Part II Astrophysical Fluids

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Preface

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CHAPTER 1

Basic Principles

1.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g., density ρ, pressure p), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples: (Fluids in the Universe)

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

In our discussion, we shall use the concept of a *fluid element*. This is a region of fluid that is

1. Small enough that there are no significant variations of any property q that interests us
$$l_{\text{region}} \ll l_{\text{scale}} \sim \frac{q}{|\boldsymbol{\nabla} q|}. \tag{1.1}$$
2. Large enough to contain sufficient particles to be considered in the continuum limit
$$n l_{\text{region}}^3 \gg 1, \tag{1.2}$$
where n is the number density of particles.

1.2 Collisional and Collisionless Fluids

In a *collisional fluid*, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.

$$l_{\text{region}} \gg \lambda, \tag{1.3}$$

where λ is the mean free path. Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p. We can relate ρ, T and p with an *equation of state*:

$$p = p(\rho, T). \tag{1.4}$$

In a *collisionless fluid*, particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$. So, distribution of particle speeds locally does not correspond to the maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples: (Collisionless Fluids)

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless).

1.2.0.1 Example of ICM

Treat as fully ionised plasma of electrons and ions. The mean free path is set by Coulomb collisions and an analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln \Lambda}, \tag{1.5}$$

where n_e is the electron number density, and Λ is the ratio of largest to smallest impact parameter. For $T \gtrsim 4 \times 10^5\,\text{K}$ we have $\ln \Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \simeq 23\,\text{kpc}\left(\frac{T_e}{10^8\,\text{K}}\right)^2 \left(\frac{n_e}{10^{-3}\,\text{cm}^{-3}}\right)^{-1}. \tag{1.6}$$

So we have

$$\overbrace{R_{\text{galaxy}} \sim \underbrace{\lambda_e}^{\text{collisionless}} \ll R_{\text{cluster}}}_{\text{collisional}} \sim 1\,\text{Mpc}. \tag{1.7}$$

Chapter 2

Formulation of the Fluid Equations

2.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow:

1. *Eulerian description*: one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like
$$\rho(\mathbf{r},t), \quad p(\mathbf{r},t), \quad T(\mathbf{r},t), \quad \mathbf{v}(\mathbf{r},t). \tag{2.1}$$
2. *Lagrangian description*: one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest. The Lagranian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals). These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position $\mathbf{r}$ and time t. At time $t+\delta t$ the element will be at position $\mathbf{r}+\delta\mathbf{r}$. The change in quantity Q of the fluid element is
$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r},t)}{\delta t}\right] \tag{2.2}$$
but
$$Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) = Q(\mathbf{r},t) + \frac{\partial Q}{\partial t}\delta t + \delta\mathbf{r}\cdot\boldsymbol{\nabla}Q + \mathcal{O}\Big(\delta t^2, |\delta\mathbf{r}|^2, \delta t|\delta\mathbf{r}|\Big), \tag{2.3}$$
so
$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{\partial Q}{\partial t} + \frac{\delta\mathbf{r}}{\delta t}\cdot\boldsymbol{\nabla}Q + \mathcal{O}(\delta t, |\delta\mathbf{r}|)\right], \tag{2.4}$$
which gives us
$$\boxed{\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian}\\ \text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian}\\ \text{time derivative}}} + \underbrace{\mathbf{u}\cdot\boldsymbol{\nabla}Q}_{\substack{\text{“convective”}\\ \text{time derivative}}}\ .} \tag{2.5}$$

2.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

- *Streamline*: families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element. Parameterise

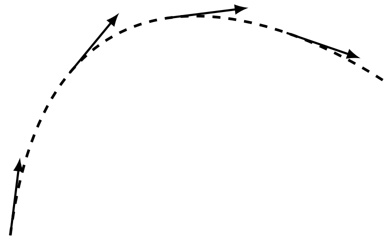

Fig. 2.1: Streamline

streamline by label s such that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}\right), \tag{2.6}$$

and demand $\mathrm{d}\mathbf{r}/\mathrm{d}s \parallel \mathbf{u}$, we get

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}. \tag{2.7}$$

- *Particle paths*: trajectories of individual fluid elements given by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t). \tag{2.8}$$

For small time intervals, particle paths follow streamlines since $\mathbf{u}$ can be treated as approximately steady.

- *Streaklines*: locus of points of all fluid that have passed through a given spatial point in the past.

$$\mathbf{r}(t) = \mathbf{r}_0 \tag{2.9}$$

for some given t in the past

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

2.3 Conservation of Mass

Consider a fixed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. If there are no sources or sinks of mass within the volume, we can say

$$\text{rate of change of mass in } \mathcal{V} = -\text{rate that mass is flowing out across } \mathcal{S} \tag{2.10}$$

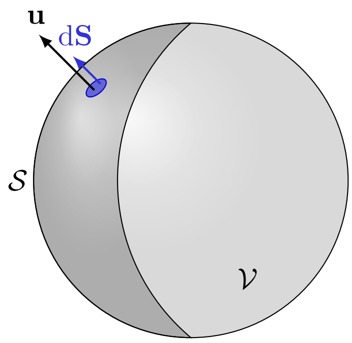

Fig. 2.2: Mass flow of a fluid element

this gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \,\mathrm{d}V = -\int_{\mathcal{S}} \rho\, \mathbf{u}\cdot \mathrm{d}\mathbf{S} \\
\implies &\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\,\mathrm{d}V \\
\implies &\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right)\mathrm{d}V = 0. \qquad (2.11)
\end{aligned}$$

This is true for all volumes $\mathcal{V}$. So we must have the *Eulerian continuity equation*,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0.} \qquad (2.12)$$

The Lagrangian expression of mass conservation is easily found:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\cdot\rho\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad (2.13)$$

Thus we have the *Lagrangian continuity equation*,

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0.} \qquad (2.14)$$

In an incompressible flow, fluid elements maintain a constant density, i.e.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \qquad (2.15)$$

We can now see that incompressible flows must be divergence free, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$.

2.4 Conservation of Momentum

2.4.1 Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of a surface element is

$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}\,. \tag{2.16}$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$\mathrm{d}F_i = \sum_j \sigma_{ij}\,\mathrm{d}S_j\,. \tag{2.17}$$

where σ_{ij} is the stress tensor – the force in direction i acting on a surface with normal along j.

Isotropic pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}. \tag{2.18}$$

2.4.2 Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field $\mathbf{g}$ and internal pressure forces. Let the fluid element have volume $\mathcal{V}$ and surface $\mathcal{S}$.

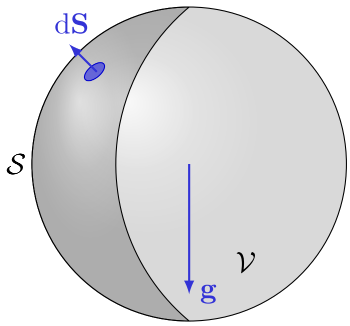

Fig. 2.3: A fluid element subject to gravity

Pressure acting on the surface element gives a force $-p\,\mathrm{d}\mathbf{S}$. The pressure force on an element projected in direction $\hat{\mathbf{n}}$ is $-p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$. So, the net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_{\mathcal{S}} p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V\,. \tag{2.19}$$

The rate of change of momentum of a fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V + \int_{\mathcal{V}} \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V\,. \tag{2.20}$$

In the limit that $\int \mathrm{d}V \to \delta V$ we have

$$\begin{aligned}
&\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}\delta V)\cdot\hat{\mathbf{u}} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}}\\
\implies\quad &\hat{\mathbf{n}}\cdot\mathbf{u}\underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta V)}_{\substack{=0\text{ by mass}\\ \text{conservation}}} + \rho\delta V\hat{\mathbf{n}}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}}\\
\implies\quad &\delta V\hat{\mathbf{n}}\cdot\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla}p - \rho\mathbf{g}\right) = 0. \qquad (2.21)
\end{aligned}$$

This must be true for all $\hat{\mathbf{n}}$ and all δV. So we arrive at the *Lagrangian momentum equation*,

$$\boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g},} \qquad (2.22)$$

or instead we have the *Eulerian momentum equation*,

$$\boxed{\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \rho\mathbf{g},} \qquad (2.23)$$

Now consider the Eulerian rate of change of momentum density $\rho\mathbf{u}$ and introduce a more compact notation

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t(\rho u_i)\\
&= \rho\partial_t u_i + u_i\partial_t\rho\\
&= -\rho u_j\partial_j u_i - \partial_j p\delta_{ij} + \rho g_i - u_i\partial_j(\rho u_j), \qquad (2.24)
\end{aligned}$$

where we have used notation

$$\partial_j \equiv \frac{\partial}{\partial x_j} \qquad (2.25)$$

and employed summation convention (summation over the repeated indices).

This gives

$$\partial_t(\rho u_i) = -\partial_j(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor}\\ \text{due to bulk flow}\\ \text{“Ram Presure”}}} + \underbrace{p\delta_{ij}}_{\substack{\text{stress tensor}\\ \text{due to random}\\ \text{thermal motions}}}) + \rho g_i = -\partial_j\sigma_{ij} + \rho g_i \qquad (2.26)$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$\sigma_{ij} = p\delta_{ij} + \rho u_i u_j. \qquad (2.27)$$

In component free language we write

$$\partial_t(\rho\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underbrace{\left(\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\underline{\mathbf{I}}}\right)}_{\substack{\text{flux of}\\ \text{momentum}\\ \text{density}}} + \rho\mathbf{g}. \qquad (2.28)$$

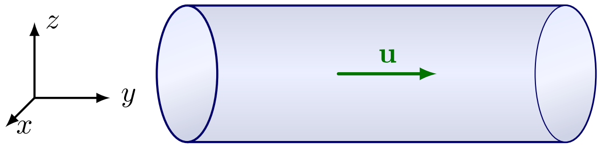

Fig. 2.4: Flow in a pipe

2.4.2.1 Example: Flow in a Pipe in the y-direction

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure.

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{2.29}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

CHAPTER 3

Gravitation

3.1 Basics

Define the gravitational potential Ψ such that the gravitational acceleration $\mathbf{g}$ is

$$\boxed{\mathbf{g} = -\boldsymbol{\nabla}\Psi.} \tag{3.1}$$

If ℓ is some closed loop, we have (using the curl theorem)

$$\oint_\ell \mathbf{g}\cdot \mathrm{d}\mathbf{l} = \int_\mathcal{S} (\boldsymbol{\nabla}\times\mathbf{g})\cdot \mathrm{d}\mathbf{S} = -\int_\mathcal{S} [\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Psi)]\cdot \mathrm{d}\mathbf{S} = 0, \tag{3.2}$$

as curl of any gradient is zero. So gravity is a conservative force – the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point $\mathbf{r}$ to ∞ is

$$-\int_\mathbf{r}^\infty \mathbf{g}\cdot \mathrm{d}\mathbf{l} = \int_\mathbf{r}^\infty \boldsymbol{\nabla}\Psi\cdot \mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r}), \tag{3.3}$$

which is independent of path.

A particular important case is the gravity of a point mass, which has

$$\Psi = -\frac{GM}{r} \qquad \text{if mass at origin} \tag{3.4}$$

$$\Psi = -\frac{GM}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{if mass at location } \mathbf{r}'. \tag{3.5}$$

For a system of point masses we have

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r}-\mathbf{r}_i'|} \tag{3.6}$$

$$\Longrightarrow \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r}-\mathbf{r}_i')}{|\mathbf{r}-\mathbf{r}_i'|^3} \tag{3.7}$$

Replacing $M_i \to \rho_i \delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G\int_\mathcal{V} \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V'\,. \tag{3.8}$$

Take divergence of both sides

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{g} &= -G\int_{\mathcal{V}}\rho(\mathbf{r}')\,\underbrace{\boldsymbol{\nabla}_{\mathbf{r}}\cdot\left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]}_{4\pi\delta(\mathbf{r}-\mathbf{r}')}\mathrm{d}V' \\ &= -4\pi G\int_{\mathcal{V}}\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' \\ &= -4\pi G\rho(\mathbf{r}). \end{aligned} \tag{3.9}$$

Thus we arrive at *Poisson's equation for gravitation*,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{g} = -\nabla^2\Psi = -4\pi G\rho.} \tag{3.10}$$

We can also express Poisson's equation in integral form: for some volume $\mathcal{V}$ bounded by surface $\mathcal{S}$ we have

$$\begin{aligned} \int_{\mathcal{V}}\boldsymbol{\nabla}\cdot\mathbf{g}\,\mathrm{d}V &= -4\pi G\int_{\mathcal{V}}\rho\,\mathrm{d}V \\ \implies \int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} &= -4\pi GM. \end{aligned} \tag{3.11}$$

This is useful for calculating g when the mass distribution obeys some symmetry.

3.1.0.1 Example: Spherical Distribution of Mass

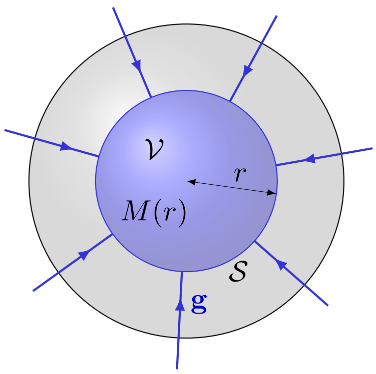

Fig. 3.1: Spherical distribution of mass

By symmetry $\mathbf{g}$ is radial and $|\mathbf{g}|$ is constant over a $r=$ const. shell. So

$$\begin{aligned} \int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} &= -4\pi G\underbrace{M(r)}_{\substack{\text{mass}\\ \text{enclosed}}} \\ \implies \quad -4\pi r^2|\mathbf{g}| &= -4\pi GM(r) \\ \implies \quad |\mathbf{g}| &= \frac{GM(r)}{r^2} \\ \therefore \mathbf{g} &= -\frac{GM(r)}{r^2}\hat{\mathbf{r}}. \end{aligned} \tag{3.12}$$

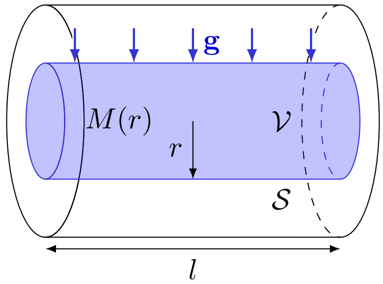

Fig. 3.2: Cylindrical distribution of mass

3.1.0.2 Example: Infinite Cylindrically Symmetric Mass

By symmetry, $\mathbf{g}$ is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \implies \quad -2\pi R l |\mathbf{g}| &= -4\pi G l \underbrace{M(r)}_{\substack{\text{enclosed mass} \\ \text{per unit length}}} \\ \therefore \mathbf{g} &= -\frac{2GM(r)}{r}\hat{\mathbf{r}}. \end{aligned} \tag{3.13}$$

3.1.0.3 Example: Infinite Planar Distribution of Mass

Assume infinite and homogeneous in x and y, $\rho = \rho(z)$.

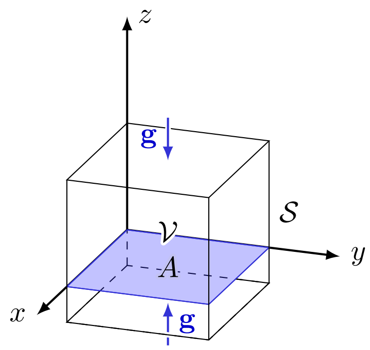

Fig. 3.3: Planar distribution of mass

By symmetry, $\mathbf{g}$ is in the $-\hat{\mathbf{z}}$ direction and is constant on a $z =$ const. surface. So, if we also have reflection symmetry about $z = 0$,

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \implies \quad -2|\mathbf{g}|A &= -4GA \int_{-z}^{z} \rho(z) \, \mathrm{d}z \\ \therefore \mathbf{g} &= -4\pi G \hat{\mathbf{z}} \int_{0}^{z} \rho z \, \mathrm{d}z \,. \end{aligned} \tag{3.14}$$

(For planar distribution of finite height $z_{\max}$, $\mathbf{g}$ is constant for $z \geq z_{\max}$.)

3.2 Potential of a Spherical Mass Distribution

We found that, for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \quad |\mathbf{g}| = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r' = \frac{\mathrm{d}\Psi}{\mathrm{d}r}, \tag{3.15}$$

so,

$$\Psi(r_0) - \Psi(\infty) = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r\,. \tag{3.16}$$

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\begin{aligned} \Psi(r_0) &= -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\Bigg|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r \\ \implies \Psi(r_0) &= -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r\,, \end{aligned} \tag{3.17}$$

where we have made an assumption that $M(r)/r \to 0$ as $r \to \infty$.

We find that $\Psi(r_0)$ is affected by matter outside of r0 through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r.

3.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}, \tag{3.18}$$

and the energy required to take a unit mass to ∞ is $-\Psi$. The energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j, \tag{3.19}$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution,

$$\Omega = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V\,. \tag{3.20}$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r)r^2\Psi(r)\,\mathrm{d}r \tag{3.21}$$

Integrate by parts, choosing parts $u \equiv \Psi$, $\mathrm{d}v \equiv 4\pi\rho r^2\,\mathrm{d}r$ so that $v = \int_0^r 4\pi\rho'^2\,\mathrm{d}r' = M(r)$, then

$$\Omega = \frac{1}{2}\left[M(r)\Psi(r)\Big|_0^\infty - \int_0^\infty M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,\mathrm{d}r\right]. \tag{3.22}$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (the boundary term) is zero. Noting further that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}, \tag{3.23}$$

we conclude

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r\,. \tag{3.24}$$

Integrate again by parts, choosing $u \equiv GM(r)^2$, $\mathrm{d}v \equiv \mathrm{d}r\,/r^2$,

$$\begin{aligned} \Omega &= \underbrace{\frac{1}{2}GM(r)^2\frac{1}{r}\Big|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r \\ \implies \quad \Omega &= -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M\,. \end{aligned} \tag{3.25}$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r}\,\mathrm{d}M(r)\,. \tag{3.26}$$

3.4 The Virial Theorem

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems.

Consider the motion of a cloud of particles (atoms, stars, galaxies, ...). A particle with mass m_i at $\mathbf{r}_i$ is acted upon by a force

$$\mathbf{F}_i = m_i\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2}. \tag{3.27}$$

Consider the 2^{nd} derivative of the scalar moment of inertia, $I_i = m_i r_i^2$,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i r_i^2\right) &= m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_i\cdot\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right) \\ &= m_i\mathbf{r}_i\cdot\frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2 \\ &= \mathbf{r}_i\cdot\mathbf{F}_i + \underbrace{m_i\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2}_{2\times\text{Kinetic Energy } T_i}\,. \end{aligned} \tag{3.28}$$

If $I \equiv \sum_i m_i r_i^2$ then we can sum the previous equation over all particles to give

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{\substack{V\text{, the Virial} \\ \text{(R. Clausius)}}} + 2T. \tag{3.29}$$

In the absence of external forces (i.e. an isolated system), we have that $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ where $\mathbf{F}_{ij}$ is the force exerted on the i^{th} particle by the j^{th} particle. Consider any two particles with m_i and m_j at $\mathbf{r}_i$ and $\mathbf{r}_j$, Newton's 3rd Law says

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \tag{3.30}$$

and so their contribution to the virial is $\mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. We then have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j). \tag{3.31}$$

If there are no non-gravitational interactions except for possibly when $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \quad \text{where} \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \tag{3.32}$$

Thus we have

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}, \tag{3.33}$$

where each term is the work done to separate each pair of particles to infinity against gravity.

And so, $V = \Omega$ and we can use the above to write

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + \Omega. \tag{3.34}$$

If the system is in a steady state ("relaxed"), then $I = \text{const.}$ and we can state the *Virial theorem*

$$\boxed{2T + \Omega = 0.} \tag{3.35}$$

Here, the kinetic energy T has contributions from local flows and random/thermal motions. The Virial theorem implies *gravitational potential sets the "temperature" or velocity dispersion of the system.*

Chapter 4

Equations of State and the Energy Equation

4.1 The Equation of State

In three dimensions, the (scalar) equation of mass conservation and the (vector) equation of momentum conservation can be written as four independent scalar equations. Given appropriate boundary conditions, these must be solved in order to find the density (scalar field), pressure (scalar field), gravitational potential (scalar field), and velocity components (3D vector field); a total of six degrees of freedom.

To close the system of equations, we need additional information. Specifically, we need to find relations between Ψ, p and the other fluid variables such as ρ and $\mathbf{u}$.

$\Psi(r)$ and ρ are related via Poisson's equation (and/or we sometimes consider an externally imposed gravitational potential).

p and the other thermodynamic properties of the system are related by the *equation of state* (EoS). This is only valid for collisional fluids.

Most astrophysical fluids are quite dilute (particle separation much larger than effective particle size) and can be well approximated as ideal gases. The corresponding EoS is

$$p = nk_B T = \frac{k_B}{\mu m_p}\rho T, \tag{4.1}$$

where μ is the mean particle mass in units of the proton mass m_p. (Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars and white dwarfs.)

The ideal gas EoS introduces another scalar field into the description of the fluid, the temperature $T(\mathbf{r}, t)$. In general, we need to solve another PDE that describes heating and cooling processes in order to close the set of equations. We shall move on to this in Section 4.2.

4.1.1 Barotropic Fluids

However, for special cases, we can relate T and ρ without the need to solve a separate energy equation. Fluids for which p is *only* a function of ρ are known as barotropic fluids.

4.1.1.1 Example: Electron Degeneracy Pressure

Important in systems with free electrons that are (relatively) cold and dense.

$$p = \frac{\pi^2 \hbar^2}{5 m_e m_{\rm ion}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}, \quad \text{(non-relativistic)} \tag{4.2}$$

e.g., interiors of white dwarfs, iron core in massive stars, deep interior of Jupiter.

4.1.1.2 Example: Isothermal Case

T is constant so that $p \propto \rho$. Valid when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance.

4.1.1.3 Example: Adiabatic Case

Ideal gas undergoes reversible thermodynamic changes such that

$$p = K\rho^\gamma \tag{4.3}$$

where K, γ are constants.

The first law of thermodynamics is

$$\underbrace{đQ}_{\substack{\text{heat absorbed by}\\ \text{unit mass of fluid}\\ \text{from surrounding}}} = \underbrace{d\mathcal{E}}_{\substack{\text{change in internal}\\ \text{energy of unit}\\ \text{mass of fluid}}} + \underbrace{p\,dV}_{\substack{\text{work done by}\\ \text{unit mass of fluid}}} . \tag{4.4}$$

Here đ is a Pfaffian operator – change in quantity depends on the path taken through the thermodynamic phase space. For an ideal gas, we can write

$$p = \frac{\mathcal{R}_*}{\mu}\rho T, \quad \mathcal{E} = \mathcal{E}(T), \tag{4.5}$$

where $\mathcal{R}_*$ is a modified gas constant.

So, the first law of thermodynamics reads

$$\begin{aligned} đQ &= \frac{d\mathcal{E}}{dT}\,dT + p\,dV \\ &= C_V\,dT + \frac{\mathcal{R}_* T}{\mu V}\,dV\,, \end{aligned} \tag{4.6}$$

where we define specific heat capacity at constant volume as $C_V \equiv d\mathcal{E}/dT$ and have noted that for unit mass we have $\rho = 1/V$.

For a reversible change we have $đQ = 0$, so

$$
\begin{aligned}
& C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\,\mathrm{d}V = 0 \\
\implies \quad & C_V\,\mathrm{d}(\ln T) + + \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}(\ln V) = 0 \\
& \implies V \propto T^{-C_V\mu/\mathcal{R}_*} \qquad (4.7)\\
& \implies p \propto T^{1+C_V\mu/\mathcal{R}_*}. \qquad (4.8)
\end{aligned}
$$

C_V depends on the number of degrees of freedom with which the gas can store kinetic energy, f such that

$$C_V = f\frac{\mathcal{R}_*}{2\mu}. \tag{4.9}$$

Monatomic gas has $f = 3 \implies C_V = 3\mathcal{R}_*/2\mu$; diatomic gas at a few$\times$100 K (two rotational modes excited) has $f = 5 \implies C_v = 5\mathcal{R}_*/2\mu$.

Returning to the ideal gas law,

$$
\begin{aligned}
p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \text{with} \quad \rho = 1/V \quad & \text{for a unit mass of fluid} \\
\implies \quad & pV = \frac{\mathcal{R}_* T}{\mu} \\
\implies \quad & p\,\mathrm{d}V + V\,\mathrm{d}p = \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}T\,, \qquad (4.10)
\end{aligned}
$$

but,

$$
\begin{aligned}
đQ &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V \\
&= \underbrace{\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} + \frac{\mathcal{R}_*}{\mu}\right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}} \mathrm{d}T - V\,\mathrm{d}p\,, \qquad (4.11)
\end{aligned}
$$

so,

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu}. \tag{4.12}$$

Let us define

$$\gamma \equiv \frac{C_p}{C_v} \tag{4.13}$$

so that, for the reversible/adiabatic processes discussed above, we have

$$
\begin{aligned}
p \propto T^{1+C_V\mu/\mathcal{R}_*} \quad &\implies \quad p \propto T^{\gamma/(\gamma+1)} \qquad (4.14)\\
V \propto T^{-C_V\mu/\mathcal{R}_*} \quad &\implies \quad V \propto T^{-1/(\gamma-1)} \qquad (4.15)
\end{aligned}
$$

which we can combine to give

$$p \propto \rho^\gamma \tag{4.16}$$

We say that a fluid element behaves *adiabatically* if $p = K\rho^\gamma$ with $K =$ constant. A fluid is *isentropic* if all fluid elements behave adiabatically with the same value of K. K is related to the entropy per unit mass

4.2 The Energy Equation

In general, the equation of state will not be barotropic and we will need to solve a separate differential equation which follows the heating and cooling processes in the gas, the *energy equation*.

From the first law of thermodynamics we have

$$đQ = \mathrm{d}\mathcal{E} + \underbrace{p\,\mathrm{d}V}_{\mathrm{d}W=-p\mathrm{d}V} \quad \text{in absence of dissipative processes,} \tag{4.17}$$

so,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{\mathrm{D}W}{\mathrm{D}t} + \frac{đQ}{\mathrm{d}t}, \tag{4.18}$$

with

$$\frac{\mathrm{D}W}{\mathrm{D}t} = -p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{4.19}$$

and

$$\frac{đQ}{\mathrm{d}t} \equiv -\dot{Q}_{\mathrm{cool}} \quad \text{rate of cooling per unit mass,} \tag{4.20}$$

therefore,

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}. \tag{4.21}$$

The total energy per unit volume is

$$E = \rho\left(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}}\right), \tag{4.22}$$

so,

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\mathrm{D}\rho}{\mathrm{D}t}\frac{E}{\rho} + \rho\left(\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}\right), \tag{4.23}$$

where,

$$\frac{\mathrm{D}E}{\mathrm{D}t} \equiv \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E \tag{4.24}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \tag{4.25}$$

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi \tag{4.26}$$

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} \equiv \frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Psi. \tag{4.27}$$

Substituting Eqs. (4.24-4.27) into Eq. (4.23) it follows that

$$\begin{aligned}\frac{\mathrm{D}E}{\mathrm{D}t} &= -\frac{E}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p - \rho\mathbf{u}\cdot\Psi + \rho\frac{\partial\Psi}{\partial t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Psi - \frac{p}{\rho}\rho\boldsymbol{\nabla}\cdot\mathbf{u} - \rho\dot{Q}_{\mathrm{cool}} \\ \implies \quad \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E &= -(E+p)\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p + \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}},\end{aligned} \tag{4.28}$$

which gives the *Energy equation*,

$$\boxed{\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \rho\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho\dot{Q}_{\mathrm{cool}}.} \tag{4.29}$$

In many settings, $\partial\Psi/\partial t = 0$, i.e. Ψ depends on position only. If, further, we have no cooling ($\dot{Q}_{\mathrm{cool}} = 0$), then this equation expresses the conservation of energy in which the Eulerian change in total energy density E is driven by the divergence of the enthalpy flux $(E+p)\mathbf{u}$.

4.3 Heating and Cooling Processes

The $\dot{Q}_{\mathrm{cool}}$ term in the energy equation describes processes that locally cool ($\dot{Q}_{\mathrm{cool}} > 0$) or locally heat ($\dot{Q}_{\mathrm{cool}} < 0$) the fluid. There are many such processes and a full discussion of them would be lengthy. Here, we discuss just a small number of important cases.

1. *Cooling by radiation*: energy carried away from fluid by photons.
   - Energy loss by recombination of an ionized gas (line emission as electrons cascade down energy levels);
   - Energy loss by free-free emission (free electrons accelerated in electric fields of ions)
   $$L_{\mathrm{ff}} \propto n_{\mathrm{e}} n_{\mathrm{p}} T^{1/2}. \tag{4.30}$$
   - Collisionally-excited atomic line radiation (electron collides with atom in ground state $\rightarrow$ produces excited atomic state which returns to ground state by emitting a photon with energy χ)
   $$L_{\mathrm{e}} \propto n_{\mathrm{e}} n_{\mathrm{ion}} e^{-\chi/kT} \chi/\sqrt{T} \tag{4.31}$$
   In cold gas clouds with $T \sim 10^4\,\mathrm{K}$, H cannot be excited so cooling occurs through trace species ($\mathrm{O}^+, \mathrm{O}^{++}, \mathrm{N}^+$).

   These are all two-body interactions $\implies$ cooling rate per unit volume proportional to ρ^2 . Recalling that $\dot{Q}_{\mathrm{cool}}$ is defined per unit mass, such processes give $\dot{Q}_{\mathrm{cool}} = \rho f(T)$.
2. *Heating by cosmic rays*: heating and energy transport via high-energy (often relativistic) particles that are diffusing/streaming through the fluid.
   - High energy particles ionise atoms in fluid, excess energy put into freed e^- which ends up as heat in fluid.
   $$\begin{aligned} &\text{ionisation rate per unit volume} \propto \text{CR flux} \times \rho \\ \implies \quad &\dot{Q}_{\mathrm{cool}} \propto \text{CR flux. (independent of } \rho) \end{aligned} \tag{4.32}$$

Combining these cases, we can parametrise $\dot{Q}_{\mathrm{cool}}$ as:

$$\dot{Q}_{\mathrm{cool}} = \underbrace{A\rho T^{\alpha}}_{\substack{\text{radiative}\\ \text{cooling}}} - \underbrace{H}_{\text{CR heating}}, \tag{4.33}$$

where α depends upon the physics of the dominant radiative cooling process.

4.4 Energy Transport Processes

Transport processes move energy through the fluid. Important examples are:

1. *Thermal conduction*: transport of thermal energy by diffusion of the hot e^- into cooler regions. Relevant in, for example

   - Interiors of white dwarfs;
   - Supernova shock fronts;
   - ICM plasma.

   There is also thermal conduction associated with ions, but it is smaller than the electron thermal conduction by a factor of $\sqrt{m_{\text{ion}}/m_e} \sim 43$.

   The energy flux per unit area is

   $$\mathbf{F}_{\text{cond}} = -\kappa \boldsymbol{\nabla} T, \tag{4.34}$$

   where κ is thermal conductivity (computed from kinetic theory).

   The rate of change of E per unit volume is

   $$-\boldsymbol{\nabla} \cdot \mathbf{F}_{\text{cond}} = \kappa \nabla^2 T. \tag{4.35}$$

2. *Convection*: transport of energy due to fluctuating or circulating fluid flows in presence of entropy gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.

3. *Radiation transport*: relevant in optically-thick systems (mean free path of photon much shorter than size of system).

   If scattering opacity dominates, then we have radiative diffusion. If ϵ_{rad} is the energy density of the radiation field, the radiative flux through the fluid is

   $$\mathbf{F}_{\text{rad}} \propto -\boldsymbol{\nabla} \epsilon_{\text{rad}}. \tag{4.36}$$

   The general topic of radiation transport through a fluid flow is very complex and beyond the scope of this course.

Chapter 5

Hydrostatic Equilibrium, Atmospheres and Stars

We now have the full set of equations describing the dynamics of an ideal (inviscid, dilute, unmagnetized) non-relativistic fluid:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \qquad \text{Continuity equation (2.12)} \tag{5.1}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g} \qquad \text{Momentum equation (2.23)} \tag{5.2}$$

$$\nabla^2 \Psi = 4\pi G \rho \qquad \text{Poisson's equation (3.10)} \tag{5.3}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho \frac{\mathrm{d}\Psi}{\mathrm{d}t} - \rho \dot{Q}_{\mathrm{cool}} \qquad \text{Energy equation (4.29)} \tag{5.4}$$

$$E = \rho \left(\frac{1}{2} u^2 + \Psi + \mathcal{E} \right) \qquad \text{Definition of total energy (4.22)} \tag{5.5}$$

$$p = \frac{k_B}{\mu m_p} \rho T \qquad \text{EoS for ideal gas (4.1)} \tag{5.6}$$

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho} \qquad \text{Internal energy for ideal gas} \tag{5.7}$$

We proceed to use those equations to explore astrophysically relevant situations.

This chapter starts with the simplest, but important, case – fluid systems that are in static equilibrium with pressure forces balancing gravity.

5.1 Hydrostatic Equilibrium

A fluid system is in a state of *hydrostatic equilibrium* if

$$\mathbf{u} = 0, \qquad \frac{\partial}{\partial t} = 0. \tag{5.8}$$

Then, the continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{5.9}$$

The momentum equation gives

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi = 0, \tag{5.10}$$

resulting in the *equation of hydrostatic equilibrium*

$$\boxed{\frac{1}{\rho} \boldsymbol{\nabla} p = -\boldsymbol{\nabla} \Psi} \tag{5.11}$$

Assuming a barotropic equation of state $p = p(\rho)$, this system of equations can be solved.

If a system is self-gravitating (rather than having an externally imposed gravitational field), we also have

$$\nabla^2 \Psi = 4\pi G \rho. \tag{5.12}$$

This must be solved together with the equation of hydrostatic equilibrium.

5.1.0.1 Example: Isothermal atmosphere with constant (externally imposed) g

Suppose $\mathbf{g} = -g\hat{\mathbf{z}}$. Then the equation of hydrostatic equilibrium with an isothermal equation of state reads

$$\begin{aligned}
\text{Isothermal} \quad \implies \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad &\implies \quad p = A\rho, \quad A = \text{const.} \\
A\frac{1}{\rho}\boldsymbol{\nabla}\rho &= -\boldsymbol{\nabla}\Psi = -g\hat{\mathbf{z}} \\
\implies \quad \ln\rho &= -\frac{gz}{A} + \text{const.} \\
\therefore \rho &= \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T} z\right),
\end{aligned} \tag{5.13}$$

i.e. exponential atmosphere.

Examples of this is the Earth's atmosphere: $T \sim 300\,\mathrm{K}$ and $\mu \sim 28 \implies$ e-folding $\sim 9\,\mathrm{km}$. The highest astronomical observatories are at $z \sim 4\,\mathrm{km}$, so have ρ and $p \sim 60\%$ of sea level.

5.1.0.2 Example: Vertical density structure of an isothermal, rotationally-supported, geometrically-thin gas disk orbiting a central mass.

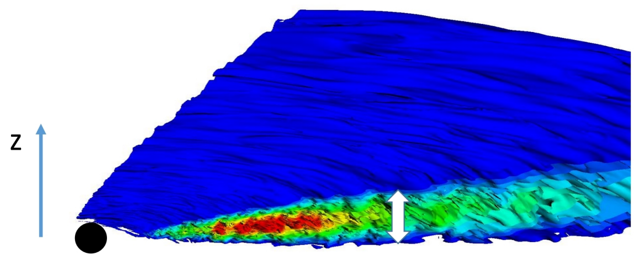

Fig. 5.1: Caption

At a given patch of the disk, transform into a locally co-moving and co-rotating frame. In z-direction, pressure forces balance z-component of gravity,

$$g_z = -\frac{GM}{r^2}\frac{z}{r} \approx -\frac{GMz}{R^3}. \tag{5.14}$$

So, hydrostatic equilibrium gives,

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial z} &= g_z \\
\implies \quad A\frac{1}{\rho}\frac{\partial p}{\partial z} &= -\frac{GM}{R^3}z \\
\implies \quad \ln\rho &= -\frac{GM}{2R^3}z^2 + \text{const.} \\
\implies \quad \rho &= \rho_0 \exp\left(-\frac{GMz^2}{2R^3A}\right) \\
\therefore \rho = \rho_0 \exp\left(-\frac{\Omega^2}{2A}z^2\right) &\quad \text{where} \quad \Omega^2 = \frac{GM}{R^3} \qquad (5.15)
\end{aligned}
$$

5.1.0.3 Example: Isothermal self-gravitating slab

Consider a static, isothermal slab in x and y which is symmetric about $z = 0$ (e.g. two clouds collide and generate a shocked slab of gas between them).

$$
\text{Isothermal} \quad \implies \quad p = \frac{\mathcal{R}_*}{\mu}\rho T \quad \implies \quad p = A\rho, \quad A = \text{const.}
$$

also, $\boldsymbol{\nabla} = \partial/\partial z$ due to symmetry, $p = p(z)$, $\Psi = \Psi(z)$.

Then the equation of hydrostatic equilibrium becomes

$$
\begin{aligned}
A\frac{1}{\rho}\boldsymbol{\nabla}\rho &= -\boldsymbol{\nabla}\Psi \\
\implies \quad A\frac{\mathrm{d}}{\mathrm{d}z}(\ln\rho) &= -\frac{\mathrm{d}\Psi}{\mathrm{d}z} \\
\implies \quad \Psi &= -A\ln(\rho/\rho_0) + \Psi_0 \quad (\rho_0 = \rho(z=0)) \\
\therefore \rho &= \rho_0 e^{-(\Psi-\Psi_0)/A}. \qquad (5.16)
\end{aligned}
$$

Since $A \propto T$, we note that this last equation has the form of a Boltzmann distribution.

Poisson's equation is

$$
\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}. \qquad (5.17)
$$

Let's change variables to $\chi = -(\Psi - \Psi_0)/A$, $Z = z\sqrt{2\pi G\rho_0/A}$ so that Poisson's equation becomes

$$
\begin{aligned}
\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} &= -2e^{\chi} \quad \chi = \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 0 \quad \text{at} \quad Z = 0 \\
\implies \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z}\frac{\mathrm{d}^2\chi}{\mathrm{d}Z^2} &= -2\frac{\mathrm{d}\chi}{\mathrm{d}Z}e^{\chi} \\
\implies \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2\right] &= -2\frac{\mathrm{d}}{\mathrm{d}Z}(e^{\chi}) \\
\implies \quad \left(\frac{\mathrm{d}\chi}{\mathrm{d}Z}\right)^2 &= C_1 - 4e^{\chi}. \qquad (5.18)
\end{aligned}
$$

But we have boundary condition $\mathrm{d}\chi/\mathrm{d}Z = 0$ when $\chi = 0 \implies C_1 = 4$.

$$\therefore \quad \frac{\mathrm{d}\chi}{\mathrm{d}Z} = 2\sqrt{1-e^\chi} \quad \implies \quad \int \frac{\mathrm{d}\chi}{\sqrt{1-e^\chi}} = 2\int \mathrm{d}Z\,. \tag{5.19}$$

Change variables $e^\chi = \sin^2\theta$

$$\implies \quad e^\chi\,\mathrm{d}\chi = 2\sin\theta\cos\theta\,\mathrm{d}\theta \quad \text{or} \quad \mathrm{d}\chi = \frac{2\cos\theta}{\sin\theta}\,\mathrm{d}\theta\,. \tag{5.20}$$

So, we can evaluate the χ integral

$$\begin{aligned}\int \frac{\mathrm{d}\chi}{\sqrt{1-e^\chi}} &= \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \\ &= \int \frac{2\,\mathrm{d}\theta}{\sin\theta} \\ &= \int 2\frac{1}{2}\frac{1+t^2}{t}\,\mathrm{d}\theta \\ &= 2\int \frac{\mathrm{d}t}{t} \\ &= 2\ln t + C_2,\end{aligned} \tag{5.21}$$

by setting

$$t = \tan\frac{\theta}{2} \quad \implies \quad \mathrm{d}t = \frac{1}{2}(1+t^2)\,\mathrm{d}\theta\,, \tag{5.22}$$

and by noting

$$\sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2t}{1+t^2} = e^{\chi/2}. \tag{5.23}$$

So, Poisson's equation becomes

$$2\ln t = 2Z + C_2. \tag{5.24}$$

Now, $\chi = 0$ at $Z = 0 \implies \theta = \pi/2$, $t = 1 \implies C_2 = 0$, so $t = e^Z$

$$\implies \quad \sin\theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}. \tag{5.25}$$

This gives

$$\boxed{\begin{aligned}\Psi - \Psi_0 &= 2A\ln\left[\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)\right] & (5.26)\\ \rho &= \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)} & (5.27)\end{aligned}}$$

5.2 Stars as Self-Gravitating Polytropes

Consider a spherically-symmetric self-gravitating system in hydrostatic equilibrium; from now on we will refer to this as a "star". We have

$$\begin{aligned} &\boldsymbol{\nabla} p = -\rho\boldsymbol{\nabla}\Psi \\ \implies \quad &\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r}. \quad \text{(spherical polar)}\end{aligned} \tag{5.28}$$

Now, $\rho > 0$ within a star $\implies$ p is a monotonic function of Ψ. Also

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad \implies \quad \rho = -\frac{\mathrm{d}p}{\mathrm{d}\Psi}. \tag{5.29}$$

So ρ is a monotonic function of Ψ.

$$\therefore p = p(\Psi), \quad \rho = \rho(\Psi) \quad \implies \quad p = p(\rho), \tag{5.30}$$

i.e. non-rotating stars are barotropes!

A barotropic EoS can be written as

$$p = K\rho^{1+1/n}, \tag{5.31}$$

where in general $n = n(\rho)$. When $n =$ constant, we say that we have a *polytropic* EoS and the structure is called a *polytrope*. Real stars are in fact well approximated as polytropes.

It is important to note that in general we will have

$$1 + \frac{1}{n} \neq \gamma. \tag{5.32}$$

We only have $1 + 1/n = \gamma$ (i.e. $p \propto \rho^\gamma$) if the star is isentropic (constant entropy throughout) due to, for example, mixing by convective motions.

Assuming a polytropic EoS, the equation of hydrostatic equilibrium gives

$$\begin{aligned} -\boldsymbol{\nabla}\Psi &= \frac{1}{\rho}\boldsymbol{\nabla}\left(K\rho^{1+1/n}\right) = (n+1)\boldsymbol{\nabla}\left(K\rho^{1/n}\right) \\ \implies \quad \rho &= \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n, \quad \Psi_T \equiv \Psi \text{ where } \rho = 0, \text{ the surface.} \end{aligned} \tag{5.33}$$

If the central density is ρ_c and central potential is Ψ_c, we have

$$\rho_c = \left(\frac{\Psi_T - \Psi_c}{(n+1)K}\right)^n, \tag{5.34}$$

so we can write,

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n. \tag{5.35}$$

Feeding this into Poisson's equation gives

$$\nabla^2\Psi = 4\pi G\rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n \tag{5.36}$$

Define $\theta = (\Psi_T - \Psi)/(\Psi_T - \Psi_c)$, we then get

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n. \tag{5.37}$$

In spherical polars, this becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n. \tag{5.38}$$

Defining a scaled radial coordinate $\xi = r\sqrt{(4\pi G\rho_c)/(\Psi_T - \Psi_c)}$, we finally get the *Lane-Emden equation of index n*,

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.} \tag{5.39}$$

The appropriate boundary conditions for the Lane-Emden equation are $\theta = 1$, $\mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$. (Zero force at $\xi = 0$, enclosed mass $\to 0$ as $\xi \to 0$.)

The Lane-Emden equation can be solved analytically for $n = 0$, 1 and 5; otherwise solve numerically.

5.2.1 Solution for $n = 0$

This is a somewhat singular case, physically corresponding to a fluid that is at constant density and incompressible.

$$\begin{aligned}
\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\theta^n = -1 \\
\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) &= -\xi^2 \\
\Longrightarrow \quad \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} &= -\frac{1}{3}\xi^3 - C \\
\therefore \theta &= -\frac{\xi^2}{6} + \frac{C}{\xi} + D.
\end{aligned} \tag{5.40}$$

We need $\theta = 1$ at $\xi = 0 \implies C = 0, D = 1$ and hence

$$\theta = 1 - \frac{\xi^2}{6}. \tag{5.41}$$

5.3 Isothermal Spheres (Case $n \to \infty$)

The isothermal case $p = K\rho$ corresponds to $n \to \infty$). Let's combine

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r} \quad &\text{and} \quad p = K\rho \\
\Longrightarrow \quad \frac{\mathrm{d}\Psi}{\mathrm{d}r} &= -\frac{K}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} \\
\Longrightarrow \quad \Psi - \Psi_c &= -K\ln(\rho/\rho_c).
\end{aligned} \tag{5.42}$$

From Poisson's equation

$$\begin{aligned}
\nabla^2\Psi &= 4\pi G\rho \\
\Longrightarrow \quad \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= 4\pi G\rho \\
\Longrightarrow \quad \frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) &= -4\pi G\rho.
\end{aligned} \tag{5.43}$$

Let $\rho = \rho_c e^{-\psi}$ (defining $\psi = \Psi/K$, and $\Psi_c = 0$), we set

$$r = a\xi, \quad a = \sqrt{\frac{K}{4\pi G\rho_c}}, \tag{5.44}$$

then,

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) = e^{-\psi}, \tag{5.45}$$

with $\psi = \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0$ at $\xi = 0$.

This replaces the Lane-Emden equation in the case where the system is isothermal.

At large radii, this has solutions of the form $\rho \propto r^{-2}$, so the enclosed mass $\propto r$. Thus, the mass of an isothermal sphere of self-gravitating gas tends to ∞ as the radius tends to ∞. This is why we cannot adopt our usual convention of defining $\Psi = 0$ at ∞.

So, to be physical, isothermal spheres need to be truncated at some finite radius. There needs to be some confining pressure by an external medium. These are called Bonnor-Ebert spheres, whose density profile depends on ξ_{cut}. E.g. dense gas cores in molecular clouds are well fitted by such Bonnor-Ebert spheres.

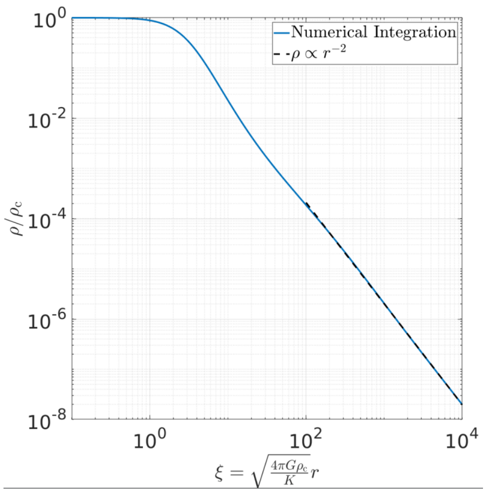

Fig. 5.2: Caption

5.4 Scaling Relations

In many circumstances, stars behave as polytropes, e.g. fully convective stars with $p(\rho)$ close to the adiabatic relation. In such a star, assuming the gas is monatomic with $\gamma = 5/3$, we have $p = K\rho^{5/3} \implies n = 3/2$.

Consider a set of stars which share a given polytropic index n and a given constant K. They will then form a one-parameter family characterised by their central density ρ_c.

Thus one can find how mass and radius vary as a function of ρ_c and, eliminating ρ_c, obtain *scaling relations* relating the mass and radius.

All stars with given n have the same $\theta(\xi)$ since the Lane-Emden equation does not depend on ρ_c. Recall the relations

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K}\right)^n \quad\implies\quad \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n} \tag{5.46}$$

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}r \quad\implies\quad \xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{(1+n)K}}r \tag{5.47}$$

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n = \rho_c\theta^n. \tag{5.48}$$

The surface of the polytrope is at $\xi = \xi_{\max}$ defined as location where we have $\theta(\xi) = 0$. Let rmax be the corresponding physical radius. Then the total mass of the polytrope is

$$\begin{aligned} M &= \int_0^{r_{\max}} 4\pi r^2\rho\,\mathrm{d}r \\ &= 4\pi\rho_c\left[\frac{4\pi G\rho_c^{1-1/n}}{(1+n)K}\right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n\xi^2\,\mathrm{d}\xi}_{\substack{\text{same for all}\\ \text{polytrope of index } n}} \\ \therefore M &\propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)}. \end{aligned} \tag{5.49}$$

From the definition of ξ above in Eq. (5.47), we also know that

$$r_{\max} \propto \rho_c^{\frac{1}{2}\left(\frac{1}{n}-1\right)}. \tag{5.50}$$

Eliminating ρ_c thus gives the *mass-radius relation for polytropic stars*

$$\boxed{M \propto R^{\frac{3-n}{1-n}}.} \tag{5.51}$$

For $\gamma = 5/3$, $n = 3/2$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EoS is due to e^- degeneracy pressure rather than gas pressure). But for most main-sequence stars we observe $M \propto R$.

The reason is that stars do not share the same polytropic constant K. Let's write the temperature at the core in terms of the central density and K

$$\left.\begin{aligned} p &= K\rho^{1+1/n} \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \quad\implies\quad T_c = \frac{\mu K}{\mathcal{R}_*}\rho_c^{1/n}. \tag{5.52}$$

Nuclear reactions in the core tend to keep T_c similar in the cores of stars of different masses. So we can say that

$$K \propto \rho_c^{-1/n}. \tag{5.53}$$

Substituting this into the above expression for mass gives

$$M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \quad \Longrightarrow \quad M \propto R. \tag{5.54}$$

When can the $K =$ const. relation be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust. The time to adjust to new hydrostatic equilibrium is roughly the time for a sound wave (Chapter 6) to propagate across the star, which is less than a day for the Sun,

$$t_{\rm hd} \sim R/c_s < 1\,\text{day}. \tag{5.55}$$

The thermal timescale, on which a star can lose a significant amount of energy, is

$$t_{\rm th} \sim \frac{\text{energy content of the star}}{\text{luminosity}} \sim \frac{GM^2}{RL}, \tag{5.56}$$

which is $\sim 30\,$Myr for the Sun. So, mass loss/gain is followed by rapid readjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time

5.4.0.1 Example: Spherical rotating star

A spherical rotating star with angular velocity Ω gains non-rotating mass. How does Ω evolve?

Conservation of angular momentum gives $MR^2\Omega =$ const. So, if $\Omega \to \Omega + \Delta\Omega$ then

$$\begin{aligned} &MR^2\Delta\Omega + \Omega\Delta(MR^2) = 0 \\ \Longrightarrow \quad &\frac{\Delta\Omega}{\Omega} = -\frac{\Delta(MR^2)}{MR^2}. \end{aligned} \tag{5.57}$$

But we can use

$$R \propto M^{\frac{1-n}{3-n}} \tag{5.58}$$

to say

$$\begin{aligned} &\frac{\Delta\Omega}{\Omega} \propto -\Delta\Big(M^{\frac{5-3n}{3-n}}\Big) \\ \Longrightarrow \quad &\frac{\Delta\Omega}{\Omega} \propto -\Big(\frac{5-3n}{3-n}\Big)\Delta M, \end{aligned} \tag{5.59}$$

so,

$$\Delta M > 0 \quad \Longrightarrow \quad \begin{cases} \Delta\Omega < 0 \quad \text{if} \quad \dfrac{5-3n}{3-n} > 0 \quad (\text{e.g. } n = \dfrac{3}{2}) \text{ Spin down} \\ \Delta\Omega > 0 \quad \text{if} \quad \dfrac{5-3n}{3-n} < 0 \quad (\text{e.g. } n = \dfrac{3}{2}) \text{ Spin up} \end{cases}. \tag{5.60}$$

5.4.0.2 Example: Star in a binary system

A star in a binary system loses mass to its companion. The donor star loses mass, $\Delta M < 0$. So since $R \propto M^{(1-n)/(3-n)}$, the radius will increase if $1 < n < 3$.

So there is the potential for unstable (runaway) mass transfer (but need to look at evolution of the size of the Roche lobe to conclusively decide whether the process is unstable).

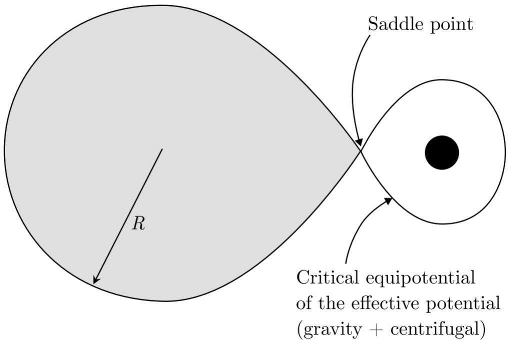

Fig. 5.3: Roche lobe overflow

Chapter 6

Sound Waves, Supersonic Flows and Shock Waves

6.1 Sound Waves

We now start discussion of how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{6.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{6.2}$$

The equilibrium around which we will perturb is

$$\begin{aligned} \rho &= \rho_0 \quad \text{(uniform and constant)} \\ p &= p_0 \quad \text{(uniform and constant)} \\ \mathbf{u} &= \mathbf{0}. \end{aligned} \tag{6.3}$$

We consider small perturbations and write in Lagrangian terms (Lagrangian meaning the change of quantities are for a *given fluid element*)

$$\begin{aligned} p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u}. \end{aligned} \tag{6.4}$$

The relation between Lagrangian and Eulerian perturbations is:

$$\underbrace{\delta \rho}_{\substack{\text{Eulerian} \\ \text{perturbation}}} = \underbrace{\Delta \rho}_{\substack{\text{Lagrangian} \\ \text{perturbation}}} - \underbrace{\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \rho_0}_{\substack{\text{Element displacement dot} \\ \text{Gradient of unperturbed state}}} \tag{6.5}$$

In the present example, $\boldsymbol{\nabla} \rho_0 = 0$ and so $\delta \rho = \Delta \rho$, but the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we will address later.

Substitute the perturbations into the fluid equations and ignore terms that are 2^{nd} order (or higher) in the perturbed quantities:

Start with continuity equation:

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla}\cdot[(\rho_0+\Delta\rho)\Delta\mathbf{u}] = 0$$

$$\implies \underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + \frac{\partial \Delta\rho}{\partial t} + \underbrace{\boldsymbol{\nabla}\rho_0\cdot\Delta\mathbf{u}}_{=0} + \underbrace{\boldsymbol{\nabla}(\Delta\rho)\cdot\Delta\mathbf{u}}_{2^{\text{nd}}\text{ order}} + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) + \underbrace{\Delta\rho\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})}_{2^{\text{nd}}\text{ order}} = 0$$

$$\therefore \frac{\partial}{\partial t}(\Delta\rho) + \rho_0\boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \tag{6.6}$$

And similarly, the momentum equation:

$$\frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta p)$$

$$\therefore \frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\frac{\boldsymbol{\nabla}(\Delta p)}{\rho_0}, \quad \text{assuming barotropic EoS.} \tag{6.7}$$

Now, taking the partial derivative of Eq. (6.6) with respect to time,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(\Delta\rho) &= -\rho_0\frac{\partial}{\partial t}[\boldsymbol{\nabla}\cdot(\Delta\mathbf{u})]\\ &= -\rho_0\boldsymbol{\nabla}\cdot\left[\frac{\partial}{\partial t}(\Delta\mathbf{u})\right]\\ &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho).\end{aligned} \tag{6.8}$$

Thus we arrive at the wave equation

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}\nabla^2(\Delta\rho).} \tag{6.9}$$

This admits solutions of the form $\Delta\rho = \Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Substituting into the wave equation we get

$$\begin{aligned}(-i\omega)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}(ik)^2\Delta\rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\\ \therefore \omega^2 &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}k^2.\end{aligned} \tag{6.10}$$

The (phase) speed of the wave is $v_p = \omega/k$, so the sound wave travels at speed determined by the derivative of $p(\rho)$

$$c_s = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}}. \tag{6.11}$$

Consider a 1D wave and substitute

$$\begin{aligned}\Delta\rho &= \Delta\rho_0 e^{i(kx-\omega t)}\\ \Delta u &= \Delta u_0 e^{i(kx-\omega t)}\end{aligned} \tag{6.12}$$

into Eq. (6.6). We get

$$
\begin{aligned}
&-i\omega\Delta\rho + \rho_0 ik\Delta u = 0 \\
\implies \qquad & \Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_s\frac{\Delta\rho}{\rho_0}.
\end{aligned} \tag{6.13}
$$

So we learn that

- Fluid velocity and density perturbations are in phase (since $\Delta u/\Delta\rho \in \mathbb{R}$);
- A disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small, since

$$
\Delta u_0 = c_s\frac{\Delta\rho_0}{\rho_0} \ll c_s. \tag{6.14}
$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes acceleration of the fluid elements, this induces further density perturbations, making disturbances propagate.

Sound speed depends on how the pressure forces react to density changes. If the EoS is "stiff" (i.e. high $\mathrm{d}p/\mathrm{d}\rho$), then restoring force is large and propagation is rapid.

6.1.1 Examples of $\mathrm{d}p/\mathrm{d}\rho$

Notes about these two examples:

- We see that $c_{s,\mathrm{I}}$ and $c_{s,\mathrm{A}}$ differ by only $\sqrt{\gamma}$;
- Thermal behaviour of the perturbations does *not* have to be the same as that of the unperturbed structure! E.g. in the Earth's atmosphere, the background is approximately isothermal but sound waves are adiabatic.
- Waves for which c_s is not a function of ω are called non-dispersive. The shape of a wave packet is preserved.

6.1.1.1 Isothermal Case

$$
c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_T \tag{6.15}
$$

In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant T. Then

$$
\begin{aligned}
p &= \frac{\mathcal{R}_*}{\mu}\rho T \\
\therefore c_{s,\mathrm{I}} &= \sqrt{\frac{\mathcal{R}_* T}{\mu}}.
\end{aligned} \tag{6.16}
$$

6.1.1.2 Adiabatic Case

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S \tag{6.17}$$

No heat exchange between fluid elements; compressions heat up and rarefactions cool down from $p\,\mathrm{d}V$ work. So

$$\begin{aligned} p &= K\rho^{\gamma} \\ \implies \quad \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S &= \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho} \\ \therefore\ c_{s,\mathrm{A}} &= \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}}. \end{aligned} \tag{6.18}$$

6.2 Sound Waves in a Stratified Atmosphere

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$.

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on z-dependent terms, taking $\mathbf{u} = u\hat{\mathbf{z}}$. The continuity and momentum equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{6.19}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{6.20}$$

and the equilibrium is

$$\begin{aligned} u_0 &= 0 \\ \rho_0(z) &= \tilde{\rho} e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu} \\ p_0(z) &= \frac{\mathcal{R}_* T}{\mu}\rho_0(z) = \tilde{p} e^{-z/H}. \end{aligned} \tag{6.21}$$

Consider a Lagrangian perturbation:

$$\begin{aligned} u &\to \Delta u \\ \rho_0 &\to \rho_0 + \Delta\rho \\ p_0 &\to p_0 + \Delta p. \end{aligned} \tag{6.22}$$

Remember from Eq. (6.5) that $\delta\rho = \Delta\rho - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho$. So we have

$$\left.\begin{aligned} \delta\rho &= \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z \frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u \end{aligned}\right\} \quad \text{Eulerian to Lagrangian perturbation relation,} \tag{6.23}$$

and

$$\Delta \mathbf{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\xi}}_{2^{\text{nd}} \text{ order}} \approx \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{6.24}$$

Substituting perturbed quantities into the Eulerian continuity equation (2.12)

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial t}[(\rho_0 + \delta\rho)\delta u_z] = 0 \\
\implies \quad & \frac{\partial}{\partial t}\left(\rho_0 + \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z}\right) + \frac{\partial}{\partial z}(\rho_0 \Delta u_z) = 0 \quad (\text{ignoring } 2^{\text{nd}} \text{ order terms}) \\
\implies \quad & \underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + \frac{\partial \Delta\rho}{\partial t} - \frac{\partial \xi_z}{\partial t}\frac{\partial \rho_0}{\partial z} - \underbrace{\xi_z \frac{\partial}{\partial t}\frac{\partial \rho_0}{\partial z}}_{=0} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\
\implies \quad & \frac{\partial \Delta\rho}{\partial t} - \underbrace{\Delta u_z}_{\partial \xi_z/\partial t} \frac{\partial \rho_0}{\partial z} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\
& \therefore \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0.
\end{aligned} \tag{6.25}$$

A similar calculation for the momentum equation gives

$$\begin{aligned}
\frac{\partial \Delta u_z}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z} \\
\implies \frac{\partial \Delta u_z}{\partial t} &= -\frac{c_u^2}{\rho_0}\frac{\partial \Delta \rho}{\partial z}, \quad c_u^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}.
\end{aligned} \tag{6.26}$$

To perform this calculation (which we leave as an exercise!), you need a relation that is obtained from the Lagrangian continuity equation:

$$\begin{aligned}
&\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \\
\implies \quad & \Delta\rho + \left(\rho_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t}\right)\Delta t = 0 \quad (\text{integrating over a short time interval } \Delta t) \\
& \therefore \Delta\rho + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0.
\end{aligned} \tag{6.27}$$

Let's now derive the wave equation and dispersion relation. Take the partial derivative of Eq. (6.25) with respect to time

$$\begin{aligned}
\frac{\partial^2 \Delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial z}\left(\frac{\partial \Delta u_z}{\partial t}\right) &= 0 \\
\implies \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}\right) &= 0,
\end{aligned} \tag{6.28}$$

where the last step involved substitution from Eq. (6.26). If the medium is isothermal, then c_u is independent of z. So,

$$\begin{aligned}
&\frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{c_u^2}{\rho_0}\frac{\partial^2 \Delta\rho}{\partial z^2} + \rho_0 \frac{c_u^2}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z} = 0 \\
\therefore \quad & \underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z}}_{\substack{\text{extra piece associated} \\ \text{with stratification}}} = 0.
\end{aligned} \tag{6.29}$$

Now,

$$\begin{aligned}\frac{\partial \rho_0}{\partial z} &= \frac{\partial}{\partial z}\Big(\tilde{\rho} e^{-z/H}\Big) \\ &= -\frac{1}{H}\tilde{\rho} e^{-z/H} \\ &= -\frac{\rho_0}{H}.\end{aligned} \tag{6.30}$$

So,

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_u^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0. \tag{6.31}$$

Look for solutions of the form $\Delta\rho \propto e^{i(kz-\omega t)}$,

$$\implies \qquad -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{ik}{H}, \tag{6.32}$$

which reveals the *dispersion relation*

$$\boxed{\omega^2 = c_u^2\Big(k^2 - \frac{ik}{H}\Big).} \tag{6.33}$$

We can also write this as

$$k^2 - \frac{ik}{H} - \frac{\omega^2}{c_u^2} = 0, \tag{6.34}$$

and solve the quadratic for $k(\omega)$ to give

$$k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}. \tag{6.35}$$

Let's take $\omega \in \mathbb{R}$. We have two cases to examine if we wish to understand the implications of this dispersion relation.

6.2.1 Case I: $\omega > c_u/2H$

Examine the real and imaginary parts of k:

$$\mathrm{Im}\, k = \frac{1}{2H} \tag{6.36}$$

$$\mathrm{Re}\, k = \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.37}$$

So the density perturbation is

$$\Delta\rho \propto \underbrace{e^{-z/2H}}_{1}\underbrace{e^{i\left(\pm\sqrt{(\omega/c_u)^2-(1/2H)^2}z-\omega t\right)}}_{2} \tag{6.38}$$

corresponding to

1. Exponentially decaying amplitude with increasing height;

2. Wave with phase velocity

$$v_{\rm ph} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \tag{6.39}$$

where $v_{\rm ph}$ is function of ω, meaning that the wave is dispersive. A wave packet consisting of different ω's will change shape as it propagates.

As before, we can relate Δu to $\Delta\rho$:

$$\Delta u_z = \frac{\Delta\rho}{\rho_0}\frac{\omega}{k}, \tag{6.40}$$

with

$$\Delta\rho \propto e^{-z/2H} \tag{6.41}$$

$$\rho_0 \propto e^{-z/H}, \tag{6.42}$$

giving

$$\Delta u_z \propto e^{+z/2H}, \quad \frac{\Delta\rho}{\rho_0} \propto e^{+z/2H}. \tag{6.43}$$

Thus the perturbed velocity and the fractional density variation both *increase* with height. In the absence of dissipation (e.g. viscosity), the kinetic energy flux ($\propto \Delta\rho\Delta u$) is conserved and the amplitude of the wave increases until

$$\Delta u \sim c_s, \quad \frac{\Delta\rho}{\rho_0} \sim 1, \tag{6.44}$$

where the linear treatment breaks down and the sound wave "steepens" into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere!

6.2.2 Case II: $\omega < c_u/2H$

In this case, we find that k is purely imaginary. So,

$$\Delta\rho \propto e^{-|k|z}e^{i\omega t}. \tag{6.45}$$

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength, giving rise to reflections.

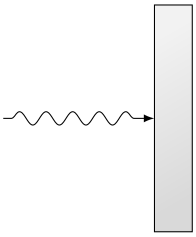

Fig. 6.1: Waves at boundary $x = 0$.

6.3 Transmission of Sound Waves at Interfaces

Consider two non-dispersive media with a boundary at $x = 0$. Suppose we have a sound wave travelling from $x < 0$ to $x > 0$. Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by r and t the amplitude of the reflected and transmitted waves, respectively.

At the boundary $x = 0$, variables must be single valued and the accelerations are finite, thus oscillations in the second medium must have the same frequency,

$$\omega_1 = \omega_2 = \omega_3 = \omega. \tag{6.46}$$

The reflected wave is in the same medium as the incident so

$$k_3 = -k_1. \quad \text{(phase speed reversed)} \tag{6.47}$$

The amplitude of a sound wave is continuous at $x = 0$ hence

$$1 + r = t, \tag{6.48}$$

and the derivative of the amplitude is continuous at $x = 0$ thus

$$k_1(1 - r) = k_2 t. \tag{6.49}$$

We can combine these relations to get

$$t = \frac{2k_1}{k_1 + k_2}, \quad r = \frac{k_1 - k_2}{k_1 + k_2}. \tag{6.50}$$

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

1. If $c_{s,2} > c_{s,1}$, then $k_2 < k_1 \implies r > 0$, i.e reflected wave in phase with incident;
2. If $c_{s,2} < c_{s,1}$, then $r < 0 \implies$ reflected wave is π out of phase with incident wave;
3. If $c_{s,2} \ll c_{s,1}$, then $k_2 \gg k_1 \implies t \ll 1$, i.e. wave almost completely reflected.

6.4 Supersonic Fluids and Shocks

Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding c_s. The linear theory applied to sound waves breaks down.

Appendix **A**

Appendix