Part II Astrophysical Fluid Dynamics

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Preface

Fluid dynamical forces drive most of the fundamental processes in the Universe and so play a crucial role in our understanding of astrophysics. Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is "frozen in" to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs.

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Chapter 1

Basic Principles

1.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g., density ρ, pressure p), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples: (Fluids in the Universe)

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

In our discussion, we shall use the concept of a *fluid element*. This is a region of fluid (size ℓ_{region} that is

1. Small enough that there are no significant variations of any property q that interests us

$$l_{\mathrm{region}} \ll l_{\mathrm{scale}} \sim \frac{q}{|\boldsymbol{\nabla} q|}. \tag{1.1}$$

2. Large enough to contain sufficient particles to be considered in the continuum limit
$$nl_{\text{region}}^3 \gg 1, \tag{1.2}$$
where n is the number density of particles.

If such fluid elements can be defined, continuum description is valid, otherwise, we must describe the system at the particle-level.

1.2 Collisional and Collisionless Fluids

In a **collisional fluid**, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.
$$l_{\text{region}} \gg \lambda, \tag{1.3}$$
where λ is the mean free path, the typical distance travelled by a particle before its direction of travel is significantly changed due to particle collisions. Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p. We can relate ρ, T and p with an *equation of state*:
$$p = p(\rho, T). \tag{1.4}$$
Almost all fluids considered in this course are collisional.

In a **collisionless fluid**, particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$. So, distribution of particle speeds locally does not correspond to the maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples: (Collisionless Fluids)

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless).

1.2.1 Example of ICM

Treat as fully ionised plasma of electrons and ions. The mean free path is set by Coulomb collisions and an analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln\Lambda}, \tag{1.5}$$

where n_e is the electron number density, and Λ is the ratio of largest to smallest impact parameter. For $T \gtrsim 4 \times 10^5\,\mathrm{K}$ we have $\ln\Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \simeq 23\,\mathrm{kpc}\left(\frac{T_e}{10^8\,\mathrm{K}}\right)^2 \left(\frac{n_e}{10^{-3}\,\mathrm{cm}^{-3}}\right)^{-1}. \tag{1.6}$$

So we have

$$\overbrace{R_{\mathrm{galaxy}} \sim \lambda_e}^{\text{collisionless}} \ll R_{\mathrm{cluster}} \sim 1\,\mathrm{Mpc}. \tag{1.7}$$

$$\text{with } \underbrace{\lambda_e \ll R_{\mathrm{cluster}}}_{\text{collisional}}$$

CHAPTER 2

Formulation of the Fluid Equations

2.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow:

1. *Eulerian description*: one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like

$$\rho(\mathbf{r}, t), \quad p(\mathbf{r}, t), \quad T(\mathbf{r}, t), \quad \mathbf{v}(\mathbf{r}, t). \tag{2.1}$$

2. *Lagrangian description*: one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest. The Lagranian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals). These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

These frameworks form the foundation of the two principal methods in computational fluid dynamics. The Eulerian approach employs grid-based codes, where space is divided into a fixed grid, and fluid flows through the grid. In contrast, the Lagrangian approach uses smoothed particle codes, treating fluid elements as smoothed particles that move through continuous space. An important application of Lagrangian methods includes grid codes designed for modeling stellar collapse and supernova explosions.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position $\mathbf{r}$ and time t. At time $t + \delta t$ the element will be at position $\mathbf{r} + \delta\mathbf{r}$. The change in quantity Q of the fluid element is

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \left[\frac{Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right] \tag{2.2}$$

but

$$Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = Q(\mathbf{r}, t) + \frac{\partial Q}{\partial t}\delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q + \mathcal{O}\Big(\delta t^2, |\delta\mathbf{r}|^2, \delta t|\delta\mathbf{r}|\Big), \tag{2.3}$$

so

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \left[\frac{\partial Q}{\partial t} + \frac{\delta \mathbf{r}}{\delta t} \cdot \boldsymbol{\nabla} Q + \mathcal{O}(\delta t, |\delta \mathbf{r}|) \right], \tag{2.4}$$

which gives us

$$\boxed{\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian} \\ \text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian} \\ \text{time derivative}}} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} Q}_{\substack{\text{“convective”} \\ \text{time derivative}}} .} \tag{2.5}$$

2.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

- **Streamline**: families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element. Parameterise streamline by label s such that

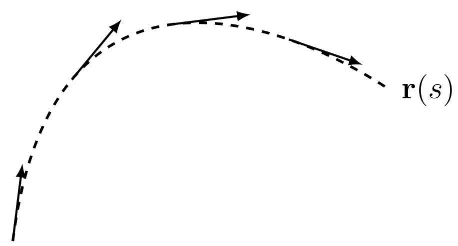


Fig. 2.1: Streamline

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s} \right), \tag{2.6}$$

and demand $\mathrm{d}\mathbf{r}/\mathrm{d}s \parallel \mathbf{u}$, we get

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}. \tag{2.7}$$

- **Particle paths**: trajectories of individual fluid elements given by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t). \tag{2.8}$$

For small time intervals, particle paths follow streamlines since $\mathbf{u}$ can be treated as approximately steady.

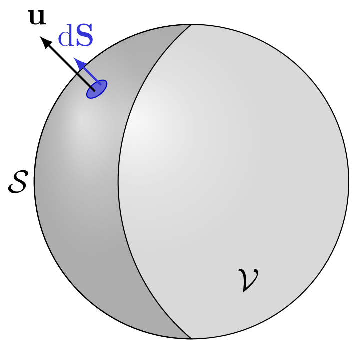


Fig. 2.2: Mass flow of a fluid element

- **Streaklines**: locus of points of all fluid that have passed through a given spatial point in the past.
$$\mathbf{r}(t) = \mathbf{r}_0 \tag{2.9}$$
for some given t in the past. Imagine the point as a source of "dye" or "smoke".

Streamlines, particle paths, and streaklines all coincide if the flow is steady (i.e. $\partial \mathbf{u}/\partial t = 0$).

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

2.3 Conservation of Mass

Consider a fixed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. If there are no sources or sinks of mass within the volume, we can say

$$\text{rate of change of mass in } \mathcal{V} = -\text{rate that mass is flowing out across } \mathcal{S} \tag{2.10}$$

this gives

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, \mathrm{d}V &= -\int_{\mathcal{S}} \rho \, \mathbf{u} \cdot \mathrm{d}\mathbf{S} \\ \implies \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V &= -\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \, \mathrm{d}V \\ \implies \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \right) \mathrm{d}V &= 0. \end{aligned} \tag{2.11}$$

This is true for all volumes $\mathcal{V}$. So we must have the *Eulerian continuity equation*,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0.} \tag{2.12}$$

The Lagrangian expression of mass conservation is easily found:

$$\begin{aligned}\frac{\mathrm{D}\rho}{\mathrm{D}t} &= \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho \\ &= -\boldsymbol{\nabla} \cdot (\rho \mathbf{u}) + \mathbf{u} \cdot \boldsymbol{\nabla} \rho \\ &= -\rho \boldsymbol{\nabla} \cdot \mathbf{u}, \end{aligned} \tag{2.13}$$

where we have used the vector identity $\boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = \rho \boldsymbol{\nabla} \cdot \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho$ for the final equality. Thus we have the *Lagrangian continuity equation*,

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0.} \tag{2.14}$$

The equations of compressible hydrodynamics are very complicated. They are difficult to integrate numerically – in 3D computing time scales like (grid size)^4. But there are useful *approximations* that give insight.

In an **incompressible flow**, fluid elements maintain a constant density, i.e.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{2.15}$$

We can now see that incompressible flows must be divergence free, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. This a good model for liquids, but also a surprisingly good approximation for gases provided the flow is subsonic.

In an **Irrotational flow**, there is no vorticity, i.e. $\boldsymbol{\nabla} \times \mathbf{u} = 0$. The vorticity field $\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{u}$ moves with the fluid, and vorticity is usually generated at boundaries, so often the bulk of a flow is irrotational. If $\boldsymbol{\nabla} \times \mathbf{u} = 0$ the velocity field can be generated from a scalar potential $\mathbf{u} = \boldsymbol{\nabla}\Phi$.[1] If the fluid is both incompressible and irrotational, the velocity potential satisfies Laplace's equation $\nabla^2 \Phi = 0$. In this important case we can use potential theory to find $\mathbf{u}$ and Bernoulli's equation to find the pressure.

2.3.1 Bernoulli Equation for Compressible Fluids

There is an important theorem which is often useful for cases involving steady flow, expressing the conservation of energy as it transported along a streamline: Bernoulli's equation.

The derivation for compressible fluids depends upon conservation of mass, and conservation of energy, ignoring viscosity, and thermal effects. For *steady flow*, consider *streamlines* connecting areas A_1 and A_2.

[1] Note: in fluid dynamics, this conventionally has a plus sign.

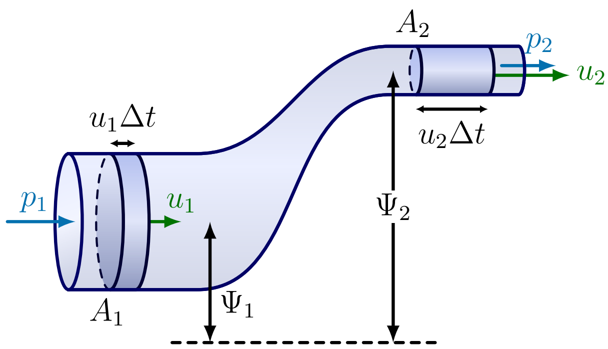


Fig. 2.3: A streamtube of fluid moving to the right. Indicated are pressure $p_{1,2}$, potential $\Psi_{1,2}$, flow speed $u_{1,2}$, distance $u_{1,2}\Delta t$, and cross-sectional area $A_{1,2}$.

Conservation of mass implies that in Fig. 2.3, in the interval of time Δt, the amount of mass passing through the boundary defined by the area A_1 is equal to the amount of mass passing outwards through the boundary defined by the area A_2:

$$0 = \Delta M_1 - \Delta M_2 = \rho_1 A_1 u_1 \Delta t - \rho_2 A_2 u_2 \Delta t. \tag{2.16}$$

Conservation of energy is applied in a similar manner: It is assumed that the change in energy of the volume of the streamtube bounded by A_1 and A_2 is due entirely to energy entering or leaving through one or the other of these two boundaries. Clearly, in a more complicated situation such as a fluid flow coupled with radiation, such conditions are not met. Nevertheless, assuming this to be the case and assuming the flow is steady so that the net change in the energy is zero,

$$\Delta E_1 - \Delta E_2 = 0, \tag{2.17}$$

where ΔE_1 and ΔE_2 are the energy entering through A_1 and leaving through A_2, respectively. The energy entering through A_1 is the sum of the kinetic energy entering, the energy entering in the form of potential gravitational energy of the fluid, the fluid thermodynamic internal energy per unit of mass ($\mathcal{E}_!$) entering, and the energy entering in the form of mechanical $p\,\mathrm{d}V$ work:

$$\Delta E_1 = \left(\frac{1}{2}\rho u_1^2 + \Psi_1 \rho_1 + \mathcal{E}_1 \rho_1 + p_1\right) A_1 u_1 \Delta t, \tag{2.18}$$

where Ψ is the potential due to gravity. A similar expression for ΔE_2 may easily be constructed. So now setting $0 = \Delta E_1 - \Delta E_2$:

$$0 = \left(\frac{1}{2}u_1^2 + \Psi_1 + \mathcal{E}_1 + \frac{p_1}{\rho_1}\right)\rho_1 A_1 u_1 \Delta t + \left(\frac{1}{2}u_2^2 + \Psi_2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}\right)\rho_2 A_2 u_2 \Delta t. \tag{2.19}$$

Now, using the previously-obtained result from conservation of mass, this may be simplified to obtain

$$\boxed{\frac{1}{2}u^2 + \Psi + \mathcal{E} + \frac{p}{\rho} = \text{const.},} \tag{2.20}$$

which is the **Bernoulli equation** for compressible flow.

For an ideal gas, the internal energy is (6.72) $\mathcal{E} = \frac{1}{\gamma-1}\frac{p}{\rho}$,

$$\frac{1}{2}u^2 + \Psi + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \text{const.} \tag{2.21}$$

So for **incompressible flow**, when ρ is constant (2.15) (i.e. $\gamma = \infty$, $\mathcal{E} = 0$), then

$$\frac{1}{2}u^2 + \Psi + \frac{p}{\rho} = \text{const.} \tag{2.22}$$

along a streamline.

Note: if a streamline is/is not curved, then a pressure gradient perpendicular to the streamline is/is not needed to provide a centripetal force. (Vertically there may be a pressure gradient due to gravity, but that is not related to the curvature of the flow).

Example 2.1 Streamline Equations
Determine the equation of a general streamline of the flow $u_\phi = a$, $u_r = b$, $u_Z = 0$ in cylindrical polar coordinates, and sketch the flow. Include more than one streamline in your sketches. Repeat for the flow $u_\phi = ar^2$, $u_r = br^2$, $u_Z = 0$. If the flows are steady, and the density at a given radius is independent of ϕ, find the radial dependence of the density in both cases.

Solution In cylindrical coordinates, the position vector $\mathbf{r} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$ can be differentiated to give the relationship between the coordinate components and the velocities,

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_r + \dot{z}\hat{\mathbf{e}}_z \\ &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\phi}\hat{\mathbf{e}}_\phi + \dot{z}\hat{\mathbf{e}}_z \quad \text{since} \quad \mathrm{d}\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\phi\,\mathrm{d}\phi\,,\end{aligned} \tag{2.23}$$

letting us make the identifications $u_r = \dot{r}$ and $u_\phi = r\dot{\phi}$. From the velocity components given in the question, we can write

$$\dot{R} = b \quad \text{and} \quad R\dot{\phi} = a, \tag{2.24}$$

thus by application of the chain rule

$$\frac{\mathrm{d}R}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = b \quad \Longrightarrow \quad a\frac{\mathrm{d}R}{R} = b\,\mathrm{d}\phi\,. \tag{2.25}$$

With the differential $\mathrm{d}(\ln R) = \frac{\mathrm{d}R}{R}$, the equation above simplifies to $a\,\mathrm{d}(\ln R) = b\,\mathrm{d}\phi$, from which integrating gives

$$R = R_0 e^{b\phi/a}, \tag{2.26}$$

for a constant R_0.

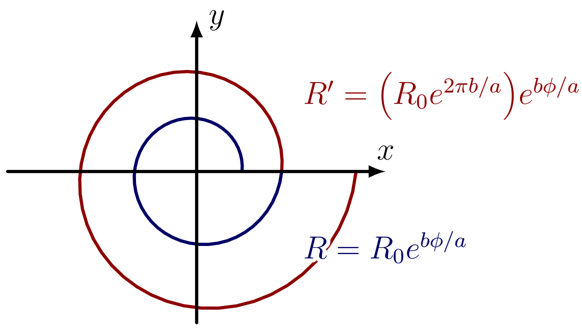


Fig. 2.4: Streamlines for $R = R_0 e^{b\phi/a}$. Different values of the constant R_0 allow different streamlines in the medium to be described, and allows the streamlines to be continuous despite the domain restriction on ϕ.

Steady flow is characterised by $\dot{\mathbf{u}} = \mathbf{0}$, let us consider the time derivative of the Eulerian continuity equation (2.12) to see what condition this implies on $\dot{\rho}$,

$$\frac{\partial}{\partial t}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right) = 0$$
$$\implies \quad \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\mathbf{u}\right)\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot\left(\rho\frac{\partial \mathbf{u}}{\partial t}\right). \tag{2.27}$$

The right hand side of the last equality vanishes for steady flow, and since in general the bracketed operator on the left hand side does *not* vanish for an arbitrary flow, then steady flow also implies $\partial\rho/\partial t = 0$, as expected.

Thus, the continuity equation reduces to $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$, or equivalently in cylindrical coordinates using the appropriate expression for $\boldsymbol{\nabla}\cdot\mathbf{u}$[2]

$$\underbrace{\left(u_r\frac{\partial}{\partial r} + u_\phi\frac{\partial}{\partial \phi}\right)}_{\mathbf{u}\cdot\boldsymbol{\nabla}}\rho + \rho\underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right)}_{\boldsymbol{\nabla}\cdot\mathbf{u}} = 0. \tag{2.29}$$

and following substitution of the given components gives the equation

$$\mathrm{d}(\ln\rho) = -\,\mathrm{d}(\ln r)\,, \tag{2.30}$$

and thus the radial dependence of the density is

$$\rho \propto r^{-1}. \tag{2.31}$$

Following the exact same procedure for the second set of conditions, we find the equation of the streamlines is the same as for the first flow (2.26), but now Eq. (2.30) instead becomes

$$\mathrm{d}(\ln\rho) = -3\,\mathrm{d}(\ln r)\,, \tag{2.32}$$

[2]In cylindrical coordinates, the divergence of a vector field $\mathbf{A} = (A_r, A_\phi, A_z)$ is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \tag{2.28}$$

and this time we get for the density dependence

$$\rho \propto r^{-3}. \tag{2.33}$$

◀

Example 2.2 Density in Steady Flow
Show that for a steady flow with $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, the density ρ is constant along the streamlines. Need ρ be constant throughout the medium?

Solution For *steady* flow, streamlines coincide with the particle paths, and so we consider the Lagrangian description. From the Lagrangian continuity equation (2.14), we see that if $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, then $\mathrm{D}\rho/\mathrm{D}t = 0$, i.e. density is constant *along* a streamline. This does not imply density is constant across different streamlines throughout the fluid. ◀

Example 2.3 Cylindrical Flow
If $\mathbf{r} = (x, y, 0)$ and $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$ and a flow velocity is given for $r \geq a$ by $\mathbf{u} = U\left(1 + \frac{a^2}{r^2}\right)\hat{\mathbf{e}}_x - 2Ua^2xr^{-3}\hat{\mathbf{e}}_r$, (where $\hat{\mathbf{e}}_x = \mathbf{x}/|\mathbf{x}|$) show that the streamlines obey $U\left(r - \frac{a^2}{r}\right)\sin\phi = \text{const.}$, where $\phi = \tan^{-1} y/x$. Sketch the streamlines and explain what the flow represents physically. (Hint: consider which coordinate system is better suited for this problem.)

Solution To rewrite the equation of flow in terms of cylindrical polars, we need $\hat{\mathbf{e}}_x$ in terms of the new basis vectors,

$$\begin{aligned}
\hat{\mathbf{e}}_x &= \boldsymbol{\nabla} x = \boldsymbol{\nabla}(r\cos\phi) \\
&= \frac{\partial(r\cos\phi)}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(r\cos\phi)}{\partial\phi}\hat{\mathbf{e}}_\phi \\
&= \cos\phi\,\hat{\mathbf{e}}_r - \sin\phi\,\hat{\mathbf{e}}_\phi.
\end{aligned} \tag{2.34}$$

Thus, we can rewrite the flow as

$$\mathbf{u} = U\left(1 - \frac{a^2}{r^2}\right)\cos\phi\,\hat{\mathbf{e}}_r - U\left(1 + \frac{a^2}{r^2}\right)\sin\phi\,\hat{\mathbf{e}}_\phi. \tag{2.35}$$

From the components $u_r = \dot{r}$ and $u_\phi = r\dot{\phi}$, we can use the chain rule to work towards finding the equation of the streamlines,

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\phi} &= \frac{\mathrm{d}r}{\mathrm{d}t}\bigg/\underbrace{\frac{\mathrm{d}\phi}{\mathrm{d}t}}_{u_\phi/r} = \frac{ru_r}{u_\phi} \\
\implies \quad \frac{\mathrm{d}r}{\mathrm{d}\phi} &= r\frac{U\left(1 - \frac{a^2}{r^2}\right)\cos\phi}{-U\left(1 + \frac{a^2}{r^2}\right)\sin\phi} \\
\implies \frac{\mathrm{d}r}{r}\frac{1 + a^2/r^2}{1 - a^2/r^2} &= -\,\mathrm{d}(\ln\sin\phi)\,.
\end{aligned} \tag{2.36}$$

Integrating this we quickly see that $\ln(r - a^2/r) = -\ln(\sin\phi) + \text{const.}$, so the streamlines obey

$$\left(r - \frac{a^2}{r}\right)\sin\phi = \text{const.} \tag{2.37}$$

◀

Example 2.4 2D Flow and Streaklines
A steady 2D flow is described by $u_x = 2/x$, $u_y = 1$. Find and sketch the streamlines. Find also a general expression for the surface density of the flow $\Sigma(x, y)$ assuming it can be written as a separable function of x and y. Radioactive nuclei are introduced in a small patch at (x_0, y_0) so as to maintain a fixed concentration there. These nuclei decay such that their number per unit mass is given by $Q = Q_0 e^{-t}$ where t is the time since introduction into the flow. Show that the surface density of radioactive nuclei (i.e. number per unit area) attains a maximum along the radioactive streakline if x_0 is less than a critical value, and determine the coordinates of this maximum.

Solution From $\dot{x} = 2/x$ and $\dot{y} = 1$, we can write $\frac{1}{2}x\,\mathrm{d}x = \mathrm{d}y$, and integrating gives

$$y = \frac{1}{4}\left(x^2 - x_0^2\right) + y_0, \tag{2.38}$$

where x_0 and y_0 are just our chosen integration constants.

Assuming a solution separable in x and y gives us $\Sigma(x, y) = X(x)Y(y)$, and we can substitute this into the continuity equation (2.12) with the steady flow condition to give

$$\boldsymbol{\nabla}\cdot(\Sigma\mathbf{u}) = 0 \quad\Longrightarrow\quad \left(\frac{2}{x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)X(x)Y(y) + X(x)Y(y)\frac{\partial}{\partial x}\left(\frac{2}{x}\right) = 0, \tag{2.39}$$

which we can evaluate and rearrange as

$$\frac{2}{x}\frac{X'}{X} - \frac{2}{x^2} = -\frac{Y'}{Y} = \text{const.}, \tag{2.40}$$

where the prime denotes differentiation with respect to the single argument of each function. We set this equal to a constant, since if a function only of x is equal to a function only of y for all allowed values of (x, y), then its partial derivative with respect to either must vanish, i.e. it is constant.

Solving Eq. (2.40) for $Y(y)$,

$$\begin{aligned} \frac{Y'}{Y} = -c \quad &\Longrightarrow \quad \frac{\mathrm{d}(\ln Y)}{\mathrm{d}y} = -c \\ &\Longrightarrow \quad Y \propto e^{-cy}, \end{aligned} \tag{2.41}$$

where c is a constant. We find for $X(x)$ that

$$\frac{\mathrm{d}(\ln X)}{\mathrm{d}x} = \frac{1}{2}cx + \frac{1}{x} \quad\Longrightarrow\quad X \propto |x|e^{cx^2/4}, \tag{2.42}$$

where the absolute value of x arises bearing in mind both signs are allowed, and $\int^x 1/x \, \mathrm{d}x = \ln|x|$.

Substituting the relationship for y along the streamline (2.38),

$$\Sigma(x, y) \propto |x| e^{\frac{1}{4}cx^2} e^{-cy} \quad \Longrightarrow \quad \Sigma = \Sigma_0 \frac{|x|}{x_0} \quad \text{given} \quad (2.38) \tag{2.43}$$

where the x^2 terms have cancelled in the exponential and the constant ones absorbed into Σ_0 and x_0.

We can write the surface density n of radioactive nuclei as

$$n = Q\Sigma = Q_0 \Sigma_0 \frac{|x|}{x_0} e^{-t}. \tag{2.44}$$

We need to find the time t since introduction of the radioactive particles along the streakline,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{x} \quad \Longrightarrow \quad \frac{1}{4}(x^2 - x_0^2) = t, \tag{2.45}$$

since at $t = 0$, the particles are at $x = x_0$. Substituting into (2.44) gives

$$n = n_0 |x| e^{-\frac{1}{4}x^2}. \tag{2.46}$$

This function is symmetric under the discrete transformation $x \to -x$, so let us find the maxima as

$$\frac{\mathrm{d}n}{\mathrm{d}|x|} = n_0 e^{-\frac{1}{4}|x|^2} - \frac{1}{2} n_0 |x|^2 e^{-\frac{1}{4}|x|^2} \quad \Longrightarrow \quad |x| = \sqrt{2}, \tag{2.47}$$

thus, if $|x_0| < \sqrt{2}$, then the surface concentration will reach this maximum before decaying again. ◀

2.4 Conservation of Momentum

2.4.1 Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of a surface element is

$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}\,. \tag{2.48}$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$\mathrm{d}F_i = \sum_j \sigma_{ij}\,\mathrm{d}S_j\,. \tag{2.49}$$

where σ_{ij} is the stress tensor – the force in direction i acting on a surface with normal along j.

Isotropic[3] pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}. \tag{2.51}$$

2.4.2 Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field $\mathbf{g}$ and internal pressure forces. Let the fluid element have volume $\mathcal{V}$ and surface $\mathcal{S}$.

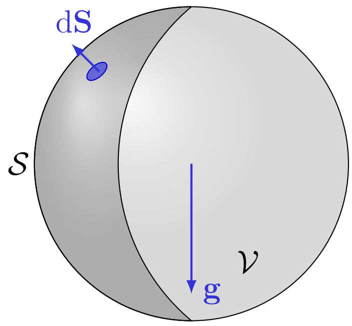


Fig. 2.5: A fluid element subject to gravity

Pressure acting on the surface element gives a force $-p\,\mathrm{d}\mathbf{S}$. The pressure force on an element projected in direction $\hat{\mathbf{n}}$ is $-p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$. So, the net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_{\mathcal{S}} p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V\,. \tag{2.52}$$

[3] An invariant tensor or invariant pseudo-tensor is one which has the same components in all frames,

$$T'_{ijk\ldots} = T_{ijk\ldots}. \tag{2.50}$$

Invariant tensors and invariant pseudo-tensors are both called isotropic tensors. All scalars are isotropic (from the transformation law for an order-zero tensor). There are no non-zero isotropic vectors or axial-vectors. The most general second-order isotropic tensor is $\lambda\delta_{ij}$ where λ is a scalar. Isotropic tensors don't have any “preferred” direction.

The rate of change of momentum of a fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}}\rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = \underbrace{-\int_{\mathcal{V}}\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V}_{(2.52)} + \int_{\mathcal{V}}\rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V\,. \tag{2.53}$$

In the limit that $\int \mathrm{d}V \to \delta V$ we have

$$\begin{aligned} &\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}\delta V)\cdot\hat{\mathbf{u}} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}} \\ \Longrightarrow\quad &\hat{\mathbf{n}}\cdot\mathbf{u}\underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta V)}_{\substack{=0\text{ by mass}\\\text{conservation}}} + \rho\delta V\hat{\mathbf{n}}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}} \\ \Longrightarrow\quad &\delta V\hat{\mathbf{n}}\cdot\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla}p - \rho\mathbf{g}\right) = 0. \end{aligned} \tag{2.54}$$

This must be true for all $\hat{\mathbf{n}}$ and all δV. So we arrive at the **Lagrangian momentum equation**,

$$\boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g},} \tag{2.55}$$

or instead from Eq. (2.5) we have the **Eulerian momentum equation**,

$$\boxed{\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \rho\mathbf{g},} \tag{2.56}$$

these are the equivalent of "$F = ma$" for the fluid element. Note the importance of the pressure *gradients*.

Now consider the Eulerian rate of change of momentum density $\rho\mathbf{u}$ and introduce a more compact notation

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t(\rho u_i) \\ &= \rho\partial_t u_i + u_i\partial_t\rho \\ &= \underbrace{-\rho u_j\partial_j u_i - \partial_j p\delta_{ij} + \rho g_i}_{(2.56)} - u_i\underbrace{\partial_j(\rho u_j)}_{(2.12)}, \end{aligned} \tag{2.57}$$

where we have used notation

$$\partial_j \equiv \frac{\partial}{\partial x_j} \tag{2.58}$$

and employed summation convention (summation over the repeated indices).

This gives

$$\partial_t(\rho u_i) = -\partial_j\Big(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor}\\\text{due to bulk flow}\\\text{“Ram Presure”}}} + \underbrace{p\delta_{ij}}_{\substack{\text{stress tensor}\\\text{due to random}\\\text{thermal motions}}}\Big) + \rho g_i = -\partial_j\sigma_{ij} + \rho g_i \tag{2.59}$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$\sigma_{ij} = p\delta_{ij} + \rho u_i u_j. \tag{2.60}$$

In component free language we write

$$\partial_t(\rho \mathbf{u}) = -\boldsymbol{\nabla} \cdot \underbrace{\left(\rho \mathbf{u} \otimes \mathbf{u} + p\underline{\underline{\mathbf{I}}}\right)}_{\substack{\text{flux of}\\ \text{momentum}\\ \text{density}}} + \rho \mathbf{g}. \tag{2.61}$$

2.4.2.1 Bernoulli's Equation Revisited

Consider the gradient of Eq. (2.22), $\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right)$. Using the identity (2.68), $\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}$. So,

$$\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = \boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Psi + \rho\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + \rho\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}). \tag{2.62}$$

We also have the Eulerian momentum equation (2.56),

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi, \tag{2.63}$$

and combining this with the right hand side of the above Eq. (2.62) gives

$$\boldsymbol{\nabla}p + \rho\boldsymbol{\nabla}\Psi + \rho\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} + \rho\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = -\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}), \tag{2.64}$$

and thus

$$\boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = -\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}). \tag{2.65}$$

If the flow is *steady*, i.e. $\rho\, \partial\mathbf{u}/\partial t = 0$, then taking the dot product of each side of the previous equation with $\mathbf{u}$ gives

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left(p + \rho\Psi + \frac{1}{2}\rho u^2\right) = 0. \tag{2.66}$$

This again proves that $p + \rho\Psi + \frac{1}{2}\rho u^2$ *is constant on a streamline* (Bernoulli's equation). If the flow is *steady and irrotational* (i.e. $\boldsymbol{\nabla} \times \mathbf{u} = 0$), then $p + \rho\Psi + \frac{1}{2}\rho u^2$ is a *constant everywhere*.

Note: this can be generalised to time-dependent flows. If the flow is irrotational and derived from a velocity potential, i.e. $\mathbf{u} = \boldsymbol{\nabla}\Phi$, then there is an important generalisation: $p + \rho\Psi + \frac{1}{2}\rho u^2 + \rho\, \partial\Phi/\partial t$ is constant everywhere, and at all times.

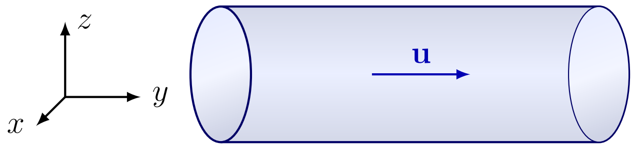


Fig. 2.6: Flow in a pipe

2.4.2.2 Example: Flow in a Pipe in the y-direction

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure. Following from equation (2.60),

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{2.67}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

2.4.3 Example 5: Vector Identities and Time Evolution of Vorticity

Example 2.5 Vector Identities and Time Evolution of Vorticity
Use the summation convention to prove:

$$\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b}) \equiv \boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{b} \cdot \mathbf{b}\right) - \mathbf{b} \cdot \boldsymbol{\nabla}\mathbf{b} \tag{2.68}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a) \equiv 0 \tag{2.69}$$

$$\boldsymbol{\nabla} \times (a\mathbf{b}) \equiv a\boldsymbol{\nabla} \times \mathbf{b} - \mathbf{b} \times \boldsymbol{\nabla} a. \tag{2.70}$$

Using the above identities and the curl of the momentum equation, show that if $\boldsymbol{\nabla} \times \mathbf{u} = 0$ everywhere at time $t = t_0$, then it remains so provided that the pressure is a function of the density only.

Solution For the first identity,

$$\begin{aligned} \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b}) &= \epsilon_{ijk} b_j (\boldsymbol{\nabla} \times \mathbf{b})_k \hat{\mathbf{e}}_i \\ &= \epsilon_{ijk}\epsilon_{klm} b_j \partial_l b_m \hat{\mathbf{e}}_i \\ &= \epsilon_{kij}\epsilon_{klm} b_j \partial_l b_m \hat{\mathbf{e}}_i \quad \text{since} \quad \epsilon_{ijk} = \epsilon_{kij} \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) b_j \partial_l b_m \hat{\mathbf{e}}_i \\ &= b_j \partial_i b_j \hat{\mathbf{e}}_i - b_j \partial_j b_i \hat{\mathbf{e}}_i \end{aligned} \tag{2.71}$$

where we have applied the very useful identity $\epsilon_{kij}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$.[4] Recognising the first term as half of a total derivative, we find

$$\begin{aligned}\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b}) &= \tfrac{1}{2}\partial_i(\mathbf{b} \cdot \mathbf{b})\hat{\mathbf{e}}_i - (\mathbf{b} \cdot \boldsymbol{\nabla}) b_i \hat{\mathbf{e}}_i \\ &= \boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{b} \cdot \mathbf{b}\right) - \mathbf{b} \cdot \boldsymbol{\nabla}\mathbf{b}. \qquad (2.75)\end{aligned}$$

For the second identity, the proof focuses on the antisymmetry of the levi-civita symbol,

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a) &= \epsilon_{ijk}\partial_j(\boldsymbol{\nabla} a)_k \hat{\mathbf{e}}_i \\ &= \epsilon_{ijk}\partial_j\partial_k a \hat{\mathbf{e}}_i \qquad (2.76) \\ &= \epsilon_{ikj}\partial_k\partial_j a \hat{\mathbf{e}}_i, \qquad (2.77)\end{aligned}$$

where we have relabelled the dummy indices $j \leftrightarrow k$. Using the commutativity of the partial derivative, we can write

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a) = \epsilon_{ikj}\partial_j\partial_k a \hat{\mathbf{e}}_i, \qquad (2.78)$$

and now using the total antisymmetry of ϵ_{ijk},

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a) = -\epsilon_{ijk}\partial_j\partial_k a \hat{\mathbf{e}}_i, \qquad (2.79)$$

which is also equal to the negative of itself (2.76), hence $\boldsymbol{\nabla} \times \boldsymbol{\nabla} a$ must vanish for all scalar fields a.

The final identity is the easiest, for it follows almost directly from application of the chain rule,

$$\begin{aligned}\boldsymbol{\nabla} \times (a\mathbf{b}) &= \epsilon_{ijk}\partial_j(a b_k)\hat{\mathbf{e}}_i \\ &= \epsilon_{ijk} a \partial_j b_k \hat{\mathbf{e}}_i + \epsilon_{ijk} b_k \partial_j a \hat{\mathbf{e}}_i \\ &= a(\boldsymbol{\nabla} \times \mathbf{b})_i \hat{\mathbf{e}}_i - \epsilon_{ikj} b_k \partial_j a \hat{\mathbf{e}}_i \quad \text{since} \quad \epsilon_{ijk} = -\epsilon_{ikj} \\ &= a(\boldsymbol{\nabla} \times \mathbf{b}) - (\mathbf{b} \times \boldsymbol{\nabla})a. \qquad (2.80)\end{aligned}$$

[4] The Levi-Civita symbol is related to the Kronecker delta. In three dimensions, the relationship is given by the following equations (vertical lines denote the determinant):[3]

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \qquad (2.72) \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). \qquad (2.73)\end{aligned}$$

A special case of this result occurs when one of the indices is repeated and summed over:

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \qquad (2.74)$$

In Einstein notation, the duplication of the i index implies the sum on i. The previous is then denoted $\epsilon_{ijk}\epsilon_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

We can write the Eulerian momentum equation (2.56) as

$$\dot{\mathbf{u}} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi, \tag{2.81}$$

where we used (3.1) to express $\mathbf{g}$ as the gradient of the potential $\mathbf{g} = -\boldsymbol{\nabla}\Psi$. Using the proven identity (2.68),

$$\dot{\mathbf{u}} = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) - \boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi. \tag{2.82}$$

Taking the curl of this equation, using Eq. (2.69) we find that multiple terms vanish, leaving us with

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla} \times \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) + \boldsymbol{\nabla} p \times \boldsymbol{\nabla}\left(\frac{1}{\rho}\right). \tag{2.83}$$

Given that the fluid is barotropic ($p = p(\rho)$), the pressure gradient can be expressed as $\boldsymbol{\nabla} p = p'(\rho)\boldsymbol{\nabla}\rho$, thus $\boldsymbol{\nabla} p \parallel \boldsymbol{\nabla}\rho$ and the cross product of these terms vanishes, since $\boldsymbol{\nabla}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\boldsymbol{\nabla}\rho$. This leaves

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla} \times \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}), \tag{2.84}$$

so, if $\boldsymbol{\nabla} \times \mathbf{u}$ initially, then the right hand side vanishes giving that $\boldsymbol{\nabla} \times \mathbf{u}$ is a conserved quantity of the flow. ◀

Chapter 3

Gravitation

3.1 Basics

Define the gravitational potential Ψ such that the gravitational acceleration $\mathbf{g}$ is

$$\boxed{\mathbf{g} = -\boldsymbol{\nabla}\Psi.} \tag{3.1}$$

If ℓ is some closed loop, we have (using the curl theorem)

$$\oint_\ell \mathbf{g}\cdot\mathrm{d}\mathbf{l} = \int_{\mathcal{S}} (\boldsymbol{\nabla}\times\mathbf{g})\cdot\mathrm{d}\mathbf{S} = -\int_{\mathcal{S}} [\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Psi)]\cdot\mathrm{d}\mathbf{S} = 0, \tag{3.2}$$

as curl of any gradient is zero. So gravity is a *conservative force* – the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point $\mathbf{r}$ to ∞ is thus

$$-\int_{\mathbf{r}}^{\infty} \mathbf{g}\cdot\mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \boldsymbol{\nabla}\Psi\cdot\mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r}), \tag{3.3}$$

which is *independent of path*.

A particular important case is the gravity of a point mass, which has

$$\Psi = -\frac{GM}{r} \qquad \text{if mass at origin} \tag{3.4}$$

$$\Psi = -\frac{GM}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{if mass at location } \mathbf{r}'. \tag{3.5}$$

For a system of point masses we have

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r}-\mathbf{r}'_i|} \tag{3.6}$$

$$\implies \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r}-\mathbf{r}'_i)}{|\mathbf{r}-\mathbf{r}'_i|^3} \tag{3.7}$$

Replacing $M_i \to \rho_i\delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G\int_{\mathcal{V}} \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V'\,. \tag{3.8}$$

Take divergence of both sides

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{g} &= -G\int_{\mathcal{V}}\rho(\mathbf{r}')\boldsymbol{\nabla}_{\mathbf{r}}\cdot\left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]\mathrm{d}V' \\
&= -G\int_{\mathcal{V}}\rho(\mathbf{r}')\boldsymbol{\nabla}_{\mathbf{r}}\cdot\left[\boldsymbol{\nabla}_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)\right]\mathrm{d}V' \\
&= -G\int_{\mathcal{V}}\rho(\mathbf{r}')\,\underbrace{\nabla^2_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right)}_{4\pi\delta(\mathbf{r}-\mathbf{r}')}\mathrm{d}V' \\
&= -4\pi G\int_{\mathcal{V}}\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' \\
&= -4\pi G\rho(\mathbf{r}).
\end{aligned}
\tag{3.9}
$$

where we have made use of our knowledge of the Green's function for the 3D laplacian operator.[1] Thus we arrive at *Poisson's equation for gravitation*,

$$
\boxed{\boldsymbol{\nabla}\cdot\mathbf{g} = -\nabla^2\Psi = -4\pi G\rho.} \tag{3.12}
$$

We can also express Poisson's equation in integral form: for some volume $\mathcal{V}$ bounded by surface $\mathcal{S}$ we have

$$
\begin{aligned}
&\int_{\mathcal{V}}\boldsymbol{\nabla}\cdot\mathbf{g}\,\mathrm{d}V = -4\pi G\int_{\mathcal{V}}\rho\,\mathrm{d}V \\
\Longrightarrow &\int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} = -4\pi GM.
\end{aligned}
\tag{3.13}
$$

This is useful for calculating $\mathbf{g}$ when the mass distribution obeys some symmetry.

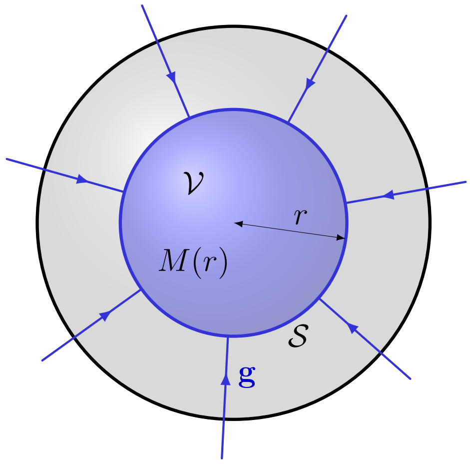


Fig. 3.1: Spherical distribution of mass

3.1.1 Spherical Distribution of Mass

By symmetry $\mathbf{g}$ is radial and $|\mathbf{g}|$ is constant over a $r = \text{const.}$ shell. So

$$
\begin{aligned}
\int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \underbrace{M(r)}_{\substack{\text{mass}\\ \text{enclosed}}} \\
\implies \quad -4\pi r^2 |\mathbf{g}| &= -4\pi G M(r) \\
\implies \quad |\mathbf{g}| &= \frac{G M(r)}{r^2} \\
\therefore \mathbf{g} &= -\frac{G M(r)}{r^2}\hat{\mathbf{r}}. \qquad (3.14)
\end{aligned}
$$

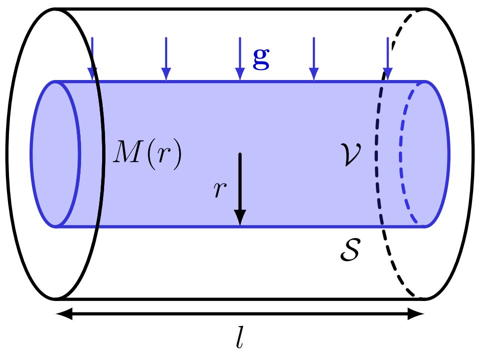


Fig. 3.2: Cylindrical distribution of mass

3.1.2 Infinite Cylindrically Symmetric Mass

By symmetry, $\mathbf{g}$ is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \Longrightarrow \qquad -2\pi r l |\mathbf{g}| &= -4\pi G l \underbrace{M(r)}_{\substack{\text{enclosed mass} \\ \text{per unit length}}} \\ \therefore \mathbf{g} &= -\frac{2GM(r)}{r}\hat{\mathbf{r}}. \end{aligned} \tag{3.15}$$

3.1.3 Infinite Planar Distribution of Mass

Assume infinite and homogeneous in x and y, $\rho = \rho(z)$.

By symmetry, $\mathbf{g}$ is in the $-\hat{\mathbf{z}}$ direction and is constant on a $z =$ const. surface.

[1]The Green's function (or fundamental solution) for the Laplacian (or Laplace operator) in three variables is used to describe the response of a particular type of physical system to a point source.

The free-space Green's function for the Laplace operator in three variables is given in terms of the reciprocal distance between two points and is known as the "Newton kernel" or "Newtonian potential". That is to say, the solution of the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{3.10}$$

is

$$G(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{3.11}$$

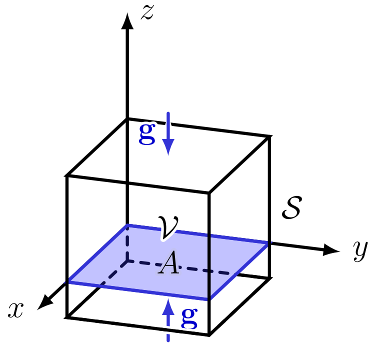


Fig. 3.3: Planar distribution of mass

So, if we also have reflection symmetry about $z = 0$,

$$
\begin{aligned}
\int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\
\implies \quad -2|\mathbf{g}|A &= -4GA \int_{-z}^{z} \rho(z) \, \mathrm{d}z \\
\therefore \mathbf{g} &= -4\pi G \hat{\mathbf{z}} \int_{0}^{z} \rho z \, \mathrm{d}z \quad \forall z \geq 0. \qquad (3.16)
\end{aligned}
$$

(For planar distribution of finite height $z_{\max}$, $\mathbf{g}$ is constant for $z \geq z_{\max}$.)

Example 3.1 Infalling Stars
A particle is released at rest at radius r_0 from the centre of a body mass M. Compute

(a) its initial acceleration,

(b) the time it takes to reach the centre of the body,

for the two cases

(i) that the body is a point mass,

(ii) that the body is a uniform sphere radius r_0.

A cluster consists initially of stars at rest, distributed in a uniform sphere. Find how long it takes a star to reach the centre as a function of its initial radius in the cluster and comment on your results.

Solution For the body that is a point mass (i), the gravitational potential is given by (3.4) $\Psi = -\frac{GM}{r}$, so considering the conservation of energy *per unit mass* at the initial radius r_0, and at an arbitrary radius r gives

$$
-\frac{GM}{r_0} = \frac{1}{2}u^2 - \frac{GM}{r}, \qquad (3.17)
$$

where u is the speed of the particle at r. Rearranging this for $\dot{r}$ gives

$$
2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \quad \implies \quad \frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}, \qquad (3.18)
$$

where we have chosen the negative sign to reflect the particles inwards trajectory, i.e. the radial distance is decreasing with time.

We can solve this integral by making the substitution $r = r_0 \cos^2\theta$, such that

$$\mathrm{d}r = -2r_0 \cos\theta \sin\theta \, \mathrm{d}\theta \,. \tag{3.19}$$

Substituting this into the right of Eq. (3.18) we see the simplification that motivated our chosen substitution,

$$\begin{aligned} 2r_0 \cos\theta \sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \sqrt{\frac{2GM}{r_0}(\sec^2\theta - 1)} \\ &= \sqrt{\frac{2GM}{r_0}} \tan\theta \end{aligned} \tag{3.20}$$

and thus we obtain the nicer looking *separable* differential equation

$$2r_0 \cos^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2GM}{r_0}} \,. \tag{3.21}$$

Finally, we can do the integral with the appropriate bounds,

$$\underbrace{\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta}_{\pi/4} = \sqrt{\frac{GM}{2r_0^3}} \int_0^t \mathrm{d}t' \,, \tag{3.22}$$

to obtain the time to the centre $t = \sqrt{\pi^2 r_0^3 / 8GM}$.

The acceleration for this case is given differentiation of Eq. (3.18), or by simply recalling the gravitational acceleration (3.14),

$$\ddot{r} = -\frac{GM}{r^2} \,, \tag{3.23}$$

so initially has a value $\ddot{r} = -GM/r_0^2$.

For the second case (ii), we again use the gravitational potential equation (3.4), $\Psi = -\frac{GM(r)}{r}$. However, since the shells outside the particle do not contribute to the potential, only the mass enclosed within radius r matters. This enclosed mass scales as $M(r) = Mr^3/r_0^3$. ◀

3.2 Potential of a Spherical Mass Distribution

We found in Eq. (3.14), for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \quad |\mathbf{g}| = \frac{G}{r^2} \underbrace{\int_0^r 4\pi\rho(r') r'^2 \, \mathrm{d}r'}_{M(r)} = \frac{\mathrm{d}\Psi}{\mathrm{d}r} \,, \tag{3.24}$$

so,

$$\Psi(r_0) - \Psi(\infty) = \int_{\infty}^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r\,. \tag{3.25}$$

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\Psi(r_0) = -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\Big|_{r=\infty}^{r_0} + \int_{\infty}^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r$$

$$\implies \Psi(r_0) = -\frac{GM(r_0)}{r_0} + \int_{\infty}^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r\,, \tag{3.26}$$

where we have made an assumption that $M(r) \to 0$ as $r \to 0$, and $M(r)/r \to 0$ as $r \to \infty$.

We find that $\Psi(r_0)$ is affected by matter outside of r_0 through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r.

3.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}, \tag{3.27}$$

and the energy required to take a unit mass to ∞ is $-\Psi$. The energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j, \tag{3.28}$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution, the natural extension is

$$\Omega = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V\,. \tag{3.29}$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2}\int_0^{\infty} 4\pi\rho(r)r^2\Psi(r)\,\mathrm{d}r \tag{3.30}$$

Integrate by parts, choosing parts $u \equiv \Psi$, $\mathrm{d}v \equiv 4\pi\rho r^2\,\mathrm{d}r$ so that $v = \int_0^r 4\pi\rho'^2\,\mathrm{d}r' = M(r)$, then

$$\Omega = \frac{1}{2}\left[M(r)\Psi(r)\Big|_0^{\infty} - \int_0^{\infty} M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,\mathrm{d}r\right]. \tag{3.31}$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (the boundary term) is zero. Noting further that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}, \tag{3.32}$$

we conclude

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r\,. \tag{3.33}$$

Integrate again by parts, choosing $u \equiv GM(r)^2$, $\mathrm{d}v \equiv \mathrm{d}r\,/r^2$,

$$\Omega = \underbrace{\frac{1}{2}GM(r)^2\frac{1}{r}\bigg|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r$$

$$\implies \quad \Omega = -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M\,. \tag{3.34}$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r}\,\mathrm{d}M(r)\,. \tag{3.35}$$

3.4 The Virial Theorem

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems. Here, we will examine the **scalar virial theorem** ($\exists$ general tensor virial theorem).

Consider the motion of a cloud of particles (atoms, stars, galaxies, ...). A particle with mass m_i at $\mathbf{r}_i$ is acted upon by a force

$$\mathbf{F}_i = m_i\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2}. \tag{3.36}$$

Consider the 2$^{\text{nd}}$ derivative of the scalar moment of inertia, $I_i = m_i r_i^2$,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i r_i^2\right) &= m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_i\cdot\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)\\
&= m_i\mathbf{r}_i\cdot\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2\\
&= \mathbf{r}_i\cdot\mathbf{F}_i + \underbrace{m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2}_{2\times\text{Kinetic Energy } T_i}\,.
\end{aligned} \tag{3.37}$$

If $I \equiv \sum_i m_i r_i^2$ then we can sum the previous equation over all particles to give

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{\substack{V\text{, the }\textbf{Virial}\\ \text{(R. Clausius)}}} + 2T. \tag{3.38}$$

The word **virial** for the first term on the right-hand side of the equation derives from *vis*, the Latin word for "force" or "energy", and was given its technical definition by Rudolf Clausius in 1870. [1]

In the absence of external forces (i.e. an isolated system), we have that $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ where $\mathbf{F}_{ij}$ is the force exerted on the i^{th} particle by the j^{th} particle. Consider any two particles with m_i and m_j at $\mathbf{r}_i$ and $\mathbf{r}_j$, Newton's 3rd Law says

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \tag{3.39}$$

and so their contribution to the virial is $\mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. We then have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j). \tag{3.40}$$

If there are no non-gravitational interactions except for possibly when $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \quad \text{where} \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \tag{3.41}$$

Thus we have

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}, \tag{3.42}$$

where each term is the work done to separate each pair of particles to infinity against gravity.

And so, $V = \Omega$ and we can use the above to write

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + \Omega. \tag{3.43}$$

If the system is in a steady state ("relaxed"), then $I =$ const. and we can state the *Virial theorem*

$$\boxed{2T + \Omega = 0.} \tag{3.44}$$

Here, the kinetic energy T has contributions from local flows and random/thermal motions. The Virial theorem implies *gravitational potential sets the "temperature" or velocity dispersion of the system.*

3.4.1 Implications of the Virial Theorem

Connects mass, velocity and size of a gravitating system

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M \left\langle v^2 \right\rangle \quad \text{and} \quad \Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r = -\int_0^\infty \frac{GM(r)}{r}\,\mathrm{d}M = -\frac{GM^2}{\bar{r}}. \tag{3.45}$$

So, invoking the virial theorem $2T = -\Omega$,

$$M\left\langle c^2 \right\rangle = \frac{GM^2}{\bar{r}} \quad \Longrightarrow \quad \left\langle v^2 \right\rangle = \frac{GM}{\bar{r}}. \tag{3.46}$$

Gravitating systems have a negative specific "heat" capacity,

$$\begin{aligned} E_{\text{total}} &= T + \Omega \\ &= -T = -\frac{1}{2} M \left\langle v^2 \right\rangle = -\frac{GM^2}{\bar{r}}. \end{aligned} \tag{3.47}$$

As a general rule, gravitationally bound systems have negative heat capacities. This is because in equilibrium (and remember we can't do classical thermodynamics without equilibrium anyway), some form of the virial theorem will apply. If the system has only kinetic energy T and potential energy Ω, then the total energy is of course $E_{\text{total}} = T + \Omega$, where $E_{\text{total}} < 0$ for bound systems. In virial equilibrium where the potential energy is purely gravitational, then we also have $T = -\Omega$. As a result, $E_{\text{total}} = -T$, and so adding more energy results in a decrease in temperature.

Broadly, this is why gravitation creates structure from initially smooth conditions. A dramatic manifestation is the gravothermal collapse (e.g. globular clusters). Note that, for a gravitating gas ball, T is directly related to the gas temperature.

Examples include stars and globular clusters. Imagine adding energy to such systems by heating up the particles in the star or giving the stars in a cluster more kinetic energy. The extra motion will work toward slightly unbinding the system, and everything will spread out. But since (negative) potential energy counts twice as much as kinetic energy in the energy budget, everything will be moving even slower in this new configuration once equilibrium is reattained.

Chapter 4

Equations of State and the Energy Equation

4.1 The Equation of State

In three dimensions, the (scalar) equation of mass conservation and the (vector) equation of momentum conservation can be written as four independent scalar equations. Given appropriate boundary conditions, these must be solved in order to find the density (scalar field), pressure (scalar field), gravitational potential (scalar field), and velocity components (3D vector field); a total of six degrees of freedom.

To close the system of equations, we need additional information. Specifically, we need to find relations between Ψ, p and the other fluid variables such as ρ and $\mathbf{u}$.

$\Psi(r)$ and ρ are related via Poisson's equation (and/or we sometimes consider an externally imposed gravitational potential).

p and the other thermodynamic properties of the system are related by the *equation of state* (EoS), depending upon microphysics of the fluid. This is only valid for collisional fluids.

Most astrophysical fluids are quite dilute (particle separation much larger than effective particle size) and can be well approximated as ideal gases. The corresponding EoS is

$$p = p(\rho, T) = nk_B T = \frac{k_B}{\mu m_p}\rho T, \tag{4.1}$$

where n is the number density of particles, and μ is the mean particle mass in units of the proton mass m_p. (Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars and white dwarfs.)

The ideal gas EoS introduces another scalar field into the description of the fluid, the temperature $T(\mathbf{r}, t)$. In general, we need to solve another PDE that describes the conservation of energy through heating and cooling processes in order to close the set of equations. We shall move on to this in Section 4.2.

4.1.1 Barotropic Fluids

However, for special cases, we can relate T and ρ without the need to solve a separate energy equation. Fluids for which p is *only* a function of ρ are known as barotropic fluids.

4.1.1.1 Electron Degeneracy Pressure

Important in systems with free electrons that are (relatively) cold and dense.

$$p = \frac{\pi^2 \hbar^2}{5 m_{\mathrm{e}} m_{\mathrm{ion}}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}, \quad \text{(non-relativistic)} \tag{4.2}$$

e.g., interiors of white dwarfs, iron core in massive stars, deep interior of Jupiter.

4.1.1.2 Isothermal Ideal Gas

$$p = A\rho, \quad A = \frac{k_B T}{\mu m_p} = \text{const.} \tag{4.3}$$

T is constant so that $p \propto \rho$. Valid most commonly when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance at some well-defined temperature.

4.1.1.3 Adiabatic Ideal Gas

Ideal gas undergoes *reversible* thermodynamic changes such that

$$p = K\rho^{\gamma} \tag{4.4}$$

where K, γ are constants.

The first law of thermodynamics is

$$\underbrace{đQ}_{\substack{\text{heat absorbed by}\\ \text{unit mass of fluid}\\ \text{from surrounding}}} = \underbrace{\mathrm{d}\mathcal{E}}_{\substack{\text{change in internal}\\ \text{energy of unit}\\ \text{mass of fluid}}} + \underbrace{p\,\mathrm{d}V}_{\substack{\text{work done by}\\ \text{unit mass of fluid}}}. \tag{4.5}$$

Here đ is a Pfaffian operator – change in quantity depends on the path taken through the thermodynamic phase space. For an ideal gas, we can write

$$p = \frac{\mathcal{R}_*}{\mu}\rho T, \quad \mathcal{E} = \mathcal{E}(T), \tag{4.6}$$

where $\mathcal{R}_*$ is a modified gas constant.

So, the first law of thermodynamics reads

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V\,, \end{aligned} \tag{4.7}$$

where we define specific heat capacity at constant volume as $C_V \equiv \mathrm{d}\mathcal{E}/\mathrm{d}T$ and have noted that for unit mass we have $\rho = 1/V$.

For a *reversible* change we have $đQ = 0$, so

$$C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V = 0$$

$$\implies \quad C_V\,\mathrm{d}(\ln T) + + \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}(\ln V) = 0$$

$$\implies V \propto T^{-C_V\mu/\mathcal{R}_*} \tag{4.8}$$

$$\implies p \propto T^{1+C_V\mu/\mathcal{R}_*} \quad \text{given} \quad (4.6). \tag{4.9}$$

C_V depends on the number of degrees of freedom with which the gas can store kinetic energy, f such that

$$C_V = f\frac{k_B}{2\mu m_p} = f\frac{\mathcal{R}_*}{2\mu}. \tag{4.10}$$

i.e. using here $\mathcal{R}_* = k_B/m_p$. From equipartition, there is an internal energy contribution of $\frac{1}{2}k_B T$ per particle per degree of freedom. Monatomic gas has $f = 3 \implies C_V = 3\mathcal{R}_*/2\mu$; diatomic gas at a few $\times$ 100 K (if two rotational modes excited) has $f = 5 \implies C_V = 5\mathcal{R}_*/2\mu$.

Returning to the ideal gas law,

$$p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \text{with} \quad \rho = 1/V \quad \text{for a unit mass of fluid}$$

$$\implies \quad pV = \frac{\mathcal{R}_* T}{\mu}$$

$$\implies \quad p\,\mathrm{d}V + V\,\mathrm{d}p = \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}T\,, \tag{4.11}$$

but,

$$\begin{aligned} đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\ &= \underbrace{\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} + \frac{\mathcal{R}_*}{\mu}\right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}}\,\mathrm{d}T - V\,\mathrm{d}p\,, \end{aligned} \tag{4.12}$$

so,

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu} = \frac{k_B}{\mu m_p}. \tag{4.13}$$

Let us define the ratio of specific heat capacities

$$\gamma \equiv \frac{C_p}{C_v} = \frac{f+2}{f}, \tag{4.14}$$

so that, for the reversible/adiabatic processes discussed above, we have

$$p \propto T^{1+C_V\mu/\mathcal{R}_*} \quad \Longrightarrow \quad p \propto T^{\gamma/(\gamma+1)} \tag{4.15}$$

$$V \propto T^{-C_V\mu/\mathcal{R}_*} \quad \Longrightarrow \quad V \propto T^{-1/(\gamma-1)} \tag{4.16}$$

which we can combine to give

$$p \propto \rho^\gamma \tag{4.17}$$

We say that a fluid element behaves *adiabatically* if $p = K\rho^\gamma$ with $K =$ constant. A fluid is **isentropic** if all fluid elements behave adiabatically with the same value of K. $\ln K$ is proportional to the entropy per unit mass.

Example 4.1 Stellar Wind
A stellar wind behaves as a steady adiabatic spherical outflow of a perfect monatomic gas (so $\gamma = c_p/c_V$) from the surface of the star, so at radius a the density is ρ_0, temperature T_0, and outflow velocity u_0. If the fluid motions are dominated by the star's gravitational potential, determine the temperature as a function of the radius from the star centre.
If the flow velocity u_0 at radius a is just the gravitational escape velocity from that point do pressure effects ever become significant?

Solution The flow is steady, so the particles follow the streamlines, along which we can apply the Bernoulli principle (see Subsection 2.3.1) to know that $\frac{1}{2}u^2 + \Psi + \frac{\gamma}{\gamma-1}\frac{p}{\rho}$ is constant (2.21), given the gas is ideal. Outside the surface of the star, the gravitational acceleration is $\Psi = -\frac{GM}{r}$ (3.4), thus we can write

$$\frac{1}{2}u^2 - \frac{GM}{r} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \text{const.} \tag{4.18}$$

In regions where gravitational effects dominate over pressure contributions, this simplifies to

$$\frac{1}{2}u^2 - \frac{GM}{r} = \text{const.} \tag{4.19}$$

Applying this equation at both the reference radius r_0 at the surface of the star and an arbitrary radial distance r, we obtain

$$u^2 - \frac{2GM}{r} = u_0^2 - \frac{2GM}{r_0}. \tag{4.20}$$

The conservation of mass (Eulerian continuity equation (2.12)), when integrated over spherical shells, implies a constant outwards mass flux,

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, \mathrm{d}V &= -\rho \int_{\mathcal{S}} \mathbf{u} \cdot \mathrm{d}\mathbf{S} \\ \implies \quad \text{const.} &= -\rho \int_{\mathcal{S}} \mathbf{u} \cdot \mathrm{d}\mathbf{S} \\ \implies \quad \rho_0 u_0 r_0^2 &= \rho u r^2. \end{aligned} \tag{4.21}$$

By eliminating $u(r)$, an explicit expression for the density profile is derived:

$$\begin{aligned} \left(\frac{\rho_0 u_0 r_0^2}{\rho r^2}\right)^2 - \frac{2GM}{r} &= u_0^2 - \frac{2GM}{r_0} \\ \implies \quad \rho &= \frac{\rho_0 u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}. \end{aligned} \tag{4.22}$$

Since gas flow is adiabatic, then we have the additional relation $p \propto \rho^\gamma$ (4.17) and the ideal gas law $p = \frac{\mathcal{R}_*}{\mu}\rho T$ (4.6), which combine to give the relation $T \propto \rho^{\gamma-1}$, so

$$T = T_0\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}. \tag{4.23}$$

Substituting ρ using (4.22) gives

$$T = T_0\left(\frac{u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\right)^{\gamma-1} = T_0\left(\frac{u_0 r_0^2}{r^2\sqrt{u_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\right)^{2/3}, \tag{4.24}$$

i.e. $T \propto r^{-4/3}$ for an ideal monatomic gas with $\gamma = 5/3$. If the gas is very hot at r_0 then thermal effects are important at first, but the rapid cooling as the gas expands means that the flow becomes very supersonic.

The escape velocity from the surface is given by

$$\frac{1}{2}u_0^2 - \frac{GM}{r_0} = 0 \quad \implies \quad u_0^2 = \frac{2GM}{r_0}, \tag{4.25}$$

so if the initial outflow velocity is equal to the escape velocity, then the density is

$$\rho = \rho_0\left(\frac{r_0}{r}\right)^{3/2}, \tag{4.26}$$

and so T is given by $T = T_0\left(\frac{r_0}{r}\right)^{\frac{3}{2}(\gamma-1)} = T_0\left(\frac{r_0}{r}\right)$, so $T \propto r^{-1}$. Using the adiabatic relation again, $p \propto \left(r^{-3/2}\right)^{5/3} \propto r^{-5/2}$ and $\boldsymbol{\nabla} p \propto r^{-7/2}$, indicating that pressure effects rapidly diminish with increasing radius. ◀

This analysis provides a physically motivated model for stellar winds, including the solar wind, where thermal expansion and gravitational forces dictate the structure of the outflow. The results demonstrate that while thermal pressure initially contributes to driving the wind, its role becomes negligible at large distances, where the flow becomes highly supersonic.

4.2 The Energy Equation

In general, the equation of state will not be barotropic and we will need to solve a separate partial differential equation which follows the energy conservation in the flow through heating and cooling processes in the gas, the *energy equation*.

From the first law of thermodynamics we have

$$đQ = \mathrm{d}\mathcal{E} + \underbrace{p\,\mathrm{d}V}_{\mathrm{d}W=-p\mathrm{d}V} \quad \text{in absence of dissipative processes,} \tag{4.27}$$

so, applying this to a given fluid element in the Lagrangian framework

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{\mathrm{D}W}{\mathrm{D}t} + \frac{đQ}{\mathrm{d}t}, \tag{4.28}$$

with, for a unit mass $V = 1/\rho$, **??**

$$\frac{\mathrm{D}W}{\mathrm{D}t} = -p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{4.29}$$

and

$$\frac{đQ}{\mathrm{d}t} \equiv -\dot{Q}_{\mathrm{cool}} \quad \text{rate of cooling per unit mass,} \tag{4.30}$$

therefore,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}. \tag{4.31}$$

The total energy per unit volume is

$$E = \rho\left(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}}\right), \tag{4.32}$$

so from a simple application of the product rule,

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\mathrm{D}\rho}{\mathrm{D}t}\frac{E}{\rho} + \rho\left(\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}\right), \tag{4.33}$$

where,

$$\frac{\mathrm{D}E}{\mathrm{D}t} \equiv \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E \qquad (2.5) \tag{4.34}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad (2.14) \tag{4.35}$$

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi \qquad (2.55) \tag{4.36}$$

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} \equiv \frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Psi. \qquad (2.5) \tag{4.37}$$

Substituting Eqs. (4.34-4.37) into Eq. (4.33) it follows that

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{E}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p - \cancel{\rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi} + \rho\frac{\partial\Psi}{\partial t} + \cancel{\rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi} - \frac{p}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \rho\dot{Q}_{\mathrm{cool}}$$

$$\Longrightarrow \quad \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E = -(E+p)\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p + \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}}, \tag{4.38}$$

which, upon using the vector identity $\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) \equiv \phi\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot(\boldsymbol{\nabla}\phi)$, gives the **Energy equation**,

$$\boxed{\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \rho\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho\dot{Q}_{\mathrm{cool}}.} \tag{4.39}$$

In many settings, $\partial\Psi/\partial t = 0$, i.e. Ψ depends on position only. If, further, we have no cooling ($\dot{Q}_{\mathrm{cool}} = 0$), then this equation expresses the conservation of energy in which the Eulerian change in total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$.

4.3 Heating and Cooling Processes

The $\dot{Q}_{\mathrm{cool}}$ term in the energy equation describes processes that locally cool ($\dot{Q}_{\mathrm{cool}} > 0$) or locally heat ($\dot{Q}_{\mathrm{cool}} < 0$) the fluid. There are many such processes and a full discussion of them would be lengthy as they depend upon the detailed microphysics of the system under consideration. Here, we discuss just a small number of important cases relevant to the energy-budget of a *thermal gas* (i.e. one in which the bulk of the particles are in thermodynamic equilibrium) in an astrophysical setting (diffuse and dominant element is hydrogen).

1. *Cooling by radiation*: energy carried away from fluid by photons.
   - Energy loss by recombination of an ionised gas. Free electron captured by ion puts ion in excited state. Electron cascades down energy levels, eventually forming ground state, so ion de-excites via line emission. Number of recombinations per ion per unit time $\propto n_{\mathrm{e}}$, and the number of recombinations per unit volume per unit time $\propto n_{\mathrm{e}} \times n_{\mathrm{ion}}$ and $\dot{Q} = \rho f(T)$;
   - Energy loss by free-free emission (free electrons accelerated in electric fields of ions)
   $$L_{\mathrm{ff}} \propto n_{\mathrm{e}} n_{\mathrm{p}} T^{1/2}. \tag{4.40}$$
   - Collisionally-excited atomic line radiation. Electron-ion collisions lead to excited electronic states (inelastic collision with ground state atom). Excited state decays back to ground state via the emission of photons at well-defined energy χ. Number of collisions per ion per unit time $\propto n_{\mathrm{e}}$, and the number of collisions per unit volume per unit time $\propto n_{\mathrm{e}} \times n_{\mathrm{ion}}$.
   $$L_{\mathrm{e}} \propto n_{\mathrm{e}} n_{\mathrm{ion}} e^{-\chi/kT} \chi/\sqrt{T} \tag{4.41}$$

In cold gas clouds with $T \sim 10^4\,\mathrm{K}$, H cannot be excited so cooling occurs through trace species ($\mathrm{O}^+, \mathrm{O}^{++}, \mathrm{N}^+$).

These are all two-body interactions $\implies$ cooling rate per unit volume proportional to ρ^2 . Recalling that $\dot{Q}_{\mathrm{cool}}$ is defined per unit mass, such processes give $\dot{Q}_{\mathrm{cool}} = \rho f(T)$.

2. *Heating by cosmic rays*: Heating can occur through the dissipation of kinetic energy via internal processes within the fluid, e.g. shocks (Chapter 6) and viscosity (Chapter 9). Heating can also occur from an external agent, heating and energy transport via high-energy (often highly relativistic) particles that are diffusing/streaming through the thermal fluid.

   - High energy particles ionise atoms in fluid, excess energy put into freed e^-. High-energy electrons proceed to collide with atoms/ions, which ends up thermalising the energy as heat in fluid.

$$\begin{aligned} &\text{ionisation rate per unit volume} \propto \text{CR flux} \times \rho \\ \implies \quad &\dot{Q}_{\mathrm{cool}} \propto \text{CR flux. (independent of } \rho\text{)} \end{aligned} \tag{4.42}$$

Combining these cases, we can parametrise $\dot{Q}_{\mathrm{cool}}$ as:

$$\dot{Q}_{\mathrm{cool}} = \underbrace{A\rho T^\alpha}_{\substack{\text{radiative}\\ \text{cooling}}} - \underbrace{H}_{\text{CR heating}}, \tag{4.43}$$

where α depends upon the physics of the dominant radiative cooling process.

4.4 Energy Transport Processes

Transport processes move energy through the fluid. Important examples are:

1. *Thermal conduction*: transport of thermal energy down temperature gradients due to diffusion of the hot e^- into cooler regions. Relevant in, for example

   - Interiors of white dwarfs;
   - Supernova shock fronts;
   - ICM plasma.

   There is also thermal conduction associated with ions, but it is smaller than the electron thermal conduction by a factor of $\sqrt{m_{\mathrm{ion}}/m_e} \sim 43$.

   The energy flux per unit area is

$$\mathbf{F}_{\mathrm{cond}} = -\kappa \boldsymbol{\nabla} T, \tag{4.44}$$

where κ is thermal conductivity (computed from kinetic theory).

The local rate of change of E per unit volume is

$$-\boldsymbol{\nabla} \cdot \mathbf{F}_{\rm cond} = \kappa \nabla^2 T. \tag{4.45}$$

2. *Convection*: transport of energy due to fluctuating or circulating fluid flows in presence of entropy gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.

3. *Radiation transport*: Transport of energy through system due to radiation, relevant in optically-thick systems (mean free path of photon much shorter than size of system).

   If scattering opacity dominates, then we have radiative diffusion. If $\epsilon_{\rm rad}$ is the energy density of the radiation field, the radiative flux through the fluid is

$$\mathbf{F}_{\rm rad} \propto -\boldsymbol{\nabla} \epsilon_{\rm rad}. \tag{4.46}$$

   The general topic of radiation transport through a fluid flow is very complex and beyond the scope of this course. Important in stellar interiors, supernova explosions, and black hole accretion disks.

Chapter 5

Hydrostatic Equilibrium, Atmospheres and Stars

We now have the full set of equations describing the dynamics of an ideal (inviscid, dilute, unmagnetised) non-relativistic fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \qquad \text{Continuity equation (2.12)} \tag{5.1}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \qquad \text{Momentum equation (2.56)} \tag{5.2}$$

$$\nabla^2 \Psi = 4\pi G \rho \qquad \text{Poisson's equation (3.12)} \tag{5.3}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho \frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho \dot{Q}_{\mathrm{cool}} \qquad \text{Energy equation (4.39)} \tag{5.4}$$

$$E = \rho\left(\frac{1}{2}u^2 + \Psi + \mathcal{E}\right) \qquad \text{Definition of total energy (4.32)} \tag{5.5}$$

$$p = \frac{k_B}{\mu m_p} \rho T \qquad \text{EoS for ideal gas (4.1)} \tag{5.6}$$

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho} \qquad \text{Internal energy for ideal gas (6.72)} \tag{5.7}$$

We proceed to use those equations to explore astrophysically relevant situations.

This chapter starts with the simplest, but important, case – fluid systems that are in static equilibrium with pressure forces balancing gravity.

5.1 Hydrostatic Equilibrium

A fluid system is in a state of **hydrostatic equilibrium** if

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial}{\partial t} = 0. \tag{5.8}$$

Then, the continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{5.9}$$

The momentum equation gives

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi = 0, \tag{5.10}$$

resulting in the *equation of hydrostatic equilibrium*

$$\boxed{\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi} \tag{5.11}$$

Assuming a barotropic equation of state $p = p(\rho)$, this system of equations can be solved.

5.1.1 Isothermal Atmosphere with Constant (Externally Imposed) g

Suppose $\mathbf{g} = -g\hat{\mathbf{z}}$. Then the equation of hydrostatic equilibrium with an isothermal equation of state reads

$$\begin{aligned}
\text{Isothermal} \quad \Longrightarrow \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad &\Longrightarrow \quad p = A\rho, \quad A = \text{const.} \\
A\frac{1}{\rho}\boldsymbol{\nabla}\rho &= -\boldsymbol{\nabla}\Psi = -g\hat{\mathbf{z}} \\
\Longrightarrow \quad \ln\rho &= -\frac{gz}{A} + \text{const.} \\
\therefore \rho &= \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T} z\right), \qquad (5.12)
\end{aligned}$$

i.e. exponential atmosphere.

Examples of this is the Earth's atmosphere: $T \sim 300\,\mathrm{K}$ and $\mu \sim 28 \implies$ e-folding $\sim 9\,\mathrm{km}$. The highest astronomical observatories are at $z \sim 4\,\mathrm{km}$, so have ρ and $p \sim 60\%$ of sea level.

5.1.2 Vertical Density Structure of an Isothermal, Rotationally-Supported, Geometrically-Thin Gas Disk Orbiting a Central Mass

At a given patch of the disk, transform into a locally co-moving and co-rotating frame. In z-direction, pressure forces balance z-component of gravity,

$$g_z \approx -\frac{GM}{r^2}\frac{z}{r} \approx -\frac{GMz}{R^3}. \tag{5.13}$$

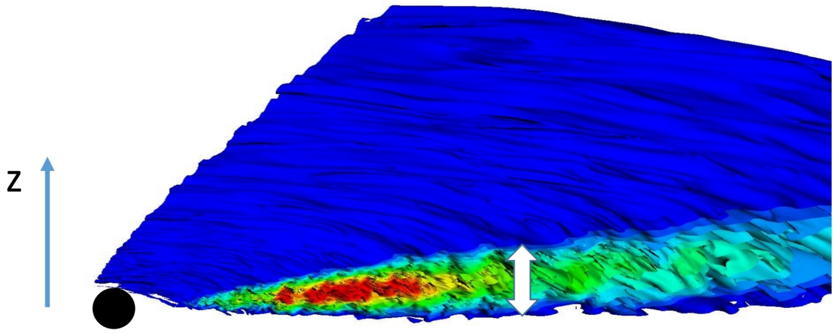


Fig. 5.1: Vertical density structure of an isothermal, rotationally- supported, geometrically-thin gas accretion disk orbiting a central mass.

So, hydrostatic equilibrium gives,

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial z} &= g_z \\
\implies \quad A\frac{1}{\rho}\frac{\partial \rho}{\partial z} &= -\frac{GM}{R^3}z \\
\implies \quad A\ln\rho &= -\frac{GM}{2R^3}z^2 + \text{const.} \\
\implies \quad \rho &= \rho_0 \exp\left(-\frac{GMz^2}{2R^3A}\right) \\
\therefore \rho = \rho_0 \exp\left(-\frac{\Omega^2}{2A}z^2\right) \quad &\text{where} \quad \Omega^2 = \frac{GM}{R^3}
\end{aligned}
\tag{5.14}
$$

If a system is self-gravitating (rather than having an externally imposed gravitational field), we also have

$$\nabla^2\Psi = 4\pi G\rho. \tag{5.15}$$

This must be solved together with the equation of hydrostatic equilibrium (5.11) $\frac{1}{\rho}\boldsymbol{\nabla}p = -\boldsymbol{\nabla}\Psi$.

Example 5.1 Self-Gravitating Fluid Slab
A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z| < a$. Find the gravitational field everywhere and the pressure distribution within the slab. [Hint: check the limits when integrating the pressure gradient.]
If a galactic disk is approximated by a uniform density slab with density 10^{-18} kg m^{-3} and $a = 10^{18}$ m, determine the velocity of a star at the midplane if it starts from rest at $z = a$, and the period of its oscillation.

Solution Using Gauss' theorem and Eq. (3.12), we proceed very similarly to

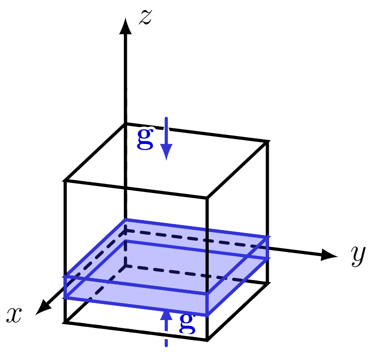


Fig. 5.2: A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z| < a$.

Subsection 3.1.3,

$$\begin{aligned}
\int_{\mathcal{V}} \nabla \cdot \mathbf{g} \, \mathrm{d}V &= -4\pi G\rho \int_{\mathcal{V}} \mathrm{d}V \\
\int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G\rho \int_{\mathcal{S}} \mathrm{d}S \int_{-z}^{z} \mathrm{d}z' \\
-2|\mathbf{g}| \int_{\mathcal{S}} \mathrm{d}S &= -4\pi G\rho \int_{\mathcal{S}} \mathrm{d}S \int_{-z}^{z} \mathrm{d}z' \\
|\mathbf{g}| &= 4\pi G\rho \int_{0}^{z} \mathrm{d}z' \,.
\end{aligned} \tag{5.16}$$

We know $\mathbf{g}$ must lie parallel to $\hat{\mathbf{z}}$, and so there are only contributions from the constant z planes as $\mathbf{g}$ lies tangential to the other planes in Fig. 5.2. For $|z| \leq a$,

$$\begin{aligned}
|\mathbf{g}| &= \quad 4\pi G\rho z \\
\mathbf{g} &= -4\pi G\rho z\hat{\mathbf{z}}.
\end{aligned} \tag{5.17}$$

For $|z| \geq a$,

$$|\mathbf{g}| = 4\pi G\rho a \qquad \Longrightarrow \qquad \mathbf{g} = \begin{cases} -4\pi G\rho a\hat{\mathbf{z}} & z \geq a \\ 4\pi G\rho a\hat{\mathbf{z}} & z \leq -a \end{cases}. \tag{5.18}$$

In hydrostatic equilibrium we can use Eqs. (5.11) and (3.1) to write

$$\nabla p = -\rho \nabla \Psi = \rho \mathbf{g}. \tag{5.19}$$

From the apparent symmetry of the situation in Fig. 5.2, $\nabla = \partial/\partial z \; \hat{\mathbf{z}}$, and so we can find the pressure distribution as

$$\begin{aligned}
\frac{\partial p}{\partial z} &= \rho\Big(-4\pi G\rho z\Big) \\
p &= -4\pi G\rho^2 \int_{a}^{z} z' \, \mathrm{d}z' && (5.20) \\
&= 2\pi G\rho^2 \Big(a^2 - z^2\Big) && (5.21)
\end{aligned}$$

Within the galactic disc, the motion can be approximated as simple harmonic oscillations with angular frequency Ω, since the gravitational acceleration is proportional to z,

$$\mathbf{g} = -4\pi G\rho z\hat{\mathbf{z}} \quad \Longrightarrow \quad \ddot{z} = -\underbrace{4\pi G\rho}_{\Omega^2} z, \tag{5.22}$$

which gives the time period of oscillation as $\sqrt{\pi/G\rho} = 6.9 \times 10^6$ yr. To find the velocity at the midpoint, we notice that $\ddot{z} = \dot{z}\frac{\mathrm{d}\dot{z}}{\mathrm{d}z}$, so it follows

$$\begin{aligned} \dot{z}\,\mathrm{d}\dot{z} &= -4\pi G\rho z\,\mathrm{d}z \\ \int_0^v \dot{z}\,\mathrm{d}\dot{z} &= -4\pi G\rho \int_a^0 z\,\mathrm{d}z \\ \Longrightarrow \quad v &= 2a\sqrt{\pi G\rho} = 29 \text{ km s}^{-1}. \end{aligned} \tag{5.23}$$

◀

5.1.3 Isothermal Self-Gravitating Slab

Consider a static, isothermal slab in x and y which is symmetric about $z = 0$ (e.g. two clouds collide and generate a shocked slab of gas between them).

$$\text{Isothermal} \quad \Longrightarrow \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \Longrightarrow \quad p = A\rho, \quad A = \text{const.}$$

also, $\boldsymbol{\nabla} = \partial/\partial z$ due to symmetry, $p = p(z)$, $\Psi = \Psi(z)$.

Then the equation of hydrostatic equilibrium becomes

$$\begin{aligned} A\frac{1}{\rho}\boldsymbol{\nabla}\rho &= -\boldsymbol{\nabla}\Psi \\ \Longrightarrow \quad A\frac{\mathrm{d}}{\mathrm{d}z}(\ln\rho) &= -\frac{\mathrm{d}\Psi}{\mathrm{d}z} \\ \Longrightarrow \quad \Psi &= -A\ln(\rho/\rho_0) + \Psi_0 \quad \text{where} \quad \rho_0 = \rho(z=0) \\ \therefore \rho &= \rho_0 e^{-(\Psi-\Psi_0)/A}. \end{aligned} \tag{5.24}$$

Since $A \propto T$, we note that this last equation has the form of a Boltzmann distribution.

Poisson's equation is

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}. \tag{5.25}$$

Let's change variables to $\chi = -(\Psi - \Psi_0)/A$, $Z = z\sqrt{2\pi G\rho_0/A}$ so that Poisson's equation becomes

$$\begin{aligned}
\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} = -2e^{\chi} \quad \text{with} \quad \chi = \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 0 \quad \text{at} \quad Z = 0 \\
\implies \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z}\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} = -2\frac{\mathrm{d}\chi}{\mathrm{d}Z}e^{\chi} \\
\implies \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2\right] = -2\frac{\mathrm{d}}{\mathrm{d}Z}(e^{\chi}) \\
\implies \quad \left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2 = C_1 - 4e^{\chi}.
\end{aligned} \tag{5.26}$$

But we have boundary condition $\mathrm{d}\chi/\mathrm{d}Z = 0$ when $\chi = 0 \implies C_1 = 4$.

$$\therefore \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 2\sqrt{1-e^{\chi}} \quad \implies \quad \int \frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} = 2\int \mathrm{d}Z\,. \tag{5.27}$$

Change variables $e^{\chi} = \sin^2\theta$

$$\implies \quad e^{\chi}\,\mathrm{d}\chi = 2\sin\theta\cos\theta\,\mathrm{d}\theta \quad \text{or} \quad \mathrm{d}\chi = \frac{2\cos\theta}{\sin\theta}\,\mathrm{d}\theta\,. \tag{5.28}$$

So, we can evaluate the χ integral

$$\begin{aligned}
\int \frac{\mathrm{d}\chi}{\sqrt{1-e^{\chi}}} &= \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \\
&= \int \frac{2\,\mathrm{d}\theta}{\sin\theta} \\
&= \int 2\frac{1}{2}\frac{1+t^2}{t}\,\mathrm{d}\theta \\
&= 2\int \frac{\mathrm{d}t}{t} \\
&= 2\ln t + C_2,
\end{aligned} \tag{5.29}$$

by setting

$$t = \tan\tfrac{\theta}{2} \quad \implies \quad \mathrm{d}t = \tfrac{1}{2}(1+t^2)\,\mathrm{d}\theta\,, \tag{5.30}$$

and by noting

$$\sin\theta \equiv 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = \frac{2t}{1+t^2} = e^{\chi/2}. \tag{5.31}$$

So, Poisson's equation becomes

$$2\ln t = 2Z + C_2. \tag{5.32}$$

Now, $\chi = 0$ at $Z = 0 \implies \theta = \pi/2$, $t = 1 \implies C_2 = 0$, so $t = e^Z$

$$\implies \quad \sin\theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}. \tag{5.33}$$

This gives

$$\boxed{\begin{aligned} \Psi - \Psi_0 &= 2A \ln\left[\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)\right] \qquad (5.34) \\ \rho &= \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)} \qquad (5.35) \end{aligned}}$$

5.2 Stars as Self-Gravitating Polytropes

Polytropes are useful as they provide simple solutions (albeit in some cases via numerical integration) for the internal structure of a star that can be tabulated and used for estimates of various quantities. They are much simpler to manipulate than the full rigorous solutions of *all* the equations of stellar structure. But the price of this simplicity is assuming a power law relationship between pressure and density which must hold (including a fixed constant) throughout the star.

Consider a spherically-symmetric self-gravitating system in hydrostatic equilibrium; from now on we will refer to this as a "star". We have

$$\begin{aligned} &\boldsymbol{\nabla} p = -\rho \boldsymbol{\nabla}\Psi \\ \implies \quad &\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r}. \quad \text{(spherical polar)} \end{aligned} \tag{5.36}$$

Now, $\rho > 0$ within a star which implies p is a monotonic function of Ψ. Also

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad \implies \quad \rho = -\frac{\mathrm{d}p}{\mathrm{d}\Psi}. \tag{5.37}$$

So ρ is a monotonic function of Ψ,

$$\therefore p = p(\Psi) \quad \text{and} \quad \rho = \rho(\Psi) \quad \implies \quad p = p(\rho), \tag{5.38}$$

i.e. non-rotating stars are barotropes!

A barotropic EoS can be written as

$$p = K\rho^{1+1/n}, \tag{5.39}$$

where in general $n = n(\rho)$. When $n =$ constant, we say that we have a *polytropic* EoS and the structure is called a **polytrope**. Real stars are in fact well approximated as polytropes.

It is important to note that in general we will have

$$1 + \frac{1}{n} \neq \gamma. \tag{5.40}$$

We only have $1 + 1/n = \gamma$ (i.e. $p \propto \rho^\gamma$ (4.17)) if the star is isentropic (constant entropy throughout) due to, for example, mixing by convective motions throughout.

Assuming a polytropic EoS, the equation of hydrostatic equilibrium gives

$$-\boldsymbol{\nabla}\Psi = \frac{1}{\rho}\boldsymbol{\nabla}\left(K\rho^{1+1/n}\right) = (n+1)\boldsymbol{\nabla}\left(K\rho^{1/n}\right)\textbf{??}$$

$$\implies \quad \rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n, \tag{5.41}$$

defining $\Psi_T \equiv \Psi$, where $\rho = 0$ on the surface. If the central density is ρ_c and central potential is Ψ_c, we have

$$\rho_c = \left(\frac{\Psi_T - \Psi_c}{(n+1)K}\right)^n, \tag{5.42}$$

so we can write,

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n. \tag{5.43}$$

Feeding this into Poisson's equation gives

$$\nabla^2\Psi = 4\pi G\rho_c\Bigg(\underbrace{\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}}_{\theta}\Bigg)^n. \tag{5.44}$$

Define a remapped potential coordinate $\theta = (\Psi_T - \Psi)/(\Psi_T - \Psi_c)$, we then get

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n. \tag{5.45}$$

Imposing spherical symmetry and writing in spherical polars, this becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n. \tag{5.46}$$

Defining a scaled radial coordinate $\xi = r\sqrt{4\pi G\rho_c/(\Psi_T - \Psi_c)}$, we finally get the **Lane-Emden equation of index $\boldsymbol{n}$**,

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.} \tag{5.47}$$

The appropriate boundary conditions for the Lane-Emden equation require that at the centre of the star where $\xi = 0$, $\theta(0)$ must be one. Furthermore, since $\mathrm{d}p/\mathrm{d}r$ approaches 0 as $r \to 0$, we need $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. (Zero force at $\xi = 0$, enclosed mass $\to 0$ as $\xi \to 0$.) The outer boundary (the surface) is the first location where $\rho = 0$, or equivalently $\theta(\xi) = 0$. That location is called ξ_1. The formal solution may have additional zeros at larger values of ξ, but $\xi > \xi_1$ is not relevant for stellar models.

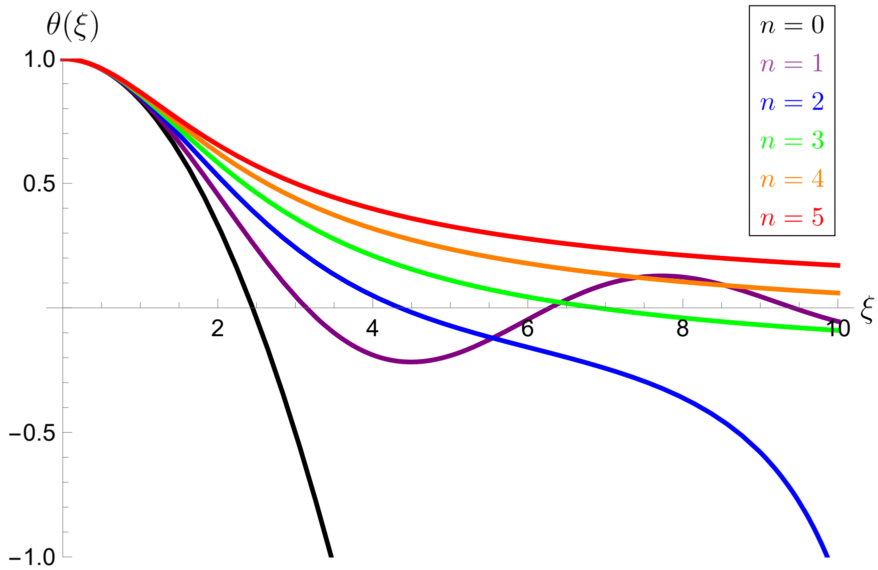


Fig. 5.3: Solutions $\theta(\xi)$ to the Lane-Emden equation for various values of n.

The Lane-Emden equation can be solved analytically for $n = 0$, 1 and 5; otherwise solve numerically. For $n = 5$, the first zero of $\theta(\xi)$, which is proportional to the radius of the polytrope, occurs at infinity. For $n > 5$, the binding energy is positive, and hence such a polytrope cannot represent a real star.

For all other polytrope indices n, a numerical solution to the Lane-Emden equation must be calculated. A display of solutions for several values of n between 0 to 6 is given in Fig. 5.3. Note that the radius of the star is defined by the first zero in the solution, and the solution at larger values of ξ is not relevant for computing stellar models

5.2.1 Solution for $n = 0$

This is a somewhat singular case, physically corresponding to a fluid that is at constant uniform density and incompressible.

$$\begin{aligned}
\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\theta^n = -1 \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\xi^2 \\
\implies \quad \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} &= -\frac{1}{3}\xi^3 - C \\
\therefore \theta = -\frac{\xi^2}{6} + \frac{C}{\xi} + D. &
\end{aligned} \tag{5.48}$$

We need $\theta = 1$ at $\xi = 0 \implies C = 0, D = 1$ and hence

$$\theta = 1 - \frac{\xi^2}{6}. \tag{5.49}$$

5.3 Isothermal Spheres (Case $n \to \infty$)

The isothermal case $p = K\rho$ corresponds to $n \to \infty$). Let's combine Eqs. (5.36) and (5.39),

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad &\text{and} \quad p = K\rho \\ \implies \quad \frac{\mathrm{d}\Psi}{\mathrm{d}r} &= -\frac{K}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} \\ \implies \quad \Psi - \Psi_c &= -K\ln(\rho/\rho_c). \end{aligned} \tag{5.50}$$

From Poisson's equation

$$\begin{aligned} \nabla^2\Psi &= 4\pi G\rho \\ \implies \quad \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= 4\pi G\rho \\ \implies \quad \frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) &= -4\pi G\rho. \end{aligned} \tag{5.51}$$

Let $\rho = \rho_c e^{-\psi}$ (defining $\psi = \Psi/K$, and $\Psi_c = 0$), we set

$$r = a\xi, \quad a = \sqrt{\frac{K}{4\pi G\rho_c}}, \tag{5.52}$$

then,

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) = e^{-\psi}, \tag{5.53}$$

with $\psi = \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0$ at $\xi = 0$.

This replaces the Lane-Emden equation in the case where the system is isothermal.

At large radii, this has solutions of the form $\rho \propto r^{-2}$ (see Fig. 5.4), so the enclosed mass $\propto r$. Thus, the mass of an isothermal sphere of self-gravitating gas tends to ∞ as the radius tends to ∞. This is why we cannot adopt our usual convention of defining $\Psi = 0$ at ∞.

So, to be physical (finite total mass), isothermal spheres need to be truncated at some finite radius. There needs to be some confining pressure by an external medium. These are called **Bonnor-Ebert spheres**, whose density profile depends on ξ_{cut}. E.g. dense gas cores in molecular clouds are well fitted by such Bonnor-Ebert spheres. Stability requires $\rho_c/\rho_{\text{ext}} < 14$.

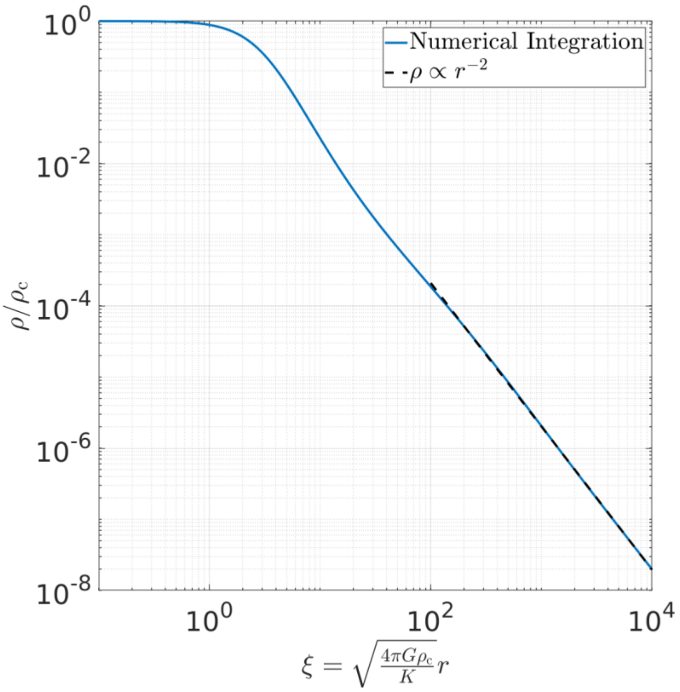


Fig. 5.4: Numerical integration solution to $1/\xi^2\ \mathrm{d}/\mathrm{d}\xi\ \left(\xi^2\ \mathrm{d}\psi/\mathrm{d}\xi\right) = e^{-\psi}$, at large radii, this has solutions of the form $\rho \propto r^{-2}$.

5.4 Scaling Relations

In many circumstances, stars behave as polytropes, e.g. fully convective stars with $p(\rho)$ close to the adiabatic relation. In such a star ideal gas pressure dominates, assuming the gas is monatomic with $\gamma = 5/3$, we have $p = K\rho^{5/3} \implies n = 3/2$. White dwarfs well below the Chandrasekhar mass also correspond to $n = 3/2$

Consider a set of stars which share a given polytropic index n and a given constant K. They will then form a one-parameter family characterised by their central density ρ_c. The *shape* of the density distribution within each star in the family is identical, given by the appropriately scaled solution of the Lane-Emden equation.

Thus one can find how mass and radius vary as a function of ρ_c and, eliminating ρ_c, obtain *scaling relations* relating the mass and radius.

To be concrete, for now, consider a family of stars with $p = K\rho^{1+1/n}$, where both n and K are fixed across the family. All stars with given n have the same $\theta(\xi)$ since the Lane-Emden equation does not depend on ρ_c. Recall the relations

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n \quad \Longrightarrow \quad \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n} \tag{5.54}$$

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}r \quad \Longrightarrow \quad \xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{(1+n)K}}r \tag{5.55}$$

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n = \rho_c\theta^n. \tag{5.56}$$

The surface of the polytropic star is at $\xi = \xi_{\max}$ defined as location where we have the first zero of the solution of the Lane-Emden equation, $\theta(\xi_{\max}) = 0$. Let $r_{\max}$ be the corresponding physical radius. Then the total mass of the polytrope is

$$\begin{aligned} M &= \int_0^{r_{\max}} 4\pi r^2 \rho \, \mathrm{d}r \\ &= 4\pi\rho_c\left[\frac{4\pi G\rho_c^{1-1/n}}{(1+n)K}\right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n \xi^2 \, \mathrm{d}\xi}_{\substack{\text{same for all} \\ \text{polytrope of index } n}} \\ \therefore M &\propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)}. \end{aligned} \tag{5.57}$$

From the definition of ξ above in Eq. (5.55), we also know that

$$r_{\max} \propto \rho_c^{\frac{1}{2}\left(\frac{1}{n}-1\right)}. \tag{5.58}$$

Eliminating density ρ_c thus gives the **mass-radius relation for polytropic stars**

$$\boxed{M \propto R^{\frac{3-n}{1-n}}.} \tag{5.59}$$

For $\gamma = 5/3$, $n = 3/2$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EoS is due to e^- degeneracy pressure rather than gas pressure). As we consider progressively more massive white dwarfs, the bulk of the electrons need to be in high energy levels (Fermi surface is higher energy). At some point, the electrons become relativistic, standard kinetic theory shows that the equation of state "softens" from $p = K\rho^{5/3}$ to $p = K'\rho^{4/3}$ (corresponding to $n = 3$). The scaling relation is Eq. (5.59). So, for $n = 3$, the mass is independent of radius, i.e. there is only one permitted mass for the configuration. This is the Chandrasekar mass, about $1.4\,M_\odot$, plays a special part in Type-Ia supernovae.

But for most main-sequence stars we do *not* observe $M \propto R^{-3}$, instead across much of the mass sequence we see $M \propto R$. The reason is that stars do not share the same polytropic constant K. Let's write the temperature at the core in terms of the central density and K

$$\left.\begin{aligned} p &= K\rho^{1+1/n} \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \quad \Longrightarrow \quad T_c = \frac{\mu K}{\mathcal{R}_*}\rho_c^{1/n}. \tag{5.60}$$

Thermonuclear reactions in the core that power stars are extremely temperature sensitive, so across the main sequence the stars will adjust as to approximately keep T_c similar in the cores. So we can say that

$$K \propto \rho_c^{-1/n}. \tag{5.61}$$

Substituting this into the above expression for mass gives

$$M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \quad \Longrightarrow \quad M \propto R. \tag{5.62}$$

We can also use these techniques to examine the behaviour of an individual star that is gaining or losing mass. In this case, when can the $K = \text{const.}$ or $T_c = \text{const.}$ relations be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust. The time to adjust to new hydrostatic equilibrium is roughly the time for a sound wave (Chapter 6) to propagate across the star, which is less than a day for the Sun,

$$t_{\text{hd}} \sim R/c_s < 1\,\text{day}. \tag{5.63}$$

The thermal timescale, on which a star can lose a significant amount of energy, is

$$t_{\text{th}} \sim \frac{\text{energy content of the star}}{\text{luminosity}} \sim \frac{GM^2}{RL}, \tag{5.64}$$

which is $\sim 30\,\text{Myr}$ for the Sun. So, mass loss/gain is followed by rapid readjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time.

5.4.1 Spherical rotating star

A spherical rotating polytropic star with angular velocity Ω gains non-rotating mass on less than the thermal timescale. How does Ω evolve?

Conservation of angular momentum gives $MR^2\Omega = \text{const.}$ So, if $\Omega \to \Omega + \Delta\Omega$ then $MR^2 \to MR^2 + \Delta(MR^2)$, and to first order in small quantities,

$$\begin{aligned} MR^2\Delta\Omega + \Omega\Delta(MR^2) &= 0 \\ \Longrightarrow \quad \frac{\Delta\Omega}{\Omega} &= -\frac{\Delta(MR^2)}{MR^2}. \end{aligned} \tag{5.65}$$

But we can use

$$R \propto M^{\frac{1-n}{3-n}} \tag{5.66}$$

to say

$$\begin{aligned} \frac{\Delta\Omega}{\Omega} &\propto -\Delta\Big(M^{\frac{5-3n}{3-n}}\Big) \\ \implies \quad \frac{\Delta\Omega}{\Omega} &\propto -\Big(\frac{5-3n}{3-n}\Big)\Delta M, \end{aligned} \tag{5.67}$$

so,

$$\Delta M > 0 \quad \implies \quad \begin{cases} \Delta\Omega < 0 & \text{if} \quad \dfrac{5-3n}{3-n} > 0 \quad (\text{e.g. } n = \frac{3}{2}) \text{ Spin down} \\ \Delta\Omega > 0 & \text{if} \quad \dfrac{5-3n}{3-n} < 0 \quad (\text{e.g. } n = 2) \text{ Spin up} \end{cases}. \tag{5.68}$$

5.4.2 Star in a binary system

A star in a binary system loses mass to its companion. The donor star loses mass, $\Delta M < 0$. So since $R \propto M^{(1-n)/(3-n)}$, the radius will increase if $1 < n < 3$.

So there is the potential for unstable (runaway) mass transfer (but need to look at evolution of the size of the Roche lobe to conclusively decide whether the process is unstable).

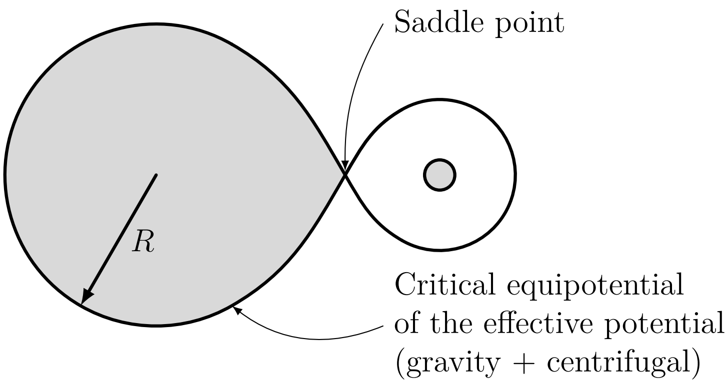


Fig. 5.5: Roche lobe overflow. Schematic of a binary star system in a semidetached configuration, the filled regions represent the two stars. The black line represents the inner critical Roche equipotential, made up of two Roche lobes that meet at the Lagrangian point L_1. In a semidetached configuration one star fills its Roche lobe.

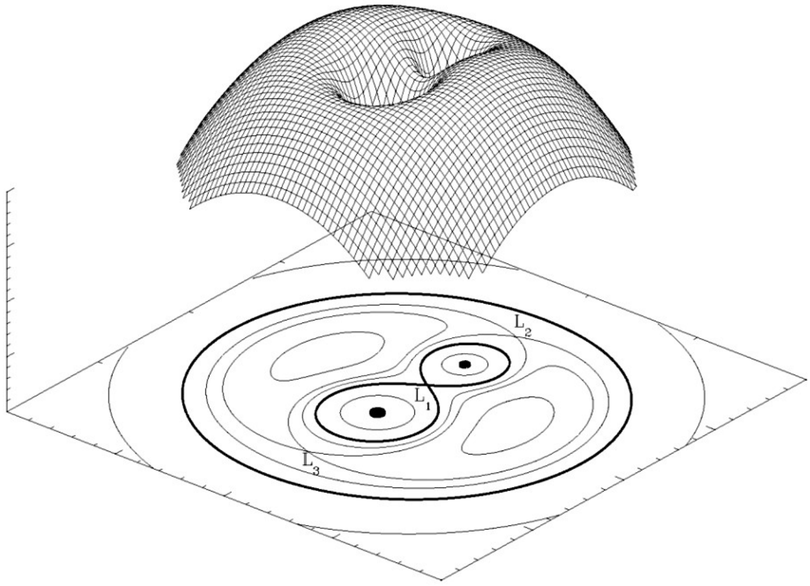


Fig. 5.6: A three-dimensional representation of the Roche potential in a binary star with a mass ratio of 2, in the co-rotating frame. The droplet-shaped figures in the equipotential plot at the bottom of the figure are called the Roche lobes of each star. L_1, L_2 and L_3 are the points of Lagrange where forces cancel out. Mass can flow through the saddle point L_1 from one star to its companion, if the donor star fills its Roche lobe. [4]

CHAPTER 6

Sound Waves, Supersonic Flows and Shock Waves

6.1 Sound Waves

We now start discussion of how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{6.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{6.2}$$

The equilibrium around which we will perturb is

$$\begin{aligned} &\rho = \rho_0 \quad \text{(uniform and constant)} \\ &p = p_0 \quad \text{(uniform and constant)} \\ &\mathbf{u} = \mathbf{0}. \end{aligned} \tag{6.3}$$

We consider small perturbations and write in Lagrangian terms (Lagrangian meaning the change of quantities are for a *given fluid element*)

$$\begin{aligned} p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u}. \end{aligned} \tag{6.4}$$

The relation between Lagrangian and Eulerian perturbations is:

$$\underbrace{\delta \rho}_{\substack{\text{Eulerian}\\ \text{perturbation}}} = \underbrace{\Delta \rho}_{\substack{\text{Lagrangian}\\ \text{perturbation}}} - \underbrace{\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \rho_0}_{\substack{\text{Element displacement dot}\\ \text{Gradient of unperturbed state}}} \tag{6.5}$$

In the present example, $\boldsymbol{\nabla} \rho_0 = 0$ and so $\delta \rho = \Delta \rho$, but the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we will address later.

Substitute the perturbations into the fluid equations and ignore terms that are 2^{nd} order (or higher) in the perturbed quantities:

Start with continuity equation (2.12):

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla}\cdot[(\rho_0+\Delta\rho)\Delta\mathbf{u}] = 0$$

$$\implies \quad \underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + \frac{\partial \Delta\rho}{\partial t} + \underbrace{\boldsymbol{\nabla}\rho_0\cdot\Delta\mathbf{u}}_{=0} + \underbrace{\boldsymbol{\nabla}(\Delta\rho)\cdot\Delta\mathbf{u}}_{2^{\text{nd}}\text{ order}} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) + \underbrace{\Delta\rho\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})}_{2^{\text{nd}}\text{ order}} = 0$$

$$\therefore \frac{\partial}{\partial t}(\Delta\rho) + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{6.6}$$

And similarly, the momentum equation (2.56):

$$\frac{\partial}{\partial t}(\Delta\mathbf{u}) + \underbrace{(\Delta\mathbf{u}\cdot\boldsymbol{\nabla})\Delta\mathbf{u}}_{2^{\text{nd}}\text{ order}} = -\frac{1}{\rho_0+\Delta\rho}\boldsymbol{\nabla}(p_0+\Delta p)$$

$$\begin{aligned}\implies \quad \frac{\partial}{\partial t}(\Delta\mathbf{u}) &= -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2}\right)\Big(\underbrace{\boldsymbol{\nabla}p_0}_{=0} + \boldsymbol{\nabla}(\Delta p)\Big) \\ &= -\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta p)\end{aligned}$$

$$\therefore \frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0}, \quad \text{assuming barotropic EoS,} \tag{6.7}$$

where we assumed a barotropic equation of state $p = p(\rho)$ to write the Taylor expansion about ρ_0 as $\Delta p = \partial p/\partial\rho\,|_{\rho=\rho_0}\Delta\rho$ to first order.

Now, taking the partial derivative of Eq. (6.6) with respect to time, and following from the commutativity of partial derivatives

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(\Delta\rho) &= -\rho_0\frac{\partial}{\partial t}[\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})] \\ &= -\rho_0\boldsymbol{\nabla}\cdot\left[\frac{\partial}{\partial t}(\Delta\mathbf{u})\right] \\ &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho) \quad \text{given} \quad (6.7).\end{aligned} \tag{6.8}$$

Thus we arrive at the wave equation

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho).} \tag{6.9}$$

This admits solutions of the form $\Delta\rho = \Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Substituting into the wave equation we get

$$\begin{aligned}(-i\omega)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}(ik)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \therefore \omega^2 &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}k^2.\end{aligned} \tag{6.10}$$

The (phase) speed of the wave is $v_p = \omega/k$, so the sound wave travels at speed determined by the derivative of $p(\rho)$

$$c_\mathrm{s} = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}}. \tag{6.11}$$

Consider a 1D wave and substitute

$$\begin{aligned} \Delta\rho &= \Delta\rho_0 e^{i(kx-\omega t)} \\ \Delta u &= \Delta u_0 e^{i(kx-\omega t)} \end{aligned} \tag{6.12}$$

into Eq. (6.6). We get

$$\begin{aligned} &-i\omega\Delta\rho + \rho_0 ik\Delta u = 0 \\ \implies \quad &\Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_\mathrm{s}\frac{\Delta\rho}{\rho_0}. \end{aligned} \tag{6.13}$$

So we learn that

- Fluid velocity and density perturbations are in phase (since $\Delta u/\Delta\rho \in \mathbb{R}$);
- A disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small, since

$$\Delta u_0 = c_\mathrm{s}\frac{\Delta\rho_0}{\rho_0} \ll c_\mathrm{s}. \tag{6.14}$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes acceleration of the fluid elements, this induces further density perturbations, making disturbances propagate.

Sound speed depends on how the pressure forces react to density changes. If the EoS is “stiff” (i.e. high $\mathrm{d}p/\mathrm{d}\rho$), then restoring force is large and propagation is rapid.

6.1.1 Examples of $\mathrm{d}p/\mathrm{d}\rho$

Notes about these two examples:

- We see that $c_\mathrm{s,I}$ and $c_\mathrm{s,A}$ differ by only $\sqrt{\gamma}$;
- Thermal behaviour of the perturbations does *not* have to be the same as that of the unperturbed structure! E.g. in the Earth’s atmosphere, the background is approximately isothermal but sound waves are adiabatic.
- Waves for which c_s is not a function of ω are called non-dispersive. The shape of a wave packet is preserved.

6.1.1.1 Isothermal Case

$$c_\mathrm{s}^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_T \tag{6.15}$$

In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant T. Then

$$p = \frac{\mathcal{R}_*}{\mu}\rho T$$
$$\therefore c_{s,\mathrm{I}} = \sqrt{\frac{\mathcal{R}_* T}{\mu}}. \tag{6.16}$$

6.1.1.2 Adiabatic Case

$$c_\mathrm{s}^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S \tag{6.17}$$

No heat exchange between fluid elements; compressions heat up and rarefactions cool down from $p\,\mathrm{d}V$ work. So

$$p = K\rho^\gamma$$
$$\Longrightarrow \quad \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S = \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho} \tag{6.18}$$
$$\therefore c_{s,\mathrm{A}} = \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}}. \tag{6.19}$$

6.2 Sound Waves in a Stratified Atmosphere

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$.

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on z-dependent terms, taking $\mathbf{u} = u\hat{\mathbf{z}}$. The continuity (2.12) and momentum equations (2.56) are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{6.20}$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{6.21}$$

and the equilibrium is (see Subsection 5.1.1)

$$\begin{aligned} u_0 &= 0 \\ \rho_0(z) &= \tilde{\rho} e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu} \\ p_0(z) &= \frac{\mathcal{R}_* T}{\mu} \rho_0(z) = \tilde{p} e^{-z/H}. \end{aligned} \tag{6.22}$$

Consider a Lagrangian perturbation:

$$\begin{aligned} u &\to \Delta u \\ \rho_0 &\to \rho_0 + \Delta\rho \\ p_0 &\to p_0 + \Delta p. \end{aligned} \tag{6.23}$$

Remember from Eq. (6.5) that $\delta\rho = \Delta\rho - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\rho$. So we have

$$\left.\begin{aligned} \delta\rho &= \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z \frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u \end{aligned}\right\} \quad \text{Eulerian to Lagrangian perturbation relation,} \tag{6.24}$$

and

$$\Delta \mathbf{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}}_{2^{\text{nd}}\text{ order}} \approx \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{6.25}$$

Substituting perturbed eulerian quantities into the Eulerian continuity equation (6.20)

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial z}[(\rho_0 + \delta\rho)\delta u_z] = 0 \\ \implies \quad &\frac{\partial}{\partial t}\left(\rho_0 + \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z}\right) + \frac{\partial}{\partial z}(\rho_0 \Delta u_z) = 0 \quad (\text{ignoring } 2^{\text{nd}} \text{ order terms}) \\ \implies \quad &\underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + \frac{\partial \Delta\rho}{\partial t} - \frac{\partial \xi_z}{\partial t}\frac{\partial \rho_0}{\partial z} - \underbrace{\xi_z \frac{\partial}{\partial t}\frac{\partial \rho_0}{\partial z}}_{=0} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ \implies \quad &\frac{\partial \Delta\rho}{\partial t} - \underbrace{\Delta u_z}_{\partial\xi_z/\partial t} \frac{\partial \rho_0}{\partial z} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ &\therefore \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0. \end{aligned} \tag{6.26}$$

To perform this next calculation, we need a relation that is obtained from the Lagrangian continuity equation:

$$\begin{aligned} &\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \\ \implies \quad &\Delta\rho + \left(\rho_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t}\right)\Delta t = 0 \quad (\text{integrating over a short time interval } \Delta t) \\ &\therefore \Delta\rho + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0. \end{aligned} \tag{6.27}$$

Following a similar method for the Eulerian momentum equation (6.21), we write in terms of the perturbed quantities

$$\frac{\partial(\delta u)}{\partial t} + \underbrace{\delta u \frac{\partial(\delta u)}{\partial z}}_{2^{\text{nd}}\text{ order}} = -\frac{1}{\rho_0 + \delta\rho}\frac{\partial}{\partial z}(p_0 + \delta p) - g, \tag{6.28}$$

discarding the second term, and using the perturbation relation (6.24),

$$\frac{\partial \Delta u}{\partial t} = -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2} + \frac{1}{\rho_0^2}\xi_z\frac{\partial\rho_0}{\partial z}\right)\frac{\partial}{\partial z}\left(p_0 + \Delta p - \xi_z\frac{\partial p_0}{\partial z}\right). \tag{6.29}$$

Given the exponential equilibrium state equations (6.22), we can evaluate the derivatives easily to give

$$\frac{\partial \Delta u}{\partial t} = -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2} - \frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-\frac{1}{H}p_0 + \frac{\partial\Delta p}{\partial z} + \frac{1}{H}\frac{\partial\xi_z}{\partial z}p - \frac{\xi_z}{H}\frac{1}{H}p_0\right) - g, \tag{6.30}$$

and using the isothermal equation of state $p_0 = gH\rho_0$,

$$\frac{\partial \Delta u}{\partial t} = -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2} - \frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-g\rho_0 + \frac{\partial\Delta p}{\partial z} + g\frac{\partial\xi_z}{\partial z}\rho_0 - g\frac{\xi_z}{H}p_0\right) - g. \tag{6.31}$$

Now, using Eq. (6.27) to write

$$\frac{\partial \Delta u}{\partial t} = -\left(\frac{1}{\rho_0} - \frac{\Delta\rho}{\rho_0^2} - \frac{1}{\rho_0^2}\frac{\xi_z}{H}\rho_0\right)\left(-g\rho_0 + \frac{\partial\Delta p}{\partial z} - g\Delta\rho - g\frac{\xi_z}{H}\rho_0 - g\right), \tag{6.32}$$

it follows to first order in small quantities (both $\Delta\rho$ and ξ_z are small here) that

$$\frac{\partial \Delta u}{\partial t} = g - \frac{1}{\rho_0}\frac{\partial\Delta p}{\partial z} + g\frac{\Delta\rho}{\rho_0} + g\frac{\xi_z}{H} - g\frac{\Delta\rho}{\rho_0} - g\frac{\xi_z}{H} - g, \tag{6.33}$$

and a large amount of these terms cancel to give

$$\begin{aligned} \frac{\partial \Delta u_z}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial\Delta p}{\partial z} \\ \implies \frac{\partial \Delta u_z}{\partial t} &= -\frac{c_u^2}{\rho_0}\frac{\partial\Delta\rho}{\partial z}, \quad c_u^2 \equiv \left.\frac{\partial p}{\partial\rho}\right|_{\rho_0}. \end{aligned} \tag{6.34}$$

Let's now derive the wave equation and dispersion relation. Take the partial derivative of Eq. (6.26) with respect to time

$$\begin{aligned} \frac{\partial^2\Delta\rho}{\partial t^2} + \rho_0\frac{\partial}{\partial z}\left(\frac{\partial\Delta u_z}{\partial t}\right) &= 0 \\ \implies \quad \frac{\partial^2\Delta\rho}{\partial t^2} - \rho_0\frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0}\frac{\partial\Delta\rho}{\partial z}\right) &= 0, \end{aligned} \tag{6.35}$$

where the last step involved substitution from Eq. (6.34). If the medium is isothermal, then c_u is independent of z. So,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{c_u^2}{\rho_0} \frac{\partial^2 \Delta\rho}{\partial z^2} + \rho_0 \frac{c_u^2}{\rho_0^2} \frac{\partial \rho_0}{\partial z} \frac{\partial \Delta\rho}{\partial z} = 0$$

$$\therefore \quad \underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0} \frac{\partial \rho_0}{\partial z} \frac{\partial \Delta\rho}{\partial z}}_{\substack{\text{extra piece associated} \\ \text{with stratification}}} = 0. \tag{6.36}$$

Now,

$$\begin{aligned} \frac{\partial \rho_0}{\partial z} &= \frac{\partial}{\partial z}\left(\tilde{\rho} e^{-z/H}\right) \\ &= -\frac{1}{H} \tilde{\rho} e^{-z/H} \\ &= -\frac{\rho_0}{H}. \end{aligned} \tag{6.37}$$

So,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_u^2}{H} \frac{\partial \Delta\rho}{\partial z} = 0. \tag{6.38}$$

Look for solutions of the form $\Delta\rho \propto e^{i(kz-\omega t)}$,

$$\implies \quad -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{ik}{H}, \tag{6.39}$$

which reveals the *dispersion relation*

$$\boxed{\omega^2 = c_u^2 \left(k^2 - \frac{ik}{H}\right).} \tag{6.40}$$

We can also write this as

$$k^2 - \frac{ik}{H} - \frac{\omega^2}{c_u^2} = 0, \tag{6.41}$$

and solve the quadratic for $k(\omega)$ to give

$$k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}. \tag{6.42}$$

Let's take $\omega \in \mathbb{R}$. We have two cases to examine if we wish to understand the implications of this dispersion relation.

6.2.1 Case I: $\omega > c_u/2H$

Examine the real and imaginary parts of k:

$$\mathrm{Im}\{k\} = \frac{1}{2H} \tag{6.43}$$

$$\mathrm{Re}\{k\} = \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.44}$$

So the density perturbation is

$$\Delta\rho \propto \underbrace{e^{-z/2H}}_{1}\underbrace{e^{i\left(\pm\sqrt{(\omega/c_u)^2-(1/2H)^2}z-\omega t\right)}}_{2} \tag{6.45}$$

corresponding to

1. Exponentially decaying amplitude with increasing height;
2. Wave with phase velocity

$$v_{\mathrm{ph}} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.46}$$

where v_{ph} is function of ω, meaning that the wave is dispersive. A wave packet consisting of different ω's will change shape as it propagates.

As before, we can relate Δu to $\Delta\rho$:

$$\Delta u_z = \frac{\Delta\rho}{\rho_0}\frac{\omega}{k}, \tag{6.47}$$

with

$$\Delta\rho \propto e^{-z/2H} \tag{6.48}$$

$$\rho_0 \propto e^{-z/H}, \tag{6.49}$$

giving

$$\Delta u_z \propto e^{+z/2H}, \quad \frac{\Delta\rho}{\rho_0} \propto e^{+z/2H}. \tag{6.50}$$

Thus the perturbed velocity and the fractional density variation both *increase* with height. In the absence of dissipation (e.g. viscosity), the kinetic energy flux ($\propto \Delta\rho\Delta u$) is conserved and the amplitude of the wave increases until

$$\Delta u \sim c_{\mathrm{s}}, \quad \frac{\Delta\rho}{\rho_0} \sim 1, \tag{6.51}$$

where the linear treatment breaks down and the sound wave "steepens" into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere!

6.2.2 Case II: $\omega < c_u/2H$

In this case, we find that k is purely imaginary. So,

$$\Delta\rho \propto e^{-|k|z} e^{i\omega t}. \tag{6.52}$$

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength, giving rise to reflections.

6.3 Transmission of Sound Waves at Interfaces

Consider two non-dispersive media with a boundary at $x = 0$. Suppose we have a sound wave travelling from $x < 0$ to $x > 0$. Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by r and t the amplitude of the reflected and transmitted waves, respectively.

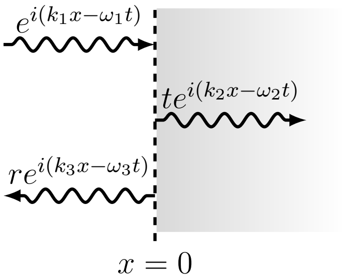


Fig. 6.1: Waves at boundary $x = 0$.

At the boundary $x = 0$, variables must be single valued and the accelerations are finite, thus oscillations in the second medium must have the same frequency,

$$\omega_1 = \omega_2 = \omega_3 = \omega. \tag{6.53}$$

The reflected wave is in the same medium as the incident so

$$k_3 = -k_1. \quad \text{(phase speed reversed)} \tag{6.54}$$

The amplitude of a sound wave is continuous at $x = 0$ hence

$$1 + r = t, \tag{6.55}$$

and the derivative of the amplitude is continuous at $x = 0$ thus

$$k_1(1 - r) = k_2 t. \tag{6.56}$$

We can combine these relations to get

$$t = \frac{2k_1}{k_1 + k_2}, \quad r = \frac{k_1 - k_2}{k_1 + k_2}. \tag{6.57}$$

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

1. If $c_{s,2} > c_{s,1}$, then $k_2 < k_1 \implies r > 0$, i.e reflected wave in phase with incident;
2. If $c_{s,2} < c_{s,1}$, then $r < 0 \implies$ reflected wave is π out of phase with incident wave;
3. If $c_{s,2} \ll c_{s,1}$, then $k_2 \gg k_1 \implies t \ll 1$, i.e. wave almost completely reflected.

6.4 Supersonic Fluids and Shocks

Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding c_s. The linear theory applied to sound waves breaks down.

When thinking about the sound speed, recall that the chemical composition of the fluid matters, $c_\mathrm{s} \propto \mu^{-1/2}$

$$c_\mathrm{s}\,\text{in}\,\underbrace{\text{atomic Hydrogen}}_{\substack{\mu\approx 1\\ \text{e.g. ISM}}} \gg c_\mathrm{s}\,\text{in}\,\underbrace{\text{diatomic Nitrogen}}_{\substack{\mu=28\\ \text{e.g. Earth's}\\ \text{atmosphere}}}, \quad \text{for a given}\,T. \tag{6.58}$$

Disturbances in a fluid always propagate at the sound speed relative to the fluid itself. Consider an observer at the centre of a spherical disturbance, watching the fluid flow past at speed v.

The velocity of the disturbance relative to the observer, v', is the vector sum of the fluid velocity and the disturbance velocity relative to the fluid.

- Subsonic case: v' sweeps 4π steradians;
- Supersonic case: disturbance always to the right. If we continuously produce a disturbance, the envelope of the disturbances will define a cone, named the *Mach cone*, with opening angle α given by

$$\sin\alpha = \frac{c_\mathrm{s}}{v}. \tag{6.59}$$

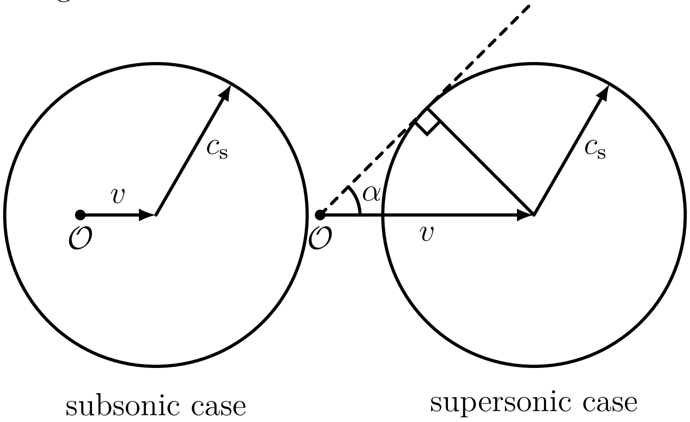


Fig. 6.2: Subsonic flow vs supersonic flow

The ratio of the flow speed to the sound speed is called the Mach number

$$M \equiv \frac{v}{c_s} \tag{6.60}$$

$$\sin\alpha = \frac{1}{M}. \tag{6.61}$$

Imagine an obstacle in a supersonic flow – disturbances cannot propagate upstream from the obstacle so the flow cannot adjust to the presence of the obstacle. The flow properties must change discontinuously once the obstacle is reached, giving a shock!

6.5 The Rankine-Hugoniot Relations

We analyse a shock by applying conservation of mass, momentum and energy across the shock front.

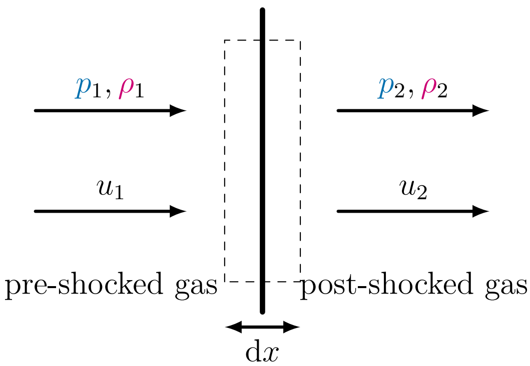


Fig. 6.3: Geometry of shock front

In the frame of the shock, lets assume the following geometry in Fig. 6.3. Continuity gives

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \\ \implies \quad \frac{\partial}{\partial t}\left(\int_{-\mathrm{d}x/2}^{\mathrm{d}x/2} \rho \,\mathrm{d}x\right) + \rho u_x\Big|_{x=\mathrm{d}x/2} - \rho u_x\Big|_{x=-\mathrm{d}x/2} = 0, \end{aligned} \tag{6.62}$$

where we have integrated over a small region $\mathrm{d}x$ around the shock.

Let's take $\mathrm{d}x \to 0$ and assume that mass does not continually accumulate at $x = 0$. Then

$$\frac{\partial}{\partial t}\left(\int \rho \,\mathrm{d}x\right) = 0, \tag{6.63}$$

which implies the 1st *Rankine-Hugoniot relation*,

$$\boxed{\rho_1 u_1 = \rho_2 u_2.} \tag{6.64}$$

Apply similar analysis to the momentum equation:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + p) - \rho \frac{\partial \Psi}{\partial x} \\ \implies \quad \frac{\partial}{\partial t}\left(\int \rho u_x \,\mathrm{d}x\right) = -\left(\rho u_x u_x + p\right)\Big|_{x=\mathrm{d}x/2} + \left(\rho u_x u_x + p\right)\Big|_{x=-\mathrm{d}x/2}, \end{aligned} \tag{6.65}$$

which gives the 2nd *Rankine-Hugoniot relation*,

$$\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2.} \tag{6.66}$$

We note that u_y and u_z do not change across the shock front (can be immediately seen by looking at the y- and z-components of the momentum equation).

Now for the energy equation. Start with the adiabatic case so that the gas cannot cool and hence we have $\dot{Q}_{\mathrm{cool}}$. Also take gravitational potential to have no time-dependence. Then

$$\begin{aligned} \frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \underbrace{-\rho \dot{Q}_{\mathrm{cool}}}_{=0} + \underbrace{\rho\frac{\partial \Psi}{\partial t}}_{=0} \\ \implies \quad \frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = 0 \\ \implies \quad \frac{\partial}{\partial t}\left(\int E \,\mathrm{d}x\right) + (E+p)u_x\Big|_{x=\mathrm{d}x/2} - (E+p)u_x\Big|_{x=-\mathrm{d}x/2} = 0 \\ \implies \quad (E_1 + p_1)u_1 = (E_2 + p_2)u_2. \end{aligned} \tag{6.67}$$

Since we know from equation (4.32) that $E + \rho\left(\frac{1}{2}u^2 + \mathcal{E} + \Psi\right)$, this becomes

$$\frac{1}{2}\rho_1 u_1^3 + \rho_1\mathcal{E}_1 u_1 + \rho_1\Psi_1 u_1 + p_1 u_1 = \frac{1}{2}\rho_2 u_2^3 + \rho_2\mathcal{E}_2 u_2 + \rho_2\Psi_2 u_2 + p_2 u_2. \tag{6.68}$$

But $\Psi_1 = \Psi_2$ and $\rho_1 u_1 = \rho_2 u_2$, so terms involving Ψ cancel out. We are left with the 3rd *Rankine-Hugoniot relation,*

$$\boxed{\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}.} \tag{6.69}$$

For an ideal gas, we have

$$\left.\begin{aligned} \mathcal{E} &= C_V T \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \implies \mathcal{E} = \frac{C_V \mu}{\mathcal{R}_*}\frac{p}{\rho} \tag{6.70}$$

$$\left.\begin{aligned} \gamma &= \frac{C_p}{C_V} \\ C_p - C_V &= \frac{\mathcal{R}_*}{\mu} \end{aligned}\right\} \implies C_V(\gamma - 1) = \frac{\mathcal{R}_*}{\mu}, \tag{6.71}$$

which combine to give the internal energy per unit mass

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}.} \tag{6.72}$$

If we assume that γ does not change across the shock (e.g. there is no disassociation of molecules), the 3rd R-H relation becomes

$$\begin{aligned} \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} &= \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} \\ \implies \quad \frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma-1} &= \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma-1}, \end{aligned} \tag{6.73}$$

since, for the adiabatic case, recall from (6.18) the sound speed is

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_S = \frac{\gamma p}{\rho}. \tag{6.74}$$

Using all three R-H relations and after some algebra we get

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}. \tag{6.75}$$

If we again define the Mach number of the incoming flow as $M_1 \equiv u_1/c_{s,1}$, then we can also now show

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \tag{6.76}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \tag{6.77}$$

$$\frac{T_2}{T_1} = \frac{((\gamma-1)M_1^2 + 2)(2\gamma M_1^2 - (\gamma-1))}{(\gamma+1)^2 M_1^2}. \tag{6.78}$$

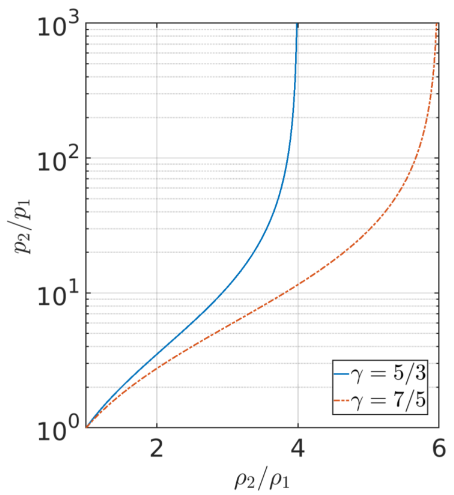


Fig. 6.4: Plot of Eq. (6.75) for various values of γ.

In the limit of strong shocks, $p_2 \gg p_1$, we get

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}. \tag{6.79}$$

For $\gamma = 5/3$, this gives $\rho_2 = 4\rho_1$. So there is a maximum possible density contrast across an adiabatic shock - with stronger and stronger shocks, the thermal pressure of the shocked gas increases and prevents further compression. Also, it follows from Eq. (6.78) that for strong shocks

$$T_2 = \frac{2\gamma(\gamma-1)M_1^2}{(\gamma+1)^2} T_1$$

$$\implies \quad k_B T_2 = \frac{4(\gamma-1)}{(\gamma+1)^2}\left(\frac{1}{2}\mu m_p u_1^2\right) \quad \text{since} \quad c_{s,1}^2 = \frac{\gamma k_B T_1}{\mu m_p} \tag{6.80}$$

$$\implies \quad k_B T_2 = \frac{3}{8}\left(\frac{1}{2}\mu m_p u_1^2\right) \quad \text{for} \quad \gamma = \frac{5}{3}. \tag{6.81}$$

This makes explicit the notion that the kinetic energy of the pre-shock fluid is being converted into random motion of the post-shock flow.

Note that, since $p_2 \gg p_1$, and $\rho_2 \leq 4\rho_1$, we have

$$\frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma} \quad \text{i.e. } K_1 \neq K_2. \tag{6.82}$$

The gas has jumped adiabats during its passage through the shock. Shocking the gas produces a non-reversible change, due to viscous processes operating within the shock.

While the R-H conditions are symmetric in the up- and down-stream quantities, the thermodynamic requirement that entropy increases dictates the direction of the jump (i.e. a fast/cold upstream flow shocking to produce a slow/fast downstream flow).

It is interesting that we can derive R-H conditions using the inviscid equations that do not explicitly include dissipation/entropy-generating terms.

Not all shocks are adiabatic! To consider the other extreme, let's discuss *isothermal shocks*. Here we have $\dot{Q}_{\text{cool}} \neq 0$ such that the shocked gas cools to produce $T_2 = T_1$. Whether a shock is isothermal or adiabatic depends on whether the "cooling length" is smaller or larger than the system size, respectively.

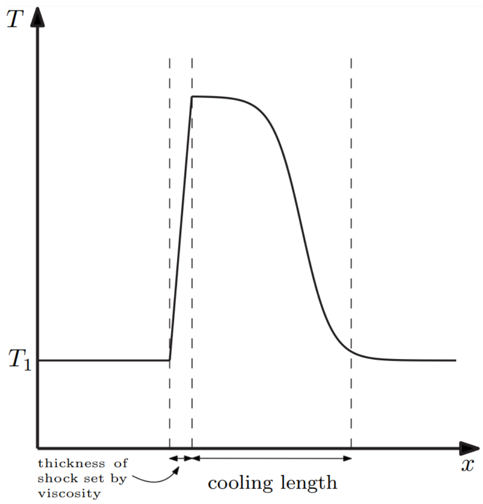


Fig. 6.5: Temperature profile through a shock

For isothermal shocks, the first two R-H equations are unchanged:

$$\rho_1 u_1 = \rho_2 u_2 \tag{6.83}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \tag{6.84}$$

but the 3rd R-H equation is replaced by

$$T_1 = T_2. \tag{6.85}$$

Now, following from Eq. (6.16),

$$c_{s,\mathrm{I}} = \sqrt{\frac{\mathcal{R}_* T}{\mu}} \quad \Longrightarrow \quad c_{s,1} = c_{s,2} \tag{6.86}$$

$$= \sqrt{\frac{p}{\rho}} \quad \Longrightarrow \quad p = c_{s,\mathrm{I}}^2 \rho. \tag{6.87}$$

So, the 2nd R-H equation becomes

$$\begin{aligned}
\rho_1(u_1^2 + c_\mathrm{s}^2) &= \rho_2(u_2^2 + c_\mathrm{s}^2) \\
\Longrightarrow \quad u_1 + \frac{c_\mathrm{s}^2}{u_1} &= u_2 + \frac{c_\mathrm{s}^2}{u_2} \quad (\text{since } \rho_1 u_1 = \rho_2 u_2) \\
\Longrightarrow \quad c_\mathrm{s}^2 \left(\frac{1}{u_1} - \frac{1}{u_2} \right) &= u_2 - u_1 \\
\Longrightarrow \quad c_\mathrm{s}^2 \frac{u_2 - u_1}{u_1 u_2} &= u_2 - u_1
\end{aligned}$$

and thus,

$$\boxed{c_s^2 = u_1 u_2.} \tag{6.88}$$

Thus we see that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s}\right)^2 = M_1^2, \tag{6.89}$$

where M_1 is the Mach number of the upstream flow. So the density compression can be very large.

Note that since $c_2^s = u_1 u_2$ and $u_1 > c_s$ (condition for a shock), we must have $u_2 < c_s$. So flow behind the shock is subsonic. In fact this is always true for any shock and is necessary to preserve causality (the post shock gas must know about the shock!).

6.6 Theory of Supernova Explosions

An important application of shock wave theory is to supernova explosions in the interstellar medium (ISM). A supernova (SN) deposits about 10^{51} erg ($= 10^{44}$ J) of energy into the surrounding medium, the shocked medium expands, sweeps up more gas, and creates large bubbles in the ISM.

Consider the following system:

- Initially uniform density interstellar medium (ISM) at rest, with density ρ_0;
- Instantaneous point-like explosion with energy E;
- Ignore temperature of the ambient ISM ($T_0 = 0$), thus no confinement of explosion by an external pressure.

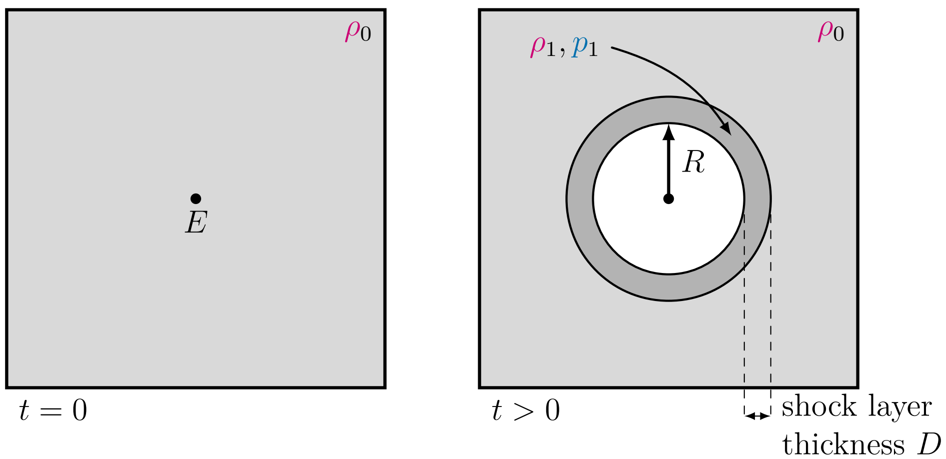


Fig. 6.6: Supernova explosion

Given that $T_0 = 0$, the shock has $M \to \infty$. Assuming an adiabatic shock, we sweep mass into a shell with density ρ, given by

$$\rho_1 = \rho_0 \frac{\gamma + 1}{\gamma - 1}. \tag{6.90}$$

If all the mass is swept up into a shell then

$$\begin{aligned} \frac{4\pi}{3}\rho_0 R^3 &= 4\pi \rho_1 R^2 D \quad (\text{assuming } D \ll R) \\ \therefore D &= \frac{1}{3}\left(\frac{\gamma - 1}{\gamma + 1}\right) R. \end{aligned} \tag{6.91}$$

For $\gamma = 5/3$, we have $D \approx 0.08R$ which justifies the assumption $D \ll R$.

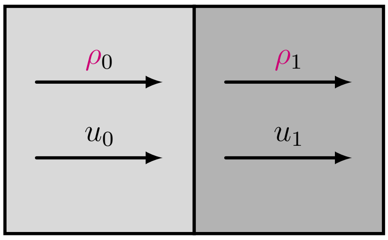


Fig. 6.7: Situation seen in shock frame.

Assume that all gas in the shell moves with a common velocity. Fig. 6.7 shows the frame of a local patch of the shock, and so

$$\begin{aligned} \rho_0 u_0 &= \rho_1 u_1 \\ \implies u_1 &= \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0. \end{aligned} \tag{6.92}$$

Thus, relative to the unshocked gas, the velocity of the shocked gas U is

$$U = u_0 - u_1 = \frac{2u_0}{\gamma + 1}. \tag{6.93}$$

Then, the rate of change of momentum of the shocked shell is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma + 1}\right]. \tag{6.94}$$

This momentum gain is provided by pressure acting on the inside surface of the shell - call this p_{in}. Let's make the ansatz that this is related to the pressure within the shell by

$$p_{\text{in}} = \alpha p_1, \tag{6.95}$$

and we now relate p_1 and u_0 using the R-H jump condition: we have

$$\begin{aligned}
p_0 + \rho_0 u_0^2 &= p_1 + \rho_1 u_1^2 \\
\implies \quad p_1 &= \rho_0 u_0^2 \left[1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2}\right] \quad \text{(since } p_0 = 0 \text{ by assumption)} \\
&= \rho_0 u_0^2 \left[1 - \frac{\gamma - 1}{\gamma + 1}\right] \quad \text{(assuming a strong shock)} \\
&= \frac{2}{\gamma + 1} \rho_0 u_0^2.
\end{aligned} \tag{6.96}$$

So, equating the rate of change of momentum in the shocked shell to the pressure acting on the inside surface of the shell, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma + 1}\right] &= 4\pi R^2 p_{\text{in}} \\
&= 4\pi R^2 \alpha p_1 \\
&= 4\pi R^2 \alpha \frac{2}{\gamma + 1} \rho_0 u_0^2
\end{aligned} \tag{6.97}$$

$$\begin{aligned}
\implies \quad & \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 u_0\right] = 3\alpha R^2 u_0^2 \\
\implies \quad & \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 \dot{R}\right] = 3\alpha R^2 \dot{R}^2 \quad \text{since} \quad u_0 \equiv \dot{R}
\end{aligned} \tag{6.98}$$

This admits solutions of the form $R \propto t^b$:

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3b} b t^{b-1}\right) = 3\alpha t^{2b}\left(b t^{b-1}\right)^2 \\
\implies \quad & b(4b - 1)t^{4b-2} = 3\alpha b^2 t^{4b-2} \quad \text{(cancellation of } t^{4b-2} \text{ justifies assumed form of solution)} \\
\implies \quad & b = 0 \quad \text{(not physical)} \quad \text{or} \quad b = \frac{1}{4 - 3\alpha} \\
\implies \quad & R \propto t^{1/(4-3\alpha)}, \quad u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \propto R^{3\alpha-3}.
\end{aligned} \tag{6.99}$$

To determine α, we need to consider energy conservation. For an adiabatic shock, the explosion energy is conserved and transformed into kinetic and internal energy:

- Kinetic energy of the shell is

$$\frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2. \tag{6.100}$$

- Internal energy per unit mass is

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho}, \tag{6.101}$$

and so the internal energy per unit volume is

$$\rho\mathcal{E} = \frac{1}{\gamma - 1} p. \tag{6.102}$$

Since the shell is very thin, it has a small volume and so most of the internal energy is in the central cavity which contains little mass

$$\text{Internal energy of cavity} \approx \frac{4\pi}{3} R^3 \frac{p_{\text{in}}}{\gamma - 1} = \frac{4\pi}{3} R^3 \alpha \frac{p_1}{\gamma - 1}. \tag{6.103}$$

So, energy conservation says that

$$\begin{aligned} E &= \frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2 + \frac{4\pi}{3} R^3 \alpha \frac{p_1}{\gamma - 1} \\ &= \frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 \underbrace{\left(\frac{2u_0}{\gamma + 1}\right)^2}_{\text{Eq. (6.93)}} + \frac{4\pi}{3} R^3 \alpha \underbrace{\frac{2}{\gamma + 1} \rho_0 u_0^2}_{\text{Eq. (6.96)}} \frac{1}{\gamma - 1} & (6.104) \\ &= \frac{4\pi}{3} R^3 u_0^2 \left[\frac{1}{2} \rho_0 \frac{4}{(\gamma + 1)^2} + \alpha \rho_0 \frac{2}{(\gamma + 1)(\gamma - 1)} \right], & (6.105) \end{aligned}$$

from which we conclude that

$$E \propto R^3 u_0^2 \propto t^{(6\alpha - 3)/(4 - 3\alpha)}. \tag{6.106}$$

But E must be conserved. So we need $\alpha = 1/2$ to remove time dependence of E. Using $\alpha = 1/2$ we find

$$\boxed{R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}.} \tag{6.107}$$

6.6.1 Similarity Solutions

The above problem only has 2 parameters, E and ρ_0. Look at their dimensions

$$[E] = \frac{ML^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}. \tag{6.108}$$

These cannot be combined to give quantities with the dimension of length or time. So, there is no natural length scale or time scale in the problem!

Given some time t, the only way to combine E, ρ_0 and t to give a length scale is

$$\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}. \tag{6.109}$$

We can define a dimensionless distance parameter

$$\xi \equiv \frac{r}{\lambda} = r \left(\frac{\rho_0}{Et^2}\right)^{1/5}. \tag{6.110}$$

Then, for any variable in the problem $X(r,t)$, we will have

$$X = X_1(t)\tilde{X}(\xi), \tag{6.111}$$

i.e. X is a function of scaled distance ξ and always has the same shape scaled up/down by the time dependence factor $X_1(t)$.

So,

$$\frac{\partial X}{\partial r} = X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial r}\right|_t \tag{6.112}$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi)\frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial t}\right|_r . \tag{6.113}$$

ξ is neither a Lagrangian nor an Eulerian coordinate. It labels a particular feature in the flow (e.g. shock wave) that can move through the fluid. So we can write

$$R_{\text{shock}} \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \tag{6.114}$$

Let's put some numbers in for the case of supernova explosions,

$$R(t) = \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \quad \text{(we will assume } \xi_0 \sim 1) \tag{6.115}$$

$$u_0(t) = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\xi_0 \left(\frac{E}{\rho_0 t^3}\right)^{1/5} = \frac{2}{5}\frac{R}{t}. \tag{6.116}$$

In a supernova we have

$$E \approx 10^{44}\,\mathrm{J} = 10^{51}\,\mathrm{erg} \tag{6.117}$$

$$\rho_0 = \rho_{\mathrm{ISM}} \approx 10^{-21}\,\mathrm{kg\,m^{-3}}. \tag{6.118}$$

So the similarity solution gives

$$\left.\begin{array}{l} R \approx 0.3 t^{2/5}\,\mathrm{pc} \\ u_0 \approx 10^5 t^{-3/5}\,\mathrm{km\,s^{-1}} \end{array}\right\} \text{where } t \text{ is measured in yrs.} \tag{6.119}$$

The original explosion injects the stellar debris at about $10^4\,\mathrm{km\,s^{-1}}$. So the above solution is valid for

$$t \gtrsim 100\,\mathrm{yr} \quad (\text{when } u_0 < u_{\mathrm{inj}}) \tag{6.120}$$

$$t \lesssim 10^5\,\mathrm{yr} \quad \text{(after which energy losses become important)} \tag{6.121}$$

6.6.2 Structure of the Blast Wave

We can, in principle, write each variable ρ, p, u, r in terms of separated functions of t and ξ. We can then substitute into the Eulerian equation of fluid dynamics (in spherical coordinates with $\partial/\partial\phi = \partial/\partial\theta = 0$, i.e. spherical symmetry).

The result is a set of ODE's where ξ is the only dependent variable - the time dependence cancels out! (Sedov 1946)

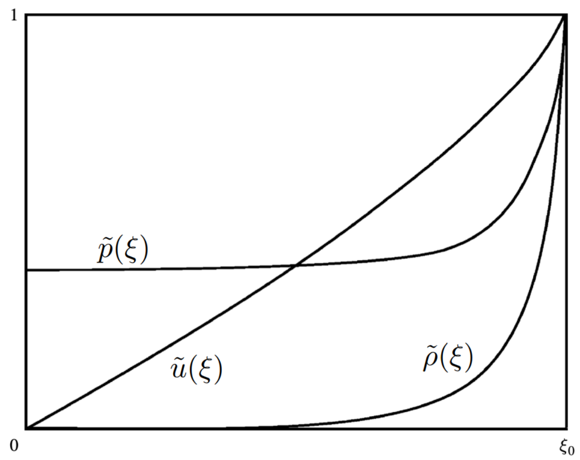


Fig. 6.8: Solution for $\gamma = 7/5$

These solutions tell us that

- Most of mass is swept up in a shell just behind the shock (from form of $\tilde{p}$);
- Post-shock pressure is indeed a multiple of p_{in} (from form of $\tilde{p}$, justifies $p_{\text{in}} = \alpha p_1$ assumption);
- Shell material is not really moving at a single velocity, but arguments above are restored by taking some weighted average (from form of $\tilde{u}$).

6.6.3 Breakdown of the Similarity Solution

The self similar solution breaks down when the surrounding medium pressure p_0 becomes significant, $p_1 \sim p_0$.

From the strong shock solution, we derived equations (6.96) and (6.18),

$$p_1 = \frac{2}{\gamma+1}\rho_0 u_0^2, \quad c_s^2 = \frac{\gamma p_0}{\rho_0}. \tag{6.122}$$

So if $p_1 \sim p_0$ then

$$\begin{aligned} \frac{2}{\gamma+1}\rho_0 u_0^2 &\sim \frac{\rho_0 c_s^2}{\gamma} \\ \implies \quad u_0 &\sim c_s, \end{aligned} \tag{6.123}$$

i.e. the shell is not moving supersonically anymore, and the blast wave weakens to a sound wave as illustrated in Fig. 6.9.

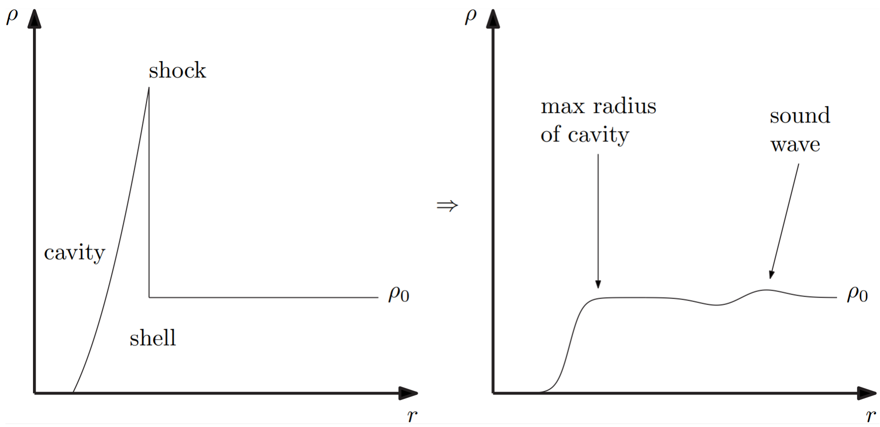


Fig. 6.9: Blast wave phase vs late phase

As a sound wave, disturbance passes into the undisturbed gas as a mild compression followed by a rarefaction. After the sound wave passes, gas returns to the original state.

For a supernova, the maximum bubble/cavity size is set by the radius when the blast wave becomes sonic and $p_1 \sim p_0$. We've just shown that this implies

$$u_0^2 \sim \frac{\gamma+1}{2\gamma} c_s^2. \tag{6.124}$$

We showed in Eq. (6.104) that energy conservation gives

$$\begin{aligned} E &= \frac{4\pi}{3} R^3 \left[\frac{1}{2}\rho_0 \left(\frac{2u_0}{\gamma+1} \right)^2 + \frac{\alpha}{\gamma-1} \frac{2\rho_0 u_0^2}{\gamma+1} \right] \quad \text{where} \quad \alpha = \frac{1}{2} \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2}{(\gamma+1)^2} + \frac{1}{(\gamma-1)(\gamma+1)} \right] \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2(\gamma-1)+(\gamma+1)}{(\gamma+1)^2(\gamma-1)} \right] \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \frac{3\gamma-1}{(\gamma+1)^2(\gamma-1)} \end{aligned} \tag{6.125}$$

$$\implies \quad u_0^2 = \frac{(\gamma+1)(\gamma^2-1)}{3\gamma-1} \cdot \frac{3E}{4\pi\rho_0 R^3} \sim \underbrace{\frac{\gamma+1}{2\gamma}c_s^2}_{\substack{\text{when blast}\\ \text{wave becomes}\\ \text{sonic and } p_1\sim p_0}} \tag{6.126}$$

$$\implies \quad E \sim \frac{4\pi}{3}\rho_0 R_{\max}^3 \frac{c_s^2}{2\gamma} \cdot \frac{3\gamma-1}{\gamma^2-1}. \tag{6.127}$$

The internal energy initially contained within $R_{\max}$ is

$$E_{\text{init}} = \frac{4\pi}{3}R_{\max}^3 \frac{p_0}{\gamma-1} = \frac{4\pi}{3}R_{\max}^3 \rho_0 \frac{c_s^2}{\gamma(\gamma-1)}. \tag{6.128}$$

So, when $p_1 \sim p_0$, we have $E \sim E_{\text{init}}$. Therefore, the blast wave propagates until the explosion energy is comparable to the internal energy in the sphere!

Some numbers:

- Timescale on which the bubble reaches $R_{\max}$ is roughly the sound crossing time
$$t_s \sim \frac{R_{\max}}{c_s}. \tag{6.129}$$
For ISM: $T \sim 10^4\,\text{K}$, $\rho \sim 10^{-21}\,\text{kg}\,\text{m}^{-3}$, giving
$$R_{\max} \sim \text{few} \times 100\,\text{pc} \tag{6.130}$$
$$t_{\max} \sim 10\,\text{Myr}. \tag{6.131}$$

- SN rate is about $10^{-7}\,\text{Myr}^{-1}\,\text{pc}^{-3}$. So, over a duration $t_{\max}$, can find 1 SN in $\sim 10^6\,\text{pc}^3$. But
$$\frac{4\pi}{3}R_{\max}^3 > 10^6\,\text{pc}^3 \tag{6.132}$$
so the filling factor of SN driven bubbles is > 1. This would seem to suggest that the entire ISM would be heated by supernovae to $> 10^6 K$, but this is *not observed*!

  We need to account for cooling and the finite height of the Galactic disk (i.e. bubble "blow out"). After 10^5 yrs, when $R \sim 20$ pc, cooling losses become important and so the bubble grows more slowly than $R \propto t^{2/5}$. Simulations show that $R \propto t^{0.3}$ and $R_{\max} \sim 50$ pc, giving a filling factor < 1. Thus, due to cooling, only a small fraction of E is deposited into the ISM.

Chapter 7

Bernoulli's Equation and Transonic Flows

7.1 Bernoulli's Equation

Let's start with the momentum equation (2.56) and substituting Eq. (3.1),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi. \tag{7.1}$$

If the fluid is barotropic, then $p = p(\rho)$ and so

$$\begin{aligned} \frac{\partial}{\partial x}\int \frac{\mathrm{d}p}{\rho} &= \frac{\partial p}{\partial x}\,\mathrm{d}p \int \frac{\mathrm{d}p}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \Longrightarrow \quad \frac{1}{\rho}\boldsymbol{\nabla} p &= \boldsymbol{\nabla}\left(\int \frac{\mathrm{d}p}{\rho}\right). \end{aligned} \tag{7.2}$$

Also, we have the vector identity

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}). \tag{7.3}$$

Using these, the momentum equation (7.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times \mathbf{w} = -\boldsymbol{\nabla}\left[\int \frac{\mathrm{d}p}{\rho} + \Psi\right]. \tag{7.4}$$

where we have defined the *vorticity*:

$$\boxed{\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}.} \tag{7.5}$$

Now, assume a steady flow ($\partial \mathbf{u}/\partial t = \mathbf{0}$) and take the dot product of (7.4) with velocity $\mathbf{u}$. Since we have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w})$ always, the result is

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left[\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi\right] = 0. \tag{7.6}$$

This gives us *Bernoulli's Principle*: For steady barotropic flows, the quantity

$$\boxed{H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi} \tag{7.7}$$

is constant along a streamline. The quantity H is called Bernoulli's constant.

- If $p = 0$, $H =$ constant is the statement that kinetic + potential energy is constant along streamlines.

- If $p \neq 0$ pressure differences accelerate or decelerate the flow as it flows along the streamline.

7.1.1 Examples of Bernoulli's Equation

7.1.1.1 The Apocryphal Aircraft Wing

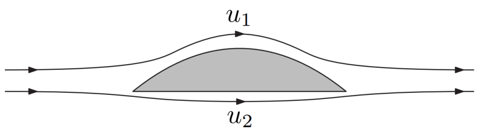


Fig. 7.1: Aircraft wing

$$
\begin{aligned}
u_1 > u_2 &\implies p_1 < p_2 \text{ from } H \\
&\implies \text{pressure difference} \\
&\implies \text{lift force.}
\end{aligned}
$$

Of course, this cannot be the whole story of how aircraft wings work or else inverted flight would be impossible!

7.1.1.2 Shower Curtain

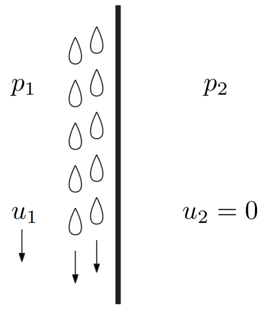


Fig. 7.2: Shower curtain

Downward flow of air on inside of curtain induced by falling water

$$
\begin{aligned}
&\implies p_1 < p_2 \\
&\implies \text{curtain blows inwards.}
\end{aligned} \tag{7.8}
$$

7.2 Rotational and Irrotational Flows

An *irrotational flow* is one in which $\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}$ everywhere, i.e. the vorticity $\mathbf{w} = \mathbf{0}$ everywhere.

For a steady irrotational flow, Eq. (7.4) gives that

$$\boldsymbol{\nabla} H = 0, \tag{7.9}$$

so, $H =$ constant everywhere (not just along streamlines).

For a general (not necessarily irrotational or steady state) flow, we have

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} H + \mathbf{u} \times \mathbf{w}. \tag{7.10}$$

Take curl:

$$\frac{\partial}{\partial t} \underbrace{(\boldsymbol{\nabla} \times \mathbf{u})}_{\mathbf{w}} = \underbrace{-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} H)}_{\equiv \mathbf{0}} + \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) \tag{7.11}$$

and from this we arrive at *Helmholtz's equation*,

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}).} \tag{7.12}$$

From Helmholtz's equation, we observe three results:

1. If $\mathbf{w} = \mathbf{0}$ initially, it will stay zero thereafter. We will see later that this is no longer true once we include viscous terms.
2. The flux of vorticity through a surface $\mathcal{S}$ that moves with the fluid is a constant, i.e.

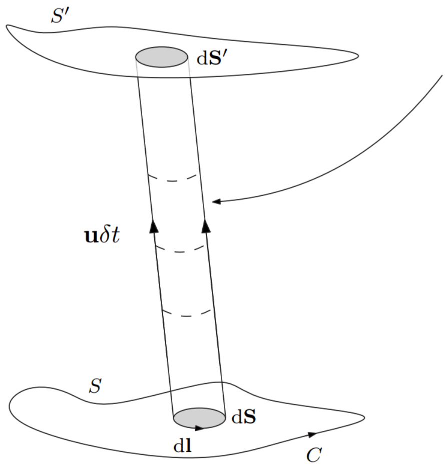
$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{\mathcal{S}} \mathbf{w} \cdot \mathrm{d}\mathbf{S} = 0. \tag{7.13}$$

For a proof, we first have

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_{\mathcal{S}} \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \underbrace{\int_{\mathcal{S}} \frac{\partial \mathbf{w}}{\partial t} \cdot \mathrm{d}\mathbf{S}}_{\text{intrinsic changes in } \mathbf{w}} + \underbrace{\int_{\mathcal{S}} \mathbf{w} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \mathrm{d}\mathbf{S}}_{\substack{\text{change in } \mathcal{S} \\ \text{caused by flow}}} . \tag{7.14}$$

So,

$$\begin{aligned}
\int_{\mathcal{S}} \mathbf{w} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \mathrm{d}\mathbf{S} &= \int_{\mathcal{S}} \oint_{\partial \mathrm{d}\mathbf{S}} \mathbf{w} \cdot (\mathbf{u} \times \mathrm{d}\mathbf{l}) \\
&= \int_{\mathcal{S}} \oint_{\partial \mathrm{d}\mathbf{S}} \mathbf{w} \times \mathbf{u} \cdot \mathrm{d}\mathbf{l} \\
&= \oint_{\mathcal{C}} \mathbf{w} \times \mathbf{u} \cdot \mathrm{d}\mathbf{l} \quad \text{since “internal loops” cancel out} \\
&= \int_{\mathcal{S}} \boldsymbol{\nabla} \times (\mathbf{w} \times \mathbf{u}) \cdot \mathrm{d}\mathbf{S} \,.
\end{aligned} \tag{7.15}$$



- vector area of element of this envelope is

$$-\delta t \mathbf{u} \times \mathrm{d}\mathbf{l}$$

- vector area of whole volume is

$$\int \mathrm{d}\mathbf{S} = 0$$

$$\mathrm{d}\mathbf{S}' - \mathrm{d}\mathbf{S} = \delta t \oint \mathbf{u} \times \mathrm{d}\mathbf{l}$$

$$\Rightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\mathbf{S} = \oint \mathbf{u} \times \mathrm{d}\mathbf{l}$$

integral around
the edge of $\mathrm{d}\mathbf{S}$

Fig. 7.3: Change of area element with time

$$\Longrightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t}\int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \int_S \mathrm{d}\mathbf{S} \cdot \underbrace{\left(\frac{\partial \mathbf{w}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{w} \times \mathbf{u})\right)}_{\substack{=\mathbf{0} \text{ from the} \\ \text{Helmholtz's equation (7.12)}}}$$

$$\Longrightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t}\int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = 0, \tag{7.16}$$

i.e. flux of vorticity is conserved and moves with the fluid. This is Kelvin's vorticity theorem.

3. For an irrotational flow, the fact that $\boldsymbol{\nabla} \times u = 0$ everywhere implies that there exists a potential function Φ_u such that

$$\mathbf{u} = -\boldsymbol{\nabla}\Phi_u. \tag{7.17}$$

If such a flow is also incompressible, then $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and so

$$\nabla^2 \Phi_u = 0, \tag{7.18}$$

i.e. can reduce the problem of finding the velocity field to that of solving Laplace's equation.

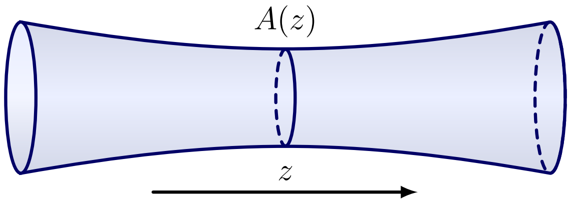


Fig. 7.4: Tube with variable cross-section

7.3 The de Laval Nozzle

Consider a steady flow in a tube with a variable cross-section A(z) as illustrated in Fig. (7.4). For a *steady flow*, mass conservation gives

$$\begin{aligned} \rho u A &= \text{const.}\, \dot{M} \quad \text{(mass flow per second)} \\ \Longrightarrow \quad & \ln \rho + \ln u + \ln A = \ln \dot{M} \\ \Longrightarrow \quad & \frac{1}{\rho}\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A = 0 \\ \Longrightarrow \quad & \frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\ln u - \boldsymbol{\nabla}\ln A, \end{aligned} \tag{7.19}$$

and the momentum equation (with no gravity) gives

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p. \tag{7.20}$$

Let's further assume a *barotropic equation* of state. Then

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\rho}\boldsymbol{\nabla}\rho. \tag{7.21}$$

So, putting these pieces together gives

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = [\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A]c_s^2 \quad \left(c_s^2 = \mathrm{d}p/\mathrm{d}\rho\right) \tag{7.22}$$

If the flow is also irrotational, we have

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) = \frac{1}{2}u^2\boldsymbol{\nabla}\left(\ln u^2\right) = u^2\boldsymbol{\nabla}\ln u, \tag{7.23}$$

and so, from Eq. (7.22) we have

$$\begin{aligned} & u^2\boldsymbol{\nabla}\ln u = [\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A]c_s^2 \\ \Longrightarrow \quad & (u^2 - c_s^2)\boldsymbol{\nabla}\ln u = c_s^2\boldsymbol{\nabla}\ln A. \end{aligned} \tag{7.24}$$

This implies that an extremum of $A(z)$ must correspond to either

1. Minimum or maximum in u, or

2. $u = c_s$.

Thus, we see that there is the potential for a transition from subsonic to supersonic flow at a minimum or maximum of the cross-sectional area of the tube.

To make progress, we apply Bernoulli's equation

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = H \quad \text{constant} \quad \text{(no gravity, steady, irrotational)} \tag{7.25}$$

and examine the two standard barotropic cases.

7.3.1 Case I: Isothermal EoS

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \quad T = \text{const.} \tag{7.26}$$

$$\begin{aligned} \implies \int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathcal{R}_* T}{\mu} \frac{\mathrm{d}\rho}{\rho} \\ &= \frac{\mathcal{R}_* T}{\mu} \ln \rho \\ &= c_s^2 \ln \rho. \end{aligned} \tag{7.27}$$

Suppose that we have a minimum or maximum in $A(z)$ that allows a flow to have a sonic transition. Let $A = A_m$ at this location. Then Bernoulli gives

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = \frac{1}{2}c_s^2 + c_s^2 \ln \rho \Big|_{A=A_m}, \tag{7.28}$$

which implies

$$\begin{aligned} u^2 &= c_s^2 \left[1 + 2 \ln \left(\frac{\rho|_{A=A_m}}{\rho} \right) \right] \\ &= c_s^2 \left[1 + 2 \ln \left(\frac{uA}{c_s A_m} \right) \right], \end{aligned} \tag{7.29}$$

where this last step has used mass conservation, i.e. $\rho u A$ = constant. Thus, given $A(z)$ we can determine $u(z)$ and $\rho(z)$, i.e. the structure of the flow everywhere subject to given $\dot{M}$ and c_s.

7.3.2 Case II: Polytropic EoS

$$p = K \rho^{1+1/n} \tag{7.30}$$

Let's examine the case where the sonic transition occurs at $A = A_m$. But now we do not know the sound speed c_s since $c_s = c_s(\rho)$ and ρ varies. We need to solve for

$$c_s^2 = \frac{n+1}{n} K \rho^{1/n}. \tag{7.31}$$

Now,

$$\begin{aligned}
\int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\rho} \\
&= \int K \frac{n+1}{n}\rho^{1/n}\frac{\mathrm{d}\rho}{\rho} \\
&= K\frac{n+1}{n}\int \rho^{1/n-1}\,\mathrm{d}\rho \\
&= K\frac{n+1}{n} n\rho^{1/n} \\
&= nc_s^2.
\end{aligned} \tag{7.32}$$

Mass conservation:

$$\begin{aligned}
\rho u A = \rho\Big|_{A_m} c_s\Big|_{A_m} A_m &= \dot{M} \\
\implies \quad \rho\Big|_{A_m}\left(\frac{n+1}{n}K\right)^{1/2}\rho^{1/2n}\Big|_{A_m} A_m &= \dot{M} \\
\implies \quad \rho^{2+1/n}\Big|_{A_m}\left(\frac{n+1}{n}K\right)A_m^2 &= \dot{M}^2 \\
\implies \quad \rho\Big|_{A_m} = \left[\left(\frac{\dot{M}}{A_m}\right)^2\frac{n}{K(n+1)}\right]^{n/(2n+1)}
\end{aligned} \tag{7.33}$$

Knowing $\rho|_{A_m}$, we can now determine c_s and A_m. Bernoulli gives:

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = \text{const.}, \tag{7.34}$$

which implies

$$\begin{aligned}
\frac{1}{2}\left(\frac{\dot{M}}{A\rho}\right)^2 + K(n+1)\rho^{1/n} &= \frac{1}{2}\,c_s^2\Big|_{A_m} + K(n+1)\,\rho^{1/n}\Big|_{A_m} \\
&= \frac{1}{2}\left(\frac{n+1}{n}\right)K\,\rho^{1/n}\Big|_{A_m} + K(n+1)\,\rho^{1/n}\Big|_{A_m} \\
&= \left(\frac{1}{2}+n\right)\left(\frac{n+1}{n}\right)K\,\rho^{1/n}\Big|_{A_m}
\end{aligned} \tag{7.35}$$

This is an implicit equation for the density structure through the flow.

General points of physical interpretation:

$$(u^2 - c_s^2)\boldsymbol{\nabla}\ln u = c_s^2\boldsymbol{\nabla}\ln A. \tag{7.36}$$

- In subsonic regime $u < c_s$

$$\begin{aligned} A \,\text{decreases} \quad &\Longrightarrow \quad \boldsymbol{\nabla} \ln u \,\text{positive} \\ &\Longrightarrow \quad u \,\text{accelerates along streamline} \end{aligned}$$

  e.g. rivers flowing through narrows;

- In supersonic regime $u > c_s$

$$\begin{aligned} A \,\text{increases} \quad &\Longrightarrow \quad \boldsymbol{\nabla} \ln u \,\text{positive} \\ &\Longrightarrow \quad u \,\text{accelerates along streamline} \end{aligned}$$

  Gas becomes very compressible. A increases, u increases, ρ is greatly reduced. $\dot{M} = A\rho u$ constant.

So, a nozzle that gets progressively narrower, reaches a minimum, and then widens again can be used to accelerate a flow from a subsonic to a supersonic regime.

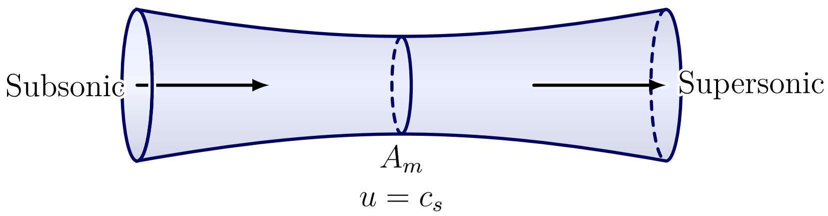


Fig. 7.5: de Laval nozzle.

Recall momentum equation:

$$\begin{aligned} (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla}\rho\frac{\mathrm{d}p}{\mathrm{d}\rho} = -c_s^2\boldsymbol{\nabla} \ln \rho \\ (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= u^2\boldsymbol{\nabla} \ln u \\ \Longrightarrow \qquad u^2\boldsymbol{\nabla} \ln u &= -c_s^2\boldsymbol{\nabla} \ln \rho \end{aligned} \tag{7.37}$$

- If $u \ll c_s$, $\nabla \ln u \gg \nabla \ln \rho$, this implies accelerations are important, pressure or density changes are small – almost incompressible;

- If $u \gg c_s$, $\nabla \ln u \ll \nabla \ln \rho$, $u \approx$ constant, pressure changes do not lead to much acceleration but there is change in ρ – compressible flow.

7.4 Spherical Accretion and Winds

We find flows with a mathematical structure when we consider steady-state and spherically-symmetric accretion flows or winds in the gravitational potential of a central body.

Consider the spherically-symmetric accretion of gas onto a star (described as a point of mass). We will assume

- gas is at rest at ∞ (reservior);
- steady state flow;
- barotropic EoS.

Mass conservation gives

$$\begin{aligned} & \rho u A = \text{constant } \dot{M} \\ \Longrightarrow \quad & 4\pi r^2 \rho u = \dot{M}, \end{aligned} \tag{7.38}$$

where, for convenience, we define u to be inward pointing.

Momentum equation gives

$$\begin{aligned} & u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} \\ \Longrightarrow \quad & u^2\frac{\mathrm{d}\ln u}{\mathrm{d}r} = -c_s^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{GM}{r^2}, \end{aligned} \tag{7.39}$$

assuming self-gravity of the accretion gas is negligible.

Now, a steady flow must have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}r}\left(\ln\dot{M}\right) = 0 \\ \Longrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho + \frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{\mathrm{d}}{\mathrm{d}r}\ln r^2 = 0 \\ \Longrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}r}\ln\rho = -\frac{\mathrm{d}}{\mathrm{d}r}\ln u - \frac{2}{r}. \end{aligned} \tag{7.40}$$

Substitute into Eq. (7.39) gives

$$u^2\frac{\mathrm{d}}{\mathrm{d}r}\ln u = c_s^2\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln u + \frac{2}{r}\right) - \frac{GM}{r^2}. \tag{7.41}$$

Therefore

$$\boxed{\left(u^2 - c_s^2\right)\frac{\mathrm{d}}{\mathrm{d}r}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right).} \tag{7.42}$$

There is a critical point in the flow at

$$r = r_s = \frac{GM}{2c_s^2}, \tag{7.43}$$

where u is either a minimum/maximum or there is a sonic transition. This is called the *sonic point*, somewhat similar to the de Laval nozzle, except with no boundaries/tubes!

Can gain insight into the general structure of such flows by plotting possible solutions on the $(r/r_s, u)$ plane.

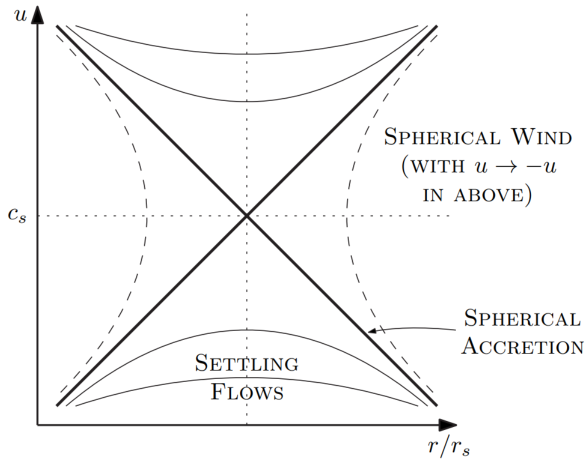


Fig. 7.6: Plot in $(r/r_s, u)$ plane

Back to accretion problem: progress requires the EoS.

7.4.1 Case I: Isothermal EoS

Equation of state is:

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \quad T = \text{const.} \tag{7.44}$$

$$\implies \quad c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \text{const.} \tag{7.45}$$

and we know from Eq. (7.43)

$$r_s = \frac{GM}{2c_s^2}. \tag{7.46}$$

Need to use Bernoulli's equation to constrain ρ and $\dot{M}$.

$$H = \frac{1}{2}u^2 + \underbrace{\int \frac{\mathrm{d}p}{\rho}}_{c_s^2 \ln \rho} + \Psi = \text{const.}$$

$$\begin{aligned}
\implies \quad & \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s} \\
\implies \quad & \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = c_s^2\left(\ln \rho_s - \frac{3}{2}\right) \\
\implies \quad & u^2 = 2c_s^2\left[\ln\left(\frac{\rho_s}{\rho}\right) - \frac{3}{2}\right] + \frac{2GM}{r},
\end{aligned} \tag{7.47}$$

where ρ_s is the density at $r = r_s$.

Now,

- as $r \to 0$, $u^2 \to 2GM/r$, i.e. free-fall;
- as $r \to \infty$ and $u \to 0$, $\rho = \rho_s e^{-3/2}$, giving

$$\rho_s = \rho_\infty e^{3/2}. \tag{7.48}$$

Thus, for a given ρ_∞, we know ρ_s and hence $\dot{M}$.

$$\begin{aligned}
& \dot{M} = 4\pi r_s^2 \rho_s c_s \\
\implies \quad & \dot{M} = \frac{\pi G^2 M^2 e^{3/2} \rho_\infty}{c_s^3}.
\end{aligned} \tag{7.49}$$

Note that:

- $\dot{M}$ proportional to M^2, more massive stars can accrete much more gas;
- $\dot{M}$ proportional to $1/c_s^3$, accretion very sensitive to temperature; can accrete more effectively from a colder medium.

7.4.2 Case II: Polytropic EoS

Equation of state is:

$$p = K\rho^{1+1/n}, \quad \int \frac{\mathrm{d}p}{\rho} = K(n+1)\rho^{1/n}. \tag{7.50}$$

At the sonic point,

$$\int \frac{\mathrm{d}p}{\rho} = nc_s^2. \tag{7.51}$$

Bernoulli then gives

$$\frac{1}{2}u^2 + K(n+1)\rho^{1/n} - \frac{GM}{r} = \frac{1}{2}c_s^2 + nc_s^2 - \frac{GM}{r_s}, \tag{7.52}$$

with $r_s = GM/(2c_s^2)$ as in Eq. (7.43). Using the mass accretion rate $\dot{M} = 4\pi r_s^2 \rho_s c_s$ we can then write

$$\begin{aligned} r_s &= \left(\frac{\dot{M}}{4\pi\rho_s c_s}\right)^{1/2} = \frac{GM}{2c_s^2} \\ \Longrightarrow \qquad c_s &= \left(\frac{GM}{2}\right)^{2/3}\left(\frac{4\pi\rho_s}{\dot{M}}\right)^{1/3}. \end{aligned} \tag{7.53}$$

Combine this with

$$c_s^2 = \frac{n+1}{n}K\rho_s^{1/n}, \tag{7.54}$$

to get

$$\begin{aligned} \left(\frac{n+1}{n}\right)K\rho_s^{1/n} &= \left(\frac{GM}{2}\right)^{4/3}\left(\frac{4\pi\rho_s}{\dot{M}}\right)^{2/3} \\ \Longrightarrow \qquad \rho_s^{1/n-2/3} = \rho_s^{(3-2n)/3n} &= \left(\frac{GM}{2}\right)^{4/3}\left(\frac{4\pi}{\dot{M}}\right)^{2/3}\frac{n}{(n+1)K} \\ \Longrightarrow \qquad \rho_s &= \left(\frac{GM}{2}\right)^{4n/(3-2n)}\left(\frac{4\pi}{\dot{M}}\right)^{2n/(3-2n)}\left(\frac{n}{(n+1)K}\right)^{3n/(3-2n)}. \end{aligned} \tag{7.55}$$

Back to Bernoulli:

$$\begin{aligned} \frac{1}{2}u^2 + (n+1)K\rho^{1/n} - \frac{GM}{r} &= c_s^2\left(n - \frac{3}{2}\right) \\ \Longrightarrow \qquad \frac{1}{2}\left(\frac{\dot{M}}{4\pi r^2\rho}\right)^2 + (n+1)K\rho^{1/n} &= c_s^2\left(n - \frac{3}{2}\right) + \frac{GM}{r}. \end{aligned} \tag{7.56}$$

As $r \to \infty$, $u \to 0$, we have

$$\rho_\infty = \left[\frac{c_s^2\left(n - \frac{3}{2}\right)}{(n+1)K}\right]^n = \left[\frac{n - \frac{3}{2}}{n}\right]^n \rho_s \tag{7.57}$$

$$c_{s,\infty}^2 = \frac{n+1}{n}K\rho_\infty^{1/n} = \frac{n - \frac{3}{2}}{n}c_s^2 \tag{7.58}$$

So, finally,

$$\begin{aligned} \dot{M} &= 4\pi r_s^2 \rho_s c_s \\ &= \frac{4\pi G^2 M^2}{4c_s^4}c_s\rho_\infty\left(\frac{n}{n - \frac{3}{2}}\right)^n \\ &= \frac{\pi G^2 M^2}{c_{s,\infty}^3}\rho_\infty\left(\frac{n}{n - \frac{3}{2}}\right)^{n-3/2}. \end{aligned} \tag{7.59}$$

Therefore,

$$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_{s,\infty}^3} \left(\frac{n}{n - \frac{3}{2}} \right)^{n-3/2}.} \tag{7.60}$$

Same functional form as in the isothermal case, but now with an additional coefficient related to the polytropic index. This is known as *Bondi Accretion*. We can recover the isothermal case by taking the limit $n \to \infty$.

The generalisation to the case of a star accreting from a medium that it is moving through is called *Bondi-Hoyle-Lyttleton Accretion*. The result is

$$\dot{M} \sim \frac{(GM)^2 \rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}}, \tag{7.61}$$

where v_∞ is the velocity of gas relative to the star at ∞.

Chapter 8

Fluid Instabilities

Consider a fluid in a steady state ($\partial/\partial t = 0$). Thus it is in a state of equilibrium.

- If a small perturbation of this configuration grows with time, the configuration is *unstable* with respect to those perturbations;
- If a small perturbation decays with time or just oscillates around the equilibrium configuration, the configuration is *stable* with respect to those perturbations.

An awful lot of interesting astrophysics is due to the action of fluid instabilities!

- Convection in stars;
- Multiphase nature of the ISM;
- Mixing of fluids that have relative motion;
- Turbulence in accretion disks;
- Formation of stars and galaxies.

In this chapter, we discuss some of the most important instabilities.

8.1 Convective Instability

This concerns the stability of a hydrostatic equilibrium. We can gain insight without doing a full perturbation analysis.

Consider the following system:

- Ideal gas in hydrostatic equilibrium;
- Uniform gravitational field in $-\hat{\mathbf{z}}$ direction.

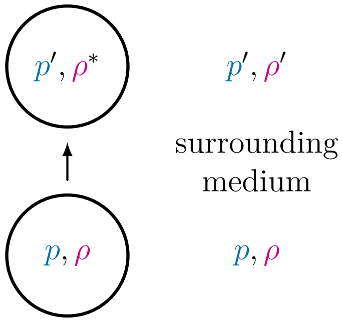


Fig. 8.1: Perturbing a fluid element upwards

Now perturb a fluid element upwards, away from its equilibrium point.

We assume that any pressure imbalances are quickly removed by acoustic waves, but that heat exchange takes longer. This implies the displaced element evolves adiabatically with a pressure p' equal to the pressure at the new location in the atmosphere.

Since we assume heat transfer is slow, initially perturbations will change adiabatically. Stability depends on the new density.

$$\begin{aligned} \rho^* < \rho' &\implies \text{perturbed element buoyant} \\ &\implies \text{system unstable;} \\ \rho^* > \rho' &\implies \text{perturbed element sinks back} \\ &\implies \text{system stable.} \end{aligned}$$

For adiabatic change,

$$\left.\begin{aligned} p &= K\rho^\gamma \\ p' &= K\rho^{*\gamma} \end{aligned}\right\} \implies \rho^* = \rho\left(\frac{p'}{p}\right)^{1/\gamma}. \tag{8.1}$$

To first order,

$$p' = p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z, \tag{8.2}$$

thus,

$$\begin{aligned} \rho^* &= \rho\left(\frac{p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z}{p}\right)^{1/\gamma} \\ &= \rho\left(1 + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z\right)^{1/\gamma} \\ &\approx \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z. \end{aligned} \tag{8.3}$$

In the surrounding medium,

$$\rho' = \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z, \tag{8.4}$$

and the system is unstable if $\rho^* < \rho'$. So instability needs

$$\begin{aligned}
\rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z &< \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z \\
\implies \quad \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} &< \frac{\mathrm{d}\rho}{\mathrm{d}z} \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\ln p &< \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln\rho \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\ln p\rho^{-\gamma}\right) &< 0 \\
\implies \quad \frac{\mathrm{d}K}{\mathrm{d}z} &< 0. \quad \text{(instability)} \qquad (8.5)
\end{aligned}$$

So, the system is unstable if the entropy of the atmosphere decreases with increasing height. This can also be related to temperature and pressure gradients.

$$\frac{\mathrm{d}K}{\mathrm{d}z} < 0 \quad \implies \quad \frac{\mathrm{d}}{\mathrm{d}z}\ln K < 0. \qquad (8.6)$$

But,

$$K = p\rho^{-\gamma} = (\text{const.})p^{1-\gamma}T^{\gamma}, \quad (p = \mathcal{R}_*\rho T/\mu) \qquad (8.7)$$

so,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\ln K = (1-\gamma)\frac{\mathrm{d}}{\mathrm{d}z}\ln p + \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln T < 0 \\
\implies \quad \frac{\mathrm{d}T}{\mathrm{d}z} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}. \quad \text{(instability)} \qquad (8.8)
\end{aligned}$$

Hence, we have the *Schwarzschild stability criterion* which reads

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}} \qquad (8.9)$$

Since hydrostatic equilibrium requires $\mathrm{d}p/\mathrm{d}z < 0$, we see that (since $\gamma > 1$)

- Always stable to convection if $\mathrm{d}T/\mathrm{d}z > 0$;
- Otherwise, can tolerate a negative temperature gradient provide

$$\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right| \qquad (8.10)$$

  So convective instability develops when T declines too steeply with increasing height.

Examples (Convectively unstable systems).

- Outer regions of low mass stars;
- Cores of high mass stars.

For stable configurations, we can examine atmosphere dynamics: equation of motion is

$$\begin{aligned}
\rho^* \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -g(\rho^* - \rho') \\
\implies \quad (\rho + \underbrace{\delta\rho}_{\text{small}})\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -g\left[\frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z \\
\implies \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z &= -\frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z. \qquad (8.11)
\end{aligned}$$

So, it is a simple harmonic motion with angular frequency N where

$$N^2 = \frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right], \qquad (8.12)$$

which defines the *Brunt-Väisälä frequency*. These oscillations are internal gravity waves.

8.2 Jeans Instability

This concerns the stability of a self-gravitating fluid against gravitational collapse. Consider the following system:

- Uniform medium initially static;
- Barotropic EoS;
- Gravitational field generated by the medium itself.

So equilibrium is

$$\begin{aligned}
p &= p_0, \quad \text{(const.)} \\
\rho &= \rho_0, \quad \text{(const.)} \qquad (8.13) \\
\mathbf{u} &= \mathbf{0},
\end{aligned}$$

and governing equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \qquad (8.14)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi \qquad (8.15)$$

$$\nabla^2 \Psi = 4\pi G \rho. \qquad (8.16)$$

Introduce a perturbation:

$$p = p_0 + \Delta p \tag{8.17}$$
$$\rho = \rho_0 + \Delta\rho \tag{8.18}$$
$$\mathbf{u} = \Delta\mathbf{u} \tag{8.19}$$
$$\Psi = \Psi_0 + \Delta\Psi \tag{8.20}$$

Note: There is an inconsistency between the assumption $\rho_0 = \text{constant} > 0$ and the assumption $\Psi_0 = \text{const}$. We proceed anyways – this is the *Jeans swindle* (1902). This is closely tied to the fact that it is impossible to construct a model of a static infinite Universe. A more complete analysis of perturbations against a background of a (relativistic) homogenous expanding Universe recovers the same local instability as that found by Jeans, hence justifying the swindle.

Linearised equations are:

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{8.21}$$
$$\frac{\partial\Delta\mathbf{u}}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta\rho) - \boldsymbol{\nabla}(\Delta\Psi) = -c_s^2\frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0} - \boldsymbol{\nabla}(\Delta\Psi) \tag{8.22}$$
$$\nabla^2(\Delta\Psi) = 4\pi G\Delta\rho. \tag{8.23}$$

Look for plane wave solutions

$$\Delta\rho = \rho_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8.24}$$
$$\Delta\Psi = \Psi_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8.25}$$
$$\Delta\mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{8.26}$$

Substitution into the linear equations gives

$$(8.21) \implies -\omega\rho_1 + \rho_0\mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{8.27}$$
$$(8.22) \implies -\rho_0\omega\mathbf{u}_1 = -c_s^2\mathbf{k}\rho_1 - \rho_1\mathbf{k}\Psi_1 \tag{8.28}$$
$$(8.23) \implies -k^2\Psi_1 = 4\pi G\rho_1. \tag{8.29}$$

Eliminating $\mathbf{u}_1$ and Ψ_1 from these

$$(8.27) + (8.28) \implies \rho_1\omega^2 = k^2\left(\rho_1 c_s^2 + \rho_0\Psi_1\right) \tag{8.30}$$
$$= k^2\rho_1 c_s^2 - 4\pi G\rho_0\rho_1$$
$$\implies \omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right). \tag{8.31}$$

Introduce the Jeans wavenumber $k_J^2 = 4\pi G\rho_0/c_s^2$ so we have the dispersion relation

$$\boxed{\omega^2 = c_s^2\left(k^2 - k_J^2\right).} \tag{8.32}$$

Notes:

- For $k \gg k_J$, we have normal sound waves $\omega^2 = c_s^2 k^2$.
- For $k \gtrsim k_J$, we have modified sound waves. Gravity leads to dispersion of the wave and a slower group velocity.
- For $k < k_J$, ω is purely imaginary (for $k \in \mathbb{R}$), giving

$$\omega = i\tilde{\omega}, \quad \tilde{\omega} \in \mathbb{R}, \tag{8.33}$$

and,

$$e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = e^{\tilde{\omega}t}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{8.34}$$

leading to exponentially growing solution: *Gravitational Instability.*

The maximum stable wavelength is the *Jeans length*,

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}. \tag{8.35}$$

The associated mass is the *Jeans mass*,

$$M_J \sim \rho_0 \lambda_J^3. \tag{8.36}$$

These are central concepts in the theory of

- *Star formation* (instability of giant molecular clouds);
- *Cosmological structure formation* (instability of the homogeneous primordial gas).

8.3 Rayleigh-Taylor and Kelvin-Helmholtz Instability

This concerns the stability of an interface with a discontinuous change in tangential velocity and/or density.

For convenience, let's assume:

- Constant gravity, ideal fluid;
- Pressure continuous across the interface;
- Incompressible flow $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$;
- Irrotational flow $\boldsymbol{\nabla} \times \mathbf{u} = 0 \implies \mathbf{u} = -\boldsymbol{\nabla}\Psi$;

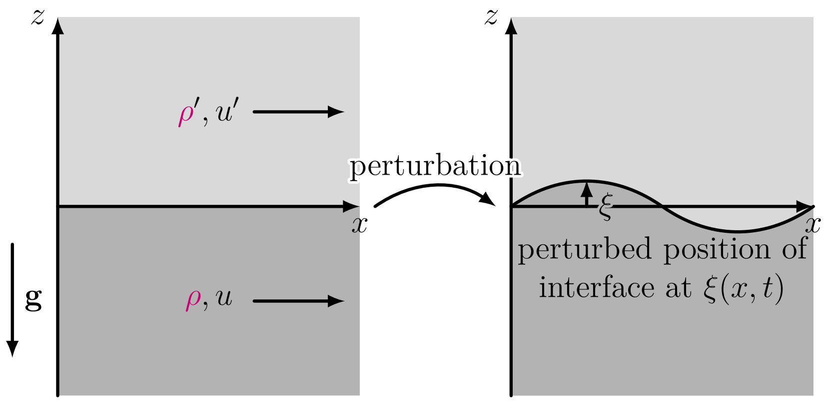


Fig. 8.2: Perturbation of interface of discontinuity

- 2D problem (symmetry direction into the page of Fig. 8.2)

The momentum equation (for either upper or lower fluid) is

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} &= -\frac{1}{\rho} \boldsymbol{\nabla} p + \mathbf{g} \\
\implies \quad -\boldsymbol{\nabla} \frac{\partial \Psi}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2} u^2\right) &= - \underbrace{\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)}_{\substack{\text{since } \rho = \text{const.} \\ \text{within each fluid}}} - \boldsymbol{\nabla} \Psi \\
\implies \quad \boldsymbol{\nabla}\left[-\frac{\partial \Psi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \Psi\right] &= 0 \\
\implies \quad -\frac{\partial \Psi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \Psi &= F(t). \qquad (8.37)
\end{aligned}$$

where $F(t)$ is a function that is constant in space but not in time.

Now consider a perturbation at the interface of these two fluids. Let us study the evolution of the perturbed position of the interface $\xi(x, t)$.

The velocity potential is $u = -\boldsymbol{\nabla} \Psi$, so if the unperturbed velocities in the fluids are U and U' we have

$$\Phi_{\text{low}} = -Ux + \phi \qquad (8.38)$$

$$\Phi_{\text{up}} = -U'x + \phi', \qquad (8.39)$$

thus,

$$\nabla^2 \phi = \nabla^2 \phi' = 0. \quad (\text{since } \boldsymbol{\nabla} \cdot \mathbf{u} = 0) \qquad (8.40)$$

ϕ and ϕ' are sourced by displacements of the interface. Consider an element of the lower fluid that is at the interface. Then

$$u_z = \frac{\mathrm{D}\xi}{\mathrm{D}t}, \qquad (8.41)$$

giving

$$\left.\begin{aligned} -\frac{\partial\phi}{\partial z} &= \frac{\partial\xi}{\partial t} + U\frac{\partial\xi}{\partial x} \\ -\frac{\partial\phi'}{\partial z} &= \frac{\partial\xi}{\partial t} + U'\frac{\partial\xi}{\partial x} \end{aligned}\right\} \quad \text{to first order} \tag{8.42}$$

Now look for plane wave solutions,

$$\xi = Ae^{i(kx-\omega t)} \tag{8.43}$$

$$\phi = Ce^{i(kx-\omega t)+k_z z} \tag{8.44}$$

$$\phi' = C'e^{i(kx-\omega t)+k_z' z}, \tag{8.45}$$

where extra terms on the exponents $k_z z$ and $k_z' z$ are there to seek solutions where the perturbed potential decays at large $|z|$.

But we know that

$$\begin{aligned} \nabla^2\phi = 0 \quad &\Longrightarrow \quad -k^2 + k_z^2 = 0 \\ &\Longrightarrow \quad k_z = |k|, \end{aligned} \tag{8.46}$$

so $\phi \to 0$ as $z \to \infty$.

For now, let's stipulate $k > 0$. So

$$\phi = Ce^{i(kx-\omega t)+kz} \tag{8.47}$$

$$\phi' = C'e^{i(kx-\omega t)-kz}. \tag{8.48}$$

From Eq. (8.42), we have

$$-kC = -i\omega A + iUkA = i(kU - \omega)A \tag{8.49}$$

$$kC' = i(kU' - \omega)A.) \tag{8.50}$$

We need one more equation if we're to solve for A, C, C'. We get that from pressure balance across the interface.

$$p = -\rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) + \rho F(t) \tag{8.51}$$

$$p' = -\rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + \rho' F'(t), \tag{8.52}$$

and equality at $z = 0$:

$$\rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) = \rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + K(t), \tag{8.53}$$

where

$$K \equiv \rho F(t) - \rho' F'(t). \tag{8.54}$$

The perturbation vanishes for $z \to \pm\infty$ at all times, so we can look at equation Eq. (8.37) for each fluid in the limit $|z| \to \infty$, taking limit carefully so that Ψ terms cancel, to get

$$\rho F(t) - \rho' F'(t) = \underbrace{\frac{1}{2}U^2\rho - \frac{1}{2}U'^2\rho'}_{\substack{\text{conditions at } \infty \\ \text{and so a constant}}} . \tag{8.55}$$

Therefore, $K(t)$ is actually a constant.

Next in our attempt to use Eq. (8.53) to match across boundary, we need to determine u and u'. Now

$$\begin{aligned} \mathbf{u} = -\boldsymbol{\nabla}\Psi &= -\boldsymbol{\nabla}(-Ux + \phi) = U\hat{\mathbf{x}} - \boldsymbol{\nabla}\phi \\ \implies \quad u^2 &= U^2 - 2U\frac{\partial\phi}{\partial x}, \quad \text{(dropping 2}^{\text{nd}}\text{ order terms)} \end{aligned} \tag{8.56}$$

and similarly

$$u'^2 = U'^2 - 2U'\frac{\partial\phi'}{\partial x}. \tag{8.57}$$

So, Eq. (8.53) reads

$$\begin{aligned} \rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}U^2 - U\frac{\partial\phi}{\partial x} + g\xi\right) &= \rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}U'^2 - U'\frac{\partial\phi'}{\partial x} + g\xi\right) + \underbrace{\frac{1}{2}U^2\rho - \frac{1}{2}U'^2\rho'}_{K} \\ \implies \quad \rho\left(-\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x} + g\xi\right) &= \rho'\left(-\frac{\partial\phi'}{\partial t} - U'\frac{\partial\phi'}{\partial x} + g\xi\right) \\ \implies \quad \rho i\omega C - \rho UikC + \rho gA &= \rho' i\omega C' - \rho' U' ikC' + \rho' gA \\ \implies \quad \rho(kU - \omega)C + i\rho gA &= \rho'(kU' - \omega)C' + i\rho' gA. \end{aligned} \tag{8.58}$$

Now eliminate C and C' from Eqs. (8.49) and (8.50) to give

$$\boxed{\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho').} \tag{8.59}$$

This is the dispersion relation for our system. Let's now look at some specific applications.

1. *Surface gravity waves*: two fluids at rest initially with $\rho' < \rho$ (i.e. denser fluid on bottom). The dispersion relation gives

$$\begin{aligned} \omega^2(\rho + \rho') &= kg(\rho - \rho') \\ \implies \quad \omega^2 &= k\frac{g(\rho - \rho')}{\rho + \rho'}. \end{aligned} \tag{8.60}$$

So, for $k \in \mathbb{R}$, we have that $\omega \in \mathbb{R}$ and hence the system displays oscillations/waves. Phase speed is

$$\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'}} = \underbrace{f(k)}_{\substack{\text{waves are} \\ \text{dispersive}}} . \tag{8.61}$$

If $\rho' \ll \rho$, then $\omega/k = \pm\sqrt{g/k}$.

Example. Surface waves on the ocean.

2. *Static stratified fluid*: two fluids at rest initially with $\rho' > \rho$ (i.e. denser fluid on top). Then

$$\omega^2 = k\frac{g(\rho - \rho')}{\rho + \rho'}. \tag{8.62}$$

So, for $k \in \mathbb{R}$ we have $\omega^2 < 0$ and so ω is purely imaginary.

$$\frac{\omega}{k} = \pm i\sqrt{\frac{g}{k}\frac{\rho' - \rho}{\rho + \rho'}}. \tag{8.63}$$

The positive root of this gives us exponentially growing solutions. This is the *Rayleigh-Taylor Instability.*

3. *Fluids in motion*: two fluids with $\rho > \rho'$ (so stable to Rayleigh-Taylor) but different velocities, non-zero U and U'. Take full dispersion relation:

$$\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho'), \tag{8.64}$$

divide by k^2 and solve the quadratic in ω/k,

$$\implies \quad \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}}. \tag{8.65}$$

There is instability if

$$\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2} < 0. \quad \text{(instability)} \tag{8.66}$$

If $g = 0$, then any relative motion gives instability, i.e. *Kelvin-Helmholtz Instability*;

If $g \neq 0$, then unstable modes are those with

$$k > \frac{(\rho^2 - \rho'^2)g}{\rho\rho'(U - U')^2}, \tag{8.67}$$

i.e. gravity is a stabilising influence.

8.4 Thermal Instability

This concerns the stability of a medium in thermal equilibrium (heating = cooling) to perturbations in temperature. Consider the following system:

- Ideal gas;

- No gravitational field;
- Static thermal equilibrium

$$\mathbf{u}_0 = \mathbf{0},\ \dot{Q}_0 = 0,\ \underbrace{\boldsymbol{\nabla} p_0 = \mathbf{0},\ \boldsymbol{\nabla}\rho_0 = \mathbf{0}}_{\boldsymbol{\nabla} K_0 = \mathbf{0}} \quad \text{where} \quad p = K\rho^\gamma. \tag{8.68}$$

Let's start by deriving an alternative form of the energy equation that involves the entropy-like variable K; this will be well suited to problems of thermal instability.

$$\begin{aligned} p = K\rho^\gamma \quad \Longrightarrow \quad \mathrm{d}p &= \rho^\gamma\,\mathrm{d}K + K\gamma\rho^{\gamma-1}\,\mathrm{d}\rho \\ &= \rho^\gamma\,\mathrm{d}K + \frac{\gamma p}{\rho}\,\mathrm{d}\rho\,, \end{aligned} \tag{8.69}$$

also,

$$\begin{aligned} p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \Longrightarrow \quad \mathrm{d}p &= \frac{\mathcal{R}_*}{\mu}T\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ &= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T\,. \end{aligned} \tag{8.70}$$

Equate Eqs. (8.69) and (8.70) to give

$$\begin{aligned} \rho^\gamma\,\mathrm{d}K + \gamma\frac{p}{\rho}\,\mathrm{d}\rho &= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ \Longrightarrow \quad \rho^\gamma\,\mathrm{d}K &= (1-\gamma)\frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \\ \Longrightarrow \quad \mathrm{d}K &= \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\right]}_{-đQ}. \end{aligned} \tag{8.71}$$

First law of thermodynamics:

$$đQ = p\,\mathrm{d}V + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T\,, \quad \text{(unit mass)} \tag{8.72}$$

and so

$$\begin{aligned} đQ &= p\,\mathrm{d}(1/\rho) + C_V\,\mathrm{d}T \\ &= -\frac{p}{\rho^2} - \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\,. \quad \text{(since we have } (\gamma-1)C_V = \mathcal{R}_*/\mu) \end{aligned} \tag{8.73}$$

Then we have

$$\mathrm{d}K = -(1-\gamma)\rho^{1-\gamma}đQ, \quad \text{(for fluid element)} \tag{8.74}$$

Turn this into Lagrangian energy equation by noting that $\dot{Q} = -đQ/\,\mathrm{d}t$,

$$\begin{aligned} \Longrightarrow \quad \frac{\mathrm{D}K}{\mathrm{D}t} &= -(1-\gamma)\rho^{1-\gamma}\dot{Q} \\ \Longrightarrow \quad \frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\ln K) &= -(\gamma-1)\frac{\rho}{p}\dot{Q}, \end{aligned} \tag{8.75}$$

and thus we derive the *Entropy form of the energy equation*,

$$\boxed{\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\frac{\rho\dot{Q}}{p}} \tag{8.76}$$

This joins our usual continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{8.77}$$

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}p. \tag{8.78}$$

Now we look at thermal stability. Consider small perturbations to the equilibrium

$$\rho \to \rho_0 + \Delta\rho \tag{8.79}$$

$$p \to p_0 + \Delta p \tag{8.80}$$

$$\mathbf{u} \to \Delta\mathbf{u} \tag{8.81}$$

$$K \to K_0 + \Delta K. \tag{8.82}$$

Linearise the equations:

$$(8.77) \implies \frac{\partial \Delta\rho}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{8.83}$$

$$(8.78) \implies \rho_0\frac{\partial \Delta\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}(\Delta p) \tag{8.84}$$

$$(8.76) \implies \frac{\partial \Delta K}{\partial t} = -\frac{\gamma-1}{\rho_0^{\gamma-1}}\Delta\dot{Q}, \tag{8.85}$$

where we can write

$$\Delta\dot{Q} = \left.\frac{\partial\dot{Q}}{\partial p}\right|_{\rho}\Delta p + \left.\frac{\partial\dot{Q}}{\partial \rho}\right|_{p}\Delta p, \tag{8.86}$$

so that

$$\frac{\partial \Delta K}{\partial t} = -A^*\Delta p - B^*\Delta\rho, \tag{8.87}$$

with

$$A^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}}{\partial p}\right|_{\rho}, \quad B^* = \frac{\gamma-1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}}{\partial \rho}\right|_{p}. \tag{8.88}$$

We also have

$$p = K\rho^\gamma \quad \implies \quad \Delta p = \rho_0^\gamma\Delta K + \gamma\frac{p_0}{\rho_0}\Delta\rho. \tag{8.89}$$

We seek solutions of the form

$$\Delta p = p_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.90}$$

$$\Delta\rho = \rho_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.91}$$

$$\Delta\mathbf{u} = \mathbf{u}_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.92}$$

$$\Delta K = K_1 e^{i\mathbf{k}\cdot\mathbf{x}+qt} \tag{8.93}$$

so, instability if $\mathrm{Re}\{q\} > 0$. Substituting into linearised equations gives

$$(8.83) \implies q\rho_1 + \rho_0 i\mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{8.94}$$
$$(8.84) \implies q\rho_0\mathbf{u}_1 = -i\mathbf{k}p_1 \tag{8.95}$$
$$(8.87) \implies qK_1 = -A^* p_1 - B^*\rho_1 \tag{8.96}$$
$$(8.89) \implies p_1 = \rho_0^\gamma K_1 + \frac{\gamma p_0}{\rho_0}\rho_1. \tag{8.97}$$

We can combine these to obtain the dispersion relation:

$$\frac{A^* q}{k^2} - \frac{B^*}{q} = -\left(\frac{q^2}{k^2} + \gamma\frac{p_0}{\rho_0}\right)\frac{1}{\rho_0^\gamma}$$
$$\implies \underbrace{q^3 + A^*\rho_0^\gamma q^2 + k^2\gamma\frac{p_0}{\rho_0}q - B^* k^2\rho_0^\gamma}_{\text{cubic in } q\text{, call } E(q)} = 0. \tag{8.98}$$

This has at least one real root – system is unstable if that real root is positive, $q > 0$.

Now $E(\infty) = \infty$, $E(0) = -B^* k^2 \rho_0^\gamma$. So the system is unstable if $B^* > 0$.

$$B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}}\left.\frac{\partial\dot{Q}}{\partial\rho}\right|_p > 0 \quad \text{(condition for stability)} \tag{8.99}$$
$$\implies \left.\frac{\partial\dot{Q}}{\partial\left(\frac{\mu p}{\mathcal{R}_* T}\right)}\right|_p > 0$$
$$\implies -\frac{T^2}{p}\left.\frac{\partial\dot{Q}}{\partial T}\right|_p > 0, \tag{8.100}$$

and thus we arrive at the *Field criterion*,

$$\boxed{\text{unstable if } \left.\frac{\partial\dot{Q}}{\partial T}\right|_p > 0.} \tag{8.101}$$

The system is always unstable if it's Field unstable (named after George Field who wrote the classic paper on thermal instability in 1965).

However, even a Field stable system can be unstable if $A^* < 0 \implies \partial\dot{Q}\big/\partial T\,\big|_\rho < 0$. From the dispersion relation, we see that this can happen for long wavelength modes, i.e. k small. Then

$$q^2(q + A^*\rho_0^\gamma) \approx 0 \quad \implies \quad q \approx -A^*\rho_0^\gamma. \tag{8.102}$$

Interpretation: : instability if net cooling rate decreases when temperature increases,

- Short wavelength perturbations are readily brought into pressure equilibrium by the action of sound waves, therefore, thermal instability proceeds at fixed pressure;

- Long wavelength perturbations: there is insufficient time for sound waves to equalise pressure with surroundings, so they tend to develop at constant density (sometimes called isochoric thermal instability).

Example. Let's assume a specific form for $\dot{Q}$,

$$\begin{aligned}\dot{Q} &= A\rho T^{\alpha} - H \\ &= \frac{A\mu}{\mathcal{R}_*} p T^{\alpha-1} - H,\end{aligned} \tag{8.103}$$

so,

$$\left.\frac{\partial \dot{Q}}{\partial T}\right|_p = (\alpha - 1)\frac{A\mu p}{\mathcal{R}_*} T^{\alpha-2}. \tag{8.104}$$

This is Field unstable, $\left.\partial\dot{Q}\middle/\partial T\right|_p < 0$ if $\alpha < 1$. Bremsstrahlung has $\alpha = 0.5 \implies$ Field unstable.

Note about gravity:

- Buoyancy interactions with thermal instability is a powerful stabilising effect;
- Thermal instability can become a subtle question of the functional dependence of the cooling/heating balance.

Chapter 9

Viscous Flows

Thus far, we have been assuming that changes in the momentum of a fluid element are due entirely to pressure forces (acting normal to the surface of the element) or gravity (acting on the bulk).

This assumption is justified in the limit $\lambda \to 0$, i.e. the particles composing the fluid have vanishingly small collisional mean-free-path.

For finite-λ, momentum can diffuse through the fluid. This brings us to a discussion of viscosity.

9.1 Basics of Viscosity

In a viscous flow, momentum can be transferred if there are velocity differences between fluid elements.

The continuity equation is unchanged

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0. \tag{9.1}$$

But the momentum equation needs to be changed

$$\frac{\partial}{\partial t}(\rho u_i) = \partial_j \sigma_{ij} + \rho g_i, \quad g_i = -\partial_i \Psi, \tag{9.2}$$

with

$$\sigma_{ij} = \rho u_i u_j + p\delta_{ij} - \underbrace{\sigma'_{ij}}_{\substack{\text{viscous} \\ \text{stress tensor}}} . \tag{9.3}$$

As we will see later, σ'_{ij} is related to velocity gradients.

The connection between the viscous stress tensor and the microphysics (i.e. the mean-free-path) is uncovered by considering a simple linear shear flow as in Fig. 9.1.

Microscopically, thermal/random motion of the particles can allow momentum to "diffuse" across streamlines. This becomes more important as gas gets less collisional.

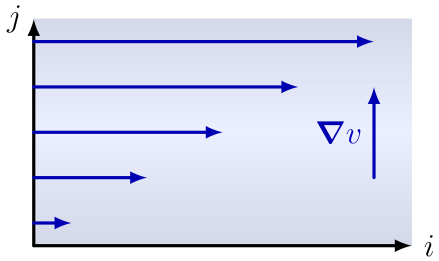


Fig. 9.1: Linear shear flow

Let's analyse the microscopic behaviour: assume the typical (thermal) velocity in the j-direction is u_j. So the momentum flux associated with this is

$$\underbrace{\rho u_i u_j}_{\substack{i\text{-component of momentum} \\ \text{carried in } j\text{-direciton}}} . \tag{9.4}$$

The typical thermal velocity is $\sim \sqrt{kT/m}$. So, flux of the i-component of momentum in the upward j-direction is

$$\rho u_i \alpha \sqrt{\frac{kT}{m}}, \quad \alpha \sim 1. \tag{9.5}$$

For the element on the other side of the surface in the j-direction, the corresponding momentum flux across the surface is

$$-\rho u_i^* \alpha \sqrt{\frac{kT}{m}}, \tag{9.6}$$

where u_i^* is the i-velocity of that element. For a j-separation of δl we have

$$u_i^* = u_i + \delta l(\partial_j u_i). \tag{9.7}$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\delta l \alpha \sqrt{\frac{kT}{m}}. \tag{9.8}$$

The relevant scale δl is the mean-free-path

$$\delta l \sim \lambda = \frac{1}{n\sigma}, \tag{9.9}$$

where n is the number density and σ is the collision cross section of the particles. If we treat the particles as hard spheres of radius a (decent approximation for neutral gas), then

$$\sigma = \pi a^2. \tag{9.10}$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\frac{m}{\rho \pi a^2}\alpha \sqrt{\frac{kT}{m}}. \tag{9.11}$$

Putting this into the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j\left[\underbrace{\frac{\alpha}{\pi a^2}\sqrt{mkT}}_{\equiv\eta,\,\text{shear viscosity}}\partial_j u_i\right] + \rho g_i. \tag{9.12}$$

A rigorous derivation shows that, for this hard-sphere model, $\alpha = 5\sqrt{\pi}/64$.

Observations about the shear viscosity:

- η is independent of density (a denser gas has more particles to transport the momentum but the mean-free-path is shorter);
- η increases with T;
- Isothermal system has $\eta = \text{const}$;
- Functional dependence on T depends on collision model, hard sphere model gives $\eta \propto T^{1/2}$ whereas coulomb collisions (relevant for fully ionised plasma) gives $\eta \propto T^{5/2}$.

For a fully ionised plasma (e.g. the ICM), the mean-free-path is set by Coulomb collisions. Then

$$\lambda \propto T^2, \quad v_{\text{th}} \propto \sqrt{T} \quad \Longrightarrow \quad \eta \propto T^{5/2}. \tag{9.13}$$

Thus the viscosity has a stronger temperature dependence than found for hard-sphere collisions.

9.2 Navier-Stokes Equation

The most general form of σ'_{ij} which is

- Galilean invariant;
- Linear in velocity components;
- Isotropic

is given by

$$\sigma'_{ij} = \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k, \tag{9.14}$$

with η and ζ independent of velocity. This is a *symmetric* tensor which ensures that there are no unbalanced torques on fluid elements.

The term associated with η relates to momentum transfer in *shear flows* (this term has zero trace). The term associated with ζ relates to momentum transfer due to bulk compression ($\partial_k u_k \equiv \boldsymbol{\nabla} \cdot \mathbf{u}$).

Putting this into the momentum equation gives

$$\frac{\partial(\rho u_i)}{\partial t} = -\partial_j(\rho u_i u_j) - \partial_j p \delta_{ij} + \partial_j \left[\eta \left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k \right) + \zeta \delta_{ij}\partial_k u_k \right] + \rho g_i, \tag{9.15}$$

which we can combine with the continuity equation to give

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \partial_j u_i\right) = -\partial_j p \delta_{ij} + \underbrace{\partial_j \left[\eta \left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k \right) + \zeta \delta_{ij}\partial_k u_k \right]}_{\text{viscous force}} + \rho g_i. \tag{9.16}$$

This is the general form of the *Navier-Stokes equation.*

Outside of shocks ($\zeta \approx 0$) and for isothermal fluids ($\eta =$ constant) we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi + \underbrace{\frac{\eta}{\rho}}_{\substack{\equiv \nu \\ \text{kinematic} \\ \text{viscosity}}} \left[\nabla^2 \mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) \right]. \tag{9.17}$$

This is the more commonly used form of the Navier-Stokes equation.

The importance of viscosity in a flow is characterized via the Reynolds number

$$\mathrm{Re} = \frac{|\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}|}{|\nu \nabla^2 \mathbf{u}|} \quad \Longrightarrow \quad \mathrm{Re} = \frac{UL}{\nu}, \tag{9.18}$$

where U and L are the characteristic velocity and length scales of the system, respectively.

Consequences of viscosity:

- Shear leads to transmission of momentum through flow (layers rub);
- Vorticity:
  - Can introduce vorticity into initially irrotational flows from the boundaries;
  - Vorticity diffuses through the flow (advection/diffusion = Re).
- Generally has stabilising effect on various fluid instabilities;
- Dissipates kinetic energy into heat.

9.3 Vorticity in Viscous Flows

Start with the Navier-Stokes equation with $\zeta = 0$ and $\eta =$ const., and take the curl of this, recalling the definition of the vorticity $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}$:

$$\frac{\partial \mathbf{w}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = \boldsymbol{\nabla} \times \left(-\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi + \frac{\eta}{\rho} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right] \right). \tag{9.19}$$

To tidy up LHS, use the vector identity and definition of vorticity:

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \frac{1}{2} \boldsymbol{\nabla} u^2 - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) \tag{9.20}$$

$$\implies \quad \boldsymbol{\nabla} \times (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = -\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}). \tag{9.21}$$

To tidy up RHS, assume a barotropic fluid, $p = p(\rho)$:

$$\begin{aligned} \implies \quad \boldsymbol{\nabla} \times \left(\frac{1}{\rho} \boldsymbol{\nabla} p \right) &= \boldsymbol{\nabla} \left(\frac{1}{\rho} \right) \times \boldsymbol{\nabla} p + \frac{1}{\rho} \boldsymbol{\nabla} \times \boldsymbol{\nabla} p \\ &= -\frac{1}{\rho^2} \underbrace{\boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} p}_{\substack{=0 \text{ since surfaces of} \\ \text{constant } \rho \text{ and } p \text{ align}}} . \end{aligned} \tag{9.22}$$

Putting the pieces together, we get

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \boldsymbol{\nabla} \times \left[\frac{\eta}{\rho} \nabla^2 \mathbf{u} \right], \tag{9.23}$$

and thus

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) + \frac{\eta}{\rho} \nabla^2 \mathbf{w}.} \tag{9.24}$$

where, in the last step, we have ignored gradients of $\nu = \eta/\rho$ (so strictly assumed isothermal and uniform density). So, vorticity is carried with the flow but also diffuses through the flow due to action of vorticity.

Lines of vorticity are advected in the flow *and* diffuse through the flow due to viscosity. Viscous term gives a way for vorticity to enter a previously irrotational flow due to boundary interactions. Relative importance of advection and diffusion given by Reynolds number.

9.4 Energy Dissipation in Incompressible Viscous Flows

Viscosity leads to dissipation of kinetic energy into heat – an irreversible process.

Let's analyse this in the case of an incompressible flow so that we don't need to worry about $p\,\mathrm{d}V$ work. Then the total kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}\int \rho u^2\,\mathrm{d}V\,. \tag{9.25}$$

Let's consider the rate of change of E_{kin} with time

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= u_i\frac{\partial}{\partial t}(\rho u_i)\\ &= -u_i\partial_j(\rho u_i u_j) - u_i\partial_j\delta_{ij}p + u_i\partial_j\sigma'_{ij}\\ &= -u_i\partial_j(\rho u_i u_j) - u_i\partial_i p + \partial_j(u_i\sigma'_{ij}) - \sigma'_{ij}\partial_j u_i.\end{aligned} \tag{9.26}$$

Look at the first term of RHS:

$$u_i\partial_j(\rho u_i u_j) = u_i\Big(u_j\partial_j(\rho u_i) + \rho u_i\underbrace{\partial_j u_j}_{=0}\Big), \tag{9.27}$$

where last term is zero due to the incompressible flow assumption,

$$\boldsymbol{\nabla}\cdot\mathbf{u} \quad\Longrightarrow\quad \partial_j u_j = 0. \tag{9.28}$$

Also note that

$$\begin{aligned}\partial_j\left(\rho u_j\cdot\frac{1}{2}u_iu_i\right) &= \frac{1}{2}\rho u_iu_i\underbrace{\partial_j u_j}_{=0} + u_j\partial_j\left(\frac{1}{2}\rho u_iu_i\right)\\ &= u_ju_i\partial_j(\rho u_i),\end{aligned} \tag{9.29}$$

therefore

$$u_i\partial_j(\rho u_iu_j) = \partial_j\left(\rho u_j\cdot\frac{1}{2}u_iu_i\right). \tag{9.30}$$

So,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= -\partial_j\left(\rho u_j\cdot\frac{1}{2}u_iu_i\right) - \partial_i(u_ip) + p\partial_iu_i + \partial_j(u_i\sigma'_{ij}) - \sigma'_{ij}\partial_ju_i\\ &= -\partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right) - \sigma'_{ij}\partial_ju_i.\end{aligned} \tag{9.31}$$

Integrating over the volume,

$$\begin{aligned}\frac{\partial E_{\text{kin}}}{\partial t} &= \frac{\partial}{\partial t}\int_{\mathcal{V}}\frac{1}{2}\rho u^2\,\mathrm{d}V\\ &= -\int_{\mathcal{V}}\partial_i\left(\rho u_i\left[\frac{1}{2}u^2+\frac{p}{\rho}\right] - u_j\sigma'_{ij}\right)\mathrm{d}V - \int_{\mathcal{V}}\sigma'_{ij}\partial_ju_i\,\mathrm{d}V\\ &= \underbrace{-\oint_{\mathcal{S}}\left(\rho\mathbf{u}\left[\frac{1}{2}u^2+\frac{p}{\rho}\right] - \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'\right)\cdot\mathrm{d}\mathbf{S}}_{\substack{\text{Energy flux into volume including}\\ \text{work done by viscous forces }\mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'}}\ \underbrace{-\int_{\mathcal{V}}\sigma'_{ij}\partial_ju_i\,\mathrm{d}V}_{\substack{\text{Rate of change of }E_{\text{kin}}\text{ due}\\ \text{to viscous dissipation}}}\,.\end{aligned} \tag{9.32}$$

Let's take the volume $\mathcal{V}$ to be the whole fluid so that the surface integral is zero (e.g. $\mathbf{u}$ at bounding surface $= 0$, or $\mathbf{u}$ at $\infty = 0$).

Then

$$\begin{aligned} \frac{\partial E_{\text{kin}}}{\partial t} &= -\int_{\mathcal{V}} \sigma'_{ij} \partial_j u_i \, \mathrm{d}V \\ &= -\frac{1}{2} \int_{\mathcal{V}} \sigma'_{ij} (\partial_j u_i + \partial_i u_j) \, \mathrm{d}V \quad \text{since } \sigma' \text{ is symmetric} \end{aligned} \tag{9.33}$$

But $\sigma'_{ij} = \eta(\partial_j u_i + \partial_i u_j)$ for an incompressible fluid. So,

$$\frac{\partial E_{\text{kin}}}{\partial t} = -\frac{1}{2} \int_{\mathcal{V}} \eta (\partial_j u_i + \partial_i u_j)^2 \, \mathrm{d}V \, . \tag{9.34}$$

2$^{\text{nd}}$ law of thermodynamics dictates that kinetic energy must "grind down" to heat rather than reverse. We see that η needs to be positive in order for us to obey the 2$^{\text{nd}}$ law of thermodynamics.

9.5 Viscous Flow through a Pipe

Now consider flow through a long pipe with a constant circular cross-section (see Fig. 9.2).

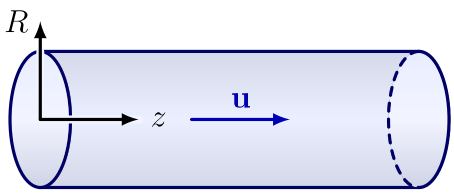


Fig. 9.2: A pipe.

Assume

- Steady flow with $u_R = u_\phi = 0$, $u_z \neq 0$;
- Incompressible, uniform density fluid;
- Neglect gravity.

The Navier-Stokes equation reads

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\substack{=0 \text{ since} \\ \text{steady state}}} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}}_{\substack{=0 \text{ due to} \\ \text{symmetry}}} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \underbrace{\boldsymbol{\nabla} \Psi}_{\substack{=0 \text{ since} \\ \text{no gravity}}} + \nu \left[\nabla^2 \mathbf{u} + \underbrace{\frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u})}_{\substack{=0 \text{ since} \\ \text{incompressible}}} \right]$$

$$\implies \quad \nu \nabla^2 \mathbf{u} = \frac{1}{\rho} \boldsymbol{\nabla} p. \tag{9.35}$$

By symmetry we have

$$u_R = u_\phi = 0 \quad \Longrightarrow \quad \frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0. \tag{9.36}$$

For the z-component

$$\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\substack{\text{function of}\\ z \text{ only}}} = \underbrace{\nu \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)}_{\text{function of } R \text{ only}} = \underbrace{-\frac{1}{\rho}\frac{\Delta p}{l}}_{\substack{\text{constant, written}\\ \text{in terms of global}\\ \text{pressure gradient}}} . \tag{9.37}$$

Integrating gives

$$u = -\frac{\Delta p}{4\rho\nu l}R^2 + a\ln R + b, \tag{9.38}$$

where a and b are constants. Apply boundary conditions:

- At $R = 0$, u finite $\implies$ $a = 0$;
- At $R = R_0$, $u = 0$ (no slip boundary condition at wall).

$$\Longrightarrow \quad u = \frac{\Delta p}{4\rho\nu l}(R_0^2 - R^2). \tag{9.39}$$

So the velocity profile is *parabolic* (see Fig. 9.3).

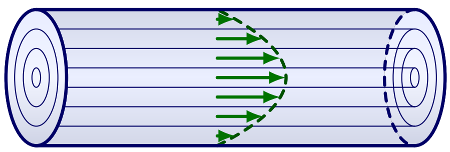


Fig. 9.3: Parabolic velocity profile for viscous flow through a pipe

The mass flux passing through an annular element $2\pi R\,\mathrm{d}R$ is $2\pi R\rho u\,\mathrm{d}R$. So, the total mass flow rate is

$$Q = \int_0^{R_0} 2\pi\rho u R\,\mathrm{d}R = \frac{\pi}{8}\frac{\Delta p}{\nu l}R_0^4. \tag{9.40}$$

Mass flux completely determined by the pressure gradient, radius of pipe, and coefficient of kinematic viscosity.

As $\eta \to 0$, i.e. $\nu \to 0$, the flow rate $\to \infty$ (or, in other words, an inviscid flow cannot be in steady state in this pipe if there is a non-zero pressure gradient).

If Δp increases sufficiently, it becomes unstable and irregular, giving turbulent motions above a critical speed.

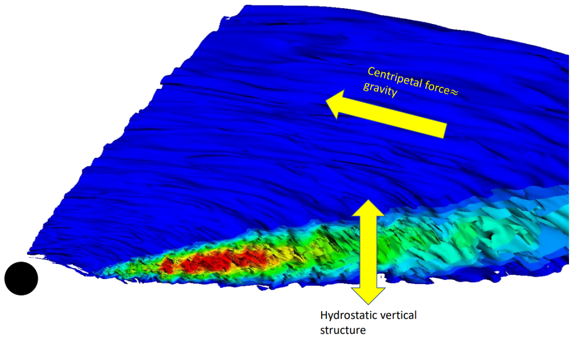


Fig. 9.4: Accretion disc, with gas settled into circular orbits in hydrostatic equilibrium with an internal vertical pressure gradient balancing the vertical component of gravity.

The actual transition to turbulence is usually phrased in terms of the *Reynolds number*

$$\mathrm{Re} \equiv \frac{LV}{\nu}, \tag{9.41}$$

where L and V are "characteristic" length and velocity scales of the system. Flow will become turbulent (not steady, $u_R \neq 0$, $u_\phi \neq 0$, motions on large range of scales), above a critical Reynolds number,

$$\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}. \tag{9.42}$$

9.6 Accretion Disks

Accretion disks are one of the most important applications of the Navier-Stokes equation in astrophysics.

Consider some gas flowing towards a central object (star, planet, black hole, ...). Almost always, the gas will have significant angular momentum about that object. If gravitationally bound to the object, the gas will settle into a plane defined by the mean angular momentum vector. Residual motions in other directions will be damped out on a free-fall timescale.

The gas will settle into circular orbits – the lowest energy configuration for a given angular momentum. In the vertical direction (parallel to the angular momentum vector) the system will come into hydrostatic equilibrium with an internal vertical pressure gradient balancing the vertical component of gravity. In the radial direction (along the direction towards the central object), the system will achieve a state where the centripetal force is supplied by gravity and the radial pressure gradient.

A very important special case is when the disk is "thin", meaning that the scale-height in the vertical direction h is much less than the radius r. Then, radial pressure gradients are negligible and we can just write

$$\Omega^2 R = \frac{GM}{R^2} \qquad \Longrightarrow \qquad \Omega = \sqrt{\frac{GM}{R^3}}, \tag{9.43}$$

where Ω is the angular velocity of the flow around the central object. This means that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} \neq 0 \qquad \Longrightarrow \qquad \text{shear flow.} \tag{9.44}$$

Viscosity will allow angular momentum to be transferred from the fast moving inner regions to the more slowly moving outer regions. This means the inner disk fluid elements lose angular momentum. We have

$$J = R^2\Omega = \sqrt{GMR}, \quad \text{(per unit mass)} \tag{9.45}$$

meaning that inner disk fluid elements drift inwards.

Ultimately, most of the mass flows inwards; a small amount of the mass carries all of the angular momentum out to large radii.

Let's set up a simple model for a geometrically-thin accretion disk. We assume:

- Cylindrical polar coordinates (R, ϕ, z);
- Axisymmetric, $\partial/\partial\phi = 0$;
- Hydrostatic equilibrium in the z-direction, $u_z = 0$;
- u_ϕ close to Keplerian velocity (i.e. thin disk);
- u_R small and set by action of viscosity;
- Bulk viscosity zero.

The continuity equation in cylindrical polar coordinates is

$$\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \underbrace{\frac{1}{R}\frac{\partial}{\partial\phi}(\rho u_\phi)}_{\substack{=0 \text{ due to}\\ \text{axisymmetry}}} + \underbrace{\frac{\partial}{\partial z}(\rho u_z)}_{\substack{=0 \text{ since}\\ \text{hydrostatic}}} = 0$$

$$\Longrightarrow \qquad \frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0. \tag{9.46}$$

Define the surface density Σ by

$$\Sigma \equiv \int_{-\infty}^{\infty} \rho \,\mathrm{d}z\,. \tag{9.47}$$

Then, integrating the above form of the continuity equation over z we have

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0.. \tag{9.48}$$

We can get the same result by thinking of the disk as a set of rings/annuli as shown in Fig. 9.5.

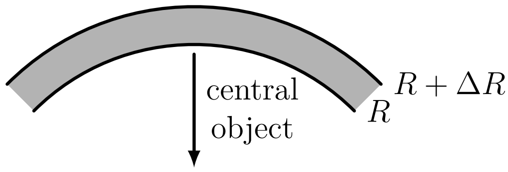


Fig. 9.5: Infinitesimal annulus element of disk

$$\begin{matrix}\text{rate of change}\\ \text{of mass in}\\ \text{the annulus}\end{matrix} = \begin{matrix}\text{flux into}\\ \text{annulus}\end{matrix} + \begin{matrix}\text{flux out}\\ \text{of annulus}\end{matrix} \tag{9.49}$$

$$\begin{aligned}
\implies \frac{\partial}{\partial t}(2\pi R\Delta R\Sigma) &= 2\pi R\Sigma(R)u_R(R) - 2\pi(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R)\\
\implies \quad R\frac{\partial \Sigma}{\partial t} &= -\left[\frac{(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R) - R\Sigma(R)u_R(R)}{\Delta R}\right]\\
\implies \quad R\frac{\partial \Sigma}{\partial t} &= -\frac{\partial}{\partial R}(R\Sigma u_R). \quad \text{taking } \Delta R \to 0.
\end{aligned} \tag{9.50}$$

Now we look at the conservation of angular momentum. A derivation starting with the Navier-Stokes equation in cylindrical coordinates appears in Appendix A.1 at the end of this chapter. Here we use the ring/annulus approach:

$$\begin{matrix}\text{rate of change}\\ \text{of ang. mtm.}\end{matrix} = \begin{matrix}\text{ang. mtm.}\\ \text{of mass}\\ \text{entering ring}\end{matrix} - \begin{matrix}\text{ang. mtm.}\\ \text{of mass}\\ \text{leaving ring}\end{matrix} + \begin{matrix}\text{net torque on ring}\\ \text{(viscour, magnetic, etc.)}\end{matrix} \tag{9.51}$$

$$\implies \frac{\partial}{\partial t}\left(2\pi R\Delta R\Sigma R^2\Omega\right) = \underbrace{f(R) - f(R+\Delta R)}_{\text{ang. mtm. advection}} + \underbrace{G(R+\Delta R) - G(R)}_{\text{viscous torques}}, \tag{9.52}$$

where

$$f(R) \equiv 2\pi R\Sigma u_R \Omega R^2, \tag{9.53}$$

and $G(R)$ is the torque exerted by the disk outside of radius R on the disk inside of radius R:

$$G(R) = 2\pi R\nu\Sigma R\frac{\mathrm{d}\Omega}{\mathrm{d}R}R = 2\pi R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}, \tag{9.54}$$

therefore

$$\frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right). \tag{9.55}$$

Now assume $\partial u_\phi/\partial t = 0$ since gas is on Keplerian orbits. Then

$$\begin{aligned}
Ru_\phi\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\implies \quad -u_\phi\frac{\partial}{\partial R}(R\Sigma u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\implies \quad -u_\phi\frac{\partial}{\partial R}(R\Sigma u_R) + \frac{u_\phi R}{R}\frac{\partial}{\partial R}(R\Sigma u_R) + \Sigma u_R\frac{\partial}{\partial R}(u_\phi R) &= \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\implies \quad R\Sigma u_R\frac{\partial}{\partial R}\left(R^2\Omega\right) &= \frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\implies \quad u_R &= \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}.
\end{aligned} \tag{9.56}$$

Substitute this into Eq. (9.55) and specialise to the case of a Newtonian point source gravitational field $\Omega = \sqrt{GM/R^3}$, yielding

$$\boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right].} \tag{9.57}$$

So the surface density $\Sigma(R,t)$ obeys a diffusion equation.

Notes on accretion disks:

- In general, $\nu = \nu(R, \Sigma, T, \cdots)$ and so this is a non-linear diffusion equation for Σ. It reduces to linear if $\nu = \nu(R)$;
- Vertical structure only enters via the temperature dependence of ν. So the geometrically-thin assumptions allows the radial and vertical problems to be mostly decoupled.
- Diffusion-like nature of the solutions of this equation show that an initial ring of matter will broaden and then "slump" inwards towards the central object (see Fig. 9.6).
- An initial narrow annulus of matter will spread out to form a full disk. This is a natural way to turn an incoming stream of gas into an accretion disk:
  - Stream flows in, swings around central object;
  - If orbit is not strictly elliptical, stream self-intersects and shocks;

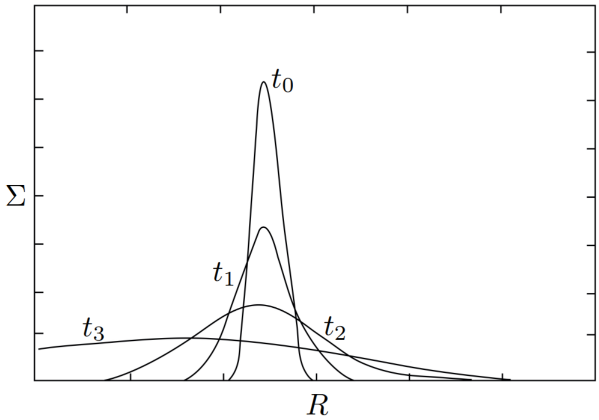


Fig. 9.6: The viscous evolution of a ring.

  - Form a narrow annulus of gas at location defined by specific angular momentum of matter in stream.

- Timescale for evolution is t_ν where

$$\begin{aligned} \frac{\Sigma}{t_\nu} &\sim \frac{1}{R}\frac{1}{R}R^{1/2}\frac{1}{R}\nu\Sigma R^{1/2} \sim \frac{\nu\Sigma}{R^2} \\ \Longrightarrow \quad t_\nu &\sim \frac{R^2}{\nu} = \frac{R}{u_\phi}\frac{Ru_\phi}{\nu} = \Omega^{-1}\mathrm{Re}, \end{aligned} \tag{9.58}$$

  where Re is the Reynolds number;

- If viscosity is due to particle thermal motions, typical values would suggest that Re $\sim 10^{14}$! This means

$$t_\nu \gg \text{age of the Universe.} \tag{9.59}$$

  There must be another source of effective viscosity: we now know that there is an *effective viscisty* due to magnetohydrodynamic turbulence driven by the *magnetorotational instability*.

  Dimensional analysis reveals $[\nu] = [L]^2[T]^{-1}$, and so turbulence gives effective viscosity $\nu_{\mathrm{eff}} \sim ul$ where l and u are the size and velocity of a typical eddy in the turbulence, respectively.

  Thinking about physics in comoving frame of orbital disk, the only characteristic velocity is c_s and the only characteristic length is the thickness H. These set the upper bounds on the velocity and size of turbulent eddies. So, we can say

$$\nu = \alpha c_s H \quad (\alpha < 1). \tag{9.60}$$

  Disk models using this prescription are known as α-disks (gives ν dependence upon disk temperature and vertical structure, Shakura & Sunyaev 1973).

9.7 Steady-State, Geometrically-Thin Disks

Consider a steady state such that $\partial/\partial t = 0$. Then

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) &= 0 \\ \Longrightarrow \qquad R\Sigma u_R = C_1 &= -\frac{\dot{m}}{2\pi} \end{aligned} \tag{9.61}$$

where $\dot{m} = -2\pi R\Sigma u_R$ is the steady state mass inflow rate. Now recall from Eq. (9.56) that

$$u_R = \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}, \tag{9.62}$$

therefore,

$$\begin{aligned} -\frac{\dot{m}}{2\pi R\Sigma} &= -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right) \quad \text{for} \quad \Omega^2 = GM/R^3 \\ \Longrightarrow \qquad \nu\Sigma &= \frac{\dot{m}}{3\pi}\left(1 - \sqrt{\frac{R_*}{R}}\right), \end{aligned} \tag{9.63}$$

where we have taken as a boundary condition that $\nu\Sigma = 0$ at $R = R_*$. This amounts to saying that there are no viscous torques at $R = R_*$. Physically R_* can be:

- Surface of an accreting star;
- Innermost circular orbit around a black hole.

Let's now calculate the viscous dissipation neglecting $p\,\mathrm{d}V$ work and bulk viscosity. Specifically, we will calculate the viscous dissipation per unit surface area of the disk:

$$\begin{aligned} F_{\mathrm{diss}} &= \int \sigma'_{ij}\partial_j u_i \frac{\mathrm{d}V}{2\pi R\,\mathrm{d}R\,\mathrm{d}\phi} \\ &= \frac{1}{2}\int \eta(\partial_j u_i + \partial_i u_j)^2\,\mathrm{d}z \\ &= \int \eta R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 \mathrm{d}z \\ &= \nu\Sigma R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2. \end{aligned} \tag{9.64}$$

Combining with our previous result (9.63) for $\nu\Sigma$ and recalling that $\Omega^2 = GM/R^3$, we have

$$F_{\mathrm{diss}} = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right). \tag{9.65}$$

Notes on dissipation in a disk:

- Total energy emitted is

$$L = \int_{R_*}^{\infty} F_{\rm diss} 2\pi R \, \mathrm{d}R = \frac{GM\dot{m}}{2R_*}. \tag{9.66}$$

  Here, $-GM/R_*$ is the gravitational potential at R_*. Therefore, $GM\dot{m}/R_*$ is the rate of gravitational energy loss of the flow. $GM\dot{m}/2R_*$ is radiated, the other half stays in the flow as kinetic energy and is dissipated in the boundary layer on an accreting star, or carried into the black hole;

- At a given location far from the inner edge ($R > R_*$) we have

$$F_{\rm diss} \approx \frac{3GM\dot{m}}{4\pi R^3}. \tag{9.67}$$

  But an elementary estimate based on the loss of gravitational potential energy would give

$$F_{\rm diss,est} = \underbrace{\frac{1}{2\pi R \, \mathrm{d}R}}_{\substack{\text{area of}\\ \text{annulus}}} \cdot \underbrace{\frac{\partial}{\partial R}\left(\frac{GM\dot{m}}{R}\right)}_{\substack{\text{change in grav.}\\ \text{potential of } \dot{m}\\ \text{over annulus}}} \cdot \underbrace{\frac{1}{2}}_{\substack{\text{half converts}\\ \text{to radiation,}\\ \text{rest to kinetic}}} = \frac{GM\dot{m}}{4\pi R^3}. \tag{9.68}$$

  The extra factor of “3” in the correct formula is due to the transport of energy through the disk by viscous torques.

9.7.1 Radiation from Steady-State Thin Disks

If a disk is optically-thick, all radiation is thermalised and it radiates locally as a black body

$$\begin{aligned} \underbrace{2}_{\substack{\text{top and}\\ \text{bottom of disk}}} \cdot \sigma_{\rm SB} T_{\rm eff}^4 &= \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right) \\ \implies \quad T_{\rm eff} &= \left[\frac{3GM\dot{m}}{8\pi\sigma_{\rm SB} R^3}\right]^{1/4}. \end{aligned} \tag{9.69}$$

So, for $R \gg R_*$, $T_{\rm eff} \propto R^{-3/4}$.

The radiation emitted at a frequency f is

$$F_f = \int_{R_*}^{\infty} \frac{2h}{c^2} \frac{f^3}{e^{hf/kT_{\rm eff}} - 1} 2\pi R \, \mathrm{d}R. \tag{9.70}$$

So, we see that all of the observables from a steady-state disk are independent of the viscosity ν, provided it is large enough to supply the necessary angular momentum transport. To constrain ν, we need to study non-steady disks.

Chapter 10

Plasmas

Plasmas are fluids composed of charged particles. Thus, electromagnetic fields become important for both microphysics and large scale dynamics.

10.1 Magnetohydrodynamic (MHD) Equations

Consider a fully ionised hydrogen plasma, containing only protons (number density n^+, bulk velocity $\mathbf{u}^+$) and electrons $(n^-, \mathbf{u}^-)$. Mass conservation for each of the proton and electron fluids is

$$\frac{\partial n^+}{\partial r} + \boldsymbol{\nabla} \cdot \left(n^+ \mathbf{u}^+\right) = 0 \tag{10.1}$$

$$\frac{\partial n^-}{\partial r} + \boldsymbol{\nabla} \cdot \left(n^- \mathbf{u}^-\right) = 0. \tag{10.2}$$

The mass density is $\rho = m^+ n^+ + m^- n^-$ and the centre-of-mass velocity is

$$\mathbf{u} = \frac{m^+ n^+ \mathbf{u}^+ + m^- n^- \mathbf{u}^-}{m^+ n^+ + m^- n^-}. \tag{10.3}$$

So, we can combine these to give the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{10.4}$$

The continuity equation is the same as found before.

The charge density is $q = n^+ e^+ + n^- e^-$ and the current density is $\mathbf{j} = e^+ n^+ \mathbf{u}^+ + e^- n^- \mathbf{u}^-$. So, the above information also gives a *conservation of charge equation*

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{10.5}$$

When we formulate the momentum equation, we have to consider the Lorentz force on each particle

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{10.6}$$

So, for the two species of particles:

$$m^+n^+\left(\frac{\partial \mathbf{u}^+}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^+\right) = e^+n^+(\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}) - f^+\boldsymbol{\nabla} p \tag{10.7}$$

$$m^-n^-\left(\frac{\partial \mathbf{u}^-}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^-\right) = e^-n^-(\mathbf{E} + \mathbf{u}^- \times \mathbf{B}) - f^-\boldsymbol{\nabla} p \tag{10.8}$$

where $f^\pm$ is the fraction of the pressure gradient that accelerates the protons/electrons.

Summing these equations gives

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla} p. \tag{10.9}$$

Ohm's law lets us relate $\mathbf{j}$ to $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}), \tag{10.10}$$

where σ is the electrical conductivity. This equation is needed to close the set of equations.

So, recapping the current set of equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{10.11}$$

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{10.12}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla} p \tag{10.13}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}). \tag{10.14}$$

We need to relate q, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ – Maxwell's equations!

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{10.15}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{q}{\epsilon_0} \tag{10.16}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{10.17}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{10.18}$$

where we note $\epsilon_0\mu_0 = 1/c^2$.

10.1.1 Simplifying MHD

Let us simplify in the case of a non-relativistic, highly conducting plasma. Suppose fields are varying over length scales l and timescales τ. Then

1.
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Longrightarrow \quad \frac{E}{B} \sim \frac{1}{\tau} \sim u. \tag{10.19}$$

2.
$$\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right| \Big/ |\nabla \times \mathbf{B}| \sim \frac{1}{c^2}\left(\frac{l}{\tau}\right)^2 \sim \frac{u^2}{c^2} \ll 1, \tag{10.20}$$

for non-relativistic flows. Therefore, displacement current can be ignored in non-relativistic MHD;

3. Look at two terms from Eq. (10.13):

$$\frac{|q\mathbf{E}|}{|\mathbf{j} \times \mathbf{B}|} \sim \frac{qE}{jB} \sim \frac{\epsilon_0 E/l}{B/l\mu_0}\frac{E}{B} \sim u^2\epsilon_0\mu_0 \sim \frac{u^2}{c^2} \ll 1. \tag{10.21}$$

Therefore, charge neutrality is preserved to a high approximation due to the strength of electrostatic forces. If there is a charge imbalance, it will oscillate with a characteristic frequency, the *plasma frequency*

$$\omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}. \tag{10.22}$$

4. Neglecting displacement current in the relevant Maxwell equation, we get

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0\sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{10.23}$$

Take curl:

$$\begin{aligned} \underbrace{\nabla \times (\nabla \times \mathbf{B})}_{=-\nabla^2\mathbf{B}-\nabla(\nabla\cdot\mathbf{B})=-\nabla^2\mathbf{B}} &= \mu_0\sigma\Big(\underbrace{\nabla \times \mathbf{E}}_{-\frac{\partial \mathbf{B}}{\partial t}} + \nabla \times (\mathbf{u} \times \mathbf{B})\Big) \\ \Longrightarrow \quad \frac{\partial \mathbf{B}}{\partial t} &= \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\substack{\text{advection of} \\ \text{the field} \\ \text{by the flow}}} + \underbrace{\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}}_{\substack{\text{dissipation of} \\ \text{the field} \\ \text{through the flow}}}. \end{aligned} \tag{10.24}$$

If the fluid is a good conductor, i.e. σ is very large, then we can ignore the diffusion term and we have an equation that is analogous to the Helmholtz equation/Kelvin's theorem:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{10.25}$$

By exact analogy with the Helmholtz equation, this says that the flux of magnetic field threading some surface $\mathcal{S}$ moving with the flow is preserved.

$$\int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \text{constant}. \tag{10.26}$$

This is the *flux-freezing condition*. Magnetic field lines are advected along in flow. We talk about the "freezing" of the magnetic flux into the plasma. In the high σ limit we must also have

$$\begin{aligned} \mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) &\quad \text{is finite} \\ \implies \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 &\quad \text{as } \sigma \to \infty \\ \implies \quad \mathbf{E} \cdot \mathbf{B} = 0 & \\ \text{i.e.} \quad \mathbf{E} \perp \mathbf{B}. & \end{aligned} \tag{10.27}$$

So, the full set of *ideal MHD equations*, i.e. equations describing a non-relativistic, perfectly conducting, charge neutral plasma are:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{10.28}$$

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p \tag{10.29}$$

$$\left.\begin{aligned} \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{E} = 0 \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} = -\,\partial \mathbf{B}/\partial t \\ p = K\rho^\gamma \end{aligned}\right\} \implies \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \end{aligned} \tag{10.30}$$

10.2 The Dynamical Effects of Magnetic Fields

The magnetic force density appearing in the above ideal MHD equations is

$$\mathbf{f}_{\text{mag}} = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}. \tag{10.31}$$

So using vector identity this is

$$\mathbf{f}_{\text{mag}} = \left[\underbrace{-\boldsymbol{\nabla}\left(\frac{B^2}{2}\right)}_{\substack{\text{magnetic pressure} \\ \text{term with} \\ p_{\text{mag}}=B^2/2\mu_0}} + \underbrace{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}_{\substack{\text{magnetic tension} \\ \text{term (vanishes for} \\ \text{straight field lines)}}} \right] \tag{10.32}$$

Since there are new force terms in the momentum equation, this will change the nature of the waves that are possible.

10.3 Waves in Plasmas

We can repeat the perturbation analysis that we conducted for sound waves but now include the effects of a magnetic field. We will perturb about an equilibrium consisting of a static ($\mathbf{u} = \mathbf{0}$) plasma with uniform density ρ_0, uniform pressure p_0, and uniform magnetic field $\mathbf{B}_0$.

We start by writing down the governing equations of ideal MHD, assuming a barotropic equation of state:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{10.33}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \right) = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \boldsymbol{\nabla} p \tag{10.34}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \tag{10.35}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{10.36}$$

$$p = p(\rho). \tag{10.37}$$

We now introduce perturbations and linearise the equations:

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\delta \mathbf{u}) = 0 \tag{10.38}$$

$$\rho_0 \frac{\partial \delta \mathbf{u}}{\partial t} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \delta \mathbf{B}) \times \mathbf{B}_0 - c_s^2 \boldsymbol{\nabla} \delta \rho \tag{10.39}$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta \mathbf{u} \times \mathbf{B}_0) = -\mathbf{B}_0 (\boldsymbol{\nabla} \cdot \delta \mathbf{u}) + (\mathbf{B}_0 \cdot \boldsymbol{\nabla}) \delta \mathbf{u} \tag{10.40}$$

$$\boldsymbol{\nabla} \cdot \delta \mathbf{B} = 0. \tag{10.41}$$

We now adopt our usual plane wave form for the perturbations,

$$\delta \rho = \delta \rho_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{10.42}$$

$$\delta p = \delta p_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{10.43}$$

$$\delta \mathbf{u} = \delta \mathbf{u}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{10.44}$$

$$\delta \mathbf{B} = \delta \mathbf{B}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}. \tag{10.45}$$

The continuity equation becomes

$$-i\omega \delta \rho + i \rho_0 \mathbf{k} \cdot \delta \mathbf{u} = 0$$

$$\implies \quad \omega \delta \rho = \rho_0 \mathbf{k} \cdot \delta \mathbf{u}. \tag{10.46}$$

The momentum equation becomes

$$-i\omega \rho_0 \delta \mathbf{u} = \frac{i}{\mu_0} (\mathbf{k} \times \delta \mathbf{B}) \times \mathbf{B}_0 - i c_s^2 \delta \rho \mathbf{k}$$

$$\implies \quad \omega \rho_0 \delta \mathbf{u} = \frac{1}{\mu_0} [(\mathbf{B}_0 \cdot \delta \mathbf{B}) \mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{B}] + c_s^2 \delta \rho \mathbf{k}. \tag{10.47}$$

Finally, the flux-freezing (induction) equation becomes

$$-i\omega \delta \mathbf{B} = -i \mathbf{B}_0 (\mathbf{k} \cdot \delta \mathbf{u}) + i (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{u}$$

$$\implies \quad \omega \delta \mathbf{B} = \mathbf{B}_0 (\mathbf{k} \cdot \delta \mathbf{u}) - (\mathbf{B}_0 \cdot \mathbf{k}) \delta \mathbf{u}. \tag{10.48}$$

The full dispersion relation for MHD waves is then derived from eliminating the perturbation amplitudes from these expressions. Here, we are going to gain insight for the physics by just focusing on some special cases.

Firstly, we consider the case of modes with wavevectors orthogonal to the background magnetic field direction, $\mathbf{k} \parallel \mathbf{B}_0$. The linearised equations then become

$$\omega \delta\rho = \rho_0 \mathbf{k} \cdot \delta\mathbf{u} \tag{10.49}$$

$$\omega \rho_0 \delta\mathbf{u} = \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \delta\mathbf{B})\mathbf{k} + c_s^2 \delta\rho \mathbf{k} \tag{10.50}$$

$$\omega \delta\mathbf{B} = \mathbf{B}_0(\mathbf{k} \cdot \delta\mathbf{u}). \tag{10.51}$$

We can immediately notice from the second of these relations that the velocity perturbations are aligned with the wavevector, $\delta\mathbf{u} \parallel \mathbf{k}$, i.e. these are longitudinal modes. Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$\omega^2 \rho_0 \delta\mathbf{u} = \frac{1}{\mu_0} B_0^2 (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{k} + c_s^2 \rho_0 (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{k}. \tag{10.52}$$

Take the dot product of this last equation with $\mathbf{k}$ and then cancel $\mathbf{k} \cdot \delta\mathbf{u}$ throughout (since we know that this must be non-zero since modes are longitudinal),

$$\omega^2 \rho_0 = \frac{k^2 B_0^2}{\mu_0} + c_s^2 \rho_0 k^2 \tag{10.53}$$

$$\implies \quad \omega^2 = \left(c_s^2 + \frac{B^2}{\mu_0 \rho_0}\right) k^2 \tag{10.54}$$

$$\omega^2 = \left(c_s^2 + v_A^2\right) k^2, \tag{10.55}$$

where we have defined the *Alfvén speed*,

$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}}. \tag{10.56}$$

This describes a compressive dispersion-free longitudinal waves with a phase speed $\sqrt{c_s^2 + v_A^2}$. The restoring force comes from both the gas pressure and magnetic pressure acting in phase. This is known as the fast magnetosonic wave.

We now consider the case of modes with $\mathbf{k} \parallel \mathbf{B}_0$. The linearised equations become

$$\omega \delta\rho = \rho_0 \mathbf{k} \cdot \delta\mathbf{u} \tag{10.57}$$

$$\omega \rho_0 \delta\mathbf{u} = \frac{1}{\mu_0}[(\mathbf{B}_0 \cdot \delta\mathbf{B}) - B_0 k \delta\mathbf{B}] + c_s^2 \delta\rho \mathbf{k} \tag{10.58}$$

$$\omega \delta\mathbf{B} = \mathbf{B}_0(\mathbf{k} \cdot \delta\mathbf{u}) - B_0 k \delta\mathbf{u}. \tag{10.59}$$

Eliminating $\delta\rho$ and $\delta\mathbf{B}$ from this set of equations in favour of $\delta\mathbf{u}$, we get

$$\omega^2 \rho_0 \delta\mathbf{u} = \frac{1}{\mu_0}\left[B_0^2 k^2 \delta\mathbf{u} - (\mathbf{B}_0 \cdot \delta\mathbf{u}) B_0 k \mathbf{k}\right] + c_s^2 (\mathbf{k} \cdot \delta\mathbf{u})\mathbf{k}. \tag{10.60}$$

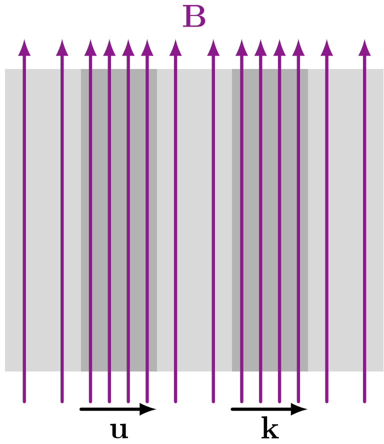


Fig. 10.1: Fast magnetosonic wave.

There are actually two distinct wave modes wrapped up in these expression, a longitudinal mode and a transverse mode. To extract the longitudinal mode, take the dot product with $\mathbf{k}$

$$\omega^2 \rho_0 (\mathbf{k} \cdot \delta\mathbf{u}) = \frac{1}{\mu_0}\left[B_0^2 k^2 (\mathbf{k} \cdot \delta\mathbf{u}) - (\mathbf{B}_0 \cdot \delta\mathbf{u}) B_0 k^3\right] + c_s^2 (\mathbf{k} \cdot \delta\mathbf{u}) k^2, \tag{10.61}$$

and cancel factor of $\mathbf{k} \cdot \delta\mathbf{u}$ to get

$$\omega^2 = c_s^2 k^2. \tag{10.62}$$

These are simply sound waves, with the magnetic field not playing a role since the velocity perturbations are directed along the magnetic field.

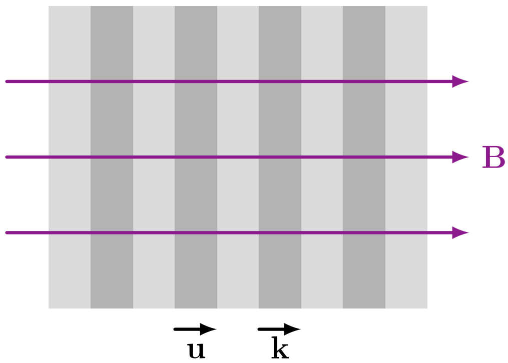


Fig. 10.2: Longitudinal wave with $\mathbf{k} \parallel \mathbf{B}$.

Return to the more general expression for the case $\mathbf{k} \parallel \mathbf{B}_0$;

$$\omega^2 \rho_0 \delta\mathbf{u} = \frac{1}{\mu_0}\left[B_0^2 k^2 \delta\mathbf{u} - (\mathbf{B}_0 \cdot \delta\mathbf{u}) B_0 k \mathbf{k}\right] + c_s^2 (\mathbf{k} \cdot \delta\mathbf{u}) \mathbf{k}. \tag{10.63}$$

Taking the cross product with $\mathbf{k}$, we get

$$\omega^2 = \frac{B_0^2}{\mu_0 \rho_0} k^2 = v_A^2 k^2. \tag{10.64}$$

This describes transverse waves with phase speed v_A where the restoring force is provided by magnetic tension. These are *Alfvén waves*. They are incompressible transverse waves due to magnetic tension.

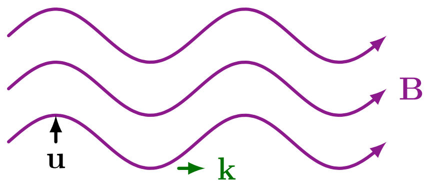


Fig. 10.3: Alfvén wave.

For a general perturbation, with $\mathbf{B}$ and $\mathbf{k}$ at some angle θ, we find three modes: Alfvén waves (phase speed goes to 0 when $\theta = \pi/2$); and fast and slow magnetosonic waves (which become degenerate at $\theta = 0$). Friedrichs diagrams showing phase speed as function of perturbation direction.

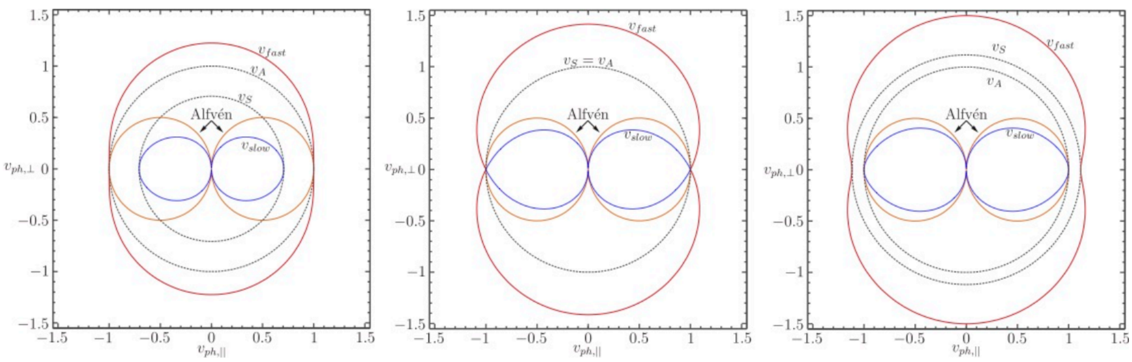


Fig. 10.4: Friedrichs diagrams for $c_s < v_A$ (left), $c_s = v_A$ (middle) and $c_s > v_A$ (right). The phase speed perturbation of the slow magnetoacoustic wave is illustrated in blue, the Alfvén wave in orange and the fast magnetoacoustic wave in red. The dotted lines correspond to the sound and Alfvén speed. The horizontal and vertical axes labelled as $v_{ph,\parallel}$ and $v_{ph,\perp}$ respectively represent the velocity perturbation components along and perpendicular to the equilibrium magnetic field, $\mathbf{B}_0$. [2]

10.4 Instabilities in Plasmas

The presence of magnetic forces can profoundly affect the nature of instabilities in plasmas. For example, we can repeat the derivation of the Rayleigh-Taylor instability including a magnetic field aligned with the interface.

We will not repeat the analysis here, but we find the new dispersion relation is

$$\omega^2 = -kg\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{2}{\mu_0}\frac{(\mathbf{k} \cdot \mathbf{B})^2}{\rho_1 + \rho_2}. \tag{10.65}$$

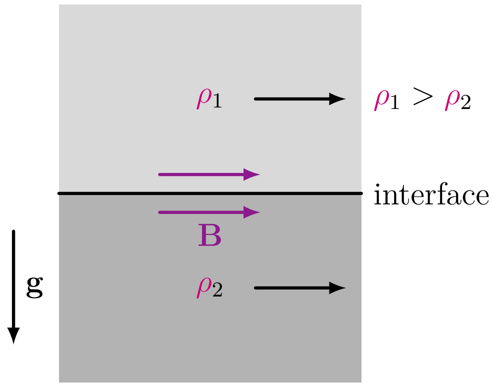


Fig. 10.5: Configuration of fluid interface

For sufficiently small wavelength (high $|\mathbf{k}|$), the second term always wins, giving stable oscillations (Alfvén waves in this case). The interpretation is that magnetic tension forces tend to *stabilise* R-T modes.

10.5 Magnetorotational Instability

We end with a discussion of an MHD instability which is extremely important for accretion disks. We examine the stability of a plasma which is in orbit about a central object.

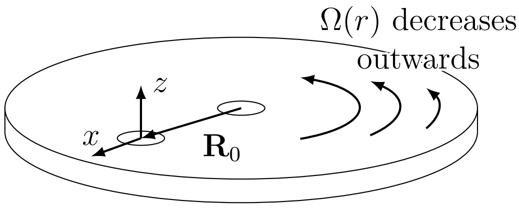


Fig. 10.6: Shear flow in an accretion disk

To uncover the essence of the instability, we simplify as much as possible. We conduct a "local analysis" meaning that we consider the dynamics in some small patch of the rotating flow at $\mathbf{R} = \mathbf{R}_0$, working in the comoving reference frame of the equilibrium flow. We assume that the equilibrium flow has an angular velocity about the central body $\Omega(R)$. We let our local frame of reference rotate at $\Omega(R_0)$ and set up a Cartesian coordinate system with $\hat{\mathbf{z}}$ pointing "upwards" (meaning aligned with the angular velocity Ω) and $\hat{\mathbf{x}}$ pointing outwards (i.e. away from the central body axis). Working in a Lagrangian picture, the momentum equation is:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \underbrace{\frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B}}_{\text{magnetic force}} + \underbrace{2\mathbf{u} \times \mathbf{\Omega}}_{\text{coriolis}} - \underbrace{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})}_{\text{centrifugal}} - \underbrace{R\Omega(R)^2\hat{\mathbf{R}}}_{\text{gravity}}, \quad (10.66)$$

where the last term is an expression of gravity. Further simplifying, let us assume that the flow is cold so that pressure forces are negligible (this assumption can be readily relaxed but will make the analysis more involved). Introducing perturbations and assuming a plane-wave form, we have

$$\begin{aligned} \frac{\mathrm{D}\Delta\mathbf{u}}{\mathrm{D}t} - 2\Delta\mathbf{u}\times\mathbf{\Omega} &= \frac{1}{\mu_0\rho}(\mathbf{B}_0\cdot\nabla)\Delta\mathbf{B} - \Delta x R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}} \\ \Longrightarrow \quad -i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\mathbf{\Omega} &= \frac{i}{\mu_0\rho}B_0k\Delta\mathbf{B} - \Delta x R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}}. \end{aligned} \tag{10.67}$$

The induction equation gives

$$\begin{aligned} \frac{\partial\Delta\mathbf{B}}{\partial t} &= \nabla\times(\Delta\mathbf{u}\times\mathbf{B}_0) = (\mathbf{B}_0\cdot\nabla)\Delta\mathbf{u} \\ \Longrightarrow \quad -i\omega\Delta\mathbf{B} &= ikB_0\Delta\mathbf{u} \\ \Longrightarrow \quad \Delta\mathbf{B} &= -\frac{kB_0}{\omega}\Delta\mathbf{u}, \end{aligned} \tag{10.68}$$

and we can relate Δx and Δu_x;

$$\begin{aligned} \frac{\mathrm{D}\Delta x}{\mathrm{D}t} &= \Delta u_x \\ \Longrightarrow \quad -i\omega\Delta x &= \Delta u_x \\ \Longrightarrow \quad \Delta x &= \frac{i\Delta u_x}{\omega}. \end{aligned} \tag{10.69}$$

So eliminating in favour of $\Delta\mathbf{u}$ in our perturbed form of the momentum equation, we have

$$-i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\mathbf{\Omega} = -\frac{i}{\mu_0\rho}B_0k\frac{kB_0}{\omega}\Delta\mathbf{u} - \frac{i\Delta u_x}{\omega}R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R}\hat{\mathbf{R}}. \tag{10.70}$$

Writing this out in components and noting that $B_0^2/\rho_0\mu_0 = v_A^2$ gives

$$\begin{aligned} \omega^2\Delta u_x - 2i\Delta u_y\Omega\omega &= (kv_A)^2\Delta u_x + \Delta u_x\frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \\ \omega^2\Delta u_y + 2i\Delta u_x\Omega\omega &= (kv_A)^2\Delta u_y, \end{aligned} \tag{10.71}$$

or in matrix form

$$\begin{pmatrix} \omega^2 - (kv_A)^2 - \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} & -2i\omega\Omega \\ 2i\omega\Omega & \omega^2 - (kv_A)^2 \end{pmatrix}\begin{pmatrix} \Delta u_x \\ \Delta u_y \end{pmatrix} = 0. \tag{10.72}$$

We obtain the dispersion relation by setting the determinant of the matrix to zero. This gives

$$\left[\omega^2 - (kv_A)^2 - \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)}\right]\left[\omega^2 - (kv_A)^2\right] - 4\Omega^2\omega^2 = 0. \tag{10.73}$$

Writing as a quadratic in ω^2 gives our final form of the dispersion relation:

$$\omega^4 - \omega^2\left[4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2\left[(kv_A)^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)}\right] = 0. \tag{10.74}$$

If we "turn off" magnetic forces by setting $v_A = 0$, the dispersion relation gives

$$\begin{aligned} \omega^2 &= 4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \\ &= \frac{1}{R^3}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^4\Omega^2\right) \equiv \kappa_R^2 \qquad (10.75) \\ &= \Omega^2 \quad \text{(Keplerian).} \qquad (10.76) \end{aligned}$$

For a Keplerian profile $\Omega^2 = GM/R^3$, or indeed any profile in which the specific angular momentum $R^2\Omega$ increases with radius, this describes local radial oscillations of the flow at the radial epicyclic frequency κ_R. If $\kappa_R^2 < 0$ (specific angular momentum decreasing with radius) then the flow is unstable.

Now turn on magnetic forces, so $v_A > 0$. There will be instability if $\omega^2 < 0$. Considering the basic properties of the dispersion relation, viewed as a quadratic in ω^2, we see that there will be instability if

$$(kv_A)^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} < 0. \tag{10.77}$$

This is the magneto-rotational instability (MRI). For sufficiently weak magnetic field or long wavelength (small k) modes, there will be instability if the angular velocity decreases outwards,

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} < 0 \quad \text{(instability).} \tag{10.78}$$

Magnetic tension will stabilise modes with $k > k_{\mathrm{crit}}$ where

$$(k_{\mathrm{crit}}v_A)^2 = -\frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} \quad (= 3\Omega^2 \text{ for Keplerian}). \tag{10.79}$$

Specialising to the Keplerian case, we find that the fastest growing mode has a growth rate

$$|\omega_{\max}| = \frac{3}{2}\Omega, \tag{10.80}$$

and wavenumber given by

$$k_{\max}v_a \approx \Omega. \tag{10.81}$$

The instability has an interesting property – while the magnetic field is essential for its existence, the maximum growth rate is independent of the magnetic field. Formally, within ideal hydrodynamics, the instability exists as $B_0 \to 0$ but not at $B_0 = 0$. Of course, the wavelength of the mode with the maximum growth rate

$k_{\mathrm{max}} \to \infty$ as $B_0 \to 0$ and so in practice finite viscosity or finite conductivity effects will kill the MRI for sufficiently small B_0.

The MRI is central to the modern theory of accretion disks. MRI drives the turbulence that, as we have described previously, is essential for the transport of angular momentum in an accretion disk.

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Appendix A

Appendix

A.1 Derivation of the Momentum Equation for Disk Evolution

The Navier–Stokes momentum equation, neglecting bulk viscosity, is

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot\left[\eta\left(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} - \frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}\right)\right] + \rho\mathbf{g}. \tag{A.1}$$

Let's look at the ϕ-component in cylindrical coordinates,

$$[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}]\cdot\hat{\mathbf{e}}_\phi = u_R\frac{\partial u_\phi}{\partial R} + u_\phi\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_z}{\partial z} + \frac{u_R u_\phi}{R}, \tag{A.2}$$

which simplifies with $u_z = 0$ and cylindrical symmetry to

$$[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}]\cdot\hat{\mathbf{e}}_\phi = u_R\frac{\partial u_\phi}{\partial R} + \frac{u_r u_\phi}{R}. \tag{A.3}$$

With cylindrical symmetry the pressure gradient in the ϕ direction is zero so we have

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_R\frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) = \left[\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{T}}}\right]\cdot\hat{\mathbf{e}}_\phi, \tag{A.4}$$

where we have defined the tensor

$$\underline{\underline{\mathbf{T}}} = \eta\left(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} - \frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{u})\underline{\underline{\mathbf{I}}}\right). \tag{A.5}$$

We will use the vector gradient tensor in cylindrical coordinates

$$\boldsymbol{\nabla}\mathbf{u} = \begin{bmatrix} \frac{\partial u_R}{\partial R} & \frac{1}{R}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} & \frac{1}{R}\frac{\partial u_\phi}{\partial \phi} + \frac{u_R}{R} & \frac{\partial u_\phi}{\partial z} \\ \frac{\partial u_z}{\partial R} & \frac{1}{R}\frac{\partial u_z}{\partial \phi} & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{A.6}$$

which simplifies with $u_z = 0$ and cylindrical symmetry to

$$\boldsymbol{\nabla}\mathbf{u} = \begin{bmatrix} \frac{\partial u_R}{\partial R} & -\frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} & \frac{u_R}{R} & \frac{\partial u_\phi}{\partial z} \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.7}$$

Then, using $\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{R}\frac{\partial}{\partial R}(Ru_R)$ ($u_z = 0$, cylindrical symmetry), we get

$$\underline{\underline{\mathbf{T}}} = \eta\begin{bmatrix} 2\frac{\partial u_R}{\partial R} - \frac{2}{3R}\frac{\partial}{\partial R}(Ru_R) & \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} & 2\frac{u_R}{R} - \frac{2}{3R}\frac{\partial}{\partial R}(Ru_R) & \frac{\partial u_\phi}{\partial z} \\ \frac{\partial u_R}{\partial z} & \frac{\partial u_\phi}{\partial z} & -\frac{2}{3R}\frac{\partial}{\partial R}(Ru_R) \end{bmatrix}, \tag{A.8}$$

and the ϕ-component of the tensor divergence is

$$\left[\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{T}}}\right]\cdot\hat{\mathbf{e}}_\phi = \frac{\partial}{\partial R}\left[\eta\left(\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R}\right)\right]+\frac{\partial}{\partial z}\left(\eta\frac{\partial u_\phi}{\partial z}\right)+\frac{2}{R}\left[\eta\left(\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R}\right)\right]. \tag{A.9}$$

Putting all together, the ϕ-component of the momentum conservation is

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_R\frac{\partial u_\phi}{\partial R}+\frac{u_R u_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\eta\frac{\partial u_\phi}{\partial R}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial u_\phi}{\partial z}\right)+\frac{1}{R}\frac{\partial}{\partial R}(\eta u_\phi)-2\frac{u_\phi}{R}\frac{\partial \eta}{\partial R}-\eta\frac{u_\phi}{R^2}. \tag{A.10}$$

Integrating over z:

$$\Sigma\left(\frac{\partial u_\phi}{\partial t}+u_R\frac{\partial u_\phi}{\partial R}+\frac{u_R u_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+\frac{1}{R}\frac{\partial}{\partial R}(\nu\Sigma u_\phi)-2\frac{u_\phi}{R}\frac{\partial}{\partial R}(\nu\Sigma)-\frac{\nu\Sigma u_\phi}{R}. \tag{A.11}$$

Adding the continuity equation (9.48) multiplied by Ru_ϕ with the momentum equation multiplied by R we get

$$\begin{aligned}
\frac{\partial}{\partial t}(R\Sigma u_\phi)+\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) &= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+\frac{\partial}{\partial R}(\nu\Sigma u_\phi)-2u_\phi\frac{\partial}{\partial R}(\nu\Sigma)-\frac{\nu\Sigma u_\phi}{R}\\
&= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+2\nu\Sigma\left(\frac{1}{\nu\Sigma}\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{u_\phi}{\nu\Sigma}\frac{\partial(\nu\Sigma)}{\partial R}\right)\\
&\quad -\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{\nu\Sigma u_\phi}{R}\\
&= R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right)+2\nu\Sigma\frac{\partial u_\phi}{\partial R}-\frac{\partial(\nu\Sigma u_\phi)}{\partial R}-\frac{\nu\Sigma u_\phi}{R}\\
&= \frac{1}{R}\frac{\partial}{\partial R}\left[R^2\nu\Sigma\frac{\partial u_\phi}{\partial R}-R\nu\Sigma u_\phi\right]\\
&= \frac{1}{R}\frac{\partial}{\partial R}\left[\nu\Sigma R^3\left(\frac{1}{R}\frac{\partial u_\phi}{\partial R}-\frac{u_\phi}{R^2}\right)\right].
\end{aligned} \tag{A.12}$$

Using $u_\phi = \Omega R$ we get

$$\frac{\partial}{\partial t}(R\Sigma u_\phi)+\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right), \tag{A.13}$$

which is the same as Eq. (9.55).