Part II Astrophysical Fluids

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Preface

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Chapter 1

Basic Principles

1.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g., density ρ, pressure p), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples: (Fluids in the Universe)

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

In our discussion, we shall use the concept of a *fluid element*. This is a region of fluid that is

1. Small enough that there are no significant variations of any property q that interests us
$$l_{\rm region} \ll l_{\rm scale} \sim \frac{q}{|\boldsymbol{\nabla} q|}. \tag{1.1}$$
2. Large enough to contain sufficient particles to be considered in the continuum limit
$$n l_{\rm region}^3 \gg 1, \tag{1.2}$$
where n is the number density of particles.

1.2 Collisional and Collisionless Fluids

In a *collisional fluid*, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.

$$l_{\text{region}} \gg \lambda, \tag{1.3}$$

where λ is the mean free path. Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p. We can relate ρ, T and p with an *equation of state*:

$$p = p(\rho, T). \tag{1.4}$$

In a *collisionless fluid*, particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$. So, distribution of particle speeds locally does not correspond to the maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples: (Collisionless Fluids)

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless).

1.2.0.1 Example of ICM

Treat as fully ionised plasma of electrons and ions. The mean free path is set by Coulomb collisions and an analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln \Lambda}, \tag{1.5}$$

where n_e is the electron number density, and Λ is the ratio of largest to smallest impact parameter. For $T \gtrsim 4 \times 10^5\,\text{K}$ we have $\ln \Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \simeq 23\,\text{kpc}\left(\frac{T_e}{10^8\,\text{K}}\right)^2 \left(\frac{n_e}{10^{-3}\,\text{cm}^{-3}}\right)^{-1}. \tag{1.6}$$

So we have

$$\overbrace{R_{\text{galaxy}} \sim \lambda_e}^{\text{collisionless}}\!\!\!\!\underbrace{ \ll R_{\text{cluster}}}_{\text{collisional}} \sim 1\,\text{Mpc}. \tag{1.7}$$

CHAPTER 2

Formulation of the Fluid Equations

2.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow:

1. *Eulerian description*: one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like

$$\rho(\mathbf{r},t), \quad p(\mathbf{r},t), \quad T(\mathbf{r},t), \quad \mathbf{v}(\mathbf{r},t). \tag{2.1}$$

2. *Lagrangian description*: one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest. The Lagranian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals). These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position $\mathbf{r}$ and time t. At time $t+\delta t$ the element will be at position $\mathbf{r}+\delta\mathbf{r}$. The change in quantity Q of the fluid element is

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{Q(\mathbf{r}+\delta\mathbf{r},t+\delta t) - Q(\mathbf{r},t)}{\delta t}\right] \tag{2.2}$$

but

$$Q(\mathbf{r}+\delta\mathbf{r},t+\delta t) = Q(\mathbf{r},t) + \frac{\partial Q}{\partial t}\delta t + \delta\mathbf{r}\cdot\boldsymbol{\nabla}Q + \mathcal{O}\Big(\delta t^2, |\delta\mathbf{r}|^2, \delta t|\delta\mathbf{r}|\Big), \tag{2.3}$$

so

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t\to 0}\left[\frac{\partial Q}{\partial t} + \frac{\delta\mathbf{r}}{\delta t}\cdot\boldsymbol{\nabla}Q + \mathcal{O}(\delta t, |\delta\mathbf{r}|)\right], \tag{2.4}$$

which gives us

$$\boxed{\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian}\\ \text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian}\\ \text{time derivative}}} + \underbrace{\mathbf{u}\cdot\boldsymbol{\nabla}Q}_{\substack{\text{“convective”}\\ \text{time derivative}}}\ .} \tag{2.5}$$

2.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

- *Streamline*: families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element. Parameterise

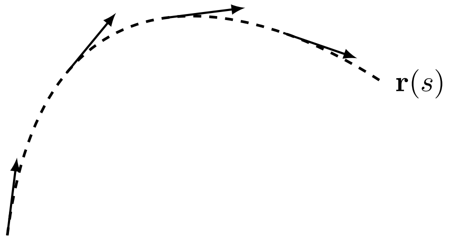

Fig. 2.1: Streamline

streamline by label s such that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}\right), \tag{2.6}$$

and demand $\mathrm{d}\mathbf{r}/\mathrm{d}s \parallel \mathbf{u}$, we get

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} = \frac{\mathrm{d}z}{u_z}. \tag{2.7}$$

- *Particle paths*: trajectories of individual fluid elements given by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{u}(\mathbf{r}, t). \tag{2.8}$$

For small time intervals, particle paths follow streamlines since $\mathbf{u}$ can be treated as approximately steady.

- *Streaklines*: locus of points of all fluid that have passed through a given spatial point in the past.

$$\mathbf{r}(t) = \mathbf{r}_0 \tag{2.9}$$

for some given t in the past

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

2.3 Conservation of Mass

Consider a fixed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. If there are no sources or sinks of mass within the volume, we can say

$$\text{rate of change of mass in } \mathcal{V} = -\text{rate that mass is flowing out across } \mathcal{S} \tag{2.10}$$

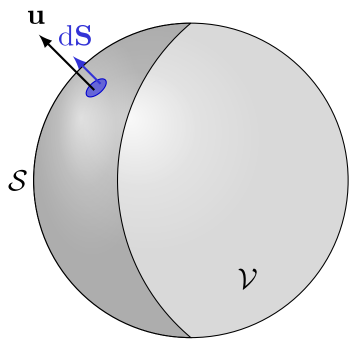

Fig. 2.2: Mass flow of a fluid element

this gives

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, \mathrm{d}V = -\int_{\mathcal{S}} \rho \, \mathbf{u} \cdot \mathrm{d}\mathbf{S} \\
\implies &\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \, \mathrm{d}V \\
\implies &\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \right) \mathrm{d}V = 0.
\end{aligned} \tag{2.11}
$$

This is true for all volumes $\mathcal{V}$. So we must have the *Eulerian continuity equation*,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0.} \tag{2.12}$$

The Lagrangian expression of mass conservation is easily found:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\boldsymbol{\nabla} \cdot \rho\mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{u} \tag{2.13}$$

Thus we have the *Lagrangian continuity equation*,

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0.} \tag{2.14}$$

In an incompressible flow, fluid elements maintain a constant density, i.e.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{2.15}$$

We can now see that incompressible flows must be divergence free, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

2.4 Conservation of Momentum

2.4.1 Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of a surface element is

$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}\,. \tag{2.16}$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$\mathrm{d}F_i = \sum_j \sigma_{ij}\,\mathrm{d}S_j\,. \tag{2.17}$$

where σ_{ij} is the stress tensor – the force in direction i acting on a surface with normal along j.

Isotropic pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}. \tag{2.18}$$

2.4.2 Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field $\mathbf{g}$ and internal pressure forces. Let the fluid element have volume $\mathcal{V}$ and surface $\mathcal{S}$.

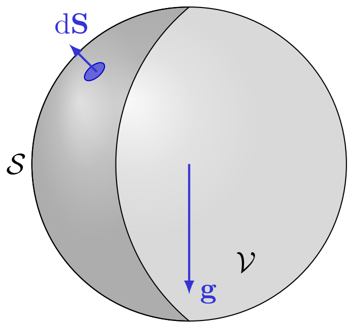

Fig. 2.3: A fluid element subject to gravity

Pressure acting on the surface element gives a force $-p\,\mathrm{d}\mathbf{S}$. The pressure force on an element projected in direction $\hat{\mathbf{n}}$ is $-p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$. So, the net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_{\mathcal{S}} p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V\,. \tag{2.19}$$

The rate of change of momentum of a fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = -\int_{\mathcal{V}} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V + \int_{\mathcal{V}} \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V\,. \tag{2.20}$$

In the limit that $\int \mathrm{d}V \to \delta V$ we have

$$
\begin{aligned}
&\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}\delta V)\cdot\hat{\mathbf{u}} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}} \\
\implies\quad &\hat{\mathbf{n}}\cdot\mathbf{u}\,\underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta V)}_{\substack{=0 \text{ by mass} \\ \text{conservation}}} + \rho\delta V\hat{\mathbf{n}}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}} \\
\implies\quad &\delta V\hat{\mathbf{n}}\cdot\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla}p - \rho\mathbf{g}\right) = 0.
\end{aligned}
\tag{2.21}
$$

This must be true for all $\hat{\mathbf{n}}$ and all δV. So we arrive at the *Lagrangian momentum equation*,

$$
\boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g},}
\tag{2.22}
$$

or instead we have the *Eulerian momentum equation*,

$$
\boxed{\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \rho\mathbf{g},}
\tag{2.23}
$$

Now consider the Eulerian rate of change of momentum density $\rho\mathbf{u}$ and introduce a more compact notation

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t(\rho u_i) \\
&= \rho\partial_t u_i + u_i\partial_t\rho \\
&= -\rho u_j\partial_j u_i - \partial_j p\delta_{ij} + \rho g_i - u_i\partial_j(\rho u_j),
\end{aligned}
\tag{2.24}
$$

where we have used notation

$$
\partial_j \equiv \frac{\partial}{\partial x_j}
\tag{2.25}
$$

and employed summation convention (summation over the repeated indices).

This gives

$$
\partial_t(\rho u_i) = -\partial_j(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor} \\ \text{due to bulk flow} \\ \text{“Ram Presure”}}} + \underbrace{p\delta_{ij}}_{\substack{\text{stress tensor} \\ \text{due to random} \\ \text{thermal motions}}}) + \rho g_i = -\partial_j\sigma_{ij} + \rho g_i
\tag{2.26}
$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$
\sigma_{ij} = p\delta_{ij} + \rho u_i u_j.
\tag{2.27}
$$

In component free language we write

$$
\partial_t(\rho\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underbrace{\left(\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\underline{\mathbf{I}}}\right)}_{\substack{\text{flux of} \\ \text{momentum} \\ \text{density}}} + \rho\mathbf{g}.
\tag{2.28}
$$

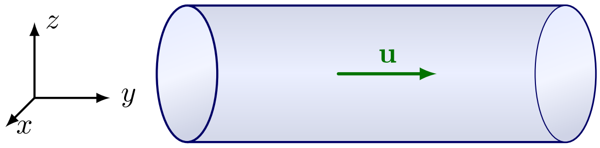

Fig. 2.4: Flow in a pipe

2.4.2.1 Example: Flow in a Pipe in the y-direction

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure.

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{2.29}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

Chapter 3

Gravitation

3.1 Basics

Define the gravitational potential Ψ such that the gravitational acceleration $\mathbf{g}$ is

$$\boxed{\mathbf{g} = -\boldsymbol{\nabla}\Psi.} \tag{3.1}$$

If ℓ is some closed loop, we have (using the curl theorem)

$$\oint_\ell \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{g}) \cdot \mathrm{d}\mathbf{S} = -\int_{\mathcal{S}} [\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Psi)] \cdot \mathrm{d}\mathbf{S} = 0, \tag{3.2}$$

as curl of any gradient is zero. So gravity is a conservative force – the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point $\mathbf{r}$ to ∞ is

$$-\int_{\mathbf{r}}^{\infty} \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \boldsymbol{\nabla}\Psi \cdot \mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r}), \tag{3.3}$$

which is independent of path.

A particular important case is the gravity of a point mass, which has

$$\Psi = -\frac{GM}{r} \qquad \text{if mass at origin} \tag{3.4}$$

$$\Psi = -\frac{GM}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{if mass at location } \mathbf{r}'. \tag{3.5}$$

For a system of point masses we have

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i'|} \tag{3.6}$$

$$\Longrightarrow \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r} - \mathbf{r}_i')}{|\mathbf{r} - \mathbf{r}_i'|^3} \tag{3.7}$$

Replacing $M_i \to \rho_i \delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G \int_{\mathcal{V}} \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V' \, . \tag{3.8}$$

Take divergence of both sides

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{g} &= -G\int_{\mathcal{V}}\rho(\mathbf{r}')\,\underbrace{\boldsymbol{\nabla}_{\mathbf{r}}\cdot\left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]}_{4\pi\delta(\mathbf{r}-\mathbf{r}')}\,\mathrm{d}V' \\
&= -4\pi G\int_{\mathcal{V}}\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' \\
&= -4\pi G\rho(\mathbf{r}). \qquad (3.9)
\end{aligned}$$

Thus we arrive at *Poisson's equation for gravitation*,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{g} = -\nabla^2\Psi = -4\pi G\rho.} \qquad (3.10)$$

We can also express Poisson's equation in integral form: for some volume $\mathcal{V}$ bounded by surface $\mathcal{S}$ we have

$$\begin{aligned}
\int_{\mathcal{V}}\boldsymbol{\nabla}\cdot\mathbf{g}\,\mathrm{d}V &= -4\pi G\int_{\mathcal{V}}\rho\,\mathrm{d}V \\
\implies \int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} &= -4\pi GM. \qquad (3.11)
\end{aligned}$$

This is useful for calculating g when the mass distribution obeys some symmetry.

3.1.0.1 Example: Spherical Distribution of Mass

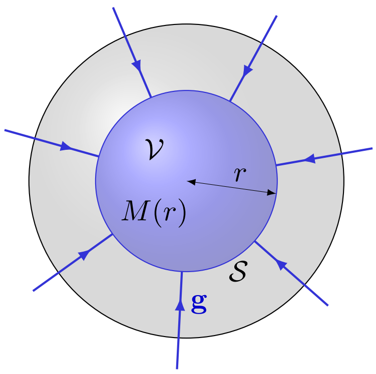

Fig. 3.1: Spherical distribution of mass

By symmetry $\mathbf{g}$ is radial and $|\mathbf{g}|$ is constant over a $r = \text{const.}$ shell. So

$$\begin{aligned}
\int_{\mathcal{S}}\mathbf{g}\cdot\mathrm{d}\mathbf{S} &= -4\pi G\underbrace{M(r)}_{\substack{\text{mass}\\\text{enclosed}}} \\
\implies \quad -4\pi r^2|\mathbf{g}| &= -4\pi GM(r) \\
\implies \quad |\mathbf{g}| &= \frac{GM(r)}{r^2} \\
\therefore \mathbf{g} &= -\frac{GM(r)}{r^2}\hat{\mathbf{r}}. \qquad (3.12)
\end{aligned}$$

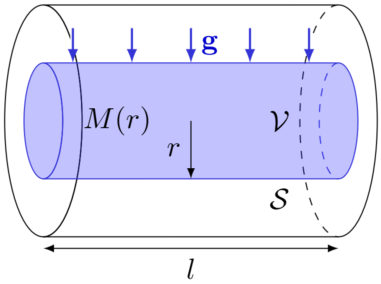

Fig. 3.2: Cylindrical distribution of mass

3.1.0.2 Example: Infinite Cylindrically Symmetric Mass

By symmetry, $\mathbf{g}$ is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \implies \quad -2\pi R l |\mathbf{g}| &= -4\pi G l \underbrace{M(r)}_{\substack{\text{enclosed mass} \\ \text{per unit length}}} \\ \therefore \mathbf{g} &= -\frac{2GM(r)}{r}\hat{\mathbf{r}}. \end{aligned} \tag{3.13}$$

3.1.0.3 Example: Infinite Planar Distribution of Mass

Assume infinite and homogeneous in x and y, $\rho = \rho(z)$.

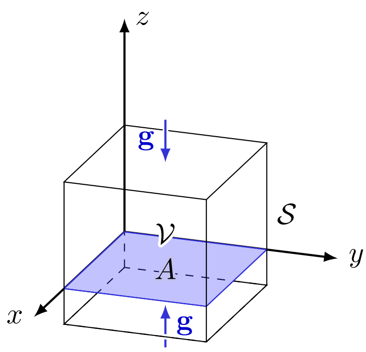

Fig. 3.3: Planar distribution of mass

By symmetry, $\mathbf{g}$ is in the $-\hat{\mathbf{z}}$ direction and is constant on a $z =$ const. surface. So, if we also have reflection symmetry about $z = 0$,

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} &= -4\pi G \int_{\mathcal{V}} \rho \, \mathrm{d}V \\ \implies \quad -2|\mathbf{g}|A &= -4GA \int_{-z}^{z} \rho(z) \, \mathrm{d}z \\ \therefore \mathbf{g} &= -4\pi G\hat{\mathbf{z}} \int_{0}^{z} \rho z \, \mathrm{d}z \, . \end{aligned} \tag{3.14}$$

(For planar distribution of finite height $z_{\max}$, $\mathbf{g}$ is constant for $z \geq z_{\max}$.)

3.2 Potential of a Spherical Mass Distribution

We found that, for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \quad |\mathbf{g}| = \frac{G}{r^2}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r' = \frac{\mathrm{d}\Psi}{\mathrm{d}r}, \tag{3.15}$$

so,

$$\Psi(r_0) - \Psi(\infty) = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r\,. \tag{3.16}$$

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\begin{aligned} \Psi(r_0) &= -\left.\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\right|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r \\ \implies \Psi(r_0) &= -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r\,, \end{aligned} \tag{3.17}$$

where we have made an assumption that $M(r)/r \to 0$ as $r \to \infty$.

We find that $\Psi(r_0)$ is affected by matter outside of r0 through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r.

3.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}, \tag{3.18}$$

and the energy required to take a unit mass to ∞ is $-\Psi$. The energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j, \tag{3.19}$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution,

$$\Omega = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V\,. \tag{3.20}$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r)r^2\Psi(r)\,\mathrm{d}r \tag{3.21}$$

Integrate by parts, choosing parts $u \equiv \Psi$, $\mathrm{d}v \equiv 4\pi\rho r^2 \,\mathrm{d}r$ so that $v = \int_0^r 4\pi\rho'^2 \,\mathrm{d}r' = M(r)$, then

$$\Omega = \frac{1}{2}\left[M(r)\Psi(r)\Big|_0^\infty - \int_0^\infty M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\,\mathrm{d}r \right]. \tag{3.22}$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (the boundary term) is zero. Noting further that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}, \tag{3.23}$$

we conclude

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r\,. \tag{3.24}$$

Integrate again by parts, choosing $u \equiv GM(r)^2$, $\mathrm{d}v \equiv \mathrm{d}r\,/r^2$,

$$\begin{aligned} \Omega &= \underbrace{\frac{1}{2}GM(r)^2\frac{1}{r}\Big|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r \\ \implies \quad \Omega &= -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M\,. \end{aligned} \tag{3.25}$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r}\,\mathrm{d}M(r)\,. \tag{3.26}$$

3.4 The Virial Theorem

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems.

Consider the motion of a cloud of particles (atoms, stars, galaxies...). A particle with mass m_i at $\mathbf{r}_i$ is acted upon by a force

Appendix **A**

Appendix