169}$$

The original explosion injects the stellar debris at about $10^4\,\mathrm{km\,s^{-1}}$. So the above solution is valid for

$$t \gtrsim 100\,\mathrm{yr} \quad (\text{when } u_0 < u_{\text{inj}}) \tag{6.170}$$

$$t \lesssim 10^5\,\mathrm{yr} \quad \text{(after which energy losses become important)} \tag{6.171}$$

6.6.2 Structure of the Blast Wave

We can, in principle, write each variable ρ, p, u, r in terms of separated functions of t and ξ. We can then substitute into the Eulerian equation of fluid dynamics (in spherical coordinates with $\partial/\partial\phi = \partial/\partial\theta = 0$, i.e. spherical symmetry).

The result is a set of ODE's where ξ is the only dependent variable - the time dependence cancels out! [6]

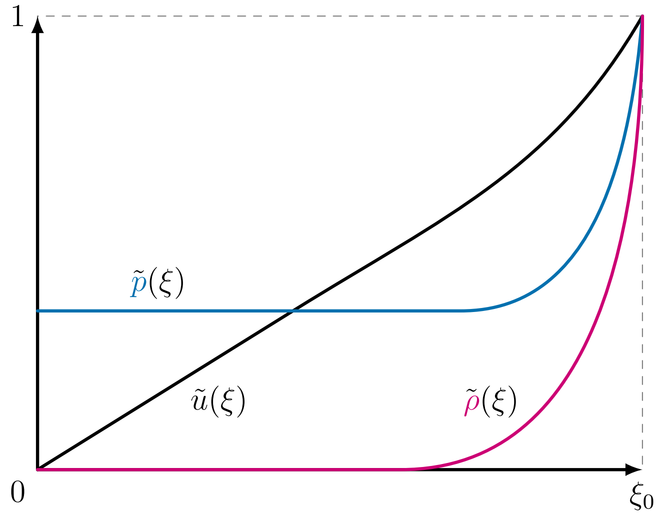

Fig. 6.10: Solution for $\gamma = 7/5$.

These solutions tell us that

- Most of mass is swept up in a shell just behind the shock (from form of $\tilde{p}$);
- Post-shock pressure is indeed a multiple of p_{in} (from form of $\tilde{p}$, justifies $p_{\text{in}} = \alpha p_1$ assumption);
- Shell material is not really moving at a single velocity, but arguments above are restored by taking some weighted average (from form of $\tilde{u}$).

6.6.3 Breakdown of the Similarity Solution

The self similar solution breaks down when the surrounding medium pressure p_0 becomes significant, $p_1 \sim p_0$.

From the strong shock solution, we derived equations (6.146) and (6.19),

$$p_1 = \frac{2}{\gamma + 1}\rho_0 u_0^2, \quad c_s^2 = \frac{\gamma p_0}{\rho_0}. \tag{6.172}$$

So if $p_1 \sim p_0$ then

$$\frac{2}{\gamma+1}\rho_0 u_0^2 \sim \frac{\rho_0 c_s^2}{\gamma}$$
$$\implies \quad u_0 \sim c_s, \tag{6.173}$$

i.e. the shell is not moving supersonically anymore, and the blast wave weakens to a sound wave as illustrated in Fig. 6.11.

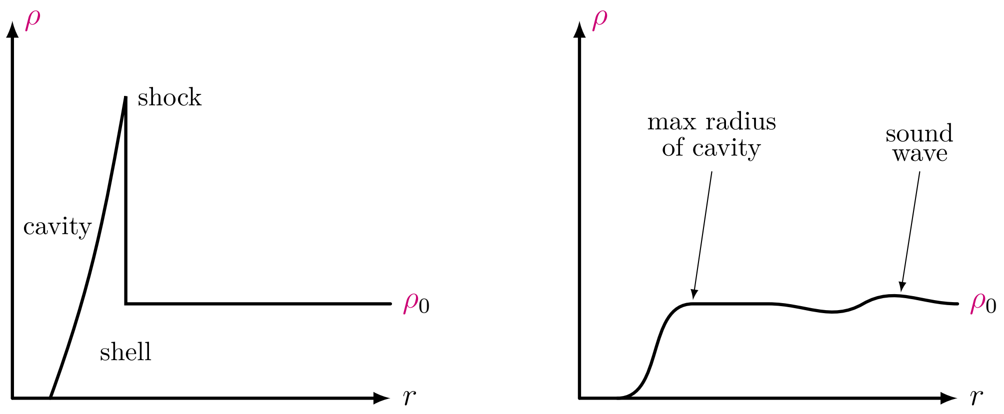

Fig. 6.11: Blast wave phase vs late phase.

As a sound wave, disturbance passes into the undisturbed gas as a mild compression followed by a rarefaction. After the sound wave passes, gas returns to the original state.

For a supernova, the maximum bubble/cavity size is set by the radius when the blast wave becomes sonic and $p_1 \sim p_0$. We've just shown that this implies

$$u_0^2 \sim \frac{\gamma+1}{2\gamma} c_s^2. \tag{6.174}$$

We showed in Eq. (6.154) that energy conservation gives

$$\begin{aligned} E &= \frac{4\pi}{3}R^3\left[\frac{1}{2}\rho_0\left(\frac{2u_0}{\gamma+1}\right)^2 + \frac{\alpha}{\gamma-1}\frac{2\rho_0 u_0^2}{\gamma+1}\right] \quad \text{where} \quad \alpha = \frac{1}{2} \\ &= \frac{4\pi}{3}R^3\rho_0 u_0^2\left[\frac{2}{(\gamma+1)^2} + \frac{1}{(\gamma-1)(\gamma+1)}\right] \\ &= \frac{4\pi}{3}R^3\rho_0 u_0^2\left[\frac{2(\gamma-1)+(\gamma+1)}{(\gamma+1)^2(\gamma-1)}\right] \\ &= \frac{4\pi}{3}R^3\rho_0 u_0^2\frac{3\gamma-1}{(\gamma+1)^2(\gamma-1)}. \end{aligned} \tag{6.175}$$

Rearranging this for u_0, we find

$$u_0^2 = \frac{(\gamma+1)(\gamma^2-1)}{3\gamma-1} \cdot \frac{3E}{4\pi\rho_0 R^3} \sim \underbrace{\frac{\gamma+1}{2\gamma}c_s^2}_{\substack{\text{when blast}\\ \text{wave becomes}\\ \text{sonic and } p_1\sim p_0}} \tag{6.176}$$

$$\implies \quad E \sim \frac{4\pi}{3}\rho_0 R_{\rm max}^3 \frac{c_s^2}{2\gamma} \cdot \frac{3\gamma-1}{\gamma^2-1}. \tag{6.177}$$

The internal energy initially contained within $R_{\rm max}$ is

$$E_{\rm init} = \frac{4\pi}{3}R_{\rm max}^3 \frac{p_0}{\gamma-1} = \frac{4\pi}{3}R_{\rm max}^3 \rho_0 \frac{c_s^2}{\gamma(\gamma-1)}. \tag{6.178}$$

So, when $p_1 \sim p_0$, we have $E \sim E_{\rm init}$. Therefore, the blast wave propagates until the explosion energy is comparable to the internal energy in the sphere!

Some numbers:

- Timescale on which the bubble reaches $R_{\rm max}$ is roughly the sound crossing time

$$t_s \sim \frac{R_{\rm max}}{c_s}. \tag{6.179}$$

  For ISM: $T \sim 10^4\,$K, $\rho \sim 10^{-21}\,$kg m^{-3}, giving

$$R_{\rm max} \sim \text{few} \times 100\,\text{pc} \tag{6.180}$$

$$t_{\rm max} \sim 10\,\text{Myr}. \tag{6.181}$$

- SN rate is about $10^{-7}\,\text{Myr}^{-1}\,\text{pc}^{-3}$. So, over a duration $t_{\rm max}$, can find 1 SN in $\sim 10^6\,\text{pc}^3$. But

$$\frac{4\pi}{3}R_{\rm max}^3 > 10^6\,\text{pc}^3 \tag{6.182}$$

  so the filling factor of SN driven bubbles is > 1. This would seem to suggest that the entire ISM would be heated by supernovae to $> 10^6 K$, but this is *not observed*!

  We need to account for cooling and the finite height of the Galactic disk (i.e. bubble "blow out"). After 10^5 yrs, when $R \sim 20$ pc, cooling losses become important and so the bubble grows more slowly than $R \propto t^{2/5}$. Simulations show that $R \propto t^{0.3}$ and $R_{\rm max} \sim 50$ pc, giving a filling factor < 1. Thus, due to cooling, only a small fraction of E is deposited into the ISM.

Chapter 7

Bernoulli's Equation and Transonic Flows

7.1 Bernoulli's Equation

Let's start with the momentum equation (2.57) and substituting Eq. (3.1),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}\rho - \boldsymbol{\nabla}\Psi. \tag{7.1}$$

If the fluid is barotropic, then $p = p(\rho)$ and so

$$\begin{aligned}\frac{\partial}{\partial x}\int \frac{\mathrm{d}p}{\rho} &= \frac{\partial p}{\partial x}\,\mathrm{d}p \int \frac{\mathrm{d}p}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \implies \quad \frac{1}{\rho}\boldsymbol{\nabla}p &= \boldsymbol{\nabla}\left(\int \frac{\mathrm{d}p}{\rho}\right).\end{aligned} \tag{7.2}$$

Also, we have the vector identity (2.70) $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u})$.

Using these, the momentum equation (7.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) - \mathbf{u} \times \mathbf{w} = -\boldsymbol{\nabla}\left[\int \frac{\mathrm{d}p}{\rho} + \Psi\right]. \tag{7.3}$$

where we have defined the *vorticity*:

Definition 7.1 Vorticity

$$\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u} \tag{7.4}$$

Now, assume a steady flow $(\partial \mathbf{u}/\partial t = \mathbf{0})$ and take the dot product of (7.3) with velocity $\mathbf{u}$. Since we have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w}) = 0$ always, the result is

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left[\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi\right] = 0. \tag{7.5}$$

This gives us *Bernoulli's Principle*:

Theorem 7.1 Bernoulli's Principle

For steady barotropic flows, the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi \tag{7.6}$$

is constant along a streamline.

The quantity H is called Bernoulli's constant.

- If $p = 0$, $H =$ constant is the statement that kinetic + potential energy is constant along streamlines.
- If $p \neq 0$ pressure differences accelerate or decelerate the flow as it flows along the streamline.

7.1.1 Examples of Bernoulli's Equation

In modern everyday life there are many observations that can be successfully explained by application of Bernoulli's principle, even though no real fluid is entirely inviscid, and a small viscosity often has a large effect on the flow.

7.1.1.1 The Apocryphal Aircraft Wing

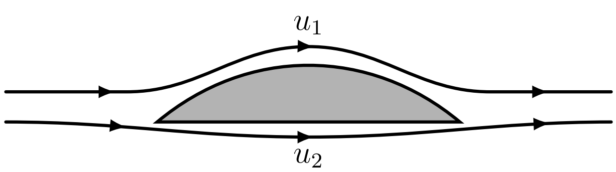

Fig. 7.1: Aircraft wing

$$\begin{aligned} u_1 > u_2 &\implies p_1 < p_2 \text{ from } H \\ &\implies \text{pressure difference} \\ &\implies \text{lift force.} \end{aligned}$$

Bernoulli's principle can be used to calculate the lift force on an airfoil, if the behaviour of the fluid flow in the vicinity of the foil is known. For example, if the air flowing past the top surface of an aircraft wing is moving faster than the air flowing past the bottom surface ($u_1 > u_2$), then Bernoulli's principle implies that the pressure on the surfaces of the wing will be lower above than below ($p_1 < p_2$). This pressure difference results in an upwards lifting force.

Whenever the distribution of speed past the top and bottom surfaces of a wing is known, the lift forces can be calculated (to a good approximation) using Bernoulli's equations, which were established by Bernoulli over a century before the first man-made wings were used for the purpose of flight.

Of course, this cannot be the whole story of how aircraft wings work or else inverted flight would be impossible!

7.1.1.2 Shower Curtain

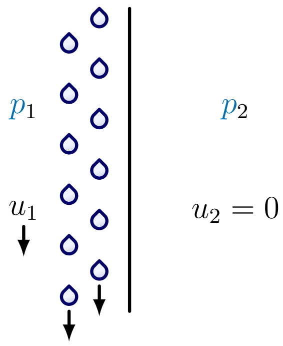

Fig. 7.2: Shower curtain

Downward flow of air on inside of curtain induced by falling water

$$\begin{aligned} &\Longrightarrow p_1 < p_2 \\ &\Longrightarrow \text{curtain blows inwards.} \end{aligned} \tag{7.7}$$

7.2 Rotational and Irrotational Flows

Definition 7.2 Irrotational Flow
An *irrotational flow* is one in which $\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}$ everywhere, i.e. the vorticity $\mathbf{w} = \mathbf{0}$ everywhere.

For a steady irrotational flow, Eq. (7.3) gives that

$$\boldsymbol{\nabla} H = 0, \tag{7.8}$$

so, $H =$ constant everywhere (not just along streamlines).

For a general (not necessarily irrotational or steady state) barotropic flow, we have from rearranging (7.3) that

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} H + \mathbf{u} \times \mathbf{w}. \tag{7.9}$$

Taking the curl of Eq. (7.9) above,

$$\frac{\partial}{\partial t}\underbrace{(\boldsymbol{\nabla}\times\mathbf{u})}_{\mathbf{w}} = \underbrace{-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}H)}_{\equiv\mathbf{0}} + \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{w}), \tag{7.10}$$

where we used the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0$ (2.71) for a scalar field ϕ, and from this we arrive at *Helmholtz's equation*,

Equation 7.1 Helmholtz's Equation

$$\frac{\partial\mathbf{w}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{w}) \tag{7.11}$$

From Helmholtz's equation, we observe three results:

1. If $\mathbf{w} = \mathbf{0}$ initially, it will stay zero thereafter (see Example 2.5). We will see later that this is no longer true once we include viscous terms.

2. The flux of vorticity through a surface $\mathcal{S}$ that moves with the fluid is a constant, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}}\mathbf{w}\cdot\mathrm{d}\mathbf{S} = 0. \tag{7.12}$$

For a proof, we first have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}}\mathbf{w}\cdot\mathrm{d}\mathbf{S} = \underbrace{\int_{\mathcal{S}}\frac{\partial\mathbf{w}}{\partial t}\cdot\mathrm{d}\mathbf{S}}_{\text{intrinsic changes in }\mathbf{w}} + \underbrace{\int_{\mathcal{S}}\mathbf{w}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\mathbf{S}}_{\substack{\text{change in }\mathcal{S}\\ \text{caused by flow}}}. \tag{7.13}$$

So,

$$\begin{aligned}\int_{\mathcal{S}}\mathbf{w}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\mathbf{S} &= \int_{\mathcal{S}}\oint_{\partial\mathrm{d}\mathbf{S}}\mathbf{w}\cdot(\mathbf{u}\times\mathrm{d}\mathbf{l})\\ &= \int_{\mathcal{S}}\oint_{\partial\mathrm{d}\mathbf{S}}\mathbf{w}\times\mathbf{u}\cdot\mathrm{d}\mathbf{l}\\ &= \oint_{\mathcal{C}}\mathbf{w}\times\mathbf{u}\cdot\mathrm{d}\mathbf{l} \quad \text{since ``internal loops'' cancel out}\\ &= \int_{\mathcal{S}}\boldsymbol{\nabla}\times(\mathbf{w}\times\mathbf{u})\cdot\mathrm{d}\mathbf{S}\,. \end{aligned} \tag{7.14}$$

$$\implies \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}}\mathbf{w}\cdot\mathrm{d}\mathbf{S} = \int_{\mathcal{S}}\mathrm{d}\mathbf{S}\cdot\underbrace{\left(\frac{\partial\mathbf{w}}{\partial t} - \boldsymbol{\nabla}\times(\mathbf{w}\times\mathbf{u})\right)}_{\substack{=\mathbf{0}\text{ from the}\\ \text{Helmholtz's equation (7.11)}}}$$

$$\implies \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{S}}\mathbf{w}\cdot\mathrm{d}\mathbf{S} = 0, \tag{7.15}$$

i.e. flux of vorticity is conserved and moves with the fluid. This is Kelvin's vorticity theorem.

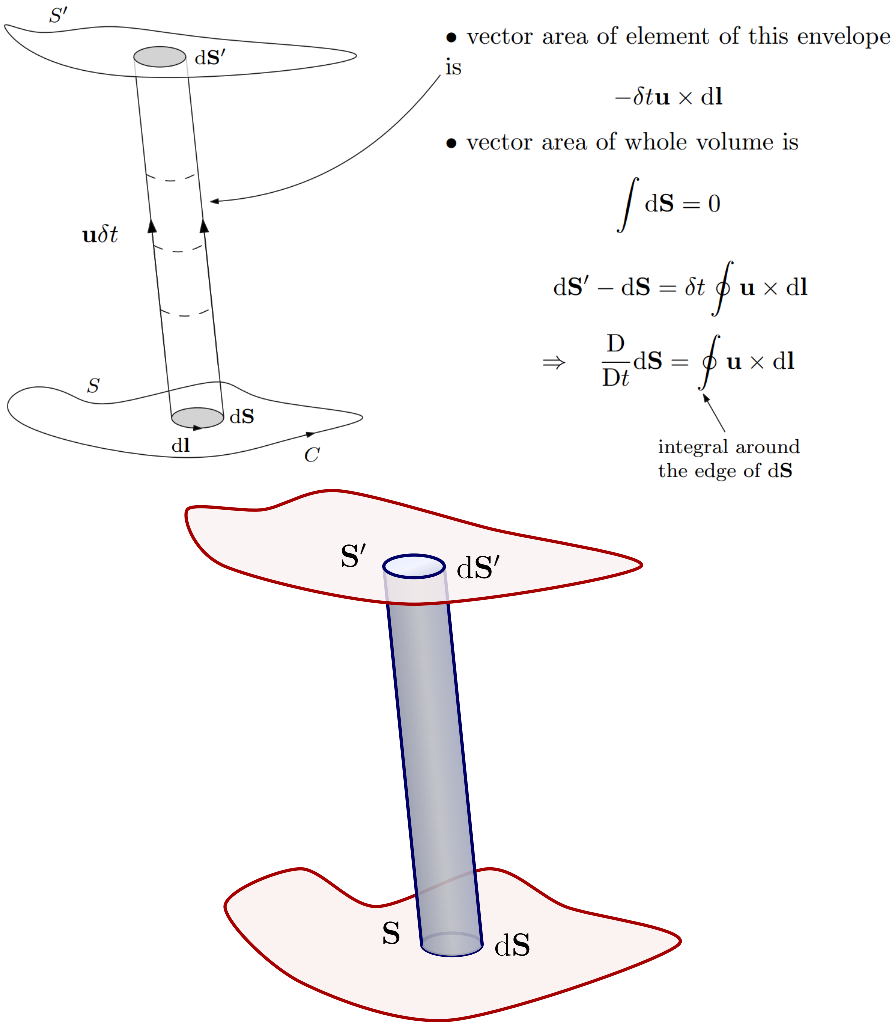

Fig. 7.3: Change of area element with time.

3. For an irrotational flow, the fact that $\boldsymbol{\nabla} \times u = \mathbf{0}$ everywhere implies that there exists a potential function Φ_u such that

$$\mathbf{u} = -\boldsymbol{\nabla}\Phi_u. \tag{7.16}$$

If such a flow is also incompressible, then $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and so

$$\nabla^2\Phi_u = 0, \tag{7.17}$$

i.e. can reduce the problem of finding the velocity field to that of solving Laplace's equation.

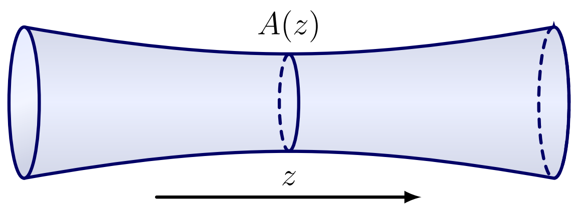

Fig. 7.4: Tube with variable cross-section

7.3 The de Laval Nozzle

Consider a steady flow in a tube with a variable cross-section $A(z)$ as illustrated in Fig. (7.4). For a *steady flow*, mass conservation gives

$$
\begin{aligned}
\rho u A = \text{const.}\ \dot{M} &\quad \text{(mass flow per second)}\\
\implies &\quad \ln\rho + \ln u + \ln A = \ln \dot{M}\\
\implies &\quad \frac{1}{\rho}\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A = 0\\
\implies &\quad \frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\ln u - \boldsymbol{\nabla}\ln A,
\end{aligned}
\tag{7.18}
$$

and the momentum equation (with no gravity) gives

$$
\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p. \tag{7.19}
$$

Let's further assume a *barotropic equation* of state. Then

$$
\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\rho}\boldsymbol{\nabla}\rho. \tag{7.20}
$$

So, putting these pieces together gives

$$
\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = [\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A]c_s^2 \quad \left(c_s^2 = \mathrm{d}p/\mathrm{d}\rho\right) \tag{7.21}
$$

If the flow is also irrotational, we have

$$
(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) = \frac{1}{2}u^2\boldsymbol{\nabla}\left(\ln u^2\right) = u^2\boldsymbol{\nabla}\ln u, \tag{7.22}
$$

and so, from Eq. (7.21) we have

$$
\begin{aligned}
u^2\boldsymbol{\nabla}\ln u &= [\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A]c_s^2\\
\implies \quad (u^2 - c_s^2)\boldsymbol{\nabla}\ln u &= c_s^2\boldsymbol{\nabla}\ln A.
\end{aligned}
\tag{7.23}
$$

This implies that an extremum of $A(z)$ must correspond to either

1. Minimum or maximum in u, or

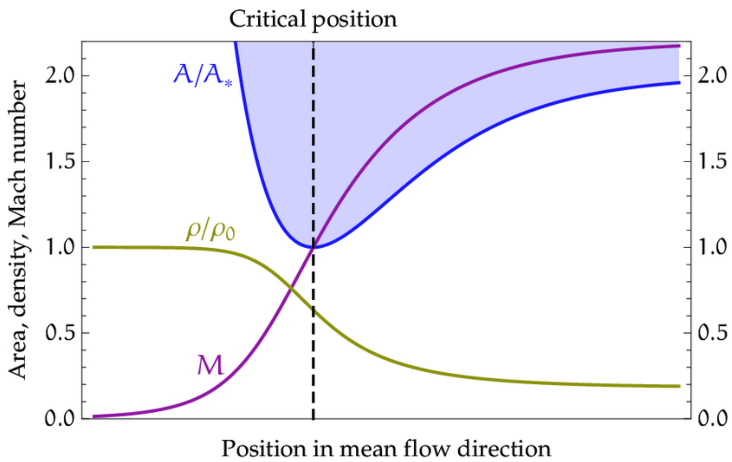

Fig. 7.5: State functions of a flow in a de Laval nozzle: density ρ in terms of its stagnation value ρ_0, Mach number $\mathcal{M}$ and the local cross sectional area A reaching A_* at its throat position. A diatomic gas with $\gamma = 7/5$ is assumed. [5]

2. $u = c_s$.

Thus, we see that there is the potential for a transition from subsonic to supersonic flow at a minimum or maximum of the cross-sectional area of the tube.

To make progress, we apply Bernoulli's equation

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = H \quad \text{constant} \quad \text{(no gravity, steady, irrotational)} \tag{7.24}$$

and examine the two standard barotropic cases.

7.3.1 Case I: Isothermal EoS

$$p = \frac{\mathcal{R}_* T}{\mu}\rho, \quad T = \text{const.} \tag{7.25}$$

$$\begin{aligned} \Longrightarrow \int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathcal{R}_* T}{\mu}\frac{\mathrm{d}\rho}{\rho} \\ &= \frac{\mathcal{R}_* T}{\mu}\ln\rho \\ &= c_s^2 \ln\rho. \end{aligned} \tag{7.26}$$

Suppose that we have a minimum or maximum in $A(z)$ that allows a flow to have a sonic transition. Let $A = A_m$ at this location. Then Bernoulli gives

$$\frac{1}{2}u^2 + c_s^2\ln\rho = \frac{1}{2}c_s^2 + c_s^2\ln\rho\bigg|_{A=A_m}, \tag{7.27}$$

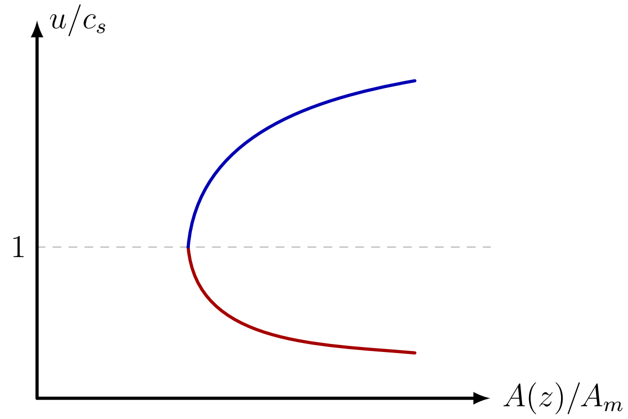

Fig. 7.6: Isothermal flow velocity u/c_s in a de Laval nozzle (7.28) plotted against the normalised cross-sectional area, A/A_m. The red curve shows the subsonic solution, while the blue curve shows the supersonic solution. At the extremum of the area, where $A/A_m = 1$, there is the possibility for the flow to be able to transition from subsonic to supersonic flow.

which implies

$$\begin{aligned} u^2 &= c_s^2\left[1 + 2\ln\left(\frac{\rho|_{A=A_m}}{\rho}\right)\right] \\ &= c_s^2\left[1 + 2\ln\left(\frac{uA}{c_s A_m}\right)\right], \end{aligned} \tag{7.28}$$

where this last step has used mass conservation, i.e. $\rho u A = \text{constant}$. Thus, given $A(z)$ we can determine $u(z)$ and $\rho(z)$, i.e. the structure of the flow everywhere subject to given $\dot{M}$ and c_s.

7.3.2 Case II: Polytropic EoS

$$p = K\rho^{1+1/n} \tag{7.29}$$

Let's examine the case where the sonic transition occurs at $A = A_m$. But now we do not know the sound speed c_s since $c_s = c_s(\rho)$ and ρ varies. We need to solve for

$$c_s^2 = \frac{n+1}{n}K\rho^{1/n}. \tag{7.30}$$

Now,

$$\begin{aligned}
\int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\rho} \\
&= \int K\frac{n+1}{n}\rho^{1/n}\frac{\mathrm{d}\rho}{\rho} \\
&= K\frac{n+1}{n}\int \rho^{1/n-1}\,\mathrm{d}\rho \\
&= K\frac{n+1}{n}n\rho^{1/n} && (7.31) \\
&= nc_s^2 \quad \text{given} \quad (7.30). && (7.32)
\end{aligned}$$

Following from the equation of mass conservation,

$$\begin{aligned}
& \rho u A = \rho|_{A_m} c_s|_{A_m} A_m = \dot{M} \\
\Longrightarrow \quad & \rho|_{A_m} \underbrace{\left(\frac{n+1}{n}K\right)^{1/2} \rho^{1/2n}|_{A_m}}_{(7.30)} A_m = \dot{M} \\
\Longrightarrow \quad & \rho^{2+1/n}|_{A_m}\left(\frac{n+1}{n}K\right)A_m^2 = \dot{M}^2 \\
\Longrightarrow \quad & \rho|_{A_m} = \left[\left(\frac{\dot{M}}{A_m}\right)^2 \frac{n}{K(n+1)}\right]^{n/(2n+1)} \qquad (7.33)
\end{aligned}$$

Knowing $\rho|_{A_m}$, we can now determine c_s and A_m. Bernoulli gives:

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = \text{const.}, \tag{7.34}$$

which implies

$$\begin{aligned}
\frac{1}{2}\left(\frac{\dot{M}}{A\rho}\right)^2 + \underbrace{K(n+1)\rho^{1/n}}_{(7.31)} &= \frac{1}{2}\left.c_s^2\right|_{A_m} + K(n+1)\left.\rho^{1/n}\right|_{A_m} \\
&= \frac{1}{2}\underbrace{\left(\frac{n+1}{n}\right)K\left.\rho^{1/n}\right|_{A_m}}_{(7.30)} + K(n+1)\left.\rho^{1/n}\right|_{A_m} \\
&= \left(\frac{1}{2}+n\right)\left(\frac{n+1}{n}\right)K\left.\rho^{1/n}\right|_{A_m} \qquad (7.35)
\end{aligned}$$

This is an implicit equation for the density structure through the flow.

General points of physical interpretation:

$$(u^2 - c_s^2)\boldsymbol{\nabla}\ln u = c_s^2\boldsymbol{\nabla}\ln A. \tag{7.36}$$

- In subsonic regime $u < c_s$

$$\begin{aligned} A \text{ decreases} &\implies \boldsymbol{\nabla} \ln u \text{ positive} \\ &\implies u \text{ accelerates along streamline} \end{aligned}$$

  e.g. rivers flowing through narrows;

- In supersonic regime $u > c_s$

$$\begin{aligned} A \text{ increases} &\implies \boldsymbol{\nabla} \ln u \text{ positive} \\ &\implies u \text{ accelerates along streamline} \end{aligned}$$

  Gas becomes very compressible. A increases, u increases, ρ is greatly reduced. $\dot{M} = A\rho u$ constant.

So, a nozzle that gets progressively narrower, reaches a minimum, and then widens again can be used to accelerate a flow from a subsonic to a supersonic regime.

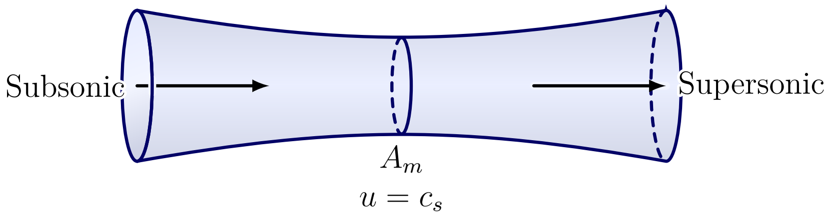

Fig. 7.7: de Laval nozzle.

Recall momentum equation:

$$\begin{aligned} (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla}\rho\frac{\mathrm{d}p}{\mathrm{d}\rho} = -c_s^2 \boldsymbol{\nabla} \ln \rho \\ (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= u^2 \boldsymbol{\nabla} \ln u \\ \implies \quad u^2 \boldsymbol{\nabla} \ln u &= -c_s^2 \boldsymbol{\nabla} \ln \rho \end{aligned} \tag{7.37}$$

- If $u \ll c_s$, $\nabla \ln u \gg \nabla \ln \rho$, this implies accelerations are important, pressure or density changes are small – almost incompressible;

- If $u \gg c_s$, $\nabla \ln u \ll \nabla \ln \rho$, $u \approx$ constant, pressure changes do not lead to much acceleration but there is change in ρ – compressible flow.

7.4 Spherical Accretion and Winds

We find flows with a mathematical structure when we consider steady-state and spherically-symmetric accretion flows or winds in the gravitational potential of a point-like central body.

Consider the spherically-symmetric accretion of gas onto a star (described as a point of mass). We will assume

- gas is at rest at infinity (reservoir);
- steady state flow;
- barotropic equation of state.

Mass conservation gives

$$\begin{aligned} \rho u A &= \text{constant } \dot{M} \\ \Longrightarrow \qquad 4\pi r^2 \rho u &= \dot{M}, \end{aligned} \tag{7.38}$$

where, for convenience, we define u to be inward pointing.

Momentum equation (2.57) gives

$$\begin{aligned} u\frac{\mathrm{d}u}{\mathrm{d}r} &= -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} \\ u^2\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r} &= -\underbrace{\frac{\mathrm{d}p}{\mathrm{d}\rho}}_{c_s^2}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2} \quad \text{given} \quad p = p(\rho) \\ \Longrightarrow \qquad u^2\frac{\mathrm{d}\ln u}{\mathrm{d}r} &= -c_s^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}, \end{aligned} \tag{7.39}$$

assuming self-gravity of the accretion gas is negligible.

Now, a steady flow must have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\dot{M}\right) &= 0 \\ \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho + \frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{\mathrm{d}}{\mathrm{d}r}\ln r^2 &= 0 \quad \text{given} \quad (7.38) \\ \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho &= -\frac{\mathrm{d}}{\mathrm{d}r}\ln u - \frac{2}{r}. \end{aligned} \tag{7.40}$$

Substitute into Eq. (7.39) gives

$$u^2\frac{\mathrm{d}}{\mathrm{d}r}\ln u = c_s^2\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{2}{r}\right) - \frac{GM}{r^2}. \tag{7.41}$$

Therefore

$$\boxed{\left(u^2 - c_s^2\right)\frac{\mathrm{d}}{\mathrm{d}r}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right).} \tag{7.42}$$

There is a critical point in the flow at

$$r = r_s = \frac{GM}{2c_s^2}, \tag{7.43}$$

where u is either a minimum/maximum or there is a sonic transition. This is called the *sonic point*, somewhat similar to the de Laval nozzle, except with no boundaries/tubes!

Can gain insight into the general structure of such flows by plotting possible solutions on the $(r/r_s, u)$ plane.

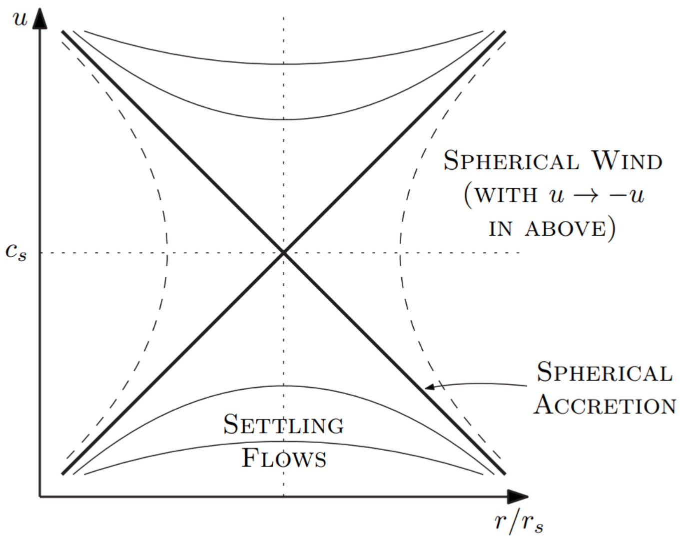

Fig. 7.8: Plot in $(r/r_s, u)$ plane.

Back to accretion problem: progress requires the EoS.

7.4.1 Case I: Isothermal EoS

Equation of state is:

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \quad T = \text{const.} \tag{7.44}$$

$$\Longrightarrow \quad c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \text{const.} \tag{7.45}$$

and we know from Eq. (7.43)

$$r_s = \frac{GM}{2c_s^2}. \tag{7.46}$$

Need to use Bernoulli's equation to constrain ρ and $\dot{M}$.

$$H = \frac{1}{2}u^2 + \underbrace{\int \frac{\mathrm{d}p}{\rho}}_{c_s^2 \ln \rho} + \Psi = \text{const.}$$

$$\Longrightarrow \quad \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s}$$

$$\Longrightarrow \quad \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = c_s^2\left(\ln \rho_s - \frac{3}{2}\right) \quad \text{given} \quad (7.46)$$

$$\Longrightarrow \quad u^2 = 2c_s^2\left[\ln\left(\frac{\rho_s}{\rho}\right) - \frac{3}{2}\right] + \frac{2GM}{r}, \tag{7.47}$$

where ρ_s is the density at $r = r_s$.

Now,

- as $r \to 0$, $u^2 \to 2GM/r$, i.e. free-fall speed once $r \ll r_s$;
- as $r \to \infty$ and $u \to 0$, $\rho = \rho_s e^{-3/2}$, giving

$$\rho_s = \rho_\infty e^{3/2}. \tag{7.48}$$

Thus, for a given ρ_∞, we know ρ_s and hence $\dot{M}$.

$$\dot{M} = 4\pi r_s^2 \rho_s c_s$$

$$\Longrightarrow \quad \dot{M} = \frac{\pi G^2 M^2 e^{3/2} \rho_\infty}{c_s^3}. \tag{7.49}$$

Note that:

- $\dot{M}$ proportional to M^2, more massive stars can accrete much more gas;
- $\dot{M}$ proportional to $1/c_s^3$, accretion very sensitive to temperature; can accrete more effectively from a colder medium.

7.4.2 Case II: Polytropic EoS

Equation of state is:

$$p = K\rho^{1+1/n}, \qquad \int \frac{\mathrm{d}p}{\rho} = K(n+1)\rho^{1/n}. \tag{7.50}$$

At the sonic point, we have (7.32)

$$\int \frac{\mathrm{d}p}{\rho} = nc_s^2. \tag{7.51}$$

Bernoulli then gives

$$\frac{1}{2}u^2 + K(n+1)\rho^{1/n} - \frac{GM}{r} = \frac{1}{2}c_s^2 + nc_s^2 - \frac{GM}{r_s}, \tag{7.52}$$

with $r_s = GM/(2c_s^2)$ as in Eq. (7.43). Using the mass accretion rate $\dot{M} = 4\pi r_s^2 \rho_s c_s$ we can then write

$$\begin{aligned} r_s &= \left(\frac{\dot{M}}{4\pi\rho_s c_s}\right)^{1/2} = \frac{GM}{2c_s^2} \\ \Longrightarrow \quad c_s &= \left(\frac{GM}{2}\right)^{2/3}\left(\frac{4\pi\rho_s}{\dot{M}}\right)^{1/3}. \end{aligned} \tag{7.53}$$

Combine this with

$$c_s^2 = \frac{n+1}{n}K\rho_s^{1/n}, \tag{7.54}$$

to get

$$\begin{aligned} \left(\frac{n+1}{n}\right)K\rho_s^{1/n} &= \left(\frac{GM}{2}\right)^{4/3}\left(\frac{4\pi\rho_s}{\dot{M}}\right)^{2/3} \\ \Longrightarrow \quad \rho_s^{1/n-2/3} = \rho_s^{(3-2n)/3n} &= \left(\frac{GM}{2}\right)^{4/3}\left(\frac{4\pi}{\dot{M}}\right)^{2/3}\frac{n}{(n+1)K} \\ \Longrightarrow \quad \rho_s &= \left(\frac{GM}{2}\right)^{4n/(3-2n)}\left(\frac{4\pi}{\dot{M}}\right)^{2n/(3-2n)}\left(\frac{n}{(n+1)K}\right)^{3n/(3-2n)}. \end{aligned} \tag{7.55}$$

Back to Bernoulli:

$$\begin{aligned} \frac{1}{2}u^2 + (n+1)K\rho^{1/n} - \frac{GM}{r} &= c_s^2\left(n - \frac{3}{2}\right) \\ \Longrightarrow \quad \frac{1}{2}\left(\frac{\dot{M}}{4\pi r^2\rho}\right)^2 + (n+1)K\rho^{1/n} &= c_s^2\left(n - \frac{3}{2}\right) + \frac{GM}{r}. \end{aligned} \tag{7.56}$$

As $r \to \infty$, $u \to 0$, we have

$$\rho_\infty = \left[\frac{c_s^2\left(n - \frac{3}{2}\right)}{(n+1)K}\right]^n = \left[\frac{n - \frac{3}{2}}{n}\right]^n \rho_s \tag{7.57}$$

$$c_{s,\infty}^2 = \frac{n+1}{n}K\rho_\infty^{1/n} = \frac{n - \frac{3}{2}}{n}c_s^2 \tag{7.58}$$

So, finally,

$$\begin{aligned} \dot{M} &= 4\pi r_s^2 \rho_s c_s \\ &= \frac{4\pi G^2 M^2}{4c_s^4}c_s\rho_\infty\left(\frac{n}{n-\frac{3}{2}}\right)^n \\ &= \frac{\pi G^2 M^2}{c_{s,\infty}^3}\rho_\infty\left(\frac{n}{n-\frac{3}{2}}\right)^{n-3/2}. \end{aligned} \tag{7.59}$$

Therefore,

$$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_{s,\infty}^3} \left(\frac{n}{n - \frac{3}{2}} \right)^{n-3/2}.} \tag{7.60}$$

Same functional form as in the isothermal case, but now with an additional coefficient related to the polytropic index. This is known as *Bondi Accretion*. We can recover the isothermal case by taking the limit $n \to \infty$.

The generalisation to the case of a star accreting from a medium that it is moving through is called *Bondi-Hoyle-Lyttleton Accretion*. The result is

$$\dot{M} \sim \frac{(GM)^2 \rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}}, \tag{7.61}$$

where v_∞ is the velocity of gas relative to the star at ∞.

Chapter 8

Fluid Instabilities

One important question that we haven't yet addressed is: are these flows stable? If we perturb them in some way, does the flow persist or does it get driven to something more complicated?

Our method to analyse instabilities will mirror the analysis of waves in Section 6. That is: we start with some background flow and perturb it. The perturbations that we call waves oscillate back and forth about the original flow. In contrast, the unstable perturbations that we will meet here grow without bound. These are known as *linear instabilities*. Our linear analysis only shows the beginning of the instability, rather than the end point of the flow, but with some imaginative thinking (and the help of experiment!) we can figure out the qualitative form of the final flow.

Consider a fluid in a steady state ($\partial/\partial t = 0$). Thus it is in a state of equilibrium.

- If a small perturbation of this configuration grows with time, the configuration is *unstable* with respect to those perturbations;
- If a small perturbation decays with time or just oscillates around the equilibrium configuration, the configuration is *stable* with respect to those perturbations.

An awful lot of interesting astrophysics is due to the action of fluid instabilities!

- Convection in stars;
- Multiphase nature of the ISM;
- Mixing of fluids that have relative motion;
- Turbulence in accretion disks;
- Formation of stars and galaxies.

In this chapter, we discuss some of the most important instabilities.

8.1 Convective Instability

Convective instability concerns the stability of a hydrostatic equilibrium. We can gain insight without doing a full perturbation analysis.

Consider the following system:

- Ideal gas in hydrostatic equilibrium;
- Uniform gravitational field in $-\hat{\mathbf{z}}$ direction.

Now perturb a fluid element upwards, away from its equilibrium point.

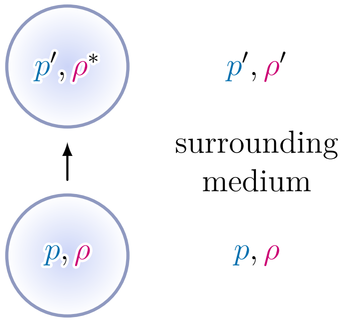

Fig. 8.1: Perturbing a fluid element upwards.

We assume that any pressure imbalances are quickly removed by acoustic waves, but that heat exchange takes longer. This implies the displaced element evolves adiabatically with a pressure p' equal to the pressure at the new location in the atmosphere.

Since we assume heat transfer is slow, initially perturbations will change adiabatically. Stability depends on the new density.

$$\begin{aligned} \rho^* < \rho' &\implies \text{perturbed element buoyant} \\ &\implies \text{system unstable;} \\ \rho^* > \rho' &\implies \text{perturbed element sinks back} \\ &\implies \text{system stable.} \end{aligned}$$

For adiabatic change,

$$\left.\begin{aligned} p &= K\rho^\gamma \\ p' &= K\rho^{*\gamma} \end{aligned}\right\} \implies \rho^* = \rho\left(\frac{p'}{p}\right)^{1/\gamma}. \tag{8.1}$$

To first order in the eulerian perturbation,

$$p' = p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z, \tag{8.2}$$

thus,

$$\begin{aligned} \rho^* &= \rho\left(\frac{p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z}{p}\right)^{1/\gamma} \\ &= \rho\left(1 + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z\right)^{1/\gamma} \\ &\approx \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z. \end{aligned} \tag{8.3}$$

In the surrounding medium,

$$\rho' = \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z, \tag{8.4}$$

and the system is unstable if $\rho^* < \rho'$. So instability needs

$$\begin{aligned}
\rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z &< \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z \\
\implies \quad \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} &< \frac{\mathrm{d}\rho}{\mathrm{d}z} \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\ln p &< \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln\rho \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\ln p\rho^{-\gamma}\right) &< 0 \\
\implies \quad \frac{\mathrm{d}K}{\mathrm{d}z} &< 0. \quad (\textit{instability})
\end{aligned} \tag{8.5}$$

So, the system is unstable if the entropy of the atmosphere decreases with increasing height. This can also be related to temperature and pressure gradients.

$$\frac{\mathrm{d}K}{\mathrm{d}z} < 0 \quad \implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\ln K < 0. \tag{8.6}$$

But,

$$K = p\rho^{-\gamma} \propto p^{1-\gamma}T^{\gamma} \quad \text{given} \quad p = \frac{\mathcal{R}_*\rho T}{\mu} \tag{8.7}$$

so,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\ln K = (1-\gamma)\frac{\mathrm{d}}{\mathrm{d}z}\ln p + \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln T &< 0 \\
\implies \quad \frac{\mathrm{d}T}{\mathrm{d}z} &< \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}. \quad (\textit{instability})
\end{aligned} \tag{8.8}$$

Hence, we have the *Schwarzschild stability criterion* which reads

Definition 8.1 Schwarzschild Stability Criterion

$$\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{8.9}$$

Since hydrostatic equilibrium requires $\mathrm{d}p/\mathrm{d}z < 0$, we see that (since $\gamma > 1$)

- Always stable to convection if $\mathrm{d}T/\mathrm{d}z > 0$;
- Otherwise, can tolerate a moderate negative temperature gradient provided

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|. \tag{8.10}$$

So convective instability develops when T declines too steeply with increasing height.

Examples (Convectively unstable systems).

- Outer regions of low mass stars;
- Cores of high mass stars.

For stable configurations, we can examine atmosphere dynamics: equation of motion for the fluid element is

$$
\begin{aligned}
\rho^* \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -g(\rho^* - \rho')\mathbf{??} \\
\implies \quad (\rho + \underbrace{\delta\rho}_{\text{small}})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -g\left[\frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z \\
\implies \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -\frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z. \qquad (8.11)
\end{aligned}
$$

So, it is a simple harmonic motion with angular frequency N where

$$N^2 = \frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right], \qquad (8.12)$$

which defines the *Brunt-Väisälä frequency*. These oscillations are *internal gravity waves*, as the restoring force is gravity.

8.2 Jeans Instability

This concerns the stability of a self-gravitating fluid against gravitational collapse. Consider the following system:

- Uniform medium initially static;
- Barotropic EoS;
- Gravitational field generated by the medium itself.

So equilibrium is

$$
\begin{aligned}
&p = p_0, \quad \text{(const.)} \\
&\rho = \rho_0, \quad \text{(const.)} \\
&\mathbf{u} = \mathbf{0} \\
&\Psi = \Psi_0, \quad \text{(const.)},
\end{aligned} \qquad (8.13)
$$

and governing equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad (8.14)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi \qquad (8.15)$$

$$\nabla^2\Psi = 4\pi G\rho. \qquad (8.16)$$

Introduce a perturbation:

$$p = p_0 + \Delta p \tag{8.17}$$
$$\rho = \rho_0 + \Delta\rho \tag{8.18}$$
$$\mathbf{u} = \Delta\mathbf{u} \tag{8.19}$$
$$\Psi = \Psi_0 + \Delta\Psi \tag{8.20}$$

Note: There is an inconsistency between the assumption ρ_0 = constant > 0 and the assumption $\Psi_0 =$ const. In a homogeneous, isotropic universe, the gravitational potential must be $\Psi_0 =$ const., by symmetry. However, Poisson's equation will yield $\Psi_0 =$ const. only if $\rho_0 = 0$. From the Newtonian point of view, in other words, an infinite, static, matter-filled universe cannot exist. The *Jeans swindle* deals with this difficulty by ignoring it.[1] This is closely tied to the fact that it is impossible to construct a model of a static infinite Universe. A more complete analysis of perturbations against a background of a (relativistic) homogenous expanding Universe recovers the same local instability as that found by Jeans, hence justifying the swindle. Let us, like Jeans, assume that $\Psi_0 = 0$ for the uniform medium.

Linearised equations are:

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{8.21}$$
$$\frac{\partial\Delta\mathbf{u}}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta\rho) - \boldsymbol{\nabla}(\Delta\Psi) = -c_s^2\frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0} - \boldsymbol{\nabla}(\Delta\Psi) \tag{8.22}$$
$$\nabla^2(\Delta\Psi) = 4\pi G\Delta\rho. \tag{8.23}$$

Look for plane wave solutions

$$\Delta\rho = \rho_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8.24}$$
$$\Delta\Psi = \Psi_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8.25}$$
$$\Delta\mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{8.26}$$

Substitution into the linear equations gives **??**

$$(8.21) \implies -\omega\rho_1 + \rho_0\mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{8.27}$$
$$(8.22) \implies -\rho_0\omega\mathbf{u}_1 = -c_s^2\mathbf{k}\rho_1 - \rho_1\mathbf{k}\Psi_1 \tag{8.28}$$
$$(8.23) \implies -k^2\Psi_1 = 4\pi G\rho_1. \tag{8.29}$$

Eliminating $\mathbf{u}_1$ and Ψ_1 from these

$$(8.27) + (8.28) \implies \rho_1\omega^2 = k^2\Big(\rho_1 c_s^2 + \rho_0\Psi_1\Big) \tag{8.30}$$
$$= k^2\rho_1 c_s^2 - 4\pi G\rho_0\rho_1$$
$$\implies \omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right). \tag{8.31}$$

[1] If you strongly object to being swindled, try reading "Mathematical Vindications of the Jeans Swindle", by M. Kiessling [4].

Introduce the Jeans wavenumber $k_J^2 = 4\pi G\rho_0/c_s^2$ so we have the dispersion relation

$$\boxed{\omega^2 = c_s^2\left(k^2 - k_J^2\right).} \tag{8.32}$$

Notes:

- For $k \gg k_J$, we have normal dispersion-free sound waves $\omega^2 = c_s^2 k^2$.
- For $k \gtrsim k_J$, we have modified sound waves. Gravity leads to dispersion of the wave and a slower group velocity.
- For $k < k_J$, ω is purely imaginary (for $k \in \mathbb{R}$), giving
$$\omega = i\tilde{\omega}, \quad \tilde{\omega} \in \mathbb{R}, \tag{8.33}$$
and,
$$e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{\tilde{\omega} t} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{8.34}$$
leading to exponentially growing perturbation solution: *Gravitational Instability.*

Definition 8.2 Jeans Length and Mass
The maximum stable wavelength is the *Jeans length*,

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}. \tag{8.35}$$

The associated mass is the *Jeans mass*,

$$M_J \sim \rho_0 \lambda_J^3. \tag{8.36}$$

Consider a collapsing system and suppose that the collapse is isothermal (cooling and heating processes balance). Noting that $M_J \propto c_s^3 \rho_0^{-1/2} \propto (T^3/\rho_0)^{1/2}$, we see the Jeans mass decreases as system collapses and the system undergoes gravitational fragmentation.

These are central concepts in the theory of

- *Star formation* (instability of giant molecular clouds);
- *Cosmological structure formation* (instability of the homogeneous primordial gas).

8.3 Rayleigh-Taylor and Kelvin-Helmholtz Instability

This concerns the stability of an interface with a discontinuous change in tangential velocity and/or density.

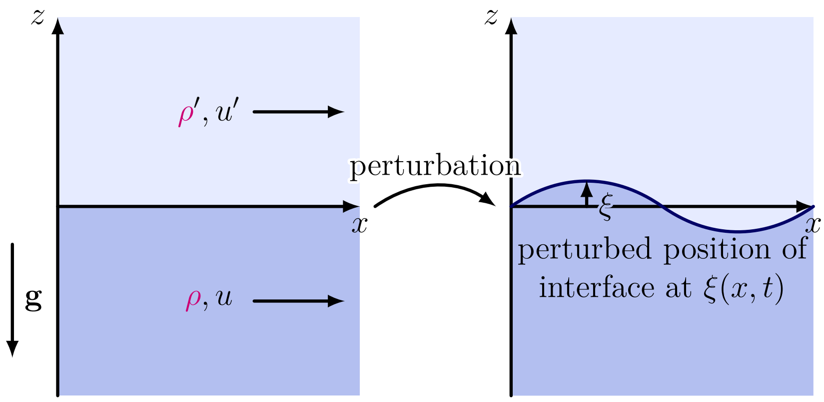

Fig. 8.2: Perturbation of interface of discontinuity

For convenience, let's assume:

- Constant gravity, ideal fluid;
- Pressure continuous across the interface;
- Incompressible flow $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$;
- Irrotational flow $\boldsymbol{\nabla} \times \mathbf{u} = 0 \implies \mathbf{u} = -\boldsymbol{\nabla}\Psi$;
- 2D problem (symmetry direction into the page of Fig. 8.2)

The momentum equation (for either upper or lower fluid) is

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla} p + \mathbf{g} \\
\implies \quad -\boldsymbol{\nabla}\frac{\partial \Psi}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) &= - \underbrace{\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)}_{\substack{\text{since } \rho = \text{const.} \\ \text{within each fluid}}} - \boldsymbol{\nabla}\Psi \\
\implies \quad \boldsymbol{\nabla}\left[-\frac{\partial \Psi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi\right] &= 0 \\
\implies \quad -\frac{\partial \Psi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi &= F(t).
\end{aligned} \tag{8.37}
$$

where $F(t)$ is a function that is constant in space but not in time.

Now consider a perturbation at the interface of these two fluids. Let us study the evolution of the perturbed position of the interface $\xi(x, t)$.

The velocity potential is $u = -\boldsymbol{\nabla}\Psi$, so if the unperturbed velocities in the fluids are U and U' we have

$$\Phi_{\text{low}} = -Ux + \phi \tag{8.38}$$

$$\Phi_{\text{up}} = -U'x + \phi', \tag{8.39}$$

thus,

$$\nabla^2\phi = \nabla^2\phi' = 0. \quad (\text{since } \boldsymbol{\nabla} \cdot \mathbf{u} = 0) \tag{8.40}$$

ϕ and ϕ' are sourced by displacements of the interface. Consider an element of the lower fluid that is at the interface. Then

$$u_z = \frac{\mathrm{D}\xi}{\mathrm{D}t}, \tag{8.41}$$

giving

$$\left.\begin{aligned} -\frac{\partial\phi}{\partial z} &= \frac{\partial\xi}{\partial t} + U\frac{\partial\xi}{\partial x} \\ -\frac{\partial\phi'}{\partial z} &= \frac{\partial\xi}{\partial t} + U'\frac{\partial\xi}{\partial x} \end{aligned}\right\} \quad \text{to first order} \tag{8.42}$$

Now look for plane wave solutions,

$$\xi = Ae^{i(kx-\omega t)} \tag{8.43}$$
$$\phi = Ce^{i(kx-\omega t)+k_z z} \tag{8.44}$$
$$\phi' = C'e^{i(kx-\omega t)+k'_z z}, \tag{8.45}$$

where extra terms on the exponents $k_z z$ and $k'_z z$ are there to seek solutions where the perturbed potential decays at large $|z|$.

But we know that

$$\begin{aligned} \nabla^2\phi = 0 \quad &\implies \quad -k^2 + k_z^2 = 0 \\ &\implies \quad k_z = |k|, \end{aligned} \tag{8.46}$$

so $\phi \to 0$ as $z \to \infty$.

For now, let's stipulate $k > 0$. So

$$\phi = Ce^{i(kx-\omega t)+kz} \tag{8.47}$$
$$\phi' = C'e^{i(kx-\omega t)-kz}. \tag{8.48}$$

From Eq. (8.42), we have

$$-kC = -i\omega A + iUkA = i(kU-\omega)A \tag{8.49}$$
$$kC' = i(kU'-\omega)A.) \tag{8.50}$$

We need one more equation if we're to solve for A, C, C'. We get that from pressure balance across the interface.

$$p = -\rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) + \rho F(t) \tag{8.51}$$
$$p' = -\rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + \rho' F'(t), \tag{8.52}$$

and equality at $z = 0$:

$$\rho\left(-\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) = \rho'\left(-\frac{\partial \phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + K(t), \tag{8.53}$$

where

$$K \equiv \rho F(t) - \rho' F'(t). \tag{8.54}$$

The perturbation vanishes for $z \to \pm\infty$ at all times, so we can look at equation Eq. (8.37) for each fluid in the limit $|z| \to \infty$, taking limit carefully so that Ψ terms cancel, to get

$$\rho F(t) - \rho' F'(t) = \underbrace{\frac{1}{2}U^2\rho - \frac{1}{2}U'^2\rho'}_{\substack{\text{conditions at } \infty \\ \text{and so a constant}}}. \tag{8.55}$$

Therefore, $K(t)$ is actually a constant.

Next in our attempt to use Eq. (8.53) to match across boundary, we need to determine u and u'. Now

$$\begin{aligned} \mathbf{u} &= -\boldsymbol{\nabla}\Psi = -\boldsymbol{\nabla}(-Ux + \phi) = U\hat{\mathbf{x}} - \boldsymbol{\nabla}\phi \\ \Longrightarrow \quad u^2 &= U^2 - 2U\frac{\partial \phi}{\partial x}, \quad \text{(dropping 2}^{\text{nd}}\text{ order terms)} \end{aligned} \tag{8.56}$$

and similarly

$$u'^2 = U'^2 - 2U'\frac{\partial \phi'}{\partial x}. \tag{8.57}$$

So, Eq. (8.53) reads

$$\rho\left(-\frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 - U\frac{\partial \phi}{\partial x} + g\xi\right) = \rho'\left(-\frac{\partial \phi'}{\partial t} + \frac{1}{2}U'^2 - U'\frac{\partial \phi'}{\partial x} + g\xi\right) + \underbrace{\frac{1}{2}U^2\rho - \frac{1}{2}U'^2\rho'}_{K}$$

$$\begin{aligned} \Longrightarrow \quad & \rho\left(-\frac{\partial \phi}{\partial t} - U\frac{\partial \phi}{\partial x} + g\xi\right) = \rho'\left(-\frac{\partial \phi'}{\partial t} - U'\frac{\partial \phi'}{\partial x} + g\xi\right) \\ \Longrightarrow \quad & \rho i\omega C - \rho U ikC + \rho gA = \rho' i\omega C' - \rho' U' ikC' + \rho' gA \\ \Longrightarrow \quad & \rho(kU - \omega)C + i\rho gA = \rho'(kU' - \omega)C' + i\rho' gA. \end{aligned} \tag{8.58}$$

Now eliminate C and C' from Eqs. (8.49) and (8.50) to give

$$\boxed{\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho').} \tag{8.59}$$

This is the dispersion relation for our system. Let's now look at some specific applications.

1. *Surface gravity waves*: two fluids at rest initially with $\rho' < \rho$ (i.e. denser fluid on bottom). The dispersion relation gives

$$\omega^2(\rho+\rho') = kg(\rho-\rho')$$
$$\implies \quad \omega^2 = k\frac{g(\rho-\rho')}{\rho+\rho'}. \tag{8.60}$$

So, for $k \in \mathbb{R}$, we have that $\omega \in \mathbb{R}$ and hence the system displays oscillations/waves. Phase speed is

$$\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}} = \underbrace{f(k)}_{\text{waves are dispersive}}. \tag{8.61}$$

If $\rho' \ll \rho$, then $\omega/k = \pm\sqrt{g/k}$.

Example. Surface waves on the ocean.

2. *Static stratified fluid*: two fluids at rest initially with $\rho' > \rho$ (i.e. denser fluid on top). Then

$$\omega^2 = k\frac{g(\rho-\rho')}{\rho+\rho'}. \tag{8.62}$$

So, for $k \in \mathbb{R}$ we have $\omega^2 < 0$ and so ω is purely imaginary.

$$\frac{\omega}{k} = \pm i\sqrt{\frac{g}{k}\frac{\rho'-\rho}{\rho+\rho'}}. \tag{8.63}$$

The positive root of this gives us exponentially growing solutions. This is the *Rayleigh-Taylor Instability.*

3. *Fluids in motion*: two fluids with $\rho > \rho'$ (so stable to Rayleigh-Taylor) but different velocities, non-zero U and U'. Take full dispersion relation:

$$\rho(kU-\omega)^2 + \rho'(kU'-\omega)^2 = kg(\rho-\rho'), \tag{8.64}$$

divide by k^2 and solve the quadratic in ω/k,

$$\implies \quad \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho+\rho'} \pm \sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'} - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}}. \tag{8.65}$$

There is instability if

$$\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'} - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2} < 0. \quad \text{(instability)} \tag{8.66}$$

If $g = 0$, then any relative motion gives instability, i.e. *Kelvin-Helmholtz Instability*;

If $g \neq 0$, then unstable modes are those with

$$k > \frac{(\rho^2-\rho'^2)g}{\rho\rho'(U-U')^2}, \tag{8.67}$$

i.e. gravity is a stabilising influence.

8.4 Thermal Instability

This concerns the stability of a medium in thermal equilibrium (heating = cooling) to perturbations in temperature. Consider the following system:

- Ideal gas;
- No gravitational field;
- Static thermal equilibrium

$$\mathbf{u}_0 = \mathbf{0},\ \dot{Q}_0 = 0,\ \underbrace{\boldsymbol{\nabla} p_0 = \mathbf{0},\ \boldsymbol{\nabla}\rho_0 = \mathbf{0}}_{\boldsymbol{\nabla} K_0 = \mathbf{0}} \quad \text{where} \quad p = K\rho^\gamma. \tag{8.68}$$

Let's start by deriving an alternative form of the energy equation that involves the entropy-like variable K; this will be well suited to problems of thermal instability.

$$\begin{aligned} p = K\rho^\gamma \quad \implies \quad \mathrm{d}p &= \rho^\gamma\,\mathrm{d}K + K\gamma\rho^{\gamma-1}\,\mathrm{d}\rho \\ &= \rho^\gamma\,\mathrm{d}K + \frac{\gamma p}{\rho}\,\mathrm{d}\rho\,, \end{aligned} \tag{8.69}$$

also,

$$\begin{aligned} p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \implies \quad \mathrm{d}p &= \frac{\mathcal{R}_*}{\mu}T\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ &= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T\,. \end{aligned} \tag{8.70}$$

Equate Eqs. (8.69) and (8.70) to give

$$\begin{aligned} \rho^\gamma\,\mathrm{d}K + \gamma\frac{p}{\rho}\,\mathrm{d}\rho &= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ \implies \quad \rho^\gamma\,\mathrm{d}K &= (1-\gamma)\frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ \implies \quad \mathrm{d}K &= \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\right]}_{-đQ}. \end{aligned} \tag{8.71}$$

First law of thermodynamics:

$$đQ = p\,\mathrm{d}V + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T\,, \quad \text{(unit mass)} \tag{8.72}$$

and so

$$\begin{aligned} đQ &= p\,\mathrm{d}(1/\rho) + C_V\,\mathrm{d}T \\ &= -\frac{p}{\rho^2} - \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\,. \quad \text{(since we have } (\gamma-1)C_V = \mathcal{R}_*/\mu) \end{aligned} \tag{8.73}$$

Then we have

$$\mathrm{d}K = -(1-\gamma)\rho^{1-\gamma}đQ, \quad \text{(for fluid element)} \tag{8.74}$$

Turn this into Lagrangian energy equation by noting that $\dot{Q} = -đQ/\,\mathrm{d}t$,

$$\implies \quad \frac{\mathrm{D}K}{\mathrm{D}t} = -(1-\gamma)\rho^{1-\gamma}\dot{Q}$$

$$\implies \quad \frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\ln K) = -(\gamma-1)\frac{\rho}{p}\dot{Q}, \tag{8.75}$$

and thus we derive the *Entropy form of the energy equation*,

Equation 8.1 Entropy Form of the Energy Equation

$$\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\frac{\rho\dot{Q}}{p} \tag{8.76}$$

This joins our usual continuity and momentum equations

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{8.77}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}p. \tag{8.78}$$

Now we look at thermal stability. Consider small perturbations to the equilibrium

$$\rho \to \rho_0 + \Delta\rho \tag{8.79}$$

$$p \to p_0 + \Delta p \tag{8.80}$$

$$\mathbf{u} \to \Delta\mathbf{u} \tag{8.81}$$

$$K \to K_0 + \Delta K. \tag{8.82}$$

Linearise the equations:

$$(8.77) \implies \frac{\partial\Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{8.83}$$

$$(8.78) \implies \rho_0\frac{\partial\Delta\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}(\Delta p) \tag{8.84}$$

$$(8.76) \implies \frac{\partial\Delta K}{\partial t} = -\frac{\gamma-1}{\rho_0^{\gamma-1}}\Delta\dot{Q}, \tag{8.85}$$

where we can write

$$\Delta\dot{Q} = \left.\frac{\partial\dot{Q}}{\partial p}\right|_{\rho}\Delta p + \left.\frac{\partial\dot{Q}}{\partial\rho}\right|_{p}\Delta p, \tag{8.86}$$

so that

$$\frac{\partial\Delta K}{\partial t} = -A^*\Delta p - B^*\Delta\rho, \tag{8.87}$$

with

$$A^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}}{\partial p}\right|_{\rho}, \quad B^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}}{\partial\rho}\right|_{p}. \tag{8.88}$$

We also have

$$p = K\rho^\gamma \quad \Longrightarrow \quad \Delta p = \rho_0^\gamma \Delta K + \gamma \frac{p_0}{\rho_0}\Delta\rho. \tag{8.89}$$

We seek solutions of the form

$$\Delta p = p_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.90}$$
$$\Delta \rho = \rho_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.91}$$
$$\Delta \mathbf{u} = \mathbf{u}_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.92}$$
$$\Delta K = K_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.93}$$

so, instability if $\mathrm{Re}\{q\} > 0$. Substituting into linearised equations gives

$$(8.83) \implies q\rho_1 + \rho_0 i\mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{8.94}$$
$$(8.84) \implies q\rho_0\mathbf{u}_1 = -i\mathbf{k}p_1 \tag{8.95}$$
$$(8.87) \implies qK_1 = -A^* p_1 - B^*\rho_1 \tag{8.96}$$
$$(8.89) \implies p_1 = \rho_0^\gamma K_1 + \frac{\gamma p_0}{\rho_0}\rho_1. \tag{8.97}$$

We can combine these to obtain the dispersion relation:

$$\frac{A^* q}{k^2} - \frac{B^*}{q} = -\left(\frac{q^2}{k^2} + \gamma\frac{p_0}{\rho_0}\right)\frac{1}{\rho_0^\gamma}$$
$$\implies \underbrace{q^3 + A^*\rho_0^\gamma q^2 + k^2\gamma\frac{p_0}{\rho_0}q - B^* k^2 \rho_0^\gamma}_{\text{cubic in } q, \text{ call } E(q)} = 0. \tag{8.98}$$

This has at least one real root – system is unstable if that real root is positive, $q > 0$.

Now $E(\infty) = \infty$, $E(0) = -B^* k^2 \rho_0^\gamma$. So the system is unstable if $B^* > 0$.

$$B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p > 0 \quad \text{(condition for stability)} \tag{8.99}$$
$$\implies \left.\frac{\partial \dot{Q}}{\partial\left(\frac{\mu p}{\mathcal{R}_* T}\right)}\right|_p > 0$$
$$\implies -\frac{T^2}{p}\left.\frac{\partial \dot{Q}}{\partial T}\right|_p > 0, \tag{8.100}$$

and thus we arrive at the *Field criterion*,

Definition 8.3 Field Criterion

$$\text{unstable if} \quad \left.\frac{\partial \dot{Q}}{\partial T}\right|_p > 0 \tag{8.101}$$

The system is always unstable if it's Field unstable (named after George Field who wrote the classic paper on thermal instability in 1965).

However, even a Field stable system can be unstable if $A^* < 0 \implies \partial\dot{Q}\big/\partial T\,\Big|_\rho < 0$. From the dispersion relation, we see that this can happen for long wavelength modes, i.e. k small. Then

$$q^2(q + A^*\rho_0^\gamma) \approx 0 \qquad \Longrightarrow \qquad q \approx -A^*\rho_0^\gamma. \tag{8.102}$$

Interpretation: : instability if net cooling rate decreases when temperature increases,

- Short wavelength perturbations are readily brought into pressure equilibrium by the action of sound waves, therefore, thermal instability proceeds at fixed pressure;
- Long wavelength perturbations: there is insufficient time for sound waves to equalise pressure with surroundings, so they tend to develop at constant density (sometimes called isochoric thermal instability).

Example. Let's assume a specific form for $\dot{Q}$,

$$\begin{aligned}\dot{Q} &= A\rho T^\alpha - H \\ &= \frac{A\mu}{\mathcal{R}_*} p T^{\alpha-1} - H,\end{aligned} \tag{8.103}$$

so,

$$\left.\frac{\partial\dot{Q}}{\partial T}\right|_p = (\alpha - 1)\frac{A\mu p}{\mathcal{R}_*}T^{\alpha-2}. \tag{8.104}$$

This is Field unstable, $\partial\dot{Q}\big/\partial T\,\Big|_p < 0$ if $\alpha < 1$. Bremsstrahlung has $\alpha = 0.5 \implies$ Field unstable.

Note about gravity:

- Buoyancy interactions with thermal instability is a powerful stabilising effect;
- Thermal instability can become a subtle question of the functional dependence of the cooling/heating balance.

CHAPTER 9

Viscous Flows

Thus far, we have been assuming that changes in the momentum of a fluid element are due entirely to pressure forces (acting normal to the surface of the element) or gravity (acting on the bulk).

This assumption is justified in the limit $\lambda \to 0$, i.e. the particles composing the fluid have vanishingly small collisional mean-free-path.

For finite-λ, momentum can diffuse through the fluid. This brings us to a discussion of viscosity.

9.1 Basics of Viscosity

In a viscous flow, momentum can be transferred if there are velocity differences between fluid elements.

The continuity equation is unchanged

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0. \tag{9.1}$$

But the momentum equation needs to be changed

$$\frac{\partial}{\partial t}(\rho u_i) = \partial_j \sigma_{ij} + \rho g_i, \quad g_i = -\partial_i \Psi, \tag{9.2}$$

with

$$\sigma_{ij} = \rho u_i u_j + p\delta_{ij} - \underbrace{\sigma'_{ij}}_{\substack{\text{viscous} \\ \text{stress tensor}}} . \tag{9.3}$$

As we will see later, σ'_{ij} is related to velocity gradients.

The connection between the viscous stress tensor and the microphysics (i.e. the mean-free-path) is uncovered by considering a simple linear shear flow as in Fig. 9.1.

Microscopically, thermal/random motion of the particles can allow momentum to "diffuse" across streamlines. This becomes more important as gas gets less collisional.

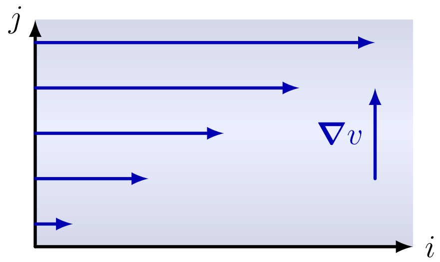

Fig. 9.1: Linear shear flow

Let's analyse the microscopic behaviour: assume the typical (thermal) velocity in the j-direction is u_j. So the momentum flux associated with this is

$$\underbrace{\rho u_i u_j}_{\substack{i\text{-component of momentum}\\ \text{carried in } j\text{-direciton}}} \quad . \tag{9.4}$$

The typical thermal velocity is $\sim \sqrt{kT/m}$. So, flux of the i-component of momentum in the upward j-direction is

$$\rho u_i \alpha \sqrt{\frac{kT}{m}}, \quad \alpha \sim 1. \tag{9.5}$$

For the element on the other side of the surface in the j-direction, the corresponding momentum flux across the surface is

$$-\rho u_i^* \alpha \sqrt{\frac{kT}{m}}, \tag{9.6}$$

where u_i^* is the i-velocity of that element. For a j-separation of δl we have

$$u_i^* = u_i + \delta l(\partial_j u_i). \tag{9.7}$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\delta l \alpha \sqrt{\frac{kT}{m}}. \tag{9.8}$$

The relevant scale δl is the mean-free-path

$$\delta l \sim \lambda = \frac{1}{n\sigma}, \tag{9.9}$$

where n is the number density and σ is the collision cross section of the particles. If we treat the particles as hard spheres of radius a (decent approximation for neutral gas), then

$$\sigma = \pi a^2. \tag{9.10}$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\frac{m}{\rho \pi a^2}\alpha\sqrt{\frac{kT}{m}}. \tag{9.11}$$

Putting this into the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j\left[\underbrace{\frac{\alpha}{\pi a^2}\sqrt{mkT}}_{\equiv\eta,\text{ shear viscosity}}\partial_j u_i\right] + \rho g_i. \tag{9.12}$$

A rigorous derivation shows that, for this hard-sphere model, $\alpha = 5\sqrt{\pi}/64$.

Observations about the shear viscosity:

- η is independent of density (a denser gas has more particles to transport the momentum but the mean-free-path is shorter);
- η increases with T;
- Isothermal system has $\eta = \text{const}$;
- Functional dependence on T depends on collision model, hard sphere model gives $\eta \propto T^{1/2}$ whereas coulomb collisions (relevant for fully ionised plasma) gives $\eta \propto T^{5/2}$.

For a fully ionised plasma (e.g. the ICM), the mean-free-path is set by Coulomb collisions. Then

$$\lambda \propto T^2, \quad v_{\text{th}} \propto \sqrt{T} \quad \Longrightarrow \quad \eta \propto T^{5/2}. \tag{9.13}$$

Thus the viscosity has a stronger temperature dependence than found for hard-sphere collisions.

9.2 Navier-Stokes Equation

The most general form of σ'_{ij} which is

- Galilean invariant;
- Linear in velocity components;
- Isotropic

is given by

$$\sigma'_{ij} = \eta\Big(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\Big) + \zeta\delta_{ij}\partial_k u_k, \tag{9.14}$$

with η and ζ independent of velocity. This is a *symmetric* tensor which ensures that there are no unbalanced torques on fluid elements.

The term associated with η relates to momentum transfer in *shear flows* (this term has zero trace). The term associated with ζ relates to momentum transfer due to bulk compression ($\partial_k u_k \equiv \boldsymbol{\nabla} \cdot \mathbf{u}$).

Putting this into the momentum equation gives

$$\frac{\partial(\rho u_i)}{\partial t} = -\partial_j(\rho u_i u_j) - \partial_j p\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i, \tag{9.15}$$

which we can combine with the continuity equation to give

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\partial_j u_i\right) = -\partial_j p\delta_{ij} + \underbrace{\partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right]}_{\text{viscous force}} + \rho g_i. \tag{9.16}$$

This is the general form of the *Navier-Stokes equation*.

Definition 9.1 Navier-Stokes Equation

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\partial_j u_i\right) = -\partial_j p\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i \tag{9.17}$$

Outside of shocks ($\zeta \approx 0$) and for isothermal fluids ($\eta =$ constant) we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi + \underbrace{\frac{\eta}{\rho}}_{\substack{\equiv\nu \\ \text{kinematic} \\ \text{viscosity}}}\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]. \tag{9.18}$$

This is the more commonly used form of the *Navier-Stokes equation*.

The importance of viscosity in a flow is characterized via the *Reynolds number*

Definition 9.2 Reynolds Number

$$\mathrm{Re} = \frac{|\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \quad\Longrightarrow\quad \mathrm{Re} = \frac{UL}{\nu}, \tag{9.19}$$

where U and L are the characteristic velocity and length scales of the system, respectively.

The Reynolds number is the ratio of the inertial forces ($\sim \rho U^2/L$) to the viscous forces ($\sim \eta U/L^2$). Thus, when $\mathrm{Re} \gg 1$, the viscous forces are negligible. As the scale of interest L becomes smaller and smaller, there is some length scale on which viscosity becomes important. Viscous flow tends to be laminar, while less-viscous flow tends to be turbulent. However, the transition between laminar and turbulent flow is not determined uniquely by the Reynolds number; the geometry of the system is also important. For fluids moving through a straight pipe, to take a well-studied example, flow with $\mathrm{Re} \lesssim 3000$ is laminar and flow with $\mathrm{Re} \gtrsim 3000$ is turbulent. For a dimpled golf ball, the critical Reynolds number for the transition from laminar to turbulent flow is $\mathrm{Re}_{\mathrm{cr}} \sim 30,000$; for a smooth golf ball, it's $\mathrm{Re}_{\mathrm{cr}} \sim 300,000$.

Consequences of viscosity:

- Shear leads to transmission of momentum through flow (layers rub);
- Vorticity:
  - Can introduce vorticity into initially irrotational flows from the boundaries;
  - Vorticity diffuses through the flow (advection/diffusion = Re).
- Generally has stabilising effect on various fluid instabilities;
- Dissipates kinetic energy into heat.

9.3 Vorticity in Viscous Flows

Start with the Navier-Stokes equation with $\zeta = 0$ and $\eta =$ const., and take the curl of this, recalling the definition of the vorticity $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$:

$$\frac{\partial \mathbf{w}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = \boldsymbol{\nabla} \times \left(-\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi + \frac{\eta}{\rho} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right] \right). \tag{9.20}$$

To tidy up LHS, use the vector identity and definition of vorticity:

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \frac{1}{2} \boldsymbol{\nabla} u^2 - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) \tag{9.21}$$

$$\implies \quad \boldsymbol{\nabla} \times (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = -\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}). \tag{9.22}$$

To tidy up RHS, assume a barotropic fluid, $p = p(\rho)$:

$$\begin{aligned} \implies \quad \boldsymbol{\nabla} \times \left(\frac{1}{\rho} \boldsymbol{\nabla} p \right) &= \boldsymbol{\nabla} \left(\frac{1}{\rho} \right) \times \boldsymbol{\nabla} p + \frac{1}{\rho} \boldsymbol{\nabla} \times \boldsymbol{\nabla} p \\ &= -\frac{1}{\rho^2} \underbrace{\boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} p}_{\substack{=0 \text{ since surfaces of} \\ \text{constant } \rho \text{ and } p \text{ align}}} . \end{aligned} \tag{9.23}$$

Putting the pieces together, we get

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \boldsymbol{\nabla} \times \left[\frac{\eta}{\rho} \nabla^2 \mathbf{u} \right], \tag{9.24}$$

and thus

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \frac{\eta}{\rho} \nabla^2 \mathbf{w}.} \tag{9.25}$$

where, in the last step, we have ignored gradients of $\nu = \eta/\rho$ (so strictly assumed isothermal and uniform density). So, vorticity is carried with the flow but also diffuses through the flow due to action of vorticity.

Lines of vorticity are advected in the flow *and* diffuse through the flow due to viscosity. Viscous term gives a way for vorticity to enter a previously irrotational flow due to boundary interactions. Relative importance of advection and diffusion given by Reynolds number.

9.4 Energy Dissipation in Incompressible Viscous Flows

Viscosity leads to dissipation of kinetic energy into heat – an irreversible process.

Let's analyse this in the case of an incompressible flow so that we don't need to worry about $p\,\mathrm{d}V$ work. Then the total kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}\int \rho u^2 \,\mathrm{d}V\,. \tag{9.26}$$

Let's consider the rate of change of E_{kin} with time

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= u_i \frac{\partial}{\partial t}(\rho u_i)\\ &= -u_i\partial_j(\rho u_i u_j) - u_i\partial_j\delta_{ij}p + u_i\partial_j\sigma'_{ij}\\ &= -u_i\partial_j(\rho u_i u_j) - u_i\partial_i p + \partial_j(u_i\sigma'_{ij}) - \sigma'_{ij}\partial_j u_i.\end{aligned} \tag{9.27}$$

Look at the first term of RHS:

$$u_i\partial_j(\rho u_i u_j) = u_i\Big(u_j\partial_j(\rho u_i) + \rho u_i \underbrace{\partial_j u_j}_{=0}\Big), \tag{9.28}$$

where last term is zero due to the incompressible flow assumption,

$$\boldsymbol{\nabla}\cdot\mathbf{u} \quad\Longrightarrow\quad \partial_j u_j = 0. \tag{9.29}$$

Also note that

$$\begin{aligned}\partial_j\left(\rho u_j\cdot\frac{1}{2}u_i u_i\right) &= \frac{1}{2}\rho u_i u_i \underbrace{\partial_j u_j}_{=0} + u_j\partial_j\left(\frac{1}{2}\rho u_i u_i\right)\\ &= u_j u_i \partial_j(\rho u_i),\end{aligned} \tag{9.30}$$

therefore

$$u_i\partial_j(\rho u_i u_j) = \partial_j\left(\rho u_j\cdot\frac{1}{2}u_i u_i\right). \tag{9.31}$$

So,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= -\partial_j\left(\rho u_j\cdot\frac{1}{2}u_i u_i\right) - \partial_i(u_i p) + p\partial_i u_i + \partial_j(u_i\sigma'_{ij}) - \sigma'_{ij}\partial_j u_i\\ &= -\partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right) - \sigma'_{ij}\partial_j u_i.\end{aligned} \tag{9.32}$$

Integrating over the volume,

$$\begin{aligned}\frac{\partial E_{\text{kin}}}{\partial t} &= \frac{\partial}{\partial t}\int_{\mathcal{V}} \frac{1}{2}\rho u^2 \,\mathrm{d}V \\ &= -\int_{\mathcal{V}} \partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right)\mathrm{d}V - \int_{\mathcal{V}} \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= \underbrace{-\oint_{\mathcal{S}}\left(\rho\mathbf{u}\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'\right)\cdot\mathrm{d}\mathbf{S}}_{\substack{\text{Energy flux into volume including} \\ \text{work done by viscous forces } \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'}} \underbrace{-\int_{\mathcal{V}} \sigma'_{ij}\partial_j u_i \,\mathrm{d}V}_{\substack{\text{Rate of change of } E_{\text{kin}} \text{ due} \\ \text{to viscous dissipation}}} \quad . \end{aligned} \tag{9.33}$$

Let's take the volume $\mathcal{V}$ to be the whole fluid so that the surface integral is zero (e.g. $\mathbf{u}$ at bounding surface $= 0$, or $\mathbf{u}$ at $\infty = 0$).

Then

$$\begin{aligned}\frac{\partial E_{\text{kin}}}{\partial t} &= -\int_{\mathcal{V}} \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= -\frac{1}{2}\int_{\mathcal{V}} \sigma'_{ij}(\partial_j u_i + \partial_i u_j)\,\mathrm{d}V \quad \text{since } \sigma' \text{ is symmetric}\end{aligned} \tag{9.34}$$

But $\sigma'_{ij} = \eta(\partial_j u_i + \partial_i u_j)$ for an incompressible fluid. So,

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\frac{1}{2}\int_{\mathcal{V}} \eta(\partial_j u_i + \partial_i u_j)^2 \,\mathrm{d}V\,. \tag{9.35}$$

2nd law of thermodynamics dictates that kinetic energy must "grind down" to heat rather than reverse. We see that η needs to be positive in order for us to obey the 2nd law of thermodynamics.

9.5 Viscous Flow through a Pipe

Now consider flow through a long pipe with a constant circular cross-section (see Fig. 9.2).

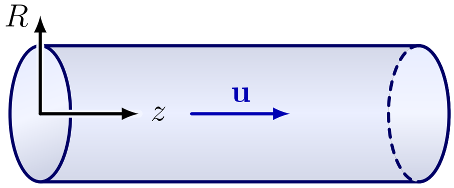

Fig. 9.2: A pipe.

Assume

- Steady flow with $u_R = u_\phi = 0$, $u_z \neq 0$;
- Incompressible, uniform density fluid;
- Neglect gravity.

The Navier-Stokes equation reads

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\substack{=0 \text{ since} \\ \text{steady state}}} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}}_{\substack{=0 \text{ due to} \\ \text{symmetry}}} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \underbrace{\boldsymbol{\nabla}\Psi}_{\substack{=0 \text{ since} \\ \text{no gravity}}} + \nu\left[\nabla^2 \mathbf{u} + \underbrace{\frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u})}_{\substack{=0 \text{ since} \\ \text{incompressible}}}\right]$$

$$\implies \quad \nu \nabla^2 \mathbf{u} = \frac{1}{\rho}\boldsymbol{\nabla} p. \tag{9.36}$$

By symmetry we have

$$u_R = u_\phi = 0 \quad \implies \quad \frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0. \tag{9.37}$$

For the z-component

$$\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\substack{\text{function of} \\ z \text{ only}}} = \underbrace{\nu \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)}_{\text{function of } R \text{ only}} = \underbrace{-\frac{1}{\rho}\frac{\Delta p}{l}}_{\substack{\text{constant, written} \\ \text{in terms of global} \\ \text{pressure gradient}}} . \tag{9.38}$$

Integrating gives

$$u = -\frac{\Delta p}{4\rho\nu l}R^2 + a \ln R + b, \tag{9.39}$$

where a and b are constants. Apply boundary conditions:

- At $R = 0$, u finite $\implies a = 0$;
- At $R = R_0$, $u = 0$ (no slip boundary condition at wall).

$$\implies \quad u = \frac{\Delta p}{4\rho\nu l}(R_0^2 - R^2). \tag{9.40}$$

So the velocity profile is *parabolic* (see Fig. 9.3).

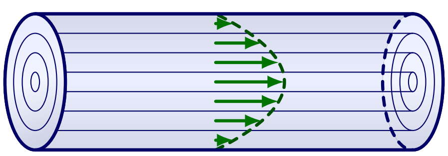

Fig. 9.3: Parabolic velocity profile for viscous flow through a pipe

The mass flux passing through an annular element $2\pi R \, \mathrm{d}R$ is $2\pi R\rho u \, \mathrm{d}R$. So, the total mass flow rate is

$$Q = \int_0^{R_0} 2\pi\rho u R \, \mathrm{d}R = \frac{\pi}{8}\frac{\Delta p}{\nu l}R_0^4. \tag{9.41}$$

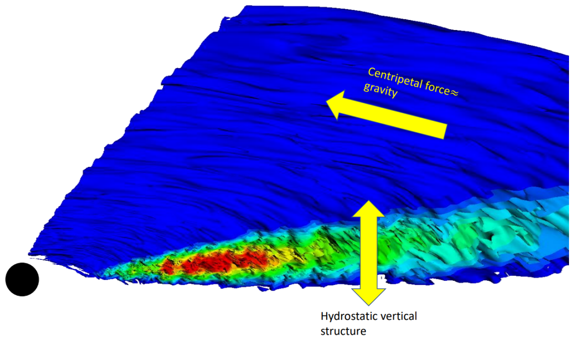

Fig. 9.4: Accretion disc, with gas settled into circular orbits in hydrostatic equilibrium with an internal vertical pressure gradient balancing the vertical component of gravity.

Mass flux completely determined by the pressure gradient, radius of pipe, and coefficient of kinematic viscosity.

As $\eta \to 0$, i.e. $\nu \to 0$, the flow rate $\to \infty$ (or, in other words, an inviscid flow cannot be in steady state in this pipe if there is a non-zero pressure gradient).

If Δp increases sufficiently, it becomes unstable and irregular, giving turbulent motions above a critical speed.

The actual transition to turbulence is usually phrased in terms of the *Reynolds number*

$$\mathrm{Re} \equiv \frac{LV}{\nu}, \tag{9.42}$$

where L and V are "characteristic" length and velocity scales of the system. Flow will become turbulent (not steady, $u_R \neq 0$, $u_\phi \neq 0$, motions on large range of scales), above a critical Reynolds number,

$$\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}. \tag{9.43}$$

9.6 Accretion Disks

Accretion disks are one of the most important applications of the Navier-Stokes equation in astrophysics.

Consider some gas flowing towards a central object (star, planet, black hole, ...). Almost always, the gas will have significant angular momentum about that object. If gravitationally bound to the object, the gas will settle into a plane defined by the mean angular momentum vector. Residual motions in other directions will be damped out on a free-fall timescale.

The gas will settle into circular orbits – the lowest energy configuration for a given angular momentum. In the vertical direction (parallel to the angular momentum vector) the system will come into hydrostatic equilibrium with an internal vertical pressure gradient balancing the vertical component of gravity. In the radial direction (along the direction towards the central object), the system will achieve a state where the centripetal force is supplied by gravity and the radial pressure gradient.

A very important special case is when the disk is "thin", meaning that the scale-height in the vertical direction h is much less than the radius r. Then, radial pressure gradients are negligible and we can just write

$$\Omega^2 R = \frac{GM}{R^2} \quad \Longrightarrow \quad \Omega = \sqrt{\frac{GM}{R^3}}, \tag{9.44}$$

where Ω is the angular velocity of the flow around the central object. This means that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} \neq 0 \quad \Longrightarrow \quad \text{shear flow.} \tag{9.45}$$

Viscosity will allow angular momentum to be transferred from the fast moving inner regions to the more slowly moving outer regions. This means the inner disk fluid elements lose angular momentum. We have

$$J = R^2\Omega = \sqrt{GMR}, \quad \text{(per unit mass)} \tag{9.46}$$

meaning that inner disk fluid elements drift inwards.

Ultimately, most of the mass flows inwards; a small amount of the mass carries all of the angular momentum out to large radii.

Let's set up a simple model for a geometrically-thin accretion disk. We assume:

- Cylindrical polar coordinates (R, ϕ, z);
- Axisymmetric, $\partial/\partial\phi = 0$;
- Hydrostatic equilibrium in the z-direction, $u_z = 0$;
- u_ϕ close to Keplerian velocity (i.e. thin disk);
- u_R small and set by action of viscosity;
- Bulk viscosity zero.

The continuity equation in cylindrical polar coordinates is

$$\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \underbrace{\frac{1}{R}\frac{\partial}{\partial\phi}(\rho u_\phi)}_{\substack{\text{=0 due to}\\ \text{axisymmetry}}} + \underbrace{\frac{\partial}{\partial z}(\rho u_z)}_{\substack{\text{=0 since}\\ \text{hydrostatic}}} = 0$$

$$\Longrightarrow \quad \frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0. \tag{9.47}$$

Define the surface density Σ by

$$\Sigma \equiv \int_{-\infty}^{\infty} \rho \,\mathrm{d}z\,. \tag{9.48}$$

Then, integrating the above form of the continuity equation over z we have

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0.. \tag{9.49}$$

We can get the same result by thinking of the disk as a set of rings/annuli as shown in Fig. 9.5.

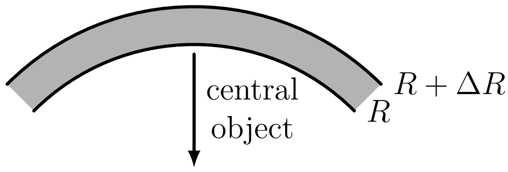

Fig. 9.5: Infinitesimal annulus element of disk

$$\begin{array}{c}\text{rate of change}\\ \text{of mass in}\\ \text{the annulus}\end{array} = \begin{array}{c}\text{flux into}\\ \text{annulus}\end{array} + \begin{array}{c}\text{flux out}\\ \text{of annulus}\end{array} \tag{9.50}$$

$$\begin{aligned}
\implies \frac{\partial}{\partial t}(2\pi R\Delta R\Sigma) &= 2\pi R\Sigma(R)u_R(R) - 2\pi(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R)\\
\implies \quad R\frac{\partial \Sigma}{\partial t} &= -\left[\frac{(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R) - R\Sigma(R)u_R(R)}{\Delta R}\right]\\
\implies \quad R\frac{\partial \Sigma}{\partial t} &= -\frac{\partial}{\partial R}(R\Sigma u_R). \quad \text{taking } \Delta R \to 0.
\end{aligned} \tag{9.51}$$

Now we look at the conservation of angular momentum. A derivation starting with the Navier-Stokes equation in cylindrical coordinates appears in Appendix A.1 at the end of this chapter. Here we use the ring/annulus approach:

$$\begin{array}{c}\text{rate of change}\\ \text{of ang. mtm.}\end{array} = \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{entering ring}\end{array} - \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{leaving ring}\end{array} + \begin{array}{c}\text{net torque on ring}\\ \text{(viscour, magnetic, etc.)}\end{array} \tag{9.52}$$

$$\implies \frac{\partial}{\partial t}\left(2\pi R\Delta R\Sigma R^2\Omega\right) = \underbrace{f(R) - f(R+\Delta R)}_{\text{ang. mtm. advection}} + \underbrace{G(R+\Delta R) - G(R)}_{\text{viscous torques}}, \tag{9.53}$$

where

$$f(R) \equiv 2\pi R\Sigma u_R \Omega R^2, \tag{9.54}$$

and $G(R)$ is the torque exerted by the disk outside of radius R on the disk inside of radius R:

$$G(R) = 2\pi R\nu\Sigma R\frac{\mathrm{d}\Omega}{\mathrm{d}R}R = 2\pi R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}, \tag{9.55}$$

therefore

$$\frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right). \tag{9.56}$$

Now assume $\partial u_\phi/\partial t = 0$ since gas is on Keplerian orbits. Then

$$\begin{aligned}
Ru_\phi\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Longrightarrow \quad -u_\phi\frac{\partial}{\partial R}(R\Sigma u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Longrightarrow \quad -u_\phi\frac{\partial}{\partial R}(R\Sigma u_R) + \frac{u_\phi R}{R}\frac{\partial}{\partial R}(R\Sigma u_R) + \Sigma u_R\frac{\partial}{\partial R}(u_\phi R) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Longrightarrow \quad R\Sigma u_R\frac{\partial}{\partial R}\left(R^2\Omega\right) &= \frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Longrightarrow \quad u_R &= \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}.
\end{aligned} \tag{9.57}$$

Substitute this into Eq. (9.56) and specialise to the case of a Newtonian point source gravitational field $\Omega = \sqrt{GM/R^3}$, yielding

$$\boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right].} \tag{9.58}$$

So the surface density $\Sigma(R,t)$ obeys a diffusion equation.

Notes on accretion disks:

- In general, $\nu = \nu(R, \Sigma, T, \cdots)$ and so this is a non-linear diffusion equation for Σ. It reduces to linear if $\nu = \nu(R)$;
- Vertical structure only enters via the temperature dependence of ν. So the geometrically-thin assumptions allows the radial and vertical problems to be mostly decoupled.
- Diffusion-like nature of the solutions of this equation show that an initial ring of matter will broaden and then "slump" inwards towards the central object (see Fig. 9.6).
- An initial narrow annulus of matter will spread out to form a full disk. This is a natural way to turn an incoming stream of gas into an accretion disk:
  - Stream flows in, swings around central object;
  - If orbit is not strictly elliptical, stream self-intersects and shocks;

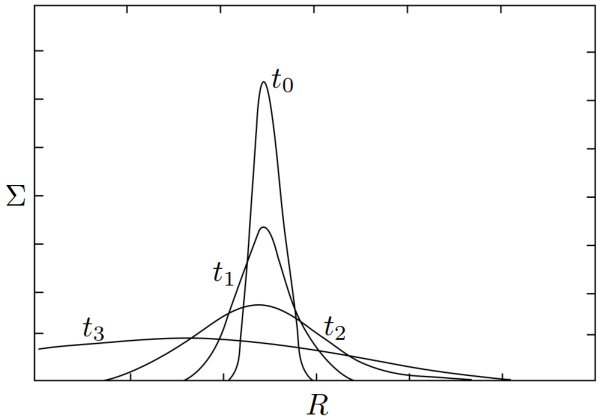

Fig. 9.6: The viscous evolution of a ring.

  – Form a narrow annulus of gas at location defined by specific angular momentum of matter in stream.

- Timescale for evolution is t_ν where

$$\frac{\Sigma}{t_\nu} \sim \frac{1}{R}\frac{1}{R}R^{1/2}\frac{1}{R}\nu\Sigma R^{1/2} \sim \frac{\nu\Sigma}{R^2}$$
$$\Longrightarrow \quad t_\nu \sim \frac{R^2}{\nu} = \frac{R}{u_\phi}\frac{Ru_\phi}{\nu} = \Omega^{-1}\mathrm{Re}, \tag{9.59}$$

where Re is the Reynolds number;

- If viscosity is due to particle thermal motions, typical values would suggest that Re $\sim 10^{14}$! This means

$$t_\nu \gg \text{age of the Universe.} \tag{9.60}$$

There must be another source of effective viscosity: we now know that there is an *effective viscisty* due to magnetohydrodynamic turbulence driven by the *magnetorotational instability*.

Dimensional analysis reveals $[\nu] = [L]^2[T]^{-1}$, and so turbulence gives effective viscosity $\nu_{\text{eff}} \sim ul$ where l and u are the size and velocity of a typical eddy in the turbulence, respectively.

Thinking about physics in comoving frame of orbital disk, the only characteristic velocity is c_s and the only characteristic length is the thickness H. These set the upper bounds on the velocity and size of turbulent eddies. So, we can say

$$\nu = \alpha c_s H \quad (\alpha < 1). \tag{9.61}$$

Disk models using this prescription are known as α-disks (gives ν dependence upon disk temperature and vertical structure, Shakura & Sunyaev 1973).

9.7 Steady-State, Geometrically-Thin Disks

Consider a steady state such that $\partial/\partial t = 0$. Then

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0$$
$$\implies \qquad R\Sigma u_R = C_1 = -\frac{\dot{m}}{2\pi} \tag{9.62}$$

where $\dot{m} = -2\pi R\Sigma u_R$ is the steady state mass inflow rate. Now recall from Eq. (9.57) that

$$u_R = \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}, \tag{9.63}$$

therefore,

$$-\frac{\dot{m}}{2\pi R\Sigma} = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right) \quad \text{for} \quad \Omega^2 = GM/R^3$$
$$\implies \qquad \nu\Sigma = \frac{\dot{m}}{3\pi}\left(1 - \sqrt{\frac{R_*}{R}}\right), \tag{9.64}$$

where we have taken as a boundary condition that $\nu\Sigma = 0$ at $R = R_*$. This amounts to saying that there are no viscous torques at $R = R_*$. Physically R_* can be:

- Surface of an accreting star;
- Innermost circular orbit around a black hole.

Let's now calculate the viscous dissipation neglecting $p\,\mathrm{d}V$ work and bulk viscosity. Specifically, we will calculate the viscous dissipation per unit surface area of the disk:

$$\begin{aligned} F_{\text{diss}} &= \int \sigma'_{ij}\partial_j u_i \frac{\mathrm{d}V}{2\pi R\,\mathrm{d}R\,\mathrm{d}\phi} \\ &= \frac{1}{2}\int \eta(\partial_j u_i + \partial_i u_j)^2\,\mathrm{d}z \\ &= \int \eta R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2\mathrm{d}z \\ &= \nu\Sigma R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2. \end{aligned} \tag{9.65}$$

Combining with our previous result (9.64) for $\nu\Sigma$ and recalling that $\Omega^2 = GM/R^3$, we have

$$F_{\text{diss}} = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right). \tag{9.66}$$

Notes on dissipation in a disk:

- Total energy emitted is

$$L = \int_{R_*}^{\infty} F_{\rm diss} 2\pi R \, \mathrm{d}R = \frac{GM\dot{m}}{2R_*}. \tag{9.67}$$

Here, $-GM/R_*$ is the gravitational potential at R_*. Therefore, $GM\dot{m}/R_*$ is the rate of gravitational energy loss of the flow. $GM\dot{m}/2R_*$ is radiated, the other half stays in the flow as kinetic energy and is dissipated in the boundary layer on an accreting star, or carried into the black hole;

- At a given location far from the inner edge ($R > R_*$) we have

$$F_{\rm diss} \approx \frac{3GM\dot{m}}{4\pi R^3}. \tag{9.68}$$

But an elementary estimate based on the loss of gravitational potential energy would give

$$F_{\rm diss,est} = \underbrace{\frac{1}{2\pi R \, \mathrm{d}R}}_{\substack{\text{area of}\\ \text{annulus}}} \cdot \underbrace{\frac{\partial}{\partial R}\left(\frac{GM\dot{m}}{R}\right)}_{\substack{\text{change in grav.}\\ \text{potential of } \dot{m}\\ \text{over annulus}}} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{half converts}\\ \text{to radiation,}\\ \text{rest to kinetic}}} = \frac{GM\dot{m}}{4\pi R^3}. \tag{9.69}$$

The extra factor of “3” in the correct formula is due to the transport of energy through the disk by viscous torques.

9.7.1 Radiation from Steady-State Thin Disks

If a disk is optically-thick, all radiation is thermalised and it radiates locally as a black body

$$\begin{aligned} \underbrace{2}_{\substack{\text{top and}\\ \text{bottom of disk}}} \cdot \sigma_{\rm SB} T_{\rm eff}^4 &= \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right) \\ \implies \quad T_{\rm eff} &= \left[\frac{3GM\dot{m}}{8\pi\sigma_{\rm SB} R^3}\right]^{1/4}. \end{aligned} \tag{9.70}$$

So, for $R \gg R_*$, $T_{\rm eff} \propto R^{-3/4}$.

The radiation emitted at a frequency f is

$$F_f = \int_{R_*}^{\infty} \frac{2h}{c^2} \frac{f^3}{e^{hf/kT_{\rm eff}} - 1} 2\pi R \, \mathrm{d}R\,. \tag{9.71}$$

So, we see that all of the observables from a steady-state disk are independent of the viscosity ν, provided it is large enough to supply the necessary angular momentum transport. To constrain ν, we need to study non-steady disks.

CHAPTER 10

Plasmas

Plasmas are fluids composed of charged particles. Thus, electromagnetic fields become important for both microphysics and large scale dynamics.

10.1 Magnetohydrodynamic (MHD) Equations

Consider a fully ionised hydrogen plasma, containing only protons (number density n^+, bulk velocity $\mathbf{u}^+$) and electrons $(n^-, \mathbf{u}^-)$. Mass conservation for each of the proton and electron fluids is

$$\frac{\partial n^+}{\partial r} + \boldsymbol{\nabla} \cdot \left(n^+ \mathbf{u}^+\right) = 0 \tag{10.1}$$

$$\frac{\partial n^-}{\partial r} + \boldsymbol{\nabla} \cdot \left(n^- \mathbf{u}^-\right) = 0. \tag{10.2}$$

The mass density is $\rho = m^+ n^+ + m^- n^-$ and the centre-of-mass velocity is

$$\mathbf{u} = \frac{m^+ n^+ \mathbf{u}^+ + m^- n^- \mathbf{u}^-}{m^+ n^+ + m^- n^-}. \tag{10.3}$$

So, we can combine these to give the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{10.4}$$

The continuity equation is the same as found before.

The charge density is $q = n^+ e^+ + n^- e^-$ and the current density is $\mathbf{j} = e^+ n^+ \mathbf{u}^+ + e^- n^- \mathbf{u}^-$. So, the above information also gives a *conservation of charge equation*

Definition 10.1 Conservation of Charge

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{10.5}$$

When we formulate the momentum equation, we have to consider the Lorentz force on each particle

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{10.6}$$

So, for the two species of particles:

$$m^+n^+\left(\frac{\partial \mathbf{u}^+}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}^+\right) = e^+n^+(\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}) - f^+\nabla p \tag{10.7}$$
$$m^-n^-\left(\frac{\partial \mathbf{u}^-}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}^-\right) = e^-n^-(\mathbf{E} + \mathbf{u}^- \times \mathbf{B}) - f^-\nabla p \tag{10.8}$$

where $f^{\pm}$ is the fraction of the pressure gradient that accelerates the protons/electrons.

Summing these equations gives

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p. \tag{10.9}$$

Ohm's law lets us relate $\mathbf{j}$ to $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}), \tag{10.10}$$

where σ is the electrical conductivity. This equation is needed to close the set of equations.

So, recapping the current set of equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{10.11}$$
$$\frac{\partial q}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{10.12}$$
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p \tag{10.13}$$
$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}). \tag{10.14}$$

We need to relate q, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ – Maxwell's equations!

Definition 10.2 Maxwell's Equations

$$\nabla\cdot\mathbf{B} = 0 \tag{10.15}$$
$$\nabla\cdot\mathbf{E} = \frac{q}{\epsilon_0} \tag{10.16}$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{10.17}$$
$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{10.18}$$

where we note $\epsilon_0\mu_0 = 1/c^2$.

10.1.1 Simplifying MHD

Let us simplify in the case of a non-relativistic, highly conducting plasma. Suppose fields are varying over length scales l and timescales τ. Then

1. $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \implies \frac{E}{B} \sim \frac{1}{\tau} \sim u. \tag{10.19}$$

2. $$\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right| \Big/ |\nabla \times \mathbf{B}| \sim \frac{1}{c^2}\left(\frac{l}{\tau}\right)^2 \sim \frac{u^2}{c^2} \ll 1, \tag{10.20}$$

   for non-relativistic flows. Therefore, displacement current can be ignored in non-relativistic MHD;

3. Look at two terms from Eq. (10.13):

   $$\frac{|q\mathbf{E}|}{|\mathbf{j} \times \mathbf{B}|} \sim \frac{qE}{jB} \sim \frac{\epsilon_0 E/l}{B/l\mu_0}\frac{E}{B} \sim u^2\epsilon_0\mu_0 \sim \frac{u^2}{c^2} \ll 1. \tag{10.21}$$

   Therefore, charge neutrality is preserved to a high approximation due to the strength of electrostatic forces. If there is a charge imbalance, it will oscillate with a characteristic frequency, the *plasma frequency*

   $$\omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}. \tag{10.22}$$

4. Neglecting displacement current in the relevant Maxwell equation, we get

   $$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0\sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{10.23}$$

   Take curl:

   $$\begin{aligned} \underbrace{\nabla \times (\nabla \times \mathbf{B})}_{=-\nabla^2\mathbf{B}-\nabla(\nabla\cdot\mathbf{B})=-\nabla^2\mathbf{B}} &= \mu_0\sigma\Big(\underbrace{\nabla \times \mathbf{E}}_{-\frac{\partial \mathbf{B}}{\partial t}} + \nabla \times (\mathbf{u} \times \mathbf{B})\Big) \\ \implies \frac{\partial \mathbf{B}}{\partial t} &= \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\substack{\text{advection of}\\\text{the field}\\\text{by the flow}}} + \underbrace{\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}}_{\substack{\text{dissipation of}\\\text{the field}\\\text{through the flow}}}. \end{aligned} \tag{10.24}$$

   If the fluid is a good conductor, i.e. σ is very large, then we can ignore the diffusion term and we have an equation that is analogous to the Helmholtz equation/Kelvin's theorem:

   $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{10.25}$$

By exact analogy with the Helmholtz equation, this says that the flux of magnetic field threading some surface $\mathcal{S}$ moving with the flow is preserved.

$$\int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \text{constant}. \tag{10.26}$$

This is the *flux-freezing condition.* Magnetic field lines are advected along in flow. We talk about the "freezing" of the magnetic flux into the plasma. In the high σ limit we must also have

$$\begin{aligned} \mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad &\text{is finite} \\ \Longrightarrow \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \quad &\text{as } \sigma \to \infty \\ \Longrightarrow \quad \mathbf{E} \cdot \mathbf{B} = 0& \\ \text{i.e.} \quad \mathbf{E} \perp \mathbf{B}.& \end{aligned} \tag{10.27}$$

So, the full set of *ideal MHD equations*, i.e. equations describing a non-relativistic, perfectly conducting, charge neutral plasma are:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{10.28}$$

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p \tag{10.29}$$

$$\left.\begin{aligned} \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{E} = 0 \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} = -\,\partial \mathbf{B}/\partial t \\ p = K \rho^{\gamma} \end{aligned}\right\} \Longrightarrow \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \end{aligned} \tag{10.30}$$

10.2 The Dynamical Effects of Magnetic Fields

The magnetic force density appearing in the above ideal MHD equations is

$$\mathbf{f}_{\mathrm{mag}} = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}. \tag{10.31}$$

So using vector identity this is

$$\mathbf{f}_{\mathrm{mag}} = \Bigg[\underbrace{-\boldsymbol{\nabla}\left(\frac{B^2}{2}\right)}_{\substack{\text{magnetic pressure} \\ \text{term with} \\ p_{\mathrm{mag}} = B^2/2\mu_0}} + \underbrace{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}_{\substack{\text{magnetic tension} \\ \text{term (vanishes for} \\ \text{straight field lines)}}} \Bigg] \tag{10.32}$$

Since there are new force terms in the momentum equation, this will change the nature of the waves that are possible.

10.3 Waves in Plasmas

We can repeat the perturbation analysis that we conducted for sound waves but now include the effects of a magnetic field. We will perturb about an equilibrium consisting of a static ($\mathbf{u} = \mathbf{0}$) plasma with uniform density ρ_0, uniform pressure p_0, and uniform magnetic field $\mathbf{B}_0$.

We start by writing down the governing equations of ideal MHD, assuming a barotropic equation of state:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{10.33}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}\right) = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \boldsymbol{\nabla} p \tag{10.34}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \tag{10.35}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{10.36}$$

$$p = p(\rho). \tag{10.37}$$

We now introduce perturbations and linearise the equations:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\delta \mathbf{u}) = 0 \tag{10.38}$$

$$\rho_0 \frac{\partial \delta \mathbf{u}}{\partial t} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \delta\mathbf{B}) \times \mathbf{B}_0 - c_s^2 \boldsymbol{\nabla} \delta\rho \tag{10.39}$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta\mathbf{u} \times \mathbf{B}_0) = -\mathbf{B}_0(\boldsymbol{\nabla} \cdot \delta\mathbf{u}) + (\mathbf{B}_0 \cdot \boldsymbol{\nabla})\delta\mathbf{u} \tag{10.40}$$

$$\boldsymbol{\nabla} \cdot \delta\mathbf{B} = 0. \tag{10.41}$$

We now adopt our usual plane wave form for the perturbations,

$$\delta\rho = \delta\rho_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.42}$$

$$\delta p = \delta p_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.43}$$

$$\delta \mathbf{u} = \delta \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{10.44}$$

$$\delta \mathbf{B} = \delta \mathbf{B}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{10.45}$$

The continuity equation becomes

$$\begin{aligned} -i\omega\delta\rho + i\rho_0 \mathbf{k} \cdot \delta\mathbf{u} &= 0 \\ \implies \quad \omega\delta\rho &= \rho_0 \mathbf{k} \cdot \delta\mathbf{u}. \end{aligned} \tag{10.46}$$

The momentum equation becomes

$$\begin{aligned} -i\omega\rho_0\delta\mathbf{u} &= \frac{i}{\mu_0}(\mathbf{k} \times \delta\mathbf{B}) \times \mathbf{B}_0 - ic_s^2\delta\rho\mathbf{k} \\ \implies \quad \omega\rho_0\delta\mathbf{u} &= \frac{1}{\mu_0}[(\mathbf{B}_0 \cdot \delta\mathbf{B})\mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k})\delta\mathbf{B}] + c_s^2\delta\rho\mathbf{k}. \end{aligned} \tag{10.47}$$

Finally, the flux-freezing (induction) equation becomes

$$\begin{aligned} -i\omega\delta\mathbf{B} &= -i\mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) + i(\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{u} \\ \implies \quad \omega\delta\mathbf{B} &= \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - (\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{u}. \end{aligned} \tag{10.48}$$

The full dispersion relation for MHD waves is then derived from eliminating the perturbation amplitudes from these expressions. Here, we are going to gain insight for the physics by just focusing on some special cases.

Firstly, we consider the case of modes with wavevectors orthogonal to the background magnetic field direction, $\mathbf{k} \parallel \mathbf{B}_0$. The linearised equations then become

$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u} \tag{10.49}$$

$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}(\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} + c_s^2\delta\rho\mathbf{k} \tag{10.50}$$

$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}). \tag{10.51}$$

We can immediately notice from the second of these relations that the velocity perturbations are aligned with the wavevector, $\delta\mathbf{u} \parallel \mathbf{k}$, i.e. these are longitudinal modes. Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}B_0^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k} + c_s^2\rho_0(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}. \tag{10.52}$$

Take the dot product of this last equation with $\mathbf{k}$ and then cancel $\mathbf{k}\cdot\delta\mathbf{u}$ throughout (since we know that this must be non-zero since modes are longitudinal),

$$\omega^2\rho_0 = \frac{k^2B_0^2}{\mu_0} + c_s^2\rho_0k^2 \tag{10.53}$$

$$\implies \quad \omega^2 = \left(c_s^2 + \frac{B^2}{\mu_0\rho_0}\right)k^2 \tag{10.54}$$

$$\omega^2 = \left(c_s^2 + v_A^2\right)k^2, \tag{10.55}$$

where we have defined the *Alfvén speed*,

$$v_A = \sqrt{\frac{B_0^2}{\mu_0\rho_0}}. \tag{10.56}$$

This describes a compressive dispersion-free longitudinal waves with a phase speed $\sqrt{c_s^2 + v_A^2}$. The restoring force comes from both the gas pressure and magnetic pressure acting in phase. This is known as the fast magnetosonic wave.

We now consider the case of modes with $\mathbf{k} \parallel \mathbf{B}_0$. The linearised equations become

$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u} \tag{10.57}$$

$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}[(\mathbf{B}_0\cdot\delta\mathbf{B}) - B_0k\delta\mathbf{B}] + c_s^2\delta\rho\mathbf{k} \tag{10.58}$$

$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - B_0k\delta\mathbf{u}. \tag{10.59}$$

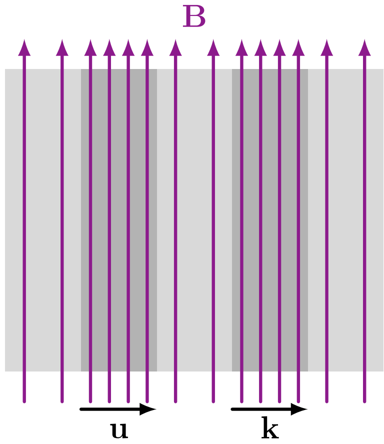

Fig. 10.1: Fast magnetosonic wave.

Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\Big[B_0^2k^2\delta\mathbf{u} - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k\mathbf{k}\Big] + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}. \tag{10.60}$$

There are actually two distinct wave modes wrapped up in these expression, a longitudinal mode and a transverse mode. To extract the longitudinal mode, take the dot product with $\mathbf{k}$

$$\omega^2\rho_0(\mathbf{k}\cdot\delta\mathbf{u}) = \frac{1}{\mu_0}\Big[B_0^2k^2(\mathbf{k}\cdot\delta\mathbf{u}) - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k^3\Big] + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})k^2, \tag{10.61}$$

and cancel factor of $\mathbf{k}\cdot\delta\mathbf{u}$ to get

$$\omega^2 = c_s^2k^2. \tag{10.62}$$

These are simply sound waves, with the magnetic field not playing a role since the velocity perturbations are directed along the magnetic field.

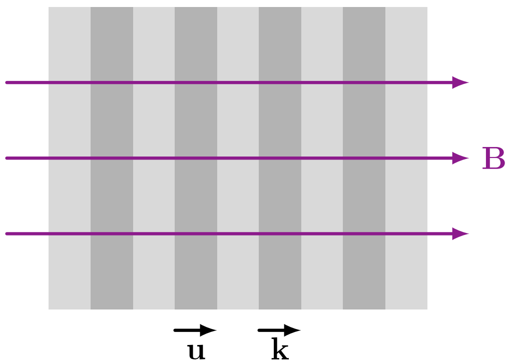

Fig. 10.2: Longitudinal wave with $\mathbf{k}\parallel\mathbf{B}$.

Return to the more general expression for the case $\mathbf{k}\parallel\mathbf{B}_0$;

$$\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\Big[B_0^2k^2\delta\mathbf{u} - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k\mathbf{k}\Big] + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}. \tag{10.63}$$

Taking the cross product with $\mathbf{k}$, we get

$$\omega^2 = \frac{B_0^2}{\mu_0 \rho_0} k^2 = v_A^2 k^2. \tag{10.64}$$

This describes transverse waves with phase speed v_A where the restoring force is provided by magnetic tension. These are *Alfvén waves.* They are incompressible transverse waves due to magnetic tension.

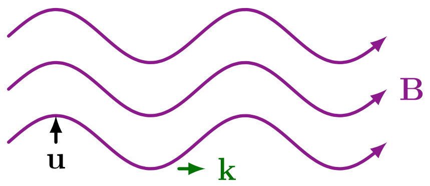

Fig. 10.3: Alfvén wave.

For a general perturbation, with $\mathbf{B}$ and $\mathbf{k}$ at some angle θ, we find three modes: Alfvén waves (phase speed goes to 0 when $\theta = \pi/2$); and fast and slow magnetosonic waves (which become degenerate at $\theta = 0$). Friedrichs diagrams showing phase speed as function of perturbation direction.

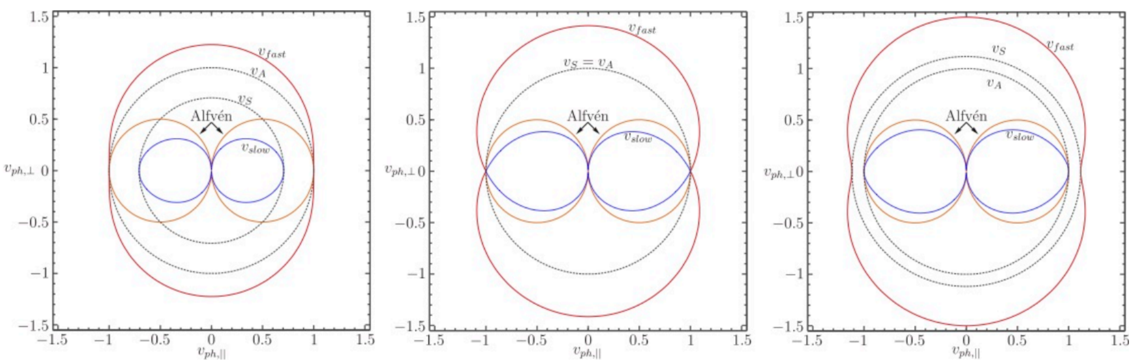

Fig. 10.4: Friedrichs diagrams for $c_s < v_A$ (left), $c_s = v_A$ (middle) and $c_s > v_A$ (right). The phase speed perturbation of the slow magnetoacoustic wave is illustrated in blue, the Alfvén wave in orange and the fast magnetoacoustic wave in red. The dotted lines correspond to the sound and Alfvén speed. The horizontal and vertical axes labelled as $v_{ph,\parallel}$ and $v_{ph,\perp}$ respectively represent the velocity perturbation components along and perpendicular to the equilibrium magnetic field, $\mathbf{B}_0$. [2]

10.4 Instabilities in Plasmas

The presence of magnetic forces can profoundly affect the nature of instabilities in plasmas. For example, we can repeat the derivation of the Rayleigh-Taylor instability including a magnetic field aligned with the interface.

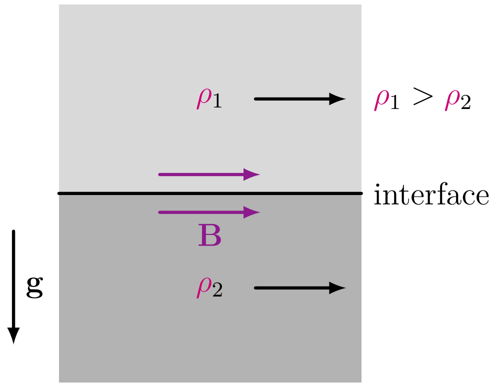

Fig. 10.5: Configuration of fluid interface

We will not repeat the analysis here, but we find the new dispersion relation is

$$\omega^2 = -kg\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{2}{\mu_0}\frac{(\mathbf{k}\cdot\mathbf{B})^2}{\rho_1 + \rho_2}. \tag{10.65}$$

For sufficiently small wavelength (high $|\mathbf{k}|$), the second term always wins, giving stable oscillations (Alfvén waves in this case). The interpretation is that magnetic tension forces tend to *stabilise* R-T modes.

10.5 Magnetorotational Instability

We end with a discussion of an MHD instability which is extremely important for accretion disks. We examine the stability of a plasma which is in orbit about a central object.

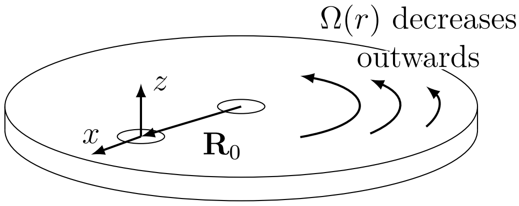

Fig. 10.6: Shear flow in an accretion disk

To uncover the essence of the instability, we simplify as much as possible. We conduct a "local analysis" meaning that we consider the dynamics in some small patch of the rotating flow at $\mathbf{R} = \mathbf{R}_0$, working in the comoving reference frame of the equilibrium flow. We assume that the equilibrium flow has an angular velocity about the central body $\Omega(R)$. We let our local frame of reference rotate at $\Omega(R_0)$ and set up a Cartesian coordinate system with $\hat{\mathbf{z}}$ pointing "upwards" (meaning aligned with the angular velocity Ω) and $\hat{\mathbf{x}}$ pointing outwards (i.e. away from the central body axis). Working in a Lagrangian picture, the momentum equation is:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \underbrace{\frac{1}{\mu_0\rho}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}}_{\text{magnetic force}} + \underbrace{2\mathbf{u}\times\boldsymbol{\Omega}}_{\text{coriolis}} - \underbrace{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})}_{\text{centrifugal}} - \underbrace{R\Omega(R)^2\hat{\mathbf{R}}}_{\text{gravity}}, \quad (10.66)$$

where the last term is an expression of gravity. Further simplifying, let us assume that the flow is cold so that pressure forces are negligible (this assumption can be readily relaxed but will make the analysis more involved). Introducing perturbations and assuming a plane-wave form, we have

$$\begin{aligned} \frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} &= \frac{1}{\mu_0\rho}(\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{B} - \Delta x R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}} \\ \implies \quad -i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} &= \frac{i}{\mu_0\rho}B_0k\Delta\mathbf{B} - \Delta x R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}}. \end{aligned} \quad (10.67)$$

The induction equation gives

$$\begin{aligned} \frac{\partial\Delta\mathbf{B}}{\partial t} &= \boldsymbol{\nabla}\times(\Delta\mathbf{u}\times\mathbf{B}_0) = (\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{u} \\ \implies \quad -i\omega\Delta\mathbf{B} &= ikB_0\Delta\mathbf{u} \\ \implies \quad \Delta\mathbf{B} &= -\frac{kB_0}{\omega}\Delta\mathbf{u}, \end{aligned} \quad (10.68)$$

and we can relate Δx and Δu_x;

$$\begin{aligned} \frac{\mathrm{D}\Delta x}{\mathrm{D}t} &= \Delta u_x \\ \implies \quad -i\omega\Delta x &= \Delta u_x \\ \implies \quad \Delta x &= \frac{i\Delta u_x}{\omega}. \end{aligned} \quad (10.69)$$

So eliminating in favour of $\Delta\mathbf{u}$ in our perturbed form of the momentum equation, we have

$$-i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = -\frac{i}{\mu_0\rho}B_0k\frac{kB_0}{\omega}\Delta\mathbf{u} - \frac{i\Delta u_x}{\omega}R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}}. \quad (10.70)$$

Writing this out in components and noting that $B_0^2/\rho_0\mu_0 = v_A^2$ gives

$$\begin{aligned} \omega^2\Delta u_x - 2i\Delta u_y\Omega\omega &= (kv_A)^2\Delta u_x + \Delta u_x\frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \\ \omega^2\Delta u_y + 2i\Delta u_x\Omega\omega &= (kv_A)^2\Delta u_y, \end{aligned} \quad (10.71)$$

or in matrix form

$$\begin{pmatrix} \omega^2 - (kv_A)^2 - \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} & -2i\omega\Omega \\ 2i\omega\Omega & \omega^2 - (kv_A)^2 \end{pmatrix}\begin{pmatrix}\Delta u_x \\ \Delta u_y\end{pmatrix} = 0. \quad (10.72)$$

We obtain the dispersion relation by setting the determinant of the matrix to zero. This gives

$$\left[\omega^2 - (kv_A)^2 - \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)}\right]\left[\omega^2 - (kv_A)^2\right] - 4\Omega^2\omega^2 = 0. \tag{10.73}$$

Writing as a quadratic in ω^2 gives our final form of the dispersion relation:

$$\omega^4 - \omega^2\left[4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2\left[(kv_A)^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)}\right] = 0. \tag{10.74}$$

If we "turn off" magnetic forces by setting $v_A = 0$, the dispersion relation gives

$$\begin{aligned}\omega^2 &= 4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \\ &= \frac{1}{R^3}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^4\Omega^2\right) \equiv \kappa_R^2 && (10.75) \\ &= \Omega^2 \quad \text{(Keplerian)}. && (10.76)\end{aligned}$$

For a Keplerian profile $\Omega^2 = GM/R^3$, or indeed any profile in which the specific angular momentum $R^2\Omega$ increases with radius, this describes local radial oscillations of the flow at the radial epicyclic frequency κ_R. If $\kappa_R^2 < 0$ (specific angular momentum decreasing with radius) then the flow is unstable.

Now turn on magnetic forces, so $v_A > 0$. There will be instability if $\omega^2 < 0$. Considering the basic properties of the dispersion relation, viewed as a quadratic in ω^2, we see that there will be instability if

$$(kv_A)^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} < 0. \tag{10.77}$$

This is the magneto-rotational instability (MRI). For sufficiently weak magnetic field or long wavelength (small k) modes, there will be instability if the angular velocity decreases outwards,

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} < 0 \quad \text{(instability)}. \tag{10.78}$$

Magnetic tension will stabilise modes with $k > k_{\mathrm{crit}}$ where

$$(k_{\mathrm{crit}}v_A)^2 = -\frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \quad (= 3\Omega^2 \text{ for Keplerian}). \tag{10.79}$$

Specialising to the Keplerian case, we find that the fastest growing mode has a growth rate

$$|\omega_{\max}| = \frac{3}{2}\Omega, \tag{10.80}$$

and wavenumber given by

$$k_{\max}v_a \approx \Omega. \tag{10.81}$$

The instability has an interesting property – while the magnetic field is essential for its existence, the maximum growth rate is independent of the magnetic field. Formally, within ideal hydrodynamics, the instability exists as $B_0 \to 0$ but not at $B_0 = 0$. Of course, the wavelength of the mode with the maximum growth rate $k_{\max} \to \infty$ as $B_0 \to 0$ and so in practice finite viscosity or finite conductivity effects will kill the MRI for sufficiently small B_0.

The MRI is central to the modern theory of accretion disks. MRI drives the turbulence that, as we have described previously, is essential for the transport of angular momentum in an accretion disk.

Bibliography

[1] Clausius, R.: XVI. On a Mechanical Theorem Applicable to Heat. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **40**(265), 122–127 (1870). https://doi.org/10.1080/14786447008640370, https://doi.org/10.1080/14786447008640370

[2] Jess, D., Morton, R., Verth, G., Fedun, V., Grant, S., Giagkiozis, I.: Multiwavelength Studies of MHD Waves in the Solar Chromosphere. Space Science Reviews (Mar 2015). https://doi.org/10.1107/s11214-015-0141-3

[3] Kay, D.: Schaum's Outline of Tensor Calculus. Schaum's Outline Series, McGraw-Hill Companies, Incorporated (1988), https://books.google.co.uk/books?id=6tUU3KruG14C

[4] Kiessling, M.K.H.: Mathematical vindications of the "jeans swindle" (1999), https://arxiv.org/abs/astro-ph/9910247

[5] Mey, T.: Radiation Characteristics of Extreme UV and Soft X-ray Sources, Göttingen series in X-ray physics, vol. 15. Universitätsverlag Göttingen (Jan 2015). https://doi.org/10.17875/gup2015-817

[6] Sedov, L.I.: Propagation of strong shock waves. Journal of Applied Mathematics and Mechanics **10**, 241–250 (Jan 1946)

[7] van der Sluys, M.V.: Formation and Evolution of Compact Binaries. Ph.D. thesis, Universiteit Utrecht, Utrecht, Netherlands (May 2006), https://www.astro.ru.nl/~sluys/Thesis/pdfs/vdSluys_thesis_screen.pdf, Promotor: Prof. Dr. Frank Verbunt, Copromotor: Dr. Onno Pols. Research supported by the Netherlands Research School for Astronomy (NOVA)

Appendix A

Appendix

A.1 Derivation of the Momentum Equation for Disk Evolution

The Navier–Stokes momentum equation, neglecting bulk viscosity, is

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right)=-\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\left[\eta\left(\boldsymbol{\nabla}\mathbf{u}+(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}-\frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}\right)\right]+\rho\mathbf{g}. \tag{A.1}$$

Let's look at the ϕ-component in cylindrical coordinates,

$$[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}]\cdot\hat{\mathbf{e}}_\phi=u_R\frac{\partial u_\phi}{\partial R}+u_\phi\frac{\partial u_\phi}{\partial \phi}+u_z\frac{\partial u_z}{\partial z}+\frac{u_R u_\phi}{R}, \tag{A.2}$$

which simplifies with $u_z=0$ and cylindrical symmetry to

$$[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}]\cdot\hat{\mathbf{e}}_\phi=u_R\frac{\partial u_\phi}{\partial R}+\frac{u_r u_\phi}{R}. \tag{A.3}$$

With cylindrical symmetry the pressure gradient in the ϕ direction is zero so we have

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_R\frac{\partial u_\phi}{\partial R}+\frac{u_R u_\phi}{R}\right)=\left[\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{T}}}\right]\cdot\hat{\mathbf{e}}_\phi, \tag{A.4}$$

where we have defined the tensor

$$\underline{\underline{\mathbf{T}}}=\eta\left(\boldsymbol{\nabla}\mathbf{u}+(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}-\frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}\right). \tag{A.5}$$

We will use the vector gradient tensor in cylindrical coordinates

$$\boldsymbol{\nabla}\mathbf{u}=\begin{bmatrix}\frac{\partial u_R}{\partial R} & \frac{1}{R}\frac{\partial u_R}{\partial \phi}-\frac{u_\phi}{R} & \frac{\partial u_R}{\partial z}\\ \frac{\partial u_\phi}{\partial R} & \frac{1}{R}\frac{\partial u_\phi}{\partial \phi}+\frac{u_R}{R} & \frac{\partial u_\phi}{\partial z}\\ \frac{\partial u_z}{\partial R} & \frac{1}{R}\frac{\partial u_z}{\partial \phi} & \frac{\partial u_z}{\partial z}\end{bmatrix}, \tag{A.6}$$

which simplifies with $u_z=0$ and cylindrical symmetry to

$$\boldsymbol{\nabla}\mathbf{u}=\begin{bmatrix}\frac{\partial u_R}{\partial R} & -\frac{u_\phi}{R} & \frac{\partial u_R}{\partial z}\\ \frac{\partial u_\phi}{\partial R} & \frac{u_R}{R} & \frac{\partial u_\phi}{\partial z}\\ 0 & 0 & 0\end{bmatrix}. \tag{A.7}$$

Then, using $\boldsymbol{\nabla}\cdot\mathbf{u}=\frac{1}{R}\frac{\partial}{\partial R}(Ru_R)$ ($u_z=0$, cylindrical symmetry), we get

$$\underline{\underline{\mathbf{T}}}=\eta\begin{bmatrix}2\frac{\partial u_R}{\partial R}-\frac{2}{3R}\frac{\partial}{\partial R}(Ru_R) & \frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R} & \frac{\partial u_R}{\partial z}\\ \frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R} & 2\frac{u_R}{R}-\frac{2}{3R}\frac{\partial}{\partial R}(Ru_R) & \frac{\partial u_\phi}{\partial z}\\ \frac{\partial u_R}{\partial z} & \frac{\partial u_\phi}{\partial z} & -\frac{2}{3R}\frac{\partial}{\partial R}(Ru_R)\end{bmatrix}, \tag{A.8}$$

and the ϕ-component of the tensor divergence is

$$\left[\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{T}}}\right]\cdot\hat{\mathbf{e}}_\phi = \frac{\partial}{\partial R}\left[\eta\left(\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R}\right)\right]+\frac{\partial}{\partial z}\left(\eta\frac{\partial u_\phi}{\partial z}\right)+\frac{2}{R}\left[\eta\left(\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R}\right)\right]. \tag{A.9}$$

Putting all together, the ϕ-component of the momentum conservation is

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_R\frac{\partial u_\phi}{\partial R}+\frac{u_R u_\phi}{R}\right)=\frac{\partial}{\partial R}\left(\eta\frac{\partial u_\phi}{\partial R}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial u_\phi}{\partial z}\right)+\frac{1}{R}\frac{\partial}{\partial R}(\eta u_\phi)-2\frac{u_\phi}{R}\frac{\partial\eta}{\partial R}-\eta\frac{u_\phi}{R^2}. \tag{A.10}$$

Integrating over z:

$$\Sigma\left(\frac{\partial u_\phi}{\partial t}+u_R\frac{\partial u_\phi}{\partial R}+\frac{u_R u_\phi}{R}\right)=\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+\frac{1}{R}\frac{\partial}{\partial R}(\nu\Sigma u_\phi)-2\frac{u_\phi}{R}\frac{\partial}{\partial R}(\nu\Sigma)-\frac{\nu\Sigma u_\phi}{R}. \tag{A.11}$$

Adding the continuity equation (9.49) multiplied by Ru_ϕ with the momentum equation multiplied by R we get

$$\begin{aligned}
\frac{\partial}{\partial t}(R\Sigma u_\phi)+\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+\frac{\partial}{\partial R}(\nu\Sigma u_\phi)-2u_\phi\frac{\partial}{\partial R}(\nu\Sigma)-\frac{\nu\Sigma u_\phi}{R}\\
&= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+2\nu\Sigma\left(\frac{1}{\nu\Sigma}\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{u_\phi}{\nu\Sigma}\frac{\partial(\nu\Sigma)}{\partial R}\right)\\
&\quad -\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{\nu\Sigma u_\phi}{R}\\
&= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+2\nu\Sigma\frac{\partial u_\phi}{\partial R}-\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{\nu\Sigma u_\phi}{R}\\
&= \frac{1}{R}\frac{\partial}{\partial R}\left[R^2\nu\Sigma\frac{\partial u_\phi}{\partial R}-R\nu\Sigma u_\phi\right]\\
&= \frac{1}{R}\frac{\partial}{\partial R}\left[\nu\Sigma R^3\left(\frac{1}{R}\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R^2}\right)\right].
\end{aligned} \tag{A.12}$$

Using $u_\phi=\Omega R$ we get

$$\frac{\partial}{\partial t}(R\Sigma u_\phi)+\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right)=\frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right), \tag{A.13}$$

which is the same as Eq. (9.56).

A.2 On Singularities and Physics

There is a general expectation that non-linear, partial differential equations will develop singularities in a finite time. When those non-linear equations describe something physical, these singularities are particularly interesting. A singularity is telling us the equations are no longer sufficient to capture the underlying physics and must be replaced by something more fundamental. This suggests that singularities may offer a window into the microscopic realm.

Within classical physics, there are two pre-eminent sets of non-linear equations. These are the Navier-Stokes equation (or its baby brother, the Euler equation) for fluids, and the Einstein equations for gravity. As we now explain, both these equations are rather special and the way in which singularities form, or fail to form, is surprising and poorly understood.

For fluids, it's useful to distinguish between the compressible and non-compressible cases. As we've seen in Section 6.5, the compressible Euler equation readily develops singularities in finite time. These are the shock waves that we've explored in this section, characterised by a discontinuity in the density ρ and other dynamical variables. As we anticipated above, the presence of the shock does mean that we have to introduce new physics. But the surprise of this section is that this new physics is the most minimal imaginable: it is just the second law of thermodynamics. Once we accept that entropy must increase when the shock develops, we have all we need to tell us what happens to the subsequent evolution. We certainly don't need to resort to any detailed microscopic description involving atoms and quantum world. This is rather remarkable. Shocks may be singular but, from a physical perspective, the singularity is very mild.

It is natural to ask: is this same property shared by all singularities of the compressible Euler equation? Or, indeed singularities of the compressible Navier-Stokes equation? The answer is: we don't know. For example, what happens when many shocks collide and start to interact with each other? Is it still the case that we can track the singular evolution of the Euler equations using only the second law as our guide? This situation is complicated and we don't know the answer. Moreover, one may worry that there are singularities worse than shocks that can arise in the compressible Euler equation. For example, it may be possible that $\rho(\mathbf{x}, t) \to \infty$ in some finite time. This kind of singularity would surely need some detailed understanding of the underlying atoms to resolve. But does such a singularity actually arise? Again, the answer is: we don't know. It can be shown that such singularities occur for very special initial data, but to be physically relevant it should happen for generic initial conditions, meaning initial conditions that lie within some open ball rather than at specific points. And it remains an open problem to show whether or not this occurs.

The situation for the incompressible Euler and Navier-Stokes equations is somewhat simpler to state, but still not well understood. Here there is a conjecture that no singularities occur in a finite time. No counter example is known, but a mathematical proof appears challenging to say the least. Indeed, proving the existence and smoothness of solutions to the Navier-Stokes equation is one of the Millennium Prize problems with a $1 million dollar prize attached. (If you're genuinely motivated by the money then I would suggest that mathematics may not be your true calling. There are easier ways to be both happy and rich.)

Finally, that leaves us with the Einstein equations of General Relativity. Here the situation is most intriguing of all. It is straightforward to show that singularities do develop in finite time (at least with a suitable definition of "time"!). This arises when matter collapses to form a black hole, with a singularity forming in the centre where the curvature of spacetime becomes infinite. The presence of such a singularity is telling us that the laws of classical gravity are breaking down and must be replaced by something quantum. This means that singularities provide a wonderful opportunity to teach us something new about the "atoms of spacetime", whatever that means. Sadly, however, nature has made these singularities very difficult to access experimentally. It appears that they are generically shielded by an event horizon, so that they can't be seen by anyone sensible who chooses not to jump into the black hole. The idea that singularities necessarily sit behind an event horizon goes by the name of the *cosmic censorship conjecture*. From a mathematical perspective, it appears utterly miraculous and a proof is generally thought to be even more challenging than the Navier-Stokes existence and smoothness conjecture.

The upshot is that the laws of physics appear to be surprisingly robust against the formation of singularities. Even when singularities do arise – as in the compressible Euler equation and the Einstein equations – some poorly understood feature of the equations means that they are more innocuous than we would have naively thought. They are either hidden behind horizons, or neatly resolved by the second law. In both cases, we can largely carry on with our lives without worrying too much about what microscopic physics lurks inside the singularity.

It feels like there is an important lesson hiding within this story. The refusal of both the Navier-Stokes and the Einstein equations to develop readily accessible singularities, that require something atomic or quantum to fully understand, is a striking mathematical fact. It should have a striking physical reason behind it. But I don't know what it is.