

Part II Astrophysical Fluids

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Preface

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is “frozen in” to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs.

Contents

1	Basic Principles	1
1.1	Introduction	1
1.2	Collisional and Collisionless Fluids	2
2	Formulation of the Fluid Equations	3
2.1	Eulerian vs Lagrangian	3
2.2	Kinematics	3
2.3	Conservation of Mass	4
2.4	Conservation of Momentum	5
2.4.1	Pressure	5
2.4.2	Momentum Equation for a Fluid	6
3	Gravitation	9
3.1	Basics	9
3.2	Potential of a Spherical Mass Distribution	12
3.3	Gravitational Potential Energy	12
3.4	The Virial Theorem	13
4	Equations of State and the Energy Equation	15
4.1	The Equation of State	15
4.1.1	Barotropic Fluids	15
4.2	The Energy Equation	18
4.3	Heating and Cooling Processes	19
4.4	Energy Transport Processes	20
5	Hydrostatic Equilibrium, Atmospheres and Stars	21
5.1	Hydrostatic Equilibrium	21
5.2	Stars as Self-Gravitating Polytropes	24
5.2.1	Solution for $n = 0$	26
5.3	Isothermal Spheres (Case $n \rightarrow \infty$)	26
5.4	Scaling Relations	27
6	Sound Waves, Supersonic Flows and Shock Waves	31
6.1	Sound Waves	31
6.1.1	Examples of $dp/d\rho$	33
6.2	Sound Waves in a Stratified Atmosphere	34
6.2.1	Case I: $\omega > c_u/2H$	36
6.2.2	Case II: $\omega < c_u/2H$	37
6.3	Transmission of Sound Waves at Interfaces	38
6.4	Supersonic Fluids and Shocks	39
6.5	The Rankine-Hugoniot Relations	40
6.6	Theory of Supernova Explosions	44
6.6.1	Similarity Solutions	47
6.6.2	Structure of the Blast Wave	48
6.6.3	Breakdown of the Similarity Solution	49

7	Bernoulli's Equation and Transonic Flows	53
7.1	Bernoulli's Equation	53
7.1.1	Examples of Bernoulli's Equation	54
7.2	Rotational and Irrotational Flows	54
7.3	The de Laval Nozzle	56
7.3.1	Case I: Isothermal EoS	57
7.3.2	Case II: Polytropic EoS	58
7.4	Spherical Accretion and Winds	60
7.4.1	Case I: Isothermal EoS	61
7.4.2	Case II: Polytropic EoS	62
8	Fluid Instabilities	65
8.1	Convective Instability	65
8.2	Jeans Instability	68
8.3	Rayleigh-Taylor and Kelvin-Helmholtz Instability	70
8.4	Thermal Instability	74
9	Viscous Flows	79
9.1	Basics of Viscosity	79
9.2	Navier-Stokes Equation	81
9.3	Vorticity in Viscous Flows	82
9.4	Energy Dissipation in Incompressible Viscous Flows	82
9.5	Viscous Flow through a Pipe	84
9.6	Accretion Disks	85
9.7	Steady-State, Geometrically-Thin Disks	89
9.7.1	Radiation from Steady-State Thin Disks	90
10	Plasmas	93
10.1	Magnetohydrodynamic (MHD) Equations	93
10.1.1	Simplifying MHD	94
10.2	The Dynamical Effects of Magnetic Fields	96
10.3	Waves in Plasmas	96
A	Appendix	A.1
A.1	Derivation of the Momentum Equation for Disk Evolution	A.1

List of Tables

List of Figures

2.1	Streamline	4
2.2	Mass flow of a fluid element	5
2.3	A fluid element subject to gravity	6
2.4	Flow in a pipe	8
3.1	Spherical distribution of mass	10
3.2	Cylindrical distribution of mass	11
3.3	Planar distribution of mass	11
5.1	Caption	22
5.2	Caption	27
5.3	Roche lobe overflow	30
6.1	Waves at boundary $x = 0$	38
6.2	Subsonic flow vs supersonic flow	39
6.3	Geometry of shock front	40
6.4	Plot of Eq. (6.69) for various values of γ	42
6.5	Temperature profile through a shock	43
6.6	Supernova explosion	45
6.7	Situation seen in shock frame	45
6.8	Solution for $\gamma = 7/5$	49
6.9	Blast wave phase vs late phase	49
7.1	Aircraft wing	54
7.2	Shower curtain	54
7.3	Change of area element with time	55
7.4	Tube with variable cross-section	56
7.5	de Laval nozzle	59
7.6	Plot in $(r/r_s, u)$ plane	61
8.1	Perturbing a fluid element upwards	66
8.2	Perturbation of interface of discontinuity	70
9.1	Linear shear flow	80
9.2	A pipe	84
9.3	Parabolic velocity profile for viscous flow through a pipe	85
9.4	Infinitesimal annulus element of disk	87
9.5	The viscous evolution of a ring	88

CHAPTER 1

Basic Principles

1.1 Introduction

Fluid Dynamics concerns itself with the dynamics of liquid, gases and (to some degree) plasmas. Phenomena considered in fluid dynamics are *macroscopic*. We describe a fluid as a *continuous medium* with well-defined macroscopic quantities (e.g., density ρ , pressure p), even though, at a microscopic level, the fluid is composed of particles.

Most of the baryonic matter in the Universe can be treated as a fluid. Fluid dynamics is thus an extremely important topic within astrophysics. Astrophysical systems can display extremes of density (both low and high) and temperature beyond those accessible in terrestrial laboratories. In addition, gravity is often a crucial component of the dynamics in astrophysical systems. Thus the subject of *Astrophysical Fluid Dynamics* encompasses but significantly extends the study of fluids relevant to terrestrial systems and/or engineers.

In the astrophysical context, the liquid state is not very common (examples are high pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples: (Fluids in the Universe)

- Interiors of stars, white dwarfs, neutron stars;
- Interstellar medium (ISM), intergalactic medium (IGM), intracluster medium (ICM);
- Stellar winds, jets, accretion disks;
- Giant planets.

In our discussion, we shall use the concept of a *fluid element*. This is a region of fluid that is

1. Small enough that there are no significant variations of any property q that interests us

$$l_{\text{region}} \ll l_{\text{scale}} \sim \frac{q}{|\nabla q|}. \quad (1.1)$$

2. Large enough to contain sufficient particles to be considered in the continuum limit

$$nl_{\text{region}}^3 \gg 1, \quad (1.2)$$

where n is the number density of particles.

1.2 Collisional and Collisionless Fluids

In a *collisional fluid*, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other, i.e.

$$l_{\text{region}} \gg \lambda, \quad (1.3)$$

where λ is the mean free path. Particles will then attain a distribution of velocities that maximises the entropy of the system at a given temperature. Thus, a collisional fluid at a given density ρ and temperature T will have a well-defined distribution of particle speeds and hence a well-defined pressure, p . We can relate ρ , T and p with an *equation of state*:

$$p = p(\rho, T). \quad (1.4)$$

In a *collisionless fluid*, particles do not interact frequently enough to satisfy $l_{\text{region}} \gg \lambda$. So, distribution of particle speeds locally does not correspond to the maximum entropy solution, instead depending on initial conditions and non-local conditions.

Examples: (Collisionless Fluids)

- Stars in a galaxy;
- Grains in Saturn's rings;
- Dark matter;
- ICM (transitional from collisional to collisionless).

1.2.0.1 Example of ICM

Treat as fully ionised plasma of electrons and ions. The mean free path is set by Coulomb collisions and an analysis gives

$$\lambda_e = \frac{3^{3/2}(k_B T_e)^2 \epsilon_0^2}{4\pi^{1/2} n_e e^4 \ln \Lambda}, \quad (1.5)$$

where n_e is the electron number density, and Λ is the ratio of largest to smallest impact parameter. For $T \gtrsim 4 \times 10^5$ K we have $\ln \Lambda \sim 40$. So, if $T_i = T_e$, we have

$$\lambda_e = \lambda_i \simeq 23 \text{ kpc} \left(\frac{T_e}{10^8 \text{ K}} \right)^2 \left(\frac{n_e}{10^{-3} \text{ cm}^{-3}} \right)^{-1}. \quad (1.6)$$

So we have

$$\overbrace{R_{\text{galaxy}} \sim \lambda_e}^{\text{collisionless}} \ll \underbrace{R_{\text{cluster}}}_{\text{collisional}} \sim 1 \text{ Mpc}. \quad (1.7)$$

CHAPTER 2

Formulation of the Fluid Equations

2.1 Eulerian vs Lagrangian

Two main frameworks for understanding fluid flow:

1. *Eulerian description*: one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like

$$\rho(\mathbf{r}, t), \quad p(\mathbf{r}, t), \quad T(\mathbf{r}, t), \quad \mathbf{v}(\mathbf{r}, t). \quad (2.1)$$

2. *Lagrangian description*: one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is co-moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest. The Lagrangian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals). These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity Q in a fluid element at position \mathbf{r} and time t . At time $t + \delta t$ the element will be at position $\mathbf{r} + \delta \mathbf{r}$. The change in quantity Q of the fluid element is

$$\frac{DQ}{Dt} = \lim_{\delta t \rightarrow 0} \left[\frac{Q(\mathbf{r} + \delta \mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right] \quad (2.2)$$

but

$$Q(\mathbf{r} + \delta \mathbf{r}, t + \delta t) = Q(\mathbf{r}, t) + \frac{\partial Q}{\partial t} \delta t + \delta \mathbf{r} \cdot \nabla Q + \mathcal{O}(\delta t^2, |\delta \mathbf{r}|^2, \delta t |\delta \mathbf{r}|), \quad (2.3)$$

so

$$\frac{DQ}{Dt} = \lim_{\delta t \rightarrow 0} \left[\frac{\partial Q}{\partial t} + \frac{\delta \mathbf{r}}{\delta t} \cdot \nabla Q + \mathcal{O}(\delta t, |\delta \mathbf{r}|) \right], \quad (2.4)$$

which gives us

$$\boxed{\underbrace{\frac{DQ}{Dt}}_{\text{Lagrangian time derivative}} = \underbrace{\frac{\partial Q}{\partial t}}_{\text{Eulerian time derivative}} + \underbrace{\mathbf{u} \cdot \nabla Q}_{\text{"convective" time derivative}}.} \quad (2.5)$$

2.2 Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

- *Streamline*: families of curves that are instantaneously tangent to the velocity vector of the flow $\mathbf{u}(\mathbf{r}, t)$. They show the direction of the fluid element. Parameterise

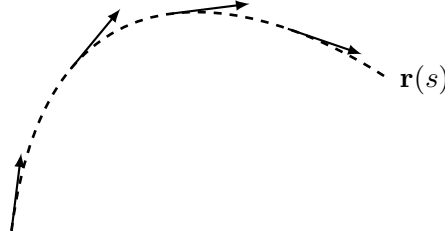


Fig. 2.1: Streamline

streamline by label s such that

$$\frac{d\mathbf{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right), \quad (2.6)$$

and demand $d\mathbf{r}/ds \parallel \mathbf{u}$, we get

$$\frac{d\mathbf{r}}{ds} \times \mathbf{u} = 0 \quad \implies \quad \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}. \quad (2.7)$$

- *Particle paths*: trajectories of individual fluid elements given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t). \quad (2.8)$$

For small time intervals, particle paths follow streamlines since \mathbf{u} can be treated as approximately steady.

- *Streaklines*: locus of points of all fluid that have passed through a given spatial point in the past.

$$\mathbf{r}(t) = \mathbf{r}_0 \quad (2.9)$$

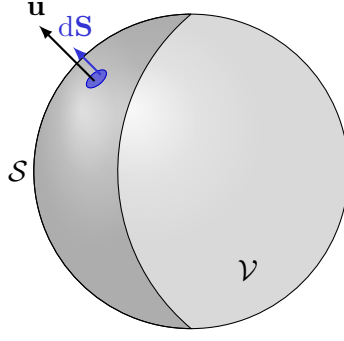
for some given t in the past

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.

2.3 Conservation of Mass

Consider a fixed volume \mathcal{V} bounded by a surface \mathcal{S} . If there are no sources or sinks of mass within the volume, we can say

$$\text{rate of change of mass in } \mathcal{V} = -\text{rate that mass is flowing out across } \mathcal{S} \quad (2.10)$$

**Fig. 2.2:** Mass flow of a fluid element

this gives

$$\begin{aligned}
 \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho dV &= - \int_{\mathcal{S}} \rho \mathbf{u} \cdot d\mathbf{S} \\
 \Rightarrow \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} dV &= - \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{u}) dV \\
 \Rightarrow \int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV &= 0.
 \end{aligned} \tag{2.11}$$

This is true for all volumes \mathcal{V} . So we must have the *Eulerian continuity equation*,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.} \tag{2.12}$$

The Lagrangian expression of mass conservation is easily found:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\nabla \cdot \rho \mathbf{u} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \tag{2.13}$$

Thus we have the *Lagrangian continuity equation*,

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.} \tag{2.14}$$

In an incompressible flow, fluid elements maintain a constant density, i.e.

$$\frac{D\rho}{Dt} = 0. \tag{2.15}$$

We can now see that incompressible flows must be divergence free, $\nabla \cdot \mathbf{u} = 0$.

2.4 Conservation of Momentum

2.4.1 Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. Therefore, there is no net acceleration even for non-zero pressure since pressure is defined as the momentum flux on *one* side of the surface.

If the particle motions within the fluid are isotropic, the momentum flux is locally independent of the orientation of the surface and the components parallel to the surface cancel out. Then, the force acting on one side of a surface element is

$$d\mathbf{F} = p d\mathbf{S}. \quad (2.16)$$

In the more general case, forces across surfaces are not perpendicular to the surface and we have

$$dF_i = \sum_j \sigma_{ij} dS_j. \quad (2.17)$$

where σ_{ij} is the stress tensor – the force in direction i acting on a surface with normal along j .

Isotropic pressure in a static fluid corresponds to

$$\sigma_{ij} = p\delta_{ij}. \quad (2.18)$$

2.4.2 Momentum Equation for a Fluid

Consider a fluid element that is subject to a gravitational field \mathbf{g} and internal pressure forces. Let the fluid element have volume \mathcal{V} and surface \mathcal{S} .

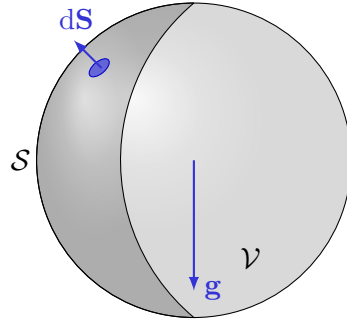


Fig. 2.3: A fluid element subject to gravity

Pressure acting on the surface element gives a force $-p d\mathbf{S}$. The pressure force on an element projected in direction $\hat{\mathbf{n}}$ is $-p \hat{\mathbf{n}} \cdot d\mathbf{S}$. So, the net pressure force in direction $\hat{\mathbf{n}}$ is

$$\mathbf{F} \cdot \hat{\mathbf{n}} = - \int_{\mathcal{S}} p \hat{\mathbf{n}} \cdot d\mathbf{S} = - \int_{\mathcal{V}} \nabla \cdot (p \hat{\mathbf{n}}) dV = - \int_{\mathcal{V}} \hat{\mathbf{n}} \cdot \nabla p dV. \quad (2.19)$$

The rate of change of momentum of a fluid element in direction $\hat{\mathbf{n}}$ is the total force in that direction:

$$\left(\frac{D}{Dt} \int_{\mathcal{V}} \rho \mathbf{u} dV \right) \cdot \hat{\mathbf{n}} = - \int_{\mathcal{V}} \hat{\mathbf{n}} \cdot \nabla p dV + \int_{\mathcal{V}} \rho \mathbf{g} \cdot \hat{\mathbf{n}} dV. \quad (2.20)$$

In the limit that $\int dV \rightarrow \delta V$ we have

$$\begin{aligned}
 & \frac{D}{Dt}(\rho \mathbf{u} \delta V) \cdot \hat{\mathbf{u}} = -\delta V \hat{\mathbf{n}} \cdot \nabla p + \delta V \rho \mathbf{g} \cdot \hat{\mathbf{n}} \\
 \Rightarrow & \quad \hat{\mathbf{n}} \cdot \mathbf{u} \underbrace{\frac{D}{Dt}(\rho \delta V)}_{=0 \text{ by mass conservation}} + \rho \delta V \hat{\mathbf{n}} \cdot \frac{D\mathbf{u}}{Dt} = -\delta V \hat{\mathbf{n}} \cdot \nabla p + \delta V \rho \mathbf{g} \cdot \hat{\mathbf{n}} \\
 \Rightarrow & \quad \delta V \hat{\mathbf{n}} \cdot \left(\rho \frac{D\mathbf{u}}{Dt} + \nabla p - \rho \mathbf{g} \right) = 0.
 \end{aligned} \tag{2.21}$$

This must be true for all $\hat{\mathbf{n}}$ and all δV . So we arrive at the *Lagrangian momentum equation*,

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}}, \tag{2.22}$$

or instead we have the *Eulerian momentum equation*,

$$\boxed{\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g}}, \tag{2.23}$$

Now consider the Eulerian rate of change of momentum density $\rho \mathbf{u}$ and introduce a more compact notation

$$\begin{aligned}
 \frac{\partial}{\partial t}(\rho u_i) & \equiv \partial_t(\rho u_i) \\
 & = \rho \partial_t u_i + u_i \partial_t \rho \\
 & = -\rho u_j \partial_j u_i - \partial_j p \delta_{ij} + \rho g_i - u_i \partial_j(\rho u_j),
 \end{aligned} \tag{2.24}$$

where we have used notation

$$\partial_j \equiv \frac{\partial}{\partial x_j} \tag{2.25}$$

and employed summation convention (summation over the repeated indices).

This gives

$$\partial_t(\rho u_i) = -\partial_j \left(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor} \\ \text{due to bulk flow} \\ \text{"Ram Pressure"}}} + \underbrace{p \delta_{ij}}_{\substack{\text{stress tensor} \\ \text{due to random} \\ \text{thermal motions}}} \right) + \rho g_i = -\partial_j \sigma_{ij} + \rho g_i \tag{2.26}$$

where we have generalised the stress tensor to include the momentum flux from the bulk flow,

$$\sigma_{ij} = p \delta_{ij} + \rho u_i u_j. \tag{2.27}$$

In component free language we write

$$\partial_t(\rho \mathbf{u}) = -\nabla \cdot \underbrace{\left(\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I} \right)}_{\substack{\text{flux of} \\ \text{momentum} \\ \text{density}}} + \rho \mathbf{g}. \tag{2.28}$$

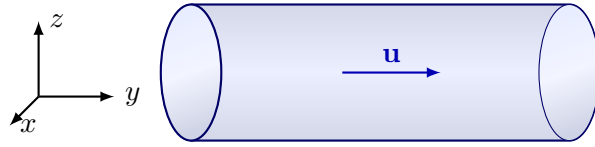


Fig. 2.4: Flow in a pipe

2.4.2.1 Example: Flow in a Pipe in the y -direction

Any surface will experience a momentum flux p due to pressure. Only surfaces with a normal that has a component parallel to flow will experience ram pressure.

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}. \quad (2.29)$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.

CHAPTER 3

Gravitation

3.1 Basics

Define the gravitational potential Ψ such that the gravitational acceleration \mathbf{g} is

$$\boxed{\mathbf{g} = -\nabla\Psi.} \quad (3.1)$$

If ℓ is some closed loop, we have (using the curl theorem)

$$\oint_{\ell} \mathbf{g} \cdot d\mathbf{l} = \int_{\mathcal{S}} (\nabla \times \mathbf{g}) \cdot d\mathbf{S} = - \int_{\mathcal{S}} [\nabla \times (\nabla\Psi)] \cdot d\mathbf{S} = 0, \quad (3.2)$$

as curl of any gradient is zero. So gravity is a conservative force – the work done around a closed loop is zero.

As a consequence, the work needed to take a mass from point \mathbf{r} to ∞ is

$$- \int_{\mathbf{r}}^{\infty} \mathbf{g} \cdot d\mathbf{l} = \int_{\mathbf{r}}^{\infty} \nabla\Psi \cdot d\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r}), \quad (3.3)$$

which is independent of path.

A particular important case is the gravity of a point mass, which has

$$\Psi = -\frac{GM}{r} \quad \text{if mass at origin} \quad (3.4)$$

$$\Psi = -\frac{GM}{|\mathbf{r} - \mathbf{r}'|} \quad \text{if mass at location } \mathbf{r}'. \quad (3.5)$$

For a system of point masses we have

$$\Psi = - \sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}'_i|} \quad (3.6)$$

$$\implies \mathbf{g} = -\nabla\Psi = - \sum_i \frac{GM_i(\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3} \quad (3.7)$$

Replacing $M_i \rightarrow \rho_i \delta V_i$ and going to the continuum limit we have

$$\mathbf{g}(\mathbf{r}) = -G \int_{\mathcal{V}} \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \quad (3.8)$$

Take divergence of both sides

$$\begin{aligned}
 \nabla \cdot \mathbf{g} &= -G \int_{\mathcal{V}} \rho(\mathbf{r}') \underbrace{\nabla_{\mathbf{r}} \cdot \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right]}_{4\pi\delta(\mathbf{r}-\mathbf{r}')} dV' \\
 &= -4\pi G \int_{\mathcal{V}} \rho(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV' \\
 &= -4\pi G \rho(\mathbf{r}).
 \end{aligned} \tag{3.9}$$

Thus we arrive at *Poisson's equation for gravitation*,

$$\boxed{\nabla \cdot \mathbf{g} = -\nabla^2 \Psi = -4\pi G \rho.} \tag{3.10}$$

We can also express Poisson's equation in integral form: for some volume \mathcal{V} bounded by surface \mathcal{S} we have

$$\begin{aligned}
 \int_{\mathcal{V}} \nabla \cdot \mathbf{g} dV &= -4\pi G \int_{\mathcal{V}} \rho dV \\
 \implies \int_{\mathcal{S}} \mathbf{g} \cdot d\mathbf{S} &= -4\pi G M.
 \end{aligned} \tag{3.11}$$

This is useful for calculating \mathbf{g} when the mass distribution obeys some symmetry.

3.1.0.1 Example: Spherical Distribution of Mass

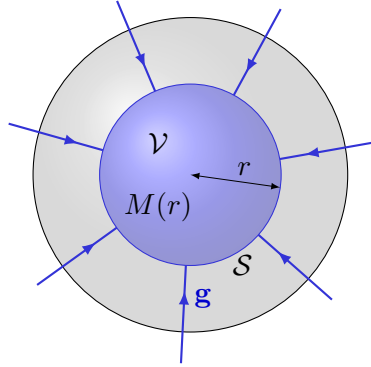


Fig. 3.1: Spherical distribution of mass

By symmetry \mathbf{g} is radial and $|\mathbf{g}|$ is constant over a $r = \text{const.}$ shell. So

$$\begin{aligned}
 \int_{\mathcal{S}} \mathbf{g} \cdot d\mathbf{S} &= -4\pi G \underbrace{M(r)}_{\text{mass enclosed}} \\
 \implies -4\pi r^2 |\mathbf{g}| &= -4\pi G M(r) \\
 \implies |\mathbf{g}| &= \frac{GM(r)}{r^2} \\
 \therefore \mathbf{g} &= -\frac{GM(r)}{r^2} \hat{\mathbf{r}}.
 \end{aligned} \tag{3.12}$$

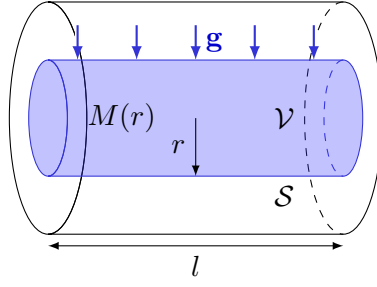


Fig. 3.2: Cylindrical distribution of mass

3.1.0.2 Example: Infinite Cylindrically Symmetric Mass

By symmetry, \mathbf{g} is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side, then

$$\begin{aligned}
 \int_S \mathbf{g} \cdot d\mathbf{S} &= -4\pi G \int_V \rho dV \\
 \Rightarrow -2\pi Rl|\mathbf{g}| &= -4\pi Gl \underbrace{M(r)}_{\text{enclosed mass per unit length}} \\
 \therefore \mathbf{g} &= -\frac{2GM(r)}{r} \hat{\mathbf{r}}.
 \end{aligned} \tag{3.13}$$

3.1.0.3 Example: Infinite Planar Distribution of Mass

Assume infinite and homogeneous in x and y , $\rho = \rho(z)$.

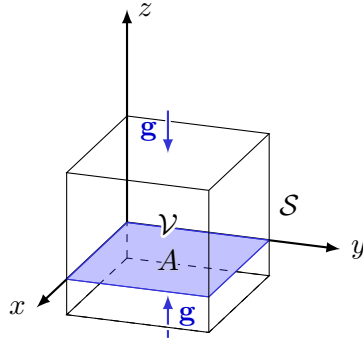


Fig. 3.3: Planar distribution of mass

By symmetry, \mathbf{g} is in the $-\hat{\mathbf{z}}$ direction and is constant on a $z = \text{const.}$ surface. So, if we also have reflection symmetry about $z = 0$,

$$\begin{aligned}
 \int_S \mathbf{g} \cdot d\mathbf{S} &= -4\pi G \int_V \rho dV \\
 \Rightarrow -2|\mathbf{g}|A &= -4GA \int_{-z}^z \rho(z) dz \\
 \therefore \mathbf{g} &= -4\pi G \hat{\mathbf{z}} \int_0^z \rho z dz.
 \end{aligned} \tag{3.14}$$

(For planar distribution of finite height z_{\max} , \mathbf{g} is constant for $z \geq z_{\max}$.)

3.2 Potential of a Spherical Mass Distribution

We found that, for a spherical distribution,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \quad |\mathbf{g}| = \frac{G}{r^2} \int_0^r 4\pi\rho(r')r'^2 dr' = \frac{d\Psi}{dr}, \quad (3.15)$$

so,

$$\Psi(r_0) - \Psi(\infty) = \int_{\infty}^{r_0} \frac{G}{r^2} \left\{ \int_0^r 4\pi\rho(r')r'^2 dr' \right\} dr. \quad (3.16)$$

Taking $\Psi(\infty) = 0$ by convention, integrate this by parts:

$$\begin{aligned} \Psi(r_0) &= - \left\{ \frac{G}{r} \int_0^r 4\pi\rho(r')r'^2 dr' \right\} \Big|_{r=\infty}^{r_0} + \int_{\infty}^{r_0} \frac{G}{r} 4\pi\rho(r)r^2 dr \\ \Rightarrow \Psi(r_0) &= -\frac{GM(r_0)}{r_0} + \int_{\infty}^{r_0} 4\pi G\rho(r)r dr, \end{aligned} \quad (3.17)$$

where we have made an assumption that $M(r)/r \rightarrow 0$ as $r \rightarrow \infty$.

We find that $\Psi(r_0)$ is affected by matter outside of r_0 through our choice of setting $\Psi = 0$ at infinity. So $\Psi \neq -GM(r)/r$ unless there is no mass outside of r .

3.3 Gravitational Potential Energy

For a given system of point masses,

$$\Psi = - \sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}, \quad (3.18)$$

and the energy required to take a unit mass to ∞ is $-\Psi$. The energy required to take a system of point masses to ∞ is

$$\Omega = -\frac{1}{2} \sum_{j \neq i} \sum_i \frac{GM_i M_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2} \sum_j M_j \Psi_j, \quad (3.19)$$

where the half is present to avoid double counting pairs.

For a continuum matter distribution,

$$\Omega = \frac{1}{2} \int_V \rho(\mathbf{r}) \Psi(\mathbf{r}) dV. \quad (3.20)$$

Specialising to the spherically symmetric case gives

$$\Omega = \frac{1}{2} \int_0^\infty 4\pi\rho(r)r^2\Psi(r) dr \quad (3.21)$$

Integrate by parts, choosing parts $u \equiv \Psi$, $dv \equiv 4\pi\rho r^2 dr$ so that $v = \int_0^r 4\pi\rho r'^2 dr' = M(r)$, then

$$\Omega = \frac{1}{2} \left[M(r)\Psi(r) \right]_0^\infty - \int_0^\infty M(r) \frac{d\Psi}{dr} dr. \quad (3.22)$$

Assuming that we have a finite distribution of mass with a non-singular behaviour at $r = 0$, the first term on the RHS (the boundary term) is zero. Noting further that

$$\frac{d\Psi}{dr} = \frac{GM(r)}{r^2}, \quad (3.23)$$

we conclude

$$\Omega = -\frac{1}{2} \int_0^\infty \frac{GM(r)^2}{r^2} dr. \quad (3.24)$$

Integrate again by parts, choosing $u \equiv GM(r)^2$, $dv \equiv dr/r^2$,

$$\begin{aligned} \Omega &= \underbrace{\frac{1}{2} GM(r)^2 \frac{1}{r}}_{=0} \Big|_0^\infty - \frac{1}{2} \int_0^\infty \frac{1}{r} 2GM \frac{dM}{dr} dr \\ \Rightarrow \quad \Omega &= -G \int_0^\infty \frac{M(r)}{r} dM. \end{aligned} \quad (3.25)$$

This is equivalent to the assembly of spherical shells of mass, each brought from ∞ with potential energy

$$\frac{GM(r)}{r} dM(r). \quad (3.26)$$

3.4 The Virial Theorem

We now come to a powerful result that greatly helps in the understanding of isolated gravitating systems.

Consider the motion of a cloud of particles (atoms, stars, galaxies, ...). A particle with mass m_i at \mathbf{r}_i is acted upon by a force

$$\mathbf{F}_i = m_i \frac{d^2 \mathbf{r}_i}{dt^2}. \quad (3.27)$$

Consider the 2nd derivative of the scalar moment of inertia, $I_i = m_i r_i^2$,

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} (m_i r_i^2) &= m_i \frac{d}{dt} \left(\mathbf{r}_i \cdot \frac{d\mathbf{r}_i}{dt} \right) \\ &= m_i \mathbf{r}_i \cdot \frac{d^2 \mathbf{r}_i}{dt^2} + m_i \left(\frac{d\mathbf{r}_i}{dt} \right)^2 \\ &= \mathbf{r}_i \cdot \mathbf{F}_i + \underbrace{m_i \left(\frac{d\mathbf{r}_i}{dt} \right)^2}_{2 \times \text{Kinetic Energy } T_i}. \end{aligned} \quad (3.28)$$

If $I \equiv \sum_i m_i r_i^2$ then we can sum the previous equation over all particles to give

$$\frac{1}{2} \frac{d^2 I}{dt^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{V, \text{ the Virial (R. Clausius)}} + 2T. \quad (3.29)$$

In the absence of external forces (i.e. an isolated system), we have that $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ where \mathbf{F}_{ij} is the force exerted on the i^{th} particle by the j^{th} particle. Consider any two particles with m_i and m_j at \mathbf{r}_i and \mathbf{r}_j , Newton's 3rd Law says

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \quad (3.30)$$

and so their contribution to the virial is $\mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. We then have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j). \quad (3.31)$$

If there are no non-gravitational interactions except for possibly when $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$\mathbf{F}_{ij} = -\frac{Gm_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad \text{where} \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j. \quad (3.32)$$

Thus we have

$$V = - \sum_i \sum_{j>i} \frac{Gm_i m_j}{r_{ij}}, \quad (3.33)$$

where each term is the work done to separate each pair of particles to infinity against gravity.

And so, $V = \Omega$ and we can use the above to write

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2T + \Omega. \quad (3.34)$$

If the system is in a steady state (“relaxed”), then $I = \text{const.}$ and we can state the *Virial theorem*

$$\boxed{2T + \Omega = 0.} \quad (3.35)$$

Here, the kinetic energy T has contributions from local flows and random/thermal motions. The Virial theorem implies *gravitational potential sets the “temperature” or velocity dispersion of the system.*

CHAPTER 4

Equations of State and the Energy Equation

4.1 The Equation of State

In three dimensions, the (scalar) equation of mass conservation and the (vector) equation of momentum conservation can be written as four independent scalar equations. Given appropriate boundary conditions, these must be solved in order to find the density (scalar field), pressure (scalar field), gravitational potential (scalar field), and velocity components (3D vector field); a total of six degrees of freedom.

To close the system of equations, we need additional information. Specifically, we need to find relations between Ψ , p and the other fluid variables such as ρ and \mathbf{u} .

$\Psi(r)$ and ρ are related via Poisson's equation (and/or we sometimes consider an externally imposed gravitational potential).

p and the other thermodynamic properties of the system are related by the *equation of state* (EoS). This is only valid for collisional fluids.

Most astrophysical fluids are quite dilute (particle separation much larger than effective particle size) and can be well approximated as ideal gases. The corresponding EoS is

$$p = nk_B T = \frac{k_B}{\mu m_p} \rho T, \quad (4.1)$$

where μ is the mean particle mass in units of the proton mass m_p . (Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars and white dwarfs.)

The ideal gas EoS introduces another scalar field into the description of the fluid, the temperature $T(\mathbf{r}, t)$. In general, we need to solve another PDE that describes heating and cooling processes in order to close the set of equations. We shall move on to this in [Section 4.2](#).

4.1.1 Barotropic Fluids

However, for special cases, we can relate T and ρ without the need to solve a separate energy equation. Fluids for which p is *only* a function of ρ are known as barotropic fluids.

4.1.1.1 Example: Electron Degeneracy Pressure

Important in systems with free electrons that are (relatively) cold and dense.

$$p = \frac{\pi^2 \hbar^2}{5m_e m_{\text{ion}}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}, \quad (\text{non-relativistic}) \quad (4.2)$$

e.g., interiors of white dwarfs, iron core in massive stars, deep interior of Jupiter.

4.1.1.2 Example: Isothermal Case

T is constant so that $p \propto \rho$. Valid when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance.

4.1.1.3 Example: Adiabatic Case

Ideal gas undergoes reversible thermodynamic changes such that

$$p = K \rho^\gamma \quad (4.3)$$

where K, γ are constants.

The first law of thermodynamics is

$$\underbrace{\mathrm{d}Q}_{\text{heat absorbed by unit mass of fluid from surrounding}} = \underbrace{\mathrm{d}\mathcal{E}}_{\text{change in internal energy of unit mass of fluid}} + \underbrace{p \mathrm{d}V}_{\text{work done by unit mass of fluid}}. \quad (4.4)$$

Here d is a Pfaffian operator – change in quantity depends on the path taken through the thermodynamic phase space. For an ideal gas, we can write

$$p = \frac{\mathcal{R}_*}{\mu} \rho T, \quad \mathcal{E} = \mathcal{E}(T), \quad (4.5)$$

where \mathcal{R}_* is a modified gas constant.

So, the first law of thermodynamics reads

$$\begin{aligned} \mathrm{d}Q &= \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} \mathrm{d}T + p \mathrm{d}V \\ &= C_V \mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V} \mathrm{d}V, \end{aligned} \quad (4.6)$$

where we define specific heat capacity at constant volume as $C_V \equiv \mathrm{d}\mathcal{E}/\mathrm{d}T$ and have noted that for unit mass we have $\rho = 1/V$.

For a reversible change we have $dQ = 0$, so

$$\begin{aligned} C_V dT + \frac{\mathcal{R}_* T}{\mu V} dV &= 0 \\ \implies C_V d(\ln T) + \frac{\mathcal{R}_*}{\mu} d(\ln V) &= 0 \\ \implies V \propto T^{-C_V \mu / \mathcal{R}_*} &\quad (4.7) \end{aligned}$$

$$\implies p \propto T^{1+C_V \mu / \mathcal{R}_*}. \quad (4.8)$$

C_V depends on the number of degrees of freedom with which the gas can store kinetic energy, f such that

$$C_V = f \frac{\mathcal{R}_*}{2\mu}. \quad (4.9)$$

Monatomic gas has $f = 3 \implies C_V = 3\mathcal{R}_*/2\mu$; diatomic gas at a few $\times 100$ K (two rotational modes excited) has $f = 5 \implies C_V = 5\mathcal{R}_*/2\mu$.

Returning to the ideal gas law,

$$\begin{aligned} p &= \frac{\mathcal{R}_*}{\mu} \rho T \quad \text{with} \quad \rho = 1/V \quad \text{for a unit mass of fluid} \\ \implies pV &= \frac{\mathcal{R}_* T}{\mu} \\ \implies p dV + V dp &= \frac{\mathcal{R}_*}{\mu} dT, \end{aligned} \quad (4.10)$$

but,

$$\begin{aligned} dQ &= \frac{d\mathcal{E}}{dT} dT + p dV \\ &= \underbrace{\left(\frac{d\mathcal{E}}{dT} + \frac{\mathcal{R}_*}{\mu} \right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}} dT - V dp, \end{aligned} \quad (4.11)$$

so,

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu}. \quad (4.12)$$

Let us define

$$\gamma \equiv \frac{C_p}{C_V} \quad (4.13)$$

so that, for the reversible/adiabatic processes discussed above, we have

$$p \propto T^{1+C_V \mu / \mathcal{R}_*} \implies p \propto T^{\gamma/(\gamma+1)} \quad (4.14)$$

$$V \propto T^{-C_V \mu / \mathcal{R}_*} \implies V \propto T^{-1/(\gamma-1)} \quad (4.15)$$

which we can combine to give

$$p \propto \rho^\gamma \quad (4.16)$$

We say that a fluid element behaves *adiabatically* if $p = K\rho^\gamma$ with $K = \text{constant}$. A fluid is *isentropic* if all fluid elements behave adiabatically with the same value of K . K is related to the entropy per unit mass

4.2 The Energy Equation

In general, the equation of state will not be barotropic and we will need to solve a separate differential equation which follows the heating and cooling processes in the gas, the *energy equation*.

From the first law of thermodynamics we have

$$dQ = d\mathcal{E} + \underbrace{p dV}_{dW = -pdV} \quad \text{in absence of dissipative processes,} \quad (4.17)$$

so,

$$\frac{D\mathcal{E}}{Dt} = \frac{DW}{Dt} + \frac{dQ}{dt}, \quad (4.18)$$

with

$$\frac{DW}{Dt} = -p \frac{D}{Dt} \left(\frac{1}{\rho} \right) = \frac{p}{\rho^2} \frac{D\rho}{Dt}, \quad (4.19)$$

and

$$\frac{dQ}{dt} \equiv -\dot{Q}_{\text{cool}} \quad \text{rate of cooling per unit mass,} \quad (4.20)$$

therefore,

$$\frac{D\mathcal{E}}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt} - \dot{Q}_{\text{cool}}. \quad (4.21)$$

The total energy per unit volume is

$$E = \rho \left(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}} \right), \quad (4.22)$$

so,

$$\frac{DE}{Dt} = \frac{D\rho}{Dt} \frac{E}{\rho} + \rho \left(\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} + \frac{D\Psi}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} - \dot{Q}_{\text{cool}} \right), \quad (4.23)$$

where,

$$\frac{DE}{Dt} \equiv \frac{\partial E}{\partial t} + \mathbf{u} \cdot \nabla E \quad (4.24)$$

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \quad (4.25)$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} = -\nabla p - \rho \nabla \Psi \quad (4.26)$$

$$\frac{D\Psi}{Dt} \equiv \frac{\partial \Psi}{\partial t} + \mathbf{u} \cdot \nabla \Psi. \quad (4.27)$$

Substituting Eqs. (4.24-4.27) into Eq. (4.23) it follows that

$$\begin{aligned} \frac{DE}{Dt} &= -\frac{E}{\rho} \rho \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p - \rho \mathbf{u} \cdot \nabla \Psi + \rho \frac{\partial \Psi}{\partial t} + \rho \mathbf{u} \cdot \nabla \Psi - \frac{p}{\rho} \rho \nabla \cdot \mathbf{u} - \rho \dot{Q}_{\text{cool}} \\ \implies \frac{\partial E}{\partial t} + \mathbf{u} \cdot \nabla E &= -(E + p) \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p + \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{\text{cool}}, \end{aligned} \quad (4.28)$$

which gives the *Energy equation*,

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] = \rho \frac{d\Psi}{dt} - \rho \dot{Q}_{\text{cool}}.} \quad (4.29)$$

In many settings, $\partial\Psi/\partial t = 0$, i.e. Ψ depends on position only. If, further, we have no cooling ($\dot{Q}_{\text{cool}} = 0$), then this equation expresses the conservation of energy in which the Eulerian change in total energy density E is driven by the divergence of the enthalpy flux $(E + p)\mathbf{u}$.

4.3 Heating and Cooling Processes

The \dot{Q}_{cool} term in the energy equation describes processes that locally cool ($\dot{Q}_{\text{cool}} > 0$) or locally heat ($\dot{Q}_{\text{cool}} < 0$) the fluid. There are many such processes and a full discussion of them would be lengthy. Here, we discuss just a small number of important cases.

1. *Cooling by radiation*: energy carried away from fluid by photons.

- Energy loss by recombination of an ionized gas (line emission as electrons cascade down energy levels);
- Energy loss by free-free emission (free electrons accelerated in electric fields of ions)

$$L_{\text{ff}} \propto n_e n_p T^{1/2}. \quad (4.30)$$

- Collisionally-excited atomic line radiation (electron collides with atom in ground state \rightarrow produces excited atomic state which returns to ground state by emitting a photon with energy χ)

$$L_e \propto n_e n_{\text{ion}} e^{-\chi/kT} \chi / \sqrt{T} \quad (4.31)$$

In cold gas clouds with $T \sim 10^4$ K, H cannot be excited so cooling occurs through trace species (O^+ , O^{++} , N^+).

These are all two-body interactions \implies cooling rate per unit volume proportional to ρ^2 . Recalling that \dot{Q}_{cool} is defined per unit mass, such processes give $\dot{Q}_{\text{cool}} = \rho f(T)$.

2. *Heating by cosmic rays*: heating and energy transport via high-energy (often relativistic) particles that are diffusing/streaming through the fluid.

- High energy particles ionise atoms in fluid, excess energy put into freed e^- which ends up as heat in fluid.

$$\begin{aligned} \text{ionisation rate per unit volume} &\propto \text{CR flux} \times \rho \\ \implies \dot{Q}_{\text{cool}} &\propto \text{CR flux. (independent of } \rho) \end{aligned} \quad (4.32)$$

Combining these cases, we can parametrise \dot{Q}_{cool} as:

$$\dot{Q}_{\text{cool}} = \underbrace{A\rho T^\alpha}_{\text{radiative cooling}} - \underbrace{H}_{\text{CR heating}}, \quad (4.33)$$

where α depends upon the physics of the dominant radiative cooling process.

4.4 Energy Transport Processes

Transport processes move energy through the fluid. Important examples are:

1. *Thermal conduction*: transport of thermal energy by diffusion of the hot e^- into cooler regions. Relevant in, for example
 - Interiors of white dwarfs;
 - Supernova shock fronts;
 - ICM plasma.

There is also thermal conduction associated with ions, but it is smaller than the electron thermal conduction by a factor of $\sqrt{m_{\text{ion}}/m_e} \sim 43$.

The energy flux per unit area is

$$\mathbf{F}_{\text{cond}} = -\kappa \nabla T, \quad (4.34)$$

where κ is thermal conductivity (computed from kinetic theory).

The rate of change of E per unit volume is

$$-\nabla \cdot \mathbf{F}_{\text{cond}} = \kappa \nabla^2 T. \quad (4.35)$$

2. *Convection*: transport of energy due to fluctuating or circulating fluid flows in presence of entropy gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.
3. *Radiation transport*: relevant in optically-thick systems (mean free path of photon much shorter than size of system).

If scattering opacity dominates, then we have radiative diffusion. If ϵ_{rad} is the energy density of the radiation field, the radiative flux through the fluid is

$$\mathbf{F}_{\text{rad}} \propto -\nabla \epsilon_{\text{rad}}. \quad (4.36)$$

The general topic of radiation transport through a fluid flow is very complex and beyond the scope of this course.

CHAPTER 5

Hydrostatic Equilibrium, Atmospheres and Stars

We now have the full set of equations describing the dynamics of an ideal (inviscid, dilute, unmagnetized) non-relativistic fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{Continuity equation (2.12)} \quad (5.1)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \rho \mathbf{g} \quad \text{Momentum equation (2.23)} \quad (5.2)$$

$$\nabla^2 \Psi = 4\pi G \rho \quad \text{Poisson's equation (3.10)} \quad (5.3)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = \rho \frac{d\Psi}{dt} - \rho \dot{Q}_{\text{cool}} \quad \text{Energy equation (4.29)} \quad (5.4)$$

$$E = \rho \left(\frac{1}{2} u^2 + \Psi + \mathcal{E} \right) \quad \text{Definition of total energy (4.22)} \quad (5.5)$$

$$p = \frac{k_B}{\mu m_p} \rho T \quad \text{EoS for ideal gas (4.1)} \quad (5.6)$$

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho} \quad \text{Internal energy for ideal gas (6.66)} \quad (5.7)$$

We proceed to use those equations to explore astrophysically relevant situations.

This chapter starts with the simplest, but important, case – fluid systems that are in static equilibrium with pressure forces balancing gravity.

5.1 Hydrostatic Equilibrium

A fluid system is in a state of *hydrostatic equilibrium* if

$$\mathbf{u} = 0, \quad \frac{\partial}{\partial t} = 0. \quad (5.8)$$

Then, the continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (5.9)$$

The momentum equation gives

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \rho \nabla \Psi = 0, \quad (5.10)$$

resulting in the *equation of hydrostatic equilibrium*

$$\boxed{\frac{1}{\rho} \nabla p = -\nabla \Psi} \quad (5.11)$$

Assuming a barotropic equation of state $p = p(\rho)$, this system of equations can be solved.

If a system is self-gravitating (rather than having an externally imposed gravitational field), we also have

$$\nabla^2 \Psi = 4\pi G \rho. \quad (5.12)$$

This must be solved together with the equation of hydrostatic equilibrium.

5.1.0.1 Example: Isothermal atmosphere with constant (externally imposed) \mathbf{g}

Suppose $\mathbf{g} = -g\hat{\mathbf{z}}$. Then the equation of hydrostatic equilibrium with an isothermal equation of state reads

$$\begin{aligned} \text{Isothermal} \quad \implies \quad p &= \frac{\mathcal{R}_*}{\mu} \rho T \quad \implies \quad p = A\rho, \quad A = \text{const.} \\ A \frac{1}{\rho} \nabla \rho &= -\nabla \Psi = -g\hat{\mathbf{z}} \\ \implies \quad \ln \rho &= -\frac{gz}{A} + \text{const.} \\ \therefore \rho &= \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T} z\right), \end{aligned} \quad (5.13)$$

i.e. exponential atmosphere.

Examples of this is the Earth's atmosphere: $T \sim 300 \text{ K}$ and $\mu \sim 28 \implies$ e-folding $\sim 9 \text{ km}$. The highest astronomical observatories are at $z \sim 4 \text{ km}$, so have ρ and $p \sim 60\%$ of sea level.

5.1.0.2 Example: Vertical density structure of an isothermal, rotationally-supported, geometrically-thin gas disk orbiting a central mass.

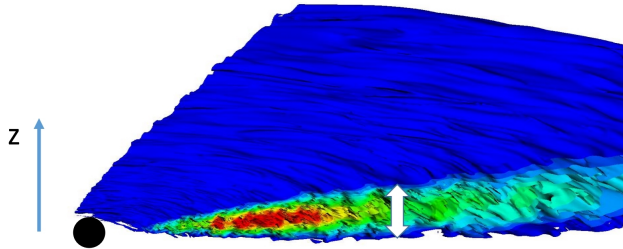


Fig. 5.1: Caption

At a given patch of the disk, transform into a locally co-moving and co-rotating frame. In z -direction, pressure forces balance z -component of gravity,

$$g_z = -\frac{GM}{r^2} \frac{z}{r} \approx -\frac{GMz}{R^3}. \quad (5.14)$$

So, hydrostatic equilibrium gives,

$$\begin{aligned}
 & \frac{1}{\rho} \frac{\partial p}{\partial z} = g_z \\
 \implies & A \frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{GM}{R^3} z \\
 \implies & \ln \rho = -\frac{GM}{2R^3} z^2 + \text{const.} \\
 \implies & \rho = \rho_0 \exp\left(-\frac{GM z^2}{2R^3 A}\right) \\
 \therefore \rho = \rho_0 \exp\left(-\frac{\Omega^2}{2A} z^2\right) \quad \text{where} \quad \Omega^2 = \frac{GM}{R^3}
 \end{aligned} \tag{5.15}$$

5.1.0.3 Example: Isothermal self-gravitating slab

Consider a static, isothermal slab in x and y which is symmetric about $z = 0$ (e.g. two clouds collide and generate a shocked slab of gas between them).

$$\text{Isothermal} \implies p = \frac{\mathcal{R}_*}{\mu} \rho T \implies p = A\rho, \quad A = \text{const.}$$

also, $\nabla = \partial/\partial z$ due to symmetry, $p = p(z)$, $\Psi = \Psi(z)$.

Then the equation of hydrostatic equilibrium becomes

$$\begin{aligned}
 & A \frac{1}{\rho} \nabla \rho = -\nabla \Psi \\
 \implies & A \frac{d}{dz} (\ln \rho) = -\frac{d\Psi}{dz} \\
 \implies & \Psi = -A \ln(\rho/\rho_0) + \Psi_0 \quad (\rho_0 = \rho(z=0)) \\
 \therefore \rho = \rho_0 e^{-(\Psi - \Psi_0)/A}.
 \end{aligned} \tag{5.16}$$

Since $A \propto T$, we note that this last equation has the form of a Boltzmann distribution.

Poisson's equation is

$$\frac{d^2 \Psi}{dz^2} = 4\pi G \rho_0 e^{-(\Psi - \Psi_0)/A}. \tag{5.17}$$

Let's change variables to $\chi = -(\Psi - \Psi_0)/A$, $Z = z\sqrt{2\pi G \rho_0/A}$ so that Poisson's equation becomes

$$\begin{aligned}
 & \frac{d^2 \chi}{dZ^2} = -2e^\chi \quad \chi = \frac{d\chi}{dZ} = 0 \quad \text{at} \quad Z = 0 \\
 \implies & \frac{d\chi}{dZ} \frac{d^2 \chi}{dZ^2} = -2 \frac{d\chi}{dZ} e^\chi \\
 \implies & \frac{1}{2} \frac{d}{dZ} \left[\left(\frac{d\chi}{dZ} \right)^2 \right] = -2 \frac{d}{dZ} (e^\chi) \\
 \implies & \left(\frac{d\chi}{dZ} \right)^2 = C_1 - 4e^\chi.
 \end{aligned} \tag{5.18}$$

But we have boundary condition $d\chi/dZ = 0$ when $\chi = 0 \implies C_1 = 4$.

$$\therefore \frac{d\chi}{dZ} = 2\sqrt{1-e^\chi} \implies \int \frac{d\chi}{\sqrt{1-e^\chi}} = 2 \int dZ. \quad (5.19)$$

Change variables $e^\chi = \sin^2 \theta$

$$\implies e^\chi d\chi = 2 \sin \theta \cos \theta d\theta \quad \text{or} \quad d\chi = \frac{2 \cos \theta}{\sin \theta} d\theta. \quad (5.20)$$

So, we can evaluate the χ integral

$$\begin{aligned} \int \frac{d\chi}{\sqrt{1-e^\chi}} &= \int \frac{2 \cos \theta d\theta}{\sin \theta \sqrt{1-\sin^2 \theta}} \\ &= \int \frac{2 d\theta}{\sin \theta} \\ &= \int 2 \frac{1}{2} \frac{1+t^2}{t} d\theta \\ &= 2 \int \frac{dt}{t} \\ &= 2 \ln t + C_2, \end{aligned} \quad (5.21)$$

by setting

$$t = \tan \frac{\theta}{2} \implies dt = \frac{1}{2}(1+t^2) d\theta, \quad (5.22)$$

and by noting

$$\sin \theta \equiv 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{2t}{1+t^2} = e^{\chi/2}. \quad (5.23)$$

So, Poisson's equation becomes

$$2 \ln t = 2Z + C_2. \quad (5.24)$$

Now, $\chi = 0$ at $Z = 0 \implies \theta = \pi/2, t = 1 \implies C_2 = 0$, so $t = e^Z$

$$\implies \sin \theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}. \quad (5.25)$$

This gives

$$\boxed{\Psi - \Psi_0 = 2A \ln \left[\cosh \left(\sqrt{\frac{2\pi G \rho_0}{A}} z \right) \right]} \quad (5.26)$$

$$\boxed{\rho = \frac{\rho_0}{\cosh^2 \left(\sqrt{\frac{2\pi G \rho_0}{A}} z \right)}} \quad (5.27)$$

5.2 Stars as Self-Gravitating Polytropes

Consider a spherically-symmetric self-gravitating system in hydrostatic equilibrium; from now on we will refer to this as a “star”. We have

$$\begin{aligned} \nabla p &= -\rho \nabla \Psi \\ \implies \frac{dp}{dr} &= -\rho \frac{d\Psi}{dr}. \quad (\text{spherical polar}) \end{aligned} \quad (5.28)$$

Now, $\rho > 0$ within a star $\implies p$ is a monotonic function of Ψ . Also

$$\frac{dp}{dr} = \frac{dp}{d\Psi} \frac{d\Psi}{dr} = -\rho \frac{d\Psi}{dr} \implies \rho = -\frac{dp}{d\Psi}. \quad (5.29)$$

So ρ is a monotonic function of Ψ .

$$\therefore p = p(\Psi), \quad \rho = \rho(\Psi) \implies p = p(\rho), \quad (5.30)$$

i.e. non-rotating stars are barotropes!

A barotropic EoS can be written as

$$p = K\rho^{1+1/n}, \quad (5.31)$$

where in general $n = n(\rho)$. When $n = \text{constant}$, we say that we have a *polytropic* EoS and the structure is called a *polytrope*. Real stars are in fact well approximated as polytropes.

It is important to note that in general we will have

$$1 + \frac{1}{n} \neq \gamma. \quad (5.32)$$

We only have $1 + 1/n = \gamma$ (i.e. $p \propto \rho^\gamma$) if the star is isentropic (constant entropy throughout) due to, for example, mixing by convective motions.

Assuming a polytropic EoS, the equation of hydrostatic equilibrium gives

$$\begin{aligned} -\nabla\Psi &= \frac{1}{\rho} \nabla(K\rho^{1+1/n}) = (n+1) \nabla(K\rho^{1/n}) \\ \implies \rho &= \left(\frac{\Psi_T - \Psi}{(n+1)K} \right)^n, \quad \Psi_T \equiv \Psi \text{ where } \rho = 0, \text{ the surface.} \end{aligned} \quad (5.33)$$

If the central density is ρ_c and central potential is Ψ_c , we have

$$\rho_c = \left(\frac{\Psi_T - \Psi_c}{(n+1)K} \right)^n, \quad (5.34)$$

so we can write,

$$\rho = \rho_c \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c} \right)^n. \quad (5.35)$$

Feeding this into Poisson's equation gives

$$\nabla^2\Psi = 4\pi G\rho_c \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c} \right)^n \quad (5.36)$$

Define $\theta = (\Psi_T - \Psi)/(\Psi_T - \Psi_c)$, we then get

$$\nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c} \theta^n. \quad (5.37)$$

In spherical polars, this becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c} \theta^n. \quad (5.38)$$

Defining a scaled radial coordinate $\xi = r\sqrt{(4\pi G\rho_c)/(\Psi_T - \Psi_c)}$, we finally get the *Lane-Emden equation of index n* ,

$$\boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.} \quad (5.39)$$

The appropriate boundary conditions for the Lane-Emden equation are $\theta = 1$, $d\theta/d\xi = 0$ at $\xi = 0$. (Zero force at $\xi = 0$, enclosed mass $\rightarrow 0$ as $\xi \rightarrow 0$.)

The Lane-Emden equation can be solved analytically for $n = 0, 1$ and 5 ; otherwise solve numerically.

5.2.1 Solution for $n = 0$

This is a somewhat singular case, physically corresponding to a fluid that is at constant density and incompressible.

$$\begin{aligned} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) &= -\theta^n = -1 \\ \implies \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) &= -\xi^2 \\ \implies \xi^2 \frac{d\theta}{d\xi} &= -\frac{1}{3}\xi^3 - C \\ \therefore \theta &= -\frac{\xi^2}{6} + \frac{C}{\xi} + D. \end{aligned} \quad (5.40)$$

We need $\theta = 1$ at $\xi = 0 \implies C = 0, D = 1$ and hence

$$\theta = 1 - \frac{\xi^2}{6}. \quad (5.41)$$

5.3 Isothermal Spheres (Case $n \rightarrow \infty$)

The isothermal case $p = K\rho$ corresponds to $n \rightarrow \infty$). Let's combine

$$\begin{aligned} \frac{dp}{dr} &= -\rho \frac{d\Psi}{dr} \quad \text{and} \quad p = K\rho \\ \implies \frac{d\Psi}{dr} &= -\frac{K}{\rho} \frac{d\rho}{dr} \\ \implies \Psi - \Psi_c &= -K \ln(\rho/\rho_c). \end{aligned} \quad (5.42)$$

From Poisson's equation

$$\begin{aligned} \nabla^2 \Psi &= 4\pi G\rho \\ \implies \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi}{dr} \right) &= 4\pi G\rho \\ \implies \frac{K}{r^2} \frac{d}{dr} \left(r^2 \frac{1}{\rho} \frac{d\rho}{dr} \right) &= -4\pi G\rho. \end{aligned} \quad (5.43)$$

Let $\rho = \rho_c e^{-\psi}$ (defining $\psi = \Psi/K$, and $\Psi_c = 0$), we set

$$r = a\xi, \quad a = \sqrt{\frac{K}{4\pi G\rho_c}}, \quad (5.44)$$

then,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi}, \quad (5.45)$$

with $\psi = \frac{d\psi}{d\xi} = 0$ at $\xi = 0$.

This replaces the Lane-Emden equation in the case where the system is isothermal.

At large radii, this has solutions of the form $\rho \propto r^{-2}$, so the enclosed mass $\propto r$. Thus, the mass of an isothermal sphere of self-gravitating gas tends to ∞ as the radius tends to ∞ . This is why we cannot adopt our usual convention of defining $\Psi = 0$ at ∞ .

So, to be physical, isothermal spheres need to be truncated at some finite radius. There needs to be some confining pressure by an external medium. These are called Bonnor-Ebert spheres, whose density profile depends on ξ_{cut} . E.g. dense gas cores in molecular clouds are well fitted by such Bonnor-Ebert spheres.

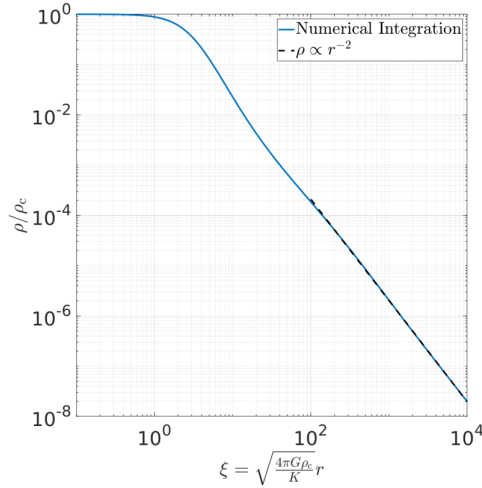


Fig. 5.2: Caption

5.4 Scaling Relations

In many circumstances, stars behave as polytropes, e.g. fully convective stars with $p(\rho)$ close to the adiabatic relation. In such a star, assuming the gas is monatomic with $\gamma = 5/3$, we have $p = K\rho^{5/3} \implies n = 3/2$.

Consider a set of stars which share a given polytropic index n and a given constant K . They will then form a one-parameter family characterised by their central density ρ_c .

Thus one can find how mass and radius vary as a function of ρ_c and, eliminating ρ_c , obtain *scaling relations* relating the mass and radius.

All stars with given n have the same $\theta(\xi)$ since the Lane-Emden equation does not depend on ρ_c . Recall the relations

$$\rho = \left(\frac{\Psi_T - \Psi}{(n+1)K} \right)^n \implies \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n} \quad (5.46)$$

$$\xi = \sqrt{\frac{4\pi G \rho_c}{\Psi_T - \Psi_c}} r \implies \xi = \sqrt{\frac{4\pi G \rho_c^{1-1/n}}{(1+n)K}} r \quad (5.47)$$

$$\rho = \rho_c \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c} \right)^n = \rho_c \theta^n. \quad (5.48)$$

The surface of the polytrope is at $\xi = \xi_{\max}$ defined as location where we have $\theta(\xi) = 0$. Let r_{\max} be the corresponding physical radius. Then the total mass of the polytrope is

$$\begin{aligned} M &= \int_0^{r_{\max}} 4\pi r^2 \rho \, dr \\ &= 4\pi \rho_c \left[\frac{4\pi G \rho_c^{1-1/n}}{(1+n)K} \right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n \xi^2 \, d\xi}_{\text{same for all polytrope of index } n} \\ \therefore M &\propto \rho_c^{\frac{1}{2}(\frac{3}{n}-1)}. \end{aligned} \quad (5.49)$$

From the definition of ξ above in Eq. (5.47), we also know that

$$r_{\max} \propto \rho_c^{\frac{1}{2}(\frac{1}{n}-1)}. \quad (5.50)$$

Eliminating ρ_c thus gives the *mass-radius relation for polytropic stars*

$$\boxed{M \propto R^{\frac{3-n}{1-n}}}. \quad (5.51)$$

For $\gamma = 5/3$, $n = 3/2$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EoS is due to e^- degeneracy pressure rather than gas pressure). But for most main-sequence stars we observe $M \propto R$.

The reason is that stars do not share the same polytropic constant K . Let's write the temperature at the core in terms of the central density and K

$$\left. \begin{aligned} p &= K \rho^{1+1/n} \\ p &= \frac{\mathcal{R}_*}{\mu} \rho T \end{aligned} \right\} \implies T_c = \frac{\mu K}{\mathcal{R}_*} \rho_c^{1/n}. \quad (5.52)$$

Nuclear reactions in the core tend to keep T_c similar in the cores of stars of different masses. So we can say that

$$K \propto \rho_c^{-1/n}. \quad (5.53)$$

Substituting this into the above expression for mass gives

$$M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \quad \implies \quad M \propto R. \quad (5.54)$$

When can the $K = \text{const.}$ relation be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust. The time to adjust to new hydrostatic equilibrium is roughly the time for a sound wave (Chapter 6) to propagate across the star, which is less than a day for the Sun,

$$t_{\text{hd}} \sim R/c_s < 1 \text{ day}. \quad (5.55)$$

The thermal timescale, on which a star can lose a significant amount of energy, is

$$t_{\text{th}} \sim \frac{\text{energy content of the star}}{\text{luminosity}} \sim \frac{GM^2}{RL}, \quad (5.56)$$

which is $\sim 30 \text{ Myr}$ for the Sun. So, mass loss/gain is followed by rapid readjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time

5.4.0.1 Example: Spherical rotating star

A spherical rotating star with angular velocity Ω gains non-rotating mass. How does Ω evolve?

Conservation of angular momentum gives $MR^2\Omega = \text{const.}$ So, if $\Omega \rightarrow \Omega + \Delta\Omega$ then

$$\begin{aligned} MR^2\Delta\Omega + \Omega\Delta(MR^2) &= 0 \\ \implies \frac{\Delta\Omega}{\Omega} &= -\frac{\Delta(MR^2)}{MR^2}. \end{aligned} \quad (5.57)$$

But we can use

$$R \propto M^{\frac{1-n}{3-n}} \quad (5.58)$$

to say

$$\begin{aligned} \frac{\Delta\Omega}{\Omega} &\propto -\Delta\left(M^{\frac{5-3n}{3-n}}\right) \\ \implies \frac{\Delta\Omega}{\Omega} &\propto -\left(\frac{5-3n}{3-n}\right)\Delta M, \end{aligned} \quad (5.59)$$

so,

$$\Delta M > 0 \quad \implies \quad \begin{cases} \Delta\Omega < 0 & \text{if } \frac{5-3n}{3-n} > 0 \quad (\text{e.g. } n = \frac{3}{2}) \text{ Spin down} \\ \Delta\Omega > 0 & \text{if } \frac{5-3n}{3-n} < 0 \quad (\text{e.g. } n = \frac{3}{2}) \text{ Spin up} \end{cases}. \quad (5.60)$$

5.4.0.2 Example: Star in a binary system

A star in a binary system loses mass to its companion. The donor star loses mass, $\Delta M < 0$. So since $R \propto M^{(1-n)/(3-n)}$, the radius will increase if $1 < n < 3$.

So there is the potential for unstable (runaway) mass transfer (but need to look at evolution of the size of the Roche lobe to conclusively decide whether the process is unstable).

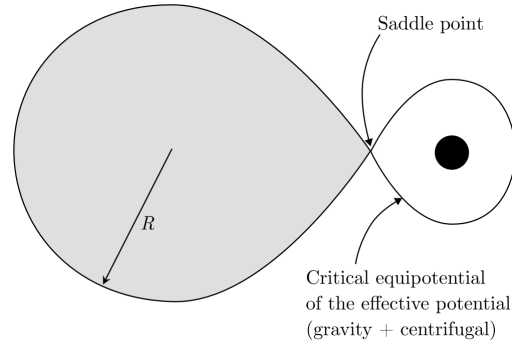


Fig. 5.3: Roche lobe overflow

CHAPTER 6

Sound Waves, Supersonic Flows and Shock Waves

6.1 Sound Waves

We now start discussion of how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (6.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p. \quad (6.2)$$

The equilibrium around which we will perturb is

$$\begin{aligned} \rho &= \rho_0 \quad (\text{uniform and constant}) \\ p &= p_0 \quad (\text{uniform and constant}) \\ \mathbf{u} &= \mathbf{0}. \end{aligned} \quad (6.3)$$

We consider small perturbations and write in Lagrangian terms (Lagrangian meaning the change of quantities are for a *given fluid element*)

$$\begin{aligned} p &= p_0 + \Delta p \\ \rho &= \rho_0 + \Delta \rho \\ \mathbf{u} &= \Delta \mathbf{u}. \end{aligned} \quad (6.4)$$

The relation between Lagrangian and Eulerian perturbations is:

$$\underbrace{\delta \rho}_{\text{Eulerian perturbation}} = \underbrace{\Delta \rho}_{\text{Lagrangian perturbation}} - \underbrace{\xi \cdot \nabla \rho_0}_{\substack{\text{Element displacement dot} \\ \text{Gradient of unperturbed state}}} \quad (6.5)$$

In the present example, $\nabla \rho_0 = 0$ and so $\delta \rho = \Delta \rho$, but the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we will address later.

Substitute the perturbations into the fluid equations and ignore terms that are 2nd order (or higher) in the perturbed quantities:

Start with continuity equation:

$$\begin{aligned}
 & \frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \nabla \cdot [(\rho_0 + \Delta\rho)\Delta\mathbf{u}] = 0 \\
 \Rightarrow & \underbrace{\frac{\partial\rho_0}{\partial t}}_{=0} + \frac{\partial\Delta\rho}{\partial t} + \underbrace{\nabla\rho_0 \cdot \Delta\mathbf{u}}_{=0} + \underbrace{\nabla(\Delta\rho) \cdot \Delta\mathbf{u}}_{\text{2nd order}} + \underbrace{\rho_0 \nabla \cdot (\Delta\mathbf{u})}_{\text{2nd order}} + \underbrace{\Delta\rho \nabla \cdot (\Delta\mathbf{u})}_{\text{2nd order}} = 0 \\
 & \therefore \frac{\partial}{\partial t}(\Delta\rho) + \rho_0 \nabla \cdot (\Delta\mathbf{u}) = 0
 \end{aligned} \tag{6.6}$$

And similarly, the momentum equation:

$$\begin{aligned}
 & \frac{\partial}{\partial t}(\Delta\mathbf{u}) = -\frac{1}{\rho_0} \nabla(\Delta p) \\
 \therefore \frac{\partial}{\partial t}(\Delta\mathbf{u}) = - \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} \frac{\nabla(\Delta p)}{\rho_0}, \quad \text{assuming barotropic EoS.}
 \end{aligned} \tag{6.7}$$

Now, taking the partial derivative of Eq. (6.6) with respect to time,

$$\begin{aligned}
 \frac{\partial^2}{\partial t^2}(\Delta\rho) &= -\rho_0 \frac{\partial}{\partial t}[\nabla \cdot (\Delta\mathbf{u})] \\
 &= -\rho_0 \nabla \cdot \left[\frac{\partial}{\partial t}(\Delta\mathbf{u}) \right] \\
 &= \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} \nabla^2(\Delta\rho).
 \end{aligned} \tag{6.8}$$

Thus we arrive at the wave equation

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} \nabla^2(\Delta\rho).} \tag{6.9}$$

This admits solutions of the form $\Delta\rho = \Delta\rho_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$. Substituting into the wave equation we get

$$\begin{aligned}
 (-i\omega)^2 \Delta\rho_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} &= \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} (ik)^2 \Delta\rho_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\
 \therefore \omega^2 &= \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} k^2.
 \end{aligned} \tag{6.10}$$

The (phase) speed of the wave is $v_p = \omega/k$, so the sound wave travels at speed determined by the derivative of $p(\rho)$

$$c_s = \sqrt{\left. \frac{dp}{d\rho} \right|_{\rho=\rho_0}}. \tag{6.11}$$

Consider a 1D wave and substitute

$$\begin{aligned}
 \Delta\rho &= \Delta\rho_0 e^{i(kx - \omega t)} \\
 \Delta u &= \Delta u_0 e^{i(kx - \omega t)}
 \end{aligned} \tag{6.12}$$

into Eq. (6.6). We get

$$\begin{aligned} -i\omega\Delta\rho + \rho_0 ik\Delta u &= 0 \\ \implies \Delta u &= \frac{\omega}{k} \frac{\Delta\rho}{\rho_0} = c_s \frac{\Delta\rho}{\rho_0}. \end{aligned} \quad (6.13)$$

So we learn that

- Fluid velocity and density perturbations are in phase (since $\Delta u/\Delta\rho \in \mathbb{R}$);
- A disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small, since

$$\Delta u_0 = c_s \frac{\Delta\rho_0}{\rho_0} \ll c_s. \quad (6.14)$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes acceleration of the fluid elements, this induces further density perturbations, making disturbances propagate.

Sound speed depends on how the pressure forces react to density changes. If the EoS is “stiff” (i.e. high $dp/d\rho$), then restoring force is large and propagation is rapid.

6.1.1 Examples of $dp/d\rho$

Notes about these two examples:

- We see that $c_{s,I}$ and $c_{s,A}$ differ by only $\sqrt{\gamma}$;
- Thermal behaviour of the perturbations does *not* have to be the same as that of the unperturbed structure! E.g. in the Earth’s atmosphere, the background is approximately isothermal but sound waves are adiabatic.
- Waves for which c_s is not a function of ω are called non-dispersive. The shape of a wave packet is preserved.

6.1.1.1 Isothermal Case

$$c_s^2 = \left. \frac{dp}{d\rho} \right|_T \quad (6.15)$$

In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant T . Then

$$\begin{aligned} p &= \frac{\mathcal{R}_*}{\mu} \rho T \\ \therefore c_{s,I} &= \sqrt{\frac{\mathcal{R}_* T}{\mu}}. \end{aligned} \quad (6.16)$$

6.1.1.2 Adiabatic Case

$$c_s^2 = \left. \frac{dp}{d\rho} \right|_S \quad (6.17)$$

No heat exchange between fluid elements; compressions heat up and rarefactions cool down from $p dV$ work. So

$$\begin{aligned} p &= K \rho^\gamma \\ \Rightarrow \left. \frac{dp}{d\rho} \right|_S &= \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho} \end{aligned} \quad (6.18)$$

$$\therefore c_{s,A} = \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}}. \quad (6.19)$$

6.2 Sound Waves in a Stratified Atmosphere

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant $\mathbf{g} = -g\hat{\mathbf{z}}$.

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on z -dependent terms, taking $\mathbf{u} = u\hat{\mathbf{z}}$. The continuity and momentum equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \quad (6.20)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \quad (6.21)$$

and the equilibrium is

$$\begin{aligned} u_0 &= 0 \\ \rho_0(z) &= \tilde{\rho} e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu} \\ p_0(z) &= \frac{\mathcal{R}_* T}{\mu} \rho_0(z) = \tilde{p} e^{-z/H}. \end{aligned} \quad (6.22)$$

Consider a Lagrangian perturbation:

$$\begin{aligned} u &\rightarrow \Delta u \\ \rho_0 &\rightarrow \rho_0 + \Delta \rho \\ p_0 &\rightarrow p_0 + \Delta p. \end{aligned} \quad (6.23)$$

Remember from Eq. (6.5) that $\delta\rho = \Delta\rho - \boldsymbol{\xi} \cdot \nabla\rho$. So we have

$$\left. \begin{aligned} \delta\rho &= \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z \frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u \end{aligned} \right\} \text{ Eulerian to Lagrangian perturbation relation,} \quad (6.24)$$

and

$$\Delta \mathbf{u} = \frac{D\boldsymbol{\xi}}{Dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla \boldsymbol{\xi}}_{2^{\text{nd}} \text{ order}} \approx \frac{\partial \boldsymbol{\xi}}{\partial t}. \quad (6.25)$$

Substituting perturbed quantities into the Eulerian continuity equation (2.12)

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial t}[(\rho_0 + \delta\rho)\delta u_z] = 0 \\ \Rightarrow & \frac{\partial}{\partial t}\left(\rho_0 + \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z}\right) + \frac{\partial}{\partial z}(\rho_0 \Delta u_z) = 0 \quad (\text{ignoring } 2^{\text{nd}} \text{ order terms}) \\ \Rightarrow & \underbrace{\frac{\partial \rho_0}{\partial t}}_{=0} + \frac{\partial \Delta\rho}{\partial t} - \frac{\partial \xi_z}{\partial t} \frac{\partial \rho_0}{\partial z} - \underbrace{\xi_z \frac{\partial}{\partial t} \frac{\partial \rho_0}{\partial z}}_{=0} + \frac{\partial \rho_0}{\partial z} \Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ \Rightarrow & \frac{\partial \Delta\rho}{\partial t} - \underbrace{\frac{\Delta u_z}{\partial \xi_z / \partial t}}_{\partial \xi_z / \partial t} \frac{\partial \rho_0}{\partial z} + \frac{\partial \rho_0}{\partial z} \Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0 \\ & \therefore \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0. \end{aligned} \quad (6.26)$$

A similar calculation for the momentum equation gives

$$\begin{aligned} & \frac{\partial \Delta u_z}{\partial t} = -\frac{1}{\rho_0} \frac{\partial \Delta p}{\partial z} \\ \Rightarrow & \frac{\partial \Delta u_z}{\partial t} = -\frac{c_u^2}{\rho_0} \frac{\partial \Delta\rho}{\partial z}, \quad c_u^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_{\rho_0}. \end{aligned} \quad (6.27)$$

To perform this calculation (which we leave as an exercise!), you need a relation that is obtained from the Lagrangian continuity equation:

$$\begin{aligned} & \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \\ \Rightarrow & \Delta\rho + \left(\rho_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \Delta t = 0 \quad (\text{integrating over a short time interval } \Delta t) \\ & \therefore \Delta\rho + \rho_0 \nabla \cdot \boldsymbol{\xi} = 0. \end{aligned} \quad (6.28)$$

Let's now derive the wave equation and dispersion relation. Take the partial derivative of Eq. (6.26) with respect to time

$$\begin{aligned} & \frac{\partial^2 \Delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial z} \left(\frac{\partial \Delta u_z}{\partial t} \right) = 0 \\ \Rightarrow & \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{\partial}{\partial z} \left(\frac{c_u^2}{\rho_0} \frac{\partial \Delta\rho}{\partial z} \right) = 0, \end{aligned} \quad (6.29)$$

where the last step involved substitution from Eq. (6.27). If the medium is isothermal, then c_u is independent of z . So,

$$\begin{aligned} & \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{c_u^2}{\rho_0} \frac{\partial^2 \Delta\rho}{\partial z^2} + \rho_0 \frac{c_u^2}{\rho_0} \frac{\partial \rho_0}{\partial z} \frac{\partial \Delta\rho}{\partial z} = 0 \\ \therefore & \underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0} \frac{\partial \rho_0}{\partial z} \frac{\partial \Delta\rho}{\partial z}}_{\text{extra piece associated with stratification}} = 0. \end{aligned} \quad (6.30)$$

Now,

$$\begin{aligned}\frac{\partial \rho_0}{\partial z} &= \frac{\partial}{\partial z} (\tilde{\rho} e^{-z/H}) \\ &= -\frac{1}{H} \tilde{\rho} e^{-z/H} \\ &= -\frac{\rho_0}{H}.\end{aligned}\tag{6.31}$$

So,

$$\frac{\partial^2 \Delta \rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta \rho}{\partial z^2} - \frac{c_u^2}{H} \frac{\partial \Delta \rho}{\partial z} = 0.\tag{6.32}$$

Look for solutions of the form $\Delta \rho \propto e^{i(kz - \omega t)}$,

$$\implies -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{ik}{H},\tag{6.33}$$

which reveals the *dispersion relation*

$$\boxed{\omega^2 = c_u^2 \left(k^2 - \frac{ik}{H} \right)}.\tag{6.34}$$

We can also write this as

$$k^2 - \frac{ik}{H} - \frac{\omega^2}{c_u^2} = 0,\tag{6.35}$$

and solve the quadratic for $k(\omega)$ to give

$$k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}.\tag{6.36}$$

Let's take $\omega \in \mathbb{R}$. We have two cases to examine if we wish to understand the implications of this dispersion relation.

6.2.1 Case I: $\omega > c_u/2H$

Examine the real and imaginary parts of k :

$$\text{Im}\{k\} = \frac{1}{2H}\tag{6.37}$$

$$\text{Re}\{k\} = \pm \sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2}\tag{6.38}$$

So the density perturbation is

$$\Delta \rho \propto \underbrace{e^{-z/2H}}_1 \underbrace{e^{i\left(\pm\sqrt{(\omega/c_u)^2 - (1/2H)^2}z - \omega t\right)}}_2\tag{6.39}$$

corresponding to

1. Exponentially decaying amplitude with increasing height;
2. Wave with phase velocity

$$v_{\text{ph}} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm \sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2} \quad (6.40)$$

where v_{ph} is function of ω , meaning that the wave is dispersive. A wave packet consisting of different ω 's will change shape as it propagates.

As before, we can relate Δu to $\Delta \rho$:

$$\Delta u_z = \frac{\Delta \rho}{\rho_0} \frac{\omega}{k}, \quad (6.41)$$

with

$$\Delta \rho \propto e^{-z/2H} \quad (6.42)$$

$$\rho_0 \propto e^{-z/H}, \quad (6.43)$$

giving

$$\Delta u_z \propto e^{+z/2H}, \quad \frac{\Delta \rho}{\rho_0} \propto e^{+z/2H}. \quad (6.44)$$

Thus the perturbed velocity and the fractional density variation both *increase* with height. In the absence of dissipation (e.g. viscosity), the kinetic energy flux ($\propto \Delta \rho \Delta u$) is conserved and the amplitude of the wave increases until

$$\Delta u \sim c_s, \quad \frac{\Delta \rho}{\rho_0} \sim 1, \quad (6.45)$$

where the linear treatment breaks down and the sound wave “steepens” into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere!

6.2.2 Case II: $\omega < c_u/2H$

In this case, we find that k is purely imaginary. So,

$$\Delta \rho \propto e^{-|k|z} e^{i\omega t}. \quad (6.46)$$

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength, giving rise to reflections.

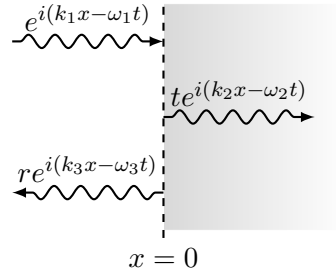


Fig. 6.1: Waves at boundary $x = 0$.

6.3 Transmission of Sound Waves at Interfaces

Consider two non-dispersive media with a boundary at $x = 0$. Suppose we have a sound wave travelling from $x < 0$ to $x > 0$. Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by r and t the amplitude of the reflected and transmitted waves, respectively.

At the boundary $x = 0$, variables must be single valued and the accelerations are finite, thus oscillations in the second medium must have the same frequency,

$$\omega_1 = \omega_2 = \omega_3 = \omega. \quad (6.47)$$

The reflected wave is in the same medium as the incident so

$$k_3 = -k_1. \quad (\text{phase speed reversed}) \quad (6.48)$$

The amplitude of a sound wave is continuous at $x = 0$ hence

$$1 + r = t, \quad (6.49)$$

and the derivative of the amplitude is continuous at $x = 0$ thus

$$k_1(1 - r) = k_2t. \quad (6.50)$$

We can combine these relations to get

$$t = \frac{2k_1}{k_1 + k_2}, \quad r = \frac{k_1 - k_2}{k_1 + k_2}. \quad (6.51)$$

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

1. If $c_{s,2} > c_{s,1}$, then $k_2 < k_1 \implies r > 0$, i.e reflected wave in phase with incident;
2. If $c_{s,2} < c_{s,1}$, then $r < 0 \implies$ reflected wave is π out of phase with incident wave;
3. If $c_{s,2} \ll c_{s,1}$, then $k_2 \gg k_1 \implies t \ll 1$, i.e. wave almost completely reflected.

6.4 Supersonic Fluids and Shocks

Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding c_s . The linear theory applied to sound waves breaks down.

When thinking about the sound speed, recall that the chemical composition of the fluid matters, $c_s \propto \mu^{-1/2}$

$$c_s \text{ in } \underbrace{\text{atomic Hydrogen}}_{\substack{\mu \approx 1 \\ \text{e.g. ISM}}} \gg c_s \text{ in } \underbrace{\text{diatomic Nitrogen}}_{\substack{\mu = 28 \\ \text{e.g. Earth's atmosphere}}}, \quad \text{for a given } T. \quad (6.52)$$

Disturbances in a fluid always propagate at the sound speed relative to the fluid itself. Consider an observer at the centre of a spherical disturbance, watching the fluid flow past at speed v .

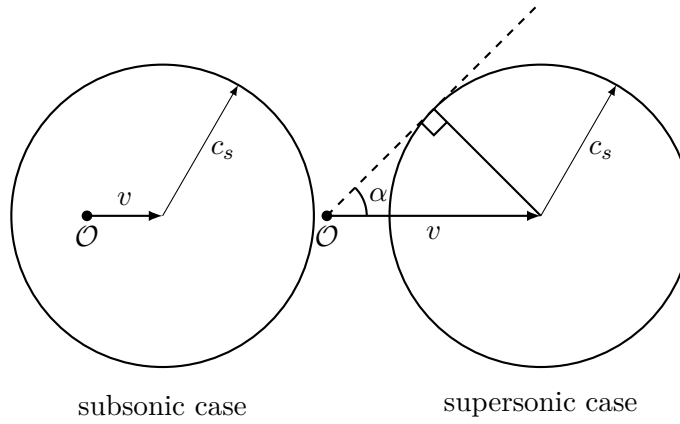


Fig. 6.2: Subsonic flow vs supersonic flow

The velocity of the disturbance relative to the observer, v' , is the vector sum of the fluid velocity and the disturbance velocity relative to the fluid.

- Subsonic case: v' sweeps 4π steradians;
- Supersonic case: disturbance always to the right. If we continuously produce a disturbance, the envelope of the disturbances will define a cone, named the *Mach cone*, with opening angle α given by

$$\sin \alpha = \frac{c_s}{v}. \quad (6.53)$$

The ratio of the flow speed to the sound speed is called the Mach number

$$M \equiv \frac{v}{c_s} \quad (6.54)$$

$$\sin \alpha = \frac{1}{M}. \quad (6.55)$$

Imagine an obstacle in a supersonic flow – disturbances cannot propagate upstream from the obstacle so the flow cannot adjust to the presence of the obstacle. The flow properties must change discontinuously once the obstacle is reached, giving a shock!

6.5 The Rankine-Hugoniot Relations

We analyse a shock by applying conservation of mass, momentum and energy across the shock front.

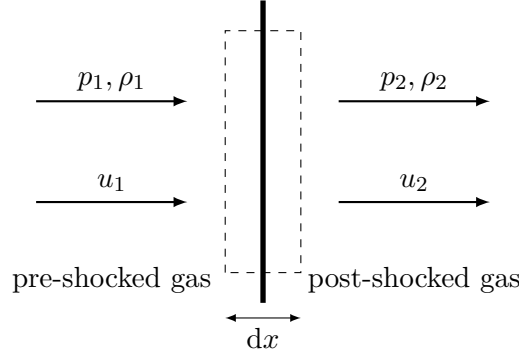


Fig. 6.3: Geometry of shock front

In the frame of the shock, let's assume the following geometry in Fig. 6.3. Continuity gives

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) &= 0 \\ \Rightarrow \frac{\partial}{\partial t} \left(\int_{-dx/2}^{dx/2} \rho dx \right) + \rho u_x \Big|_{x=dx/2} - \rho u_x \Big|_{x=-dx/2} &= 0, \end{aligned} \quad (6.56)$$

where we have integrated over a small region dx around the shock.

Let's take $dx \rightarrow 0$ and assume that mass does not continually accumulate at $x = 0$. Then

$$\frac{\partial}{\partial t} \left(\int \rho dx \right) = 0, \quad (6.57)$$

which implies the 1st *Rankine-Hugoniot relation*,

$$\boxed{\rho_1 u_1 = \rho_2 u_2.} \quad (6.58)$$

Apply similar analysis to the momentum equation:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_x) &= -\frac{\partial}{\partial x}(\rho u_x u_x + p) - \rho \frac{\partial \Psi}{\partial x} \\ \Rightarrow \frac{\partial}{\partial t} \left(\int \rho u_x dx \right) &= -(\rho u_x u_x + p) \Big|_{x=dx/2} + (\rho u_x u_x + p) \Big|_{x=-dx/2}, \end{aligned} \quad (6.59)$$

which gives the 2nd *Rankine-Hugoniot relation*,

$$\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2.} \quad (6.60)$$

We note that u_y and u_z do not change across the shock front (can be immediately seen by looking at the y - and z -components of the momentum equation).

Now for the energy equation. Start with the adiabatic case so that the gas cannot cool and hence we have $\dot{Q}_{\text{cool}} = 0$. Also take gravitational potential to have no time-dependence. Then

$$\begin{aligned}
 \frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] &= \underbrace{-\rho\dot{Q}_{\text{cool}}}_{=0} + \underbrace{\rho\frac{\partial \Psi}{\partial t}}_{=0} \\
 \implies \frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] &= 0 \\
 \implies \frac{\partial}{\partial t} \left(\int E \, dx \right) + (E + p)u_x \Big|_{x=dx/2} - (E + p)u_x \Big|_{x=-dx/2} &= 0 \\
 \implies (E_1 + p_1)u_1 &= (E_2 + p_2)u_2.
 \end{aligned} \tag{6.61}$$

Since we know from equation (4.22) that $E + p\left(\frac{1}{2}u^2 + \mathcal{E} + \Psi\right)$, this becomes

$$\frac{1}{2}\rho_1 u_1^3 + \rho_1 \mathcal{E}_1 u_1 + \rho_1 \Psi_1 u_1 + p_1 u_1 = \frac{1}{2}\rho_2 u_2^3 + \rho_2 \mathcal{E}_2 u_2 + \rho_2 \Psi_2 u_2 + p_2 u_2. \tag{6.62}$$

But $\Psi_1 = \Psi_2$ and $\rho_1 u_1 = \rho_2 u_2$, so terms involving Ψ cancel out. We are left with the 3rd Rankine-Hugoniot relation,

$$\boxed{\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}}. \tag{6.63}$$

For an ideal gas, we have

$$\left. \begin{aligned} \mathcal{E} &= C_V T \\ p &= \frac{\mathcal{R}_*}{\mu} \rho T \end{aligned} \right\} \implies \mathcal{E} = \frac{C_V \mu}{\mathcal{R}_*} \frac{p}{\rho} \tag{6.64}$$

$$\left. \begin{aligned} \gamma &= \frac{C_p}{C_V} \\ C_p - C_V &= \frac{\mathcal{R}_*}{\mu} \end{aligned} \right\} \implies C_V(\gamma - 1) = \frac{\mathcal{R}_*}{\mu}, \tag{6.65}$$

which combine to give the internal energy per unit mass

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho}}. \tag{6.66}$$

If we assume that γ does not change across the shock (e.g. there is no disassociation of molecules), the 3rd R-H relation becomes

$$\begin{aligned}
 \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} &= \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} \\
 \implies \frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} &= \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma - 1},
 \end{aligned} \tag{6.67}$$

since, for the adiabatic case, recall from (6.18) the sound speed is

$$c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_S = \frac{\gamma p}{\rho}. \tag{6.68}$$

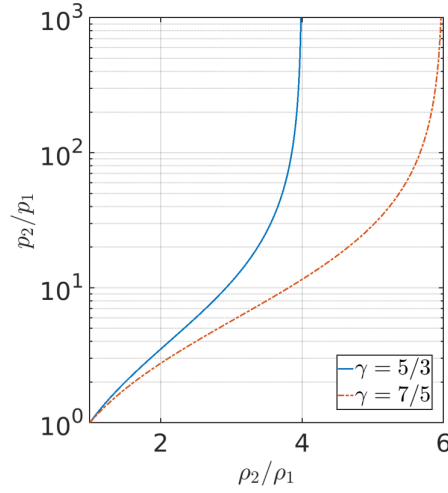


Fig. 6.4: Plot of Eq. (6.69) for various values of γ .

Using all three R-H relations and after some algebra we get

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma + 1)p_1 + (\gamma - 1)p_2}. \quad (6.69)$$

If we again define the Mach number of the incoming flow as $M_1 \equiv u_1/c_{s,1}$, then we can also now show

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \quad (6.70)$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \quad (6.71)$$

$$\frac{T_2}{T_1} = \frac{((\gamma - 1)M_1^2 + 2)(2\gamma M_1^2 - (\gamma - 1))}{(\gamma + 1)^2 M_1^2}. \quad (6.72)$$

In the limit of strong shocks, $p_2 \gg p_1$, we get

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma + 1}{\gamma - 1}. \quad (6.73)$$

For $\gamma = 5/3$, this gives $\rho_2 = 4\rho_1$. So there is a maximum possible density contrast across an adiabatic shock - with stronger and stronger shocks, the thermal pressure of the shocked gas increases and prevents further compression. Also, it follows from Eq. (6.72) that for strong shocks

$$T_2 = \frac{2\gamma(\gamma - 1)M_1^2}{(\gamma + 1)^2} T_1$$

$$\implies k_B T_2 = \frac{4(\gamma - 1)}{(\gamma + 1)^2} \left(\frac{1}{2} \mu m_p u_1^2 \right) \quad \text{since} \quad c_{s,1}^2 = \frac{\gamma k_B T_1}{\mu m_p} \quad (6.74)$$

$$\implies k_B T_2 = \frac{3}{8} \left(\frac{1}{2} \mu m_p u_1^2 \right) \quad \text{for} \quad \gamma = \frac{5}{3}. \quad (6.75)$$

This makes explicit the notion that the kinetic energy of the pre-shock fluid is being converted into random motion of the post-shock flow.

Note that, since $p_2 \gg p_1$, and $\rho_2 \leq 4\rho_1$, we have

$$\frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma} \quad \text{i.e. } K_1 \neq K_2. \quad (6.76)$$

The gas has jumped adiabats during its passage through the shock. Shocking the gas produces a non-reversible change, due to viscous processes operating within the shock.

While the R-H conditions are symmetric in the up- and down-stream quantities, the thermodynamic requirement that entropy increases dictates the direction of the jump (i.e. a fast/cold upstream flow shocking to produce a slow/fast downstream flow).

It is interesting that we can derive R-H conditions using the inviscid equations that do not explicitly include dissipation/entropy-generating terms.

Not all shocks are adiabatic! To consider the other extreme, let's discuss *isothermal shocks*. Here we have $\dot{Q}_{\text{cool}} \neq 0$ such that the shocked gas cools to produce $T_2 = T_1$. Whether a shock is isothermal or adiabatic depends on whether the “cooling length” is smaller or larger than the system size, respectively.

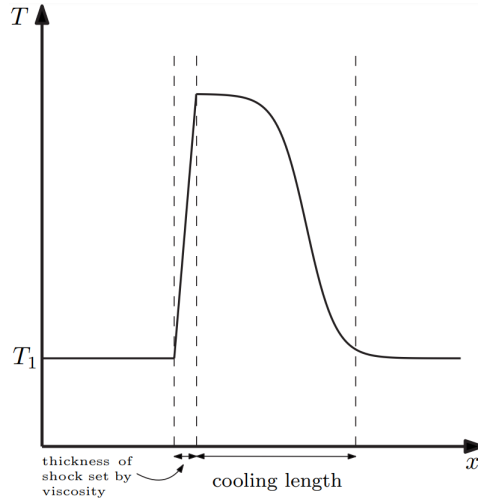


Fig. 6.5: Temperature profile through a shock

For isothermal shocks, the first two R-H equations are unchanged:

$$\rho_1 u_1 = \rho_2 u_2 \quad (6.77)$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \quad (6.78)$$

but the 3rd R-H equation is replaced by

$$T_1 = T_2. \quad (6.79)$$

Now, following from Eq. (6.16),

$$c_{s,I} = \sqrt{\frac{\mathcal{R}_* T}{\mu}} \quad \Rightarrow \quad c_{s,1} = c_{s,2} \quad (6.80)$$

$$= \sqrt{\frac{p}{\rho}} \quad \Rightarrow \quad p = c_{s,I}^2 \rho. \quad (6.81)$$

So, the 2nd R-H equation becomes

$$\begin{aligned}
 \rho_1(u_1^2 + c_s^2) &= \rho_2(u_2^2 + c_s^2) \\
 \implies u_1 + \frac{c_s^2}{u_1} &= u_2 + \frac{c_s^2}{u_2} \quad (\text{since } \rho_1 u_1 = \rho_2 u_2) \\
 \implies c_s^2 \left(\frac{1}{u_1} - \frac{1}{u_2} \right) &= u_2 - u_1 \\
 \implies c_s^2 \frac{u_2 - u_1}{u_1 u_2} &= u_2 - u_1
 \end{aligned}$$

and thus,

$$\boxed{c_s^2 = u_1 u_2.} \tag{6.82}$$

Thus we see that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \left(\frac{u_1}{c_s} \right)^2 = M_1^2, \tag{6.83}$$

where M_1 is the Mach number of the upstream flow. So the density compression can be very large.

Note that since $c_s^2 = u_1 u_2$ and $u_1 > c_s$ (condition for a shock), we must have $u_2 < c_s$. So flow behind the shock is subsonic. In fact this is always true for any shock and is necessary to preserve causality (the post shock gas must know about the shock!).

6.6 Theory of Supernova Explosions

An important application of shock wave theory is to supernova explosions in the interstellar medium (ISM). A supernova (SN) deposits about 10^{51} erg ($= 10^{44}$ J) of energy into the surrounding medium, the shocked medium expands, sweeps up more gas, and creates large bubbles in the ISM.

Consider the following system:

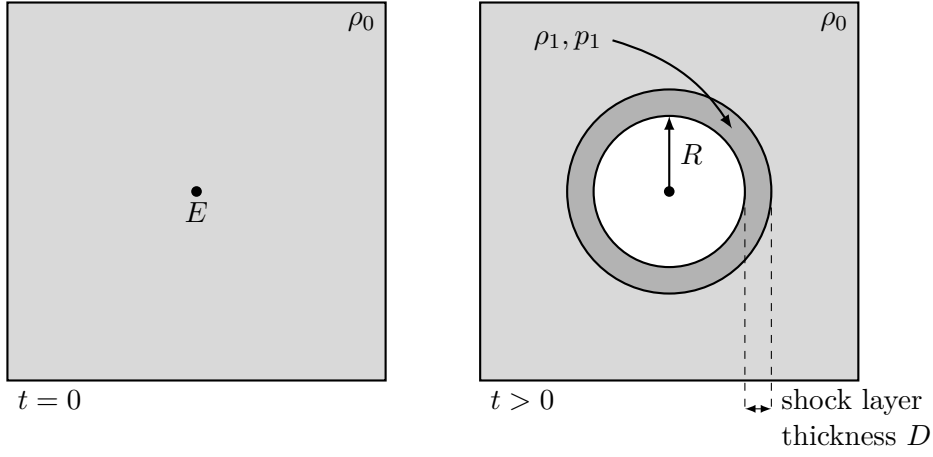
- Initially uniform density interstellar medium (ISM) at rest, with density ρ_0 ;
- Instantaneous point-like explosion with energy E ;
- Ignore temperature of the ambient ISM ($T_0 = 0$), thus no confinement of explosion by an external pressure.

Given that $T_0 = 0$, the shock has $M \rightarrow \infty$. Assuming an adiabatic shock, we sweep mass into a shell with density ρ , given by

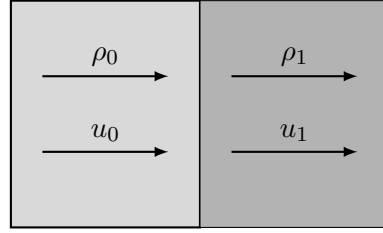
$$\rho_1 = \rho_0 \frac{\gamma + 1}{\gamma - 1}. \tag{6.84}$$

If all the mass is swept up into a shell then

$$\begin{aligned}
 \frac{4\pi}{3} \rho_0 R^3 &= 4\pi \rho_1 R^2 D \quad (\text{assuming } D \ll R) \\
 \therefore D &= \frac{1}{3} \left(\frac{\gamma + 1}{\gamma - 1} \right) R.
 \end{aligned} \tag{6.85}$$

**Fig. 6.6:** Supernova explosion

For $\gamma = 5/3$, we have $D \approx 0.08R$ which justifies the assumption $D \ll R$.

**Fig. 6.7:** Situation seen in shock frame

Assume that all gas in the shell moves with a common velocity. Fig. 6.7 shows the frame of a local patch of the shock, and so

$$\begin{aligned} \rho_0 u_0 &= \rho_1 u_1 \\ \implies u_1 &= \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0. \end{aligned} \quad (6.86)$$

Thus, relative to the unshocked gas, the velocity of the shocked gas U is

$$U = u_0 - u_1 = \frac{2u_0}{\gamma + 1}. \quad (6.87)$$

Then, the rate of change of momentum of the shocked shell is

$$\frac{d}{dt} \left[\frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma + 1} \right]. \quad (6.88)$$

This momentum gain is provided by pressure acting on the inside surface of the shell - call this p_{in} . Let's make the ansatz that this is related to the pressure within the shell by

$$p_{\text{in}} = \alpha p_1, \quad (6.89)$$

and we now relate p_1 and u_0 using the R-H jump condition: we have

$$\begin{aligned}
 p_0 + \rho_0 u_0^2 &= p_1 + \rho_1 u_1^2 \\
 \Rightarrow p_1 &= \rho_0 u_0^2 \left[1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2} \right] \quad (\text{since } p_0 = 0 \text{ by assumption}) \\
 &= \rho_0 u_0^2 \left[1 - \frac{\gamma - 1}{\gamma + 1} \right] \quad (\text{assuming a strong shock}) \\
 &= \frac{2}{\gamma + 1} \rho_0 u_0^2.
 \end{aligned} \tag{6.90}$$

So, equating the rate of change of momentum in the shocked shell to the pressure acting on the inside surface of the shell, we have

$$\begin{aligned}
 \frac{d}{dt} \left[\frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma + 1} \right] &= 4\pi R^2 p_{\text{in}} \\
 &= 4\pi R^2 \alpha p_1 \\
 &= 4\pi R^2 \alpha \frac{2}{\gamma + 1} \rho_0 u_0^2
 \end{aligned} \tag{6.91}$$

$$\begin{aligned}
 \Rightarrow \frac{d}{dt} [R^3 u_0] &= 3\alpha R^2 u_0^2 \\
 \Rightarrow \frac{d}{dt} [R^3 \dot{R}] &= 3\alpha R^2 \dot{R}^2 \quad \text{since } u_0 \equiv \dot{R}
 \end{aligned} \tag{6.92}$$

This admits solutions of the form $R \propto t^b$:

$$\begin{aligned}
 \frac{d}{dt} (t^{3b} b t^{b-1}) &= 3\alpha t^{2b} (b t^{b-1})^2 \\
 \Rightarrow b(4b - 1)t^{4b-2} &= 3\alpha b^2 t^{4b-2} \quad (\text{cancellation of } t^{4b-2} \text{ justifies assumed form of solution}) \\
 \Rightarrow b &= 0 \quad (\text{not physical}) \quad \text{or} \quad b = \frac{1}{4 - 3\alpha} \\
 \Rightarrow R \propto t^{1/(4-3\alpha)}, \quad u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \propto R^{3\alpha-3}.
 \end{aligned} \tag{6.93}$$

To determine α , we need to consider energy conservation. For an adiabatic shock, the explosion energy is conserved and transformed into kinetic and internal energy:

- Kinetic energy of the shell is

$$\frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2. \tag{6.94}$$

- Internal energy per unit mass is

$$\mathcal{E} = \frac{1}{\gamma - 1} \frac{p}{\rho}, \tag{6.95}$$

and so the internal energy per unit volume is

$$\rho \mathcal{E} = \frac{1}{\gamma - 1} p. \tag{6.96}$$

Since the shell is very thin, it has a small volume and so most of the internal energy is in the central cavity which contains little mass

$$\text{Internal energy of cavity} \approx \frac{4\pi}{3} R^3 \frac{p_{\text{in}}}{\gamma - 1} = \frac{4\pi}{3} R^3 \alpha \frac{p_1}{\gamma - 1}. \tag{6.97}$$

So, energy conservation says that

$$\begin{aligned} E &= \frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2 + \frac{4\pi}{3} R^3 \alpha \frac{p_1}{\gamma - 1} \\ &= \frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 \underbrace{\left(\frac{2u_0}{\gamma + 1} \right)^2}_{\text{Eq. (6.87)}} + \frac{4\pi}{3} R^3 \alpha \underbrace{\frac{2}{\gamma + 1} \rho_0 u_0^2}_{\text{Eq. (6.90)}} \frac{1}{\gamma - 1} \end{aligned} \quad (6.98)$$

$$= \frac{4\pi}{3} R^3 u_0^2 \left[\frac{1}{2} \rho_0 \frac{4}{(\gamma + 1)^2} + \alpha \rho_0 \frac{2}{(\gamma + 1)(\gamma - 1)} \right], \quad (6.99)$$

from which we conclude that

$$E \propto R^3 u_0^2 \propto t^{(6\alpha - 3)/(4 - 3\alpha)}. \quad (6.100)$$

But E must be conserved. So we need $\alpha = 1/2$ to remove time dependence of E . Using $\alpha = 1/2$ we find

$$\boxed{R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}.} \quad (6.101)$$

6.6.1 Similarity Solutions

The above problem only has 2 parameters, E and ρ_0 . Look at their dimensions

$$[E] = \frac{ML^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}. \quad (6.102)$$

These cannot be combined to give quantities with the dimension of length or time. So, there is no natural length scale or time scale in the problem!

Given some time t , the only way to combine E , ρ_0 and t to give a length scale is

$$\lambda = \left(\frac{Et^2}{\rho_0} \right)^{1/5}. \quad (6.103)$$

We can define a dimensionless distance parameter

$$\xi \equiv \frac{r}{\lambda} = r \left(\frac{\rho_0}{Et^2} \right)^{1/5}. \quad (6.104)$$

Then, for any variable in the problem $X(r, t)$, we will have

$$X = X_1(t) \tilde{X}(\xi), \quad (6.105)$$

i.e. X is a function of scaled distance ξ and always has the same shape scaled up/down by the time dependence factor $X_1(t)$.

So,

$$\frac{\partial X}{\partial r} = X_1 \frac{d\tilde{X}}{d\xi} \frac{\partial \xi}{\partial r} \Big|_t \quad (6.106)$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi) \frac{dX_1}{dt} + X_1 \frac{d\tilde{X}}{d\xi} \frac{\partial \xi}{\partial t} \Big|_r. \quad (6.107)$$

ξ is neither a Lagrangian nor an Eulerian coordinate. It labels a particular feature in the flow (e.g. shock wave) that can move through the fluid. So we can write

$$R_{\text{shock}} \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \quad (6.108)$$

Let's put some numbers in for the case of supernova explosions,

$$R(t) = \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} \quad (\text{we will assume } \xi_0 \sim 1) \quad (6.109)$$

$$u_0(t) = \frac{dR}{dt} = \frac{2}{5} \xi_0 \left(\frac{E}{\rho_0 t^3}\right)^{1/5} = \frac{2}{5} \frac{R}{t}. \quad (6.110)$$

In a supernova we have

$$E \approx 10^{44} \text{ J} = 10^{51} \text{ erg} \quad (6.111)$$

$$\rho_0 = \rho_{\text{ISM}} \approx 10^{-21} \text{ kg m}^{-3}. \quad (6.112)$$

So the similarity solution gives

$$\left. \begin{aligned} R &\approx 0.3 t^{2/5} \text{ pc} \\ u_0 &\approx 10^5 t^{-3/5} \text{ km s}^{-1} \end{aligned} \right\} \text{ where } t \text{ is measured in yrs.} \quad (6.113)$$

The original explosion injects the stellar debris at about 10^4 km s^{-1} . So the above solution is valid for

$$t \gtrsim 100 \text{ yr} \quad (\text{when } u_0 < u_{\text{inj}}) \quad (6.114)$$

$$t \lesssim 10^5 \text{ yr} \quad (\text{after which energy losses become important}) \quad (6.115)$$

6.6.2 Structure of the Blast Wave

We can, in principle, write each variable ρ , p , u , r in terms of separated functions of t and ξ . We can then substitute into the Eulerian equation of fluid dynamics (in spherical coordinates with $\partial/\partial\phi = \partial/\partial\theta = 0$, i.e. spherical symmetry).

The result is a set of ODE's where ξ is the only dependent variable - the time dependence cancels out! (Sedov 1946)

These solutions tell us that

- Most of mass is swept up in a shell just behind the shock (from form of \tilde{p});
- Post-shock pressure is indeed a multiple of p_{in} (from form of \tilde{p} , justifies $p_{\text{in}} = \alpha p_1$ assumption);
- Shell material is not really moving at a single velocity, but arguments above are restored by taking some weighted average (from form of \tilde{u}).

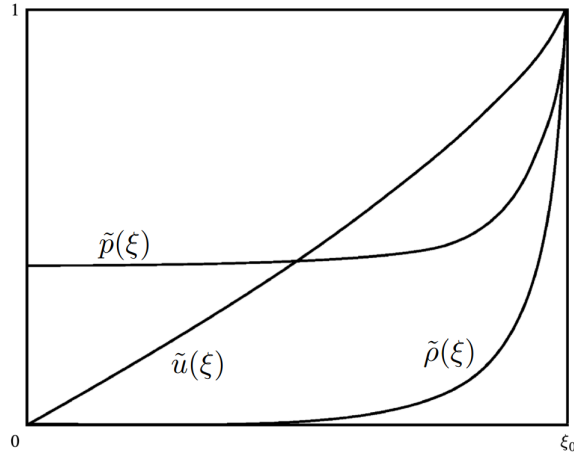


Fig. 6.8: Solution for $\gamma = 7/5$

6.6.3 Breakdown of the Similarity Solution

The self similar solution breaks down when the surrounding medium pressure p_0 becomes significant, $p_1 \sim p_0$.

From the strong shock solution, we derived equations (6.90) and (6.18),

$$p_1 = \frac{2}{\gamma + 1} \rho_0 u_0^2, \quad c_s^2 = \frac{\gamma p_0}{\rho_0}. \quad (6.116)$$

So if $p_1 \sim p_0$ then

$$\begin{aligned} \frac{2}{\gamma + 1} \rho_0 u_0^2 &\sim \frac{\rho_0 c_s^2}{\gamma} \\ \Rightarrow u_0 &\sim c_s, \end{aligned} \quad (6.117)$$

i.e. the shell is not moving supersonically anymore, and the blast wave weakens to a sound wave as illustrated in Fig. 6.9.

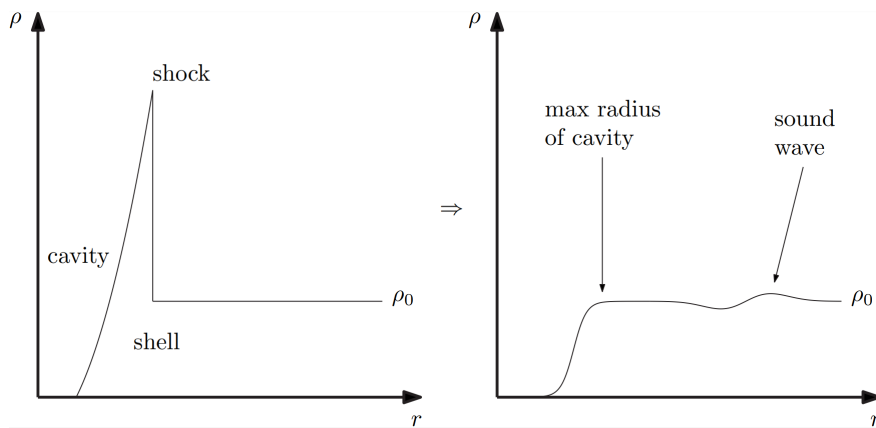


Fig. 6.9: Blast wave phase vs late phase

As a sound wave, disturbance passes into the undisturbed gas as a mild compression followed by a rarefaction. After the sound wave passes, gas returns to the original state.

For a supernova, the maximum bubble/cavity size is set by the radius when the blast wave becomes sonic and $p_1 \sim p_0$. We've just shown that this implies

$$u_0^2 \sim \frac{\gamma+1}{2\gamma} c_s^2. \quad (6.118)$$

We showed in Eq. (6.98) that energy conservation gives

$$\begin{aligned} E &= \frac{4\pi}{3} R^3 \left[\frac{1}{2} \rho_0 \left(\frac{2u_0}{\gamma+1} \right)^2 + \frac{\alpha}{\gamma-1} \frac{2\rho_0 u_0^2}{\gamma+1} \right] \quad \text{where } \alpha = \frac{1}{2} \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2}{(\gamma+1)^2} + \frac{1}{(\gamma-1)(\gamma+1)} \right] \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2(\gamma-1) + (\gamma+1)}{(\gamma+1)^2(\gamma-1)} \right] \\ &= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \frac{3\gamma-1}{(\gamma+1)^2(\gamma-1)} \end{aligned} \quad (6.119)$$

$$\Rightarrow u_0^2 = \frac{(\gamma+1)(\gamma^2-1)}{3\gamma-1} \cdot \frac{3E}{4\pi\rho_0 R^3} \sim \underbrace{\frac{\gamma+1}{2\gamma} c_s^2}_{\substack{\text{when blast} \\ \text{wave becomes} \\ \text{sonic and } p_1 \sim p_0}} \quad (6.120)$$

$$\Rightarrow E \sim \frac{4\pi}{3} \rho_0 R_{\max}^3 \frac{c_s^2}{2\gamma} \cdot \frac{3\gamma-1}{\gamma^2-1}. \quad (6.121)$$

The internal energy initially contained within R_{\max} is

$$E_{\text{init}} = \frac{4\pi}{3} R_{\max}^3 \frac{p_0}{\gamma-1} = \frac{4\pi}{3} R_{\max}^3 \rho_0 \frac{c_s^2}{\gamma(\gamma-1)}. \quad (6.122)$$

So, when $p_1 \sim p_0$, we have $E \sim E_{\text{init}}$. Therefore, the blast wave propagates until the explosion energy is comparable to the internal energy in the sphere!

Some numbers:

- Timescale on which the bubble reaches R_{\max} is roughly the sound crossing time

$$t_s \sim \frac{R_{\max}}{c_s}. \quad (6.123)$$

For ISM: $T \sim 10^4$ K, $\rho \sim 10^{-21}$ kg m⁻³, giving

$$R_{\max} \sim \text{few} \times 100 \text{ pc} \quad (6.124)$$

$$t_{\max} \sim 10 \text{ Myr}. \quad (6.125)$$

- SN rate is about $10^{-7} \text{ Myr}^{-1} \text{ pc}^{-3}$. So, over a duration t_{max} , can find 1 SN in $\sim 10^6 \text{ pc}^3$. But

$$\frac{4\pi}{3} R_{\text{max}}^3 > 10^6 \text{ pc}^3 \quad (6.126)$$

so the filling factor of SN driven bubbles is > 1 . This would seem to suggest that the entire ISM would be heated by supernovae to $> 10^6 K$, but this is *not observed*!

We need to account for cooling and the finite height of the Galactic disk (i.e. bubble “blow out”). After 10^5 yrs, when $R \sim 20$ pc, cooling losses become important and so the bubble grows more slowly than $R \propto t^{2/5}$. Simulations show that $R \propto t^{0.3}$ and $R_{\text{max}} \sim 50$ pc, giving a filling factor < 1 . Thus, due to cooling, only a small fraction of E is deposited into the ISM.

Bernoulli's Equation and Transonic Flows

7.1 Bernoulli's Equation

Let's start with the momentum equation (2.23) and substituting Eq. (3.1),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Psi. \quad (7.1)$$

If the fluid is barotropic, then $p = p(\rho)$ and so

$$\begin{aligned} \frac{\partial}{\partial x} \int \frac{dp}{\rho} &= \frac{\partial p}{\partial x} \int \frac{dp}{\rho} = \frac{1}{\rho} \frac{\partial p}{\partial x} \\ \implies \frac{1}{\rho} \nabla p &= \nabla \left(\int \frac{dp}{\rho} \right). \end{aligned} \quad (7.2)$$

Also, we have the vector identity

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} u^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u}). \quad (7.3)$$

Using these, the momentum equation (7.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} u^2 \right) - \mathbf{u} \times \mathbf{w} = -\nabla \left[\int \frac{dp}{\rho} + \Psi \right]. \quad (7.4)$$

where we have defined the *vorticity*:

$$\boxed{\mathbf{w} = \nabla \times \mathbf{u}.} \quad (7.5)$$

Now, assume a steady flow ($\partial \mathbf{u} / \partial t = \mathbf{0}$) and take the dot product of (7.4) with velocity \mathbf{u} . Since we have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w})$ always, the result is

$$\mathbf{u} \cdot \nabla \left[\frac{1}{2} u^2 + \int \frac{dp}{\rho} + \Psi \right] = 0. \quad (7.6)$$

This gives us *Bernoulli's Principle*: For steady barotropic flows, the quantity

$$\boxed{H = \frac{1}{2} u^2 + \int \frac{dp}{\rho} + \Psi} \quad (7.7)$$

is constant along a streamline. The quantity H is called Bernoulli's constant.

- If $p = 0$, $H = \text{constant}$ is the statement that kinetic + potential energy is constant along streamlines.
- If $p \neq 0$ pressure differences accelerate or decelerate the flow as it flows along the streamline.

7.1.1 Examples of Bernoulli's Equation

7.1.1.1 The Apocryphal Aircraft Wing

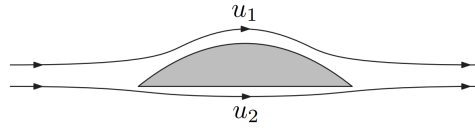


Fig. 7.1: Aircraft wing

$$\begin{aligned}
 u_1 > u_2 &\implies p_1 < p_2 \text{ from } H \\
 &\implies \text{pressure difference} \\
 &\implies \text{lift force.}
 \end{aligned}$$

Of course, this cannot be the whole story of how aircraft wings work or else inverted flight would be impossible!

7.1.1.2 Shower Curtain

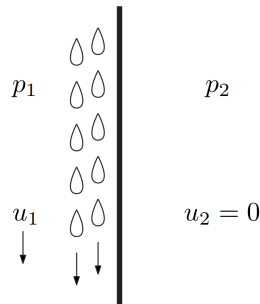


Fig. 7.2: Shower curtain

Downward flow of air on inside of curtain induced by falling water

$$\begin{aligned}
 &\implies p_1 < p_2 \\
 &\implies \text{curtain blows inwards.}
 \end{aligned} \tag{7.8}$$

7.2 Rotational and Irrotational Flows

An *irrotational flow* is one in which $\nabla \times \mathbf{u} = \mathbf{0}$ everywhere, i.e. the vorticity $\mathbf{w} = \mathbf{0}$ everywhere.

For a steady irrotational flow, Eq. (7.4) gives that

$$\nabla H = 0, \tag{7.9}$$

so, $H = \text{constant}$ everywhere (not just along streamlines).

For a general (not necessarily irrotational or steady state) flow, we have

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla H + \mathbf{u} \times \mathbf{w}. \quad (7.10)$$

Take curl:

$$\frac{\partial}{\partial t} \underbrace{(\nabla \times \mathbf{u})}_{\mathbf{w}} = \underbrace{-\nabla \times (\nabla H)}_{\equiv 0} + \nabla \times (\mathbf{u} \times \mathbf{w}) \quad (7.11)$$

and from this we arrive at *Helmholtz's equation*,

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w})}. \quad (7.12)$$

From Helmholtz's equation, we observe three results:

1. If $\mathbf{w} = \mathbf{0}$ initially, it will stay zero thereafter. We will see later that this is no longer true once we include viscous terms.
2. The flux of vorticity through a surface \mathcal{S} that moves with the fluid is a constant, i.e.

$$\frac{D}{Dt} \int_S \mathbf{w} \cdot d\mathbf{S} = 0. \quad (7.13)$$

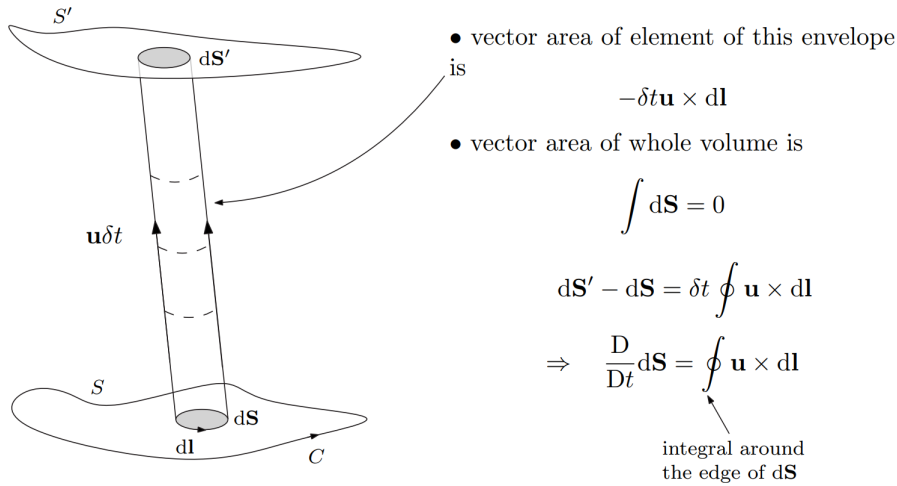


Fig. 7.3: Change of area element with time

For a proof, we first have

$$\frac{D}{Dt} \int_S \mathbf{w} \cdot d\mathbf{S} = \underbrace{\int_S \frac{\partial \mathbf{w}}{\partial t} \cdot d\mathbf{S}}_{\text{intrinsic changes in } \mathbf{w}} + \underbrace{\int_S \mathbf{w} \cdot \frac{D}{Dt} d\mathbf{S}}_{\text{change in } S \text{ caused by flow}}. \quad (7.14)$$

So,

$$\begin{aligned}
 \int_S \mathbf{w} \cdot \frac{D}{Dt} d\mathbf{S} &= \int_S \oint_{\partial d\mathbf{S}} \mathbf{w} \cdot (\mathbf{u} \times d\mathbf{l}) \\
 &= \int_S \oint_{\partial d\mathbf{S}} \mathbf{w} \times \mathbf{u} \cdot d\mathbf{l} \\
 &= \oint_C \mathbf{w} \times \mathbf{u} \cdot d\mathbf{l} \quad \text{since "internal loops" cancel out} \\
 &= \int_S \nabla \times (\mathbf{w} \times \mathbf{u}) \cdot d\mathbf{S}.
 \end{aligned} \tag{7.15}$$

$$\begin{aligned}
 \Rightarrow \frac{D}{Dt} \int_S \mathbf{w} \cdot d\mathbf{S} &= \int_S d\mathbf{S} \cdot \underbrace{\left(\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{w} \times \mathbf{u}) \right)}_{=0 \text{ from the Helmholtz's equation (7.12)}} \\
 &\Rightarrow \frac{D}{Dt} \int_S \mathbf{w} \cdot d\mathbf{S} = 0,
 \end{aligned} \tag{7.16}$$

i.e. flux of vorticity is conserved and moves with the fluid. This is Kelvin's vorticity theorem.

3. For an irrotational flow, the fact that $\nabla \times \mathbf{u} = 0$ everywhere implies that there exists a potential function Φ_u such that

$$\mathbf{u} = -\nabla \Phi_u. \tag{7.17}$$

If such a flow is also incompressible, then $\nabla \cdot \mathbf{u} = 0$ and so

$$\nabla^2 \Phi_u = 0, \tag{7.18}$$

i.e. can reduce the problem of finding the velocity field to that of solving Laplace's equation.

7.3 The de Laval Nozzle

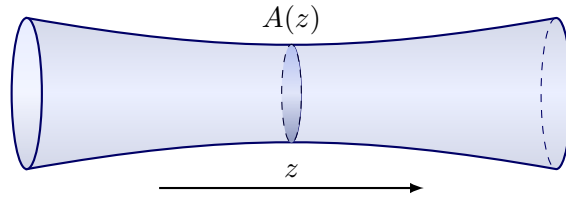


Fig. 7.4: Tube with variable cross-section

Consider a steady flow in a tube with a variable cross-section $A(z)$ as illustrated in Fig. (7.4). For a *steady flow*, mass conservation gives

$$\begin{aligned}
 \rho u A &= \text{const. } \dot{M} \quad (\text{mass flow per second}) \\
 \Rightarrow \ln \rho + \ln u + \ln A &= \ln \dot{M} \\
 \Rightarrow \frac{1}{\rho} \nabla \rho + \nabla \ln u + \nabla \ln A &= 0 \\
 \Rightarrow \frac{1}{\rho} \nabla \rho &= -\nabla \ln u - \nabla \ln A,
 \end{aligned} \tag{7.19}$$

and the momentum equation (with no gravity) gives

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p. \quad (7.20)$$

Let's further assume a *barotropic equation* of state. Then

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \frac{dp}{d\rho} \nabla \rho. \quad (7.21)$$

So, putting these pieces together gives

$$\mathbf{u} \cdot \nabla \mathbf{u} = [\nabla \ln u + \nabla \ln A] c_s^2 \quad (c_s^2 = dp/d\rho) \quad (7.22)$$

If the flow is also irrotational, we have

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} u^2 \right) = \frac{1}{2} u^2 \nabla (\ln u^2) = u^2 \nabla \ln u, \quad (7.23)$$

and so, from Eq. (7.22) we have

$$\begin{aligned} u^2 \nabla \ln u &= qty[\nabla \ln u + \nabla \ln A] c_s^2 \\ \implies (u^2 - c_s^2) \nabla \ln u &= c_s^2 \nabla \ln A. \end{aligned} \quad (7.24)$$

This implies that an extremum of $A(z)$ must correspond to either

1. Minimum or maximum in u , or
2. $u = c_s$.

Thus, we see that there is the potential for a transition from subsonic to supersonic flow at a minimum or maximum of the cross-sectional area of the tube.

To make progress, we apply Bernoulli's equation

$$\frac{1}{2} u^2 + \int \frac{dp}{\rho} = H \quad \text{constant} \quad (\text{no gravity, steady, irrotational}) \quad (7.25)$$

and examine the two standard barotropic cases.

7.3.1 Case I: Isothermal EoS

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \quad T = \text{const.} \quad (7.26)$$

$$\begin{aligned} \implies \int \frac{dp}{\rho} &= \int \frac{\mathcal{R}_* T}{\mu} \frac{d\rho}{\rho} \\ &= \frac{\mathcal{R}_* T}{\mu} \ln \rho \\ &= c_s^2 \ln \rho. \end{aligned} \quad (7.27)$$

Suppose that we have a minimum or maximum in $A(z)$ that allows a flow to have a sonic transition. Let $A = A_m$ at this location. Then Bernoulli gives

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = \frac{1}{2}c_s^2 + c_s^2 \ln \rho \Big|_{A=A_m}, \quad (7.28)$$

which implies

$$\begin{aligned} u^2 &= c_s^2 \left[1 + 2 \ln \left(\frac{\rho|_{A=A_m}}{\rho} \right) \right] \\ &= c_s^2 \left[1 + 2 \ln \left(\frac{uA}{c_s A_m} \right) \right], \end{aligned} \quad (7.29)$$

where this last step has used mass conservation, i.e. $\rho u A = \text{constant}$. Thus, given $A(z)$ we can determine $u(z)$ and $\rho(z)$, i.e. the structure of the flow everywhere subject to given \dot{M} and c_s .

7.3.2 Case II: Polytopic EoS

$$p = K \rho^{1+1/n} \quad (7.30)$$

Let's examine the case where the sonic transition occurs at $A = A_m$. But now we do not know the sound speed c_s since $c_s = c_s(\rho)$ and ρ varies. We need to solve for

$$c_s^2 = \frac{n+1}{n} K \rho^{1/n}. \quad (7.31)$$

Now,

$$\begin{aligned} \int \frac{dp}{\rho} &= \int \frac{dp}{d\rho} \frac{d\rho}{\rho} \\ &= \int K \frac{n+1}{n} \rho^{1/n} \frac{d\rho}{\rho} \\ &= K \frac{n+1}{n} \int \rho^{1/n-1} d\rho \\ &= K \frac{n+1}{n} n \rho^{1/n} \\ &= n c_s^2. \end{aligned} \quad (7.32)$$

Mass conservation:

$$\begin{aligned} \rho u A &= \rho \Big|_{A_m} c_s \Big|_{A_m} A_m = \dot{M} \\ \Rightarrow \rho \Big|_{A_m} \left(\frac{n+1}{n} K \right)^{1/2} \rho^{1/2n} \Big|_{A_m} A_m &= \dot{M} \\ \Rightarrow \rho^{2+1/n} \Big|_{A_m} \left(\frac{n+1}{n} K \right) A_m^2 &= \dot{M}^2 \\ \Rightarrow \rho \Big|_{A_m} &= \left[\left(\frac{\dot{M}}{A_m} \right)^2 \frac{n}{K(n+1)} \right]^{n/(2n+1)} \end{aligned} \quad (7.33)$$

Knowing $\rho|_{A_m}$, we can now determine c_s and A_m . Bernoulli gives:

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} = \text{const.}, \quad (7.34)$$

which implies

$$\begin{aligned} \frac{1}{2} \left(\frac{\dot{M}}{A\rho} \right)^2 + K(n+1)\rho^{1/n} &= \frac{1}{2} c_s^2 \Big|_{A_m} + K(n+1) \rho^{1/n} \Big|_{A_m} \\ &= \frac{1}{2} \left(\frac{n+1}{n} \right) K \rho^{1/n} \Big|_{A_m} + K(n+1) \rho^{1/n} \Big|_{A_m} \\ &= \left(\frac{1}{2} + n \right) \left(\frac{n+1}{n} \right) K \rho^{1/n} \Big|_{A_m} \end{aligned} \quad (7.35)$$

This is an implicit equation for the density structure through the flow.

General points of physical interpretation:

$$(u^2 - c_s^2) \nabla \ln u = c_s^2 \nabla \ln A. \quad (7.36)$$

- In subsonic regime $u < c_s$

$$\begin{aligned} A \text{ decreases} &\implies \nabla \ln u \text{ positive} \\ &\implies u \text{ accelerates along streamline} \end{aligned}$$

e.g. rivers flowing through narrows;

- In supersonic regime $u > c_s$

$$\begin{aligned} A \text{ increases} &\implies \nabla \ln u \text{ positive} \\ &\implies u \text{ accelerates along streamline} \end{aligned}$$

Gas becomes very compressible. A increases, u increases, ρ is greatly reduced.
 $\dot{M} = A\rho u$ constant.

So, a nozzle that gets progressively narrower, reaches a minimum, and then widens again can be used to accelerate a flow from a subsonic to a supersonic regime.

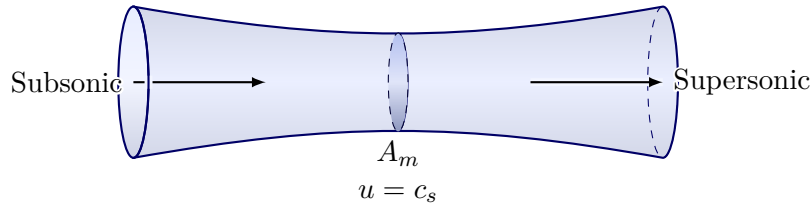


Fig. 7.5: de Laval nozzle

Recall momentum equation:

$$\begin{aligned} (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{\rho} \nabla \rho \frac{dp}{d\rho} = -c_s^2 \nabla \ln \rho \\ (\mathbf{u} \cdot \nabla) \mathbf{u} &= u^2 \nabla \ln u \\ \implies u^2 \nabla \ln u &= -c_s^2 \nabla \ln \rho \end{aligned} \quad (7.37)$$

- If $u \ll c_s$, $\nabla \ln u \gg \nabla \ln \rho$, this implies accelerations are important, pressure or density changes are small – almost incompressible;
- If $u \gg c_s$, $\nabla \ln u \ll \nabla \ln \rho$, $u \approx \text{constant}$, pressure changes do not lead to much acceleration but there is change in ρ – compressible flow.

7.4 Spherical Accretion and Winds

We find flows with a mathematical structure when we consider steady-state and spherically-symmetric accretion flows or winds in the gravitational potential of a central body.

Consider the spherically-symmetric accretion of gas onto a star (described as a point of mass). We will assume

- gas is at rest at ∞ (reservoir);
- steady state flow;
- barotropic EoS.

Mass conservation gives

$$\begin{aligned} \rho u A &= \text{constant } \dot{M} \\ \implies 4\pi r^2 \rho u &= \dot{M}, \end{aligned} \quad (7.38)$$

where, for convenience, we define u to be inward pointing.

Momentum equation gives

$$\begin{aligned} u \frac{du}{dr} &= -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM}{r^2} \\ \implies u^2 \frac{d \ln u}{dr} &= -c_s^2 \frac{d \ln \rho}{dr} - \frac{GM}{r^2}, \end{aligned} \quad (7.39)$$

assuming self-gravity of the accretion gas is negligible.

Now, a steady flow must have

$$\begin{aligned} \frac{d}{dr} (\ln \dot{M}) &= 0 \\ \implies \frac{d}{dr} \ln \rho + \frac{d}{dr} \ln u + \frac{d}{dr} \ln r^2 &= 0 \\ \implies \frac{d}{dr} \ln \rho &= -\frac{d}{dr} \ln u - \frac{2}{r}. \end{aligned} \quad (7.40)$$

Substitute into Eq. (7.39) gives

$$u^2 \frac{d}{dr} \ln u = c_s^2 \left(\frac{d}{dr} \ln u + \frac{2}{r} \right) - \frac{GM}{r^2}. \quad (7.41)$$

Therefore

$$\left(u^2 - c_s^2 \right) \frac{d}{dr} \ln u = \frac{2c_s^2}{r} \left(1 - \frac{GM}{2c_s^2 r} \right). \quad (7.42)$$

There is a critical point in the flow at

$$r = r_s = \frac{GM}{2c_s^2}, \quad (7.43)$$

where u is either a minimum/maximum or there is a sonic transition. This is called the *sonic point*, somewhat similar to the de Laval nozzle, except with no boundaries/tubes!

Can gain insight into the general structure of such flows by plotting possible solutions on the $(r/r_s, u)$ plane.

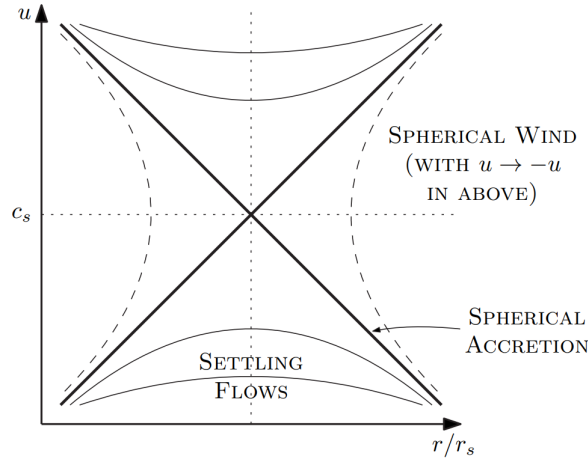


Fig. 7.6: Plot in $(r/r_s, u)$ plane

Back to accretion problem: progress requires the EoS.

7.4.1 Case I: Isothermal EoS

Equation of state is:

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \quad T = \text{const.} \quad (7.44)$$

$$\Rightarrow c_s = \sqrt{\frac{\mathcal{R}_* T}{\mu}} = \text{const.} \quad (7.45)$$

and we know from Eq. (7.43)

$$r_s = \frac{GM}{2c_s^2}. \quad (7.46)$$

Need to use Bernoulli's equation to constrain ρ and \dot{M} .

$$\begin{aligned}
 H &= \frac{1}{2}u^2 + \underbrace{\int \frac{dp}{\rho}}_{c_s^2 \ln \rho} + \Psi = \text{const.} \\
 \implies \quad \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} &= \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s} \\
 \implies \quad \frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} &= c_s^2 \left(\ln \rho_s - \frac{3}{2} \right) \\
 \implies \quad u^2 &= 2c_s^2 \left[\ln \left(\frac{\rho_s}{\rho} \right) - \frac{3}{2} \right] + \frac{2GM}{r}, \tag{7.47}
 \end{aligned}$$

where ρ_s is the density at $r = r_s$.

Now,

- as $r \rightarrow 0$, $u^2 \rightarrow 2GM/r$, i.e. free-fall;
- as $r \rightarrow \infty$ and $u \rightarrow 0$, $\rho = \rho_s e^{-3/2}$, giving

$$\rho_s = \rho_\infty e^{3/2}. \tag{7.48}$$

Thus, for a given ρ_∞ , we know ρ_s and hence \dot{M} .

$$\begin{aligned}
 \dot{M} &= 4\pi r_s^2 \rho_s c_s \\
 \implies \quad \dot{M} &= \frac{\pi G^2 M^2 e^{3/2} \rho_\infty}{c_s^3}. \tag{7.49}
 \end{aligned}$$

Note that:

- \dot{M} proportional to M^2 , more massive stars can accrete much more gas;
- \dot{M} proportional to $1/c_s^3$, accretion very sensitive to temperature; can accrete more effectively from a colder medium.

7.4.2 Case II: Polytropic EoS

Equation of state is:

$$p = K\rho^{1+1/n}, \quad \int \frac{dp}{\rho} = K(n+1)\rho^{1/n}. \tag{7.50}$$

At the sonic point,

$$\int \frac{dp}{\rho} = nc_s^2. \tag{7.51}$$

Bernoulli then gives

$$\frac{1}{2}u^2 + K(n+1)\rho^{1/n} - \frac{GM}{r} = \frac{1}{2}c_s^2 + nc_s^2 - \frac{GM}{r_s}, \tag{7.52}$$

with $r_s = GM/(2c_s^2)$ as in Eq. (7.43). Using the mass accretion rate $\dot{M} = 4\pi r_s^2 \rho_s c_s$ we can then write

$$\begin{aligned} r_s &= \left(\frac{\dot{M}}{4\pi \rho_s c_s} \right)^{1/2} = \frac{GM}{2c_s^2} \\ \Rightarrow c_s &= \left(\frac{GM}{2} \right)^{2/3} \left(\frac{4\pi \rho_s}{\dot{M}} \right)^{1/3}. \end{aligned} \quad (7.53)$$

Combine this with

$$c_s^2 = \frac{n+1}{n} K \rho_s^{1/n}, \quad (7.54)$$

to get

$$\begin{aligned} \left(\frac{n+1}{n} \right) K \rho_s^{1/n} &= \left(\frac{GM}{2} \right)^{4/3} \left(\frac{4\pi \rho_s}{\dot{M}} \right)^{2/3} \\ \Rightarrow \rho_s^{1/n-2/3} &= \rho_s^{(3-2n)/3n} = \left(\frac{GM}{2} \right)^{4/3} \left(\frac{4\pi}{\dot{M}} \right)^{2/3} \frac{n}{(n+1)K} \\ \Rightarrow \rho_s &= \left(\frac{GM}{2} \right)^{4n/(3-2n)} \left(\frac{4\pi}{\dot{M}} \right)^{2n/(3-2n)} \left(\frac{n}{(n+1)K} \right)^{3n/(3-2n)}. \end{aligned} \quad (7.55)$$

Back to Bernoulli:

$$\begin{aligned} \frac{1}{2} u^2 + (n+1) K \rho^{1/n} - \frac{GM}{r} &= c_s^2 \left(n - \frac{3}{2} \right) \\ \Rightarrow \frac{1}{2} \left(\frac{\dot{M}}{4\pi r^2 \rho} \right)^2 + (n+1) K \rho^{1/n} &= c_s^2 \left(n - \frac{3}{2} \right) + \frac{GM}{r}. \end{aligned} \quad (7.56)$$

As $r \rightarrow \infty$, $u \rightarrow 0$, we have

$$\rho_\infty = \left[\frac{c_s^2 \left(n - \frac{3}{2} \right)}{(n+1)K} \right]^n = \left[\frac{n - \frac{3}{2}}{n} \right]^n \rho_s \quad (7.57)$$

$$c_{s,\infty}^2 = \frac{n+1}{n} K \rho_\infty^{1/n} = \frac{n - \frac{3}{2}}{n} c_s^2 \quad (7.58)$$

So, finally,

$$\begin{aligned} \dot{M} &= 4\pi r_s^2 \rho_s c_s \\ &= \frac{4\pi G^2 M^2}{4c_s^4} c_s \rho_\infty \left(\frac{n}{n - \frac{3}{2}} \right)^n \\ &= \frac{\pi G^2 M^2}{c_{s,\infty}^3} \rho_\infty \left(\frac{n}{n - \frac{3}{2}} \right)^{n-3/2}. \end{aligned} \quad (7.59)$$

Therefore,

$$\boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_{s,\infty}^3} \left(\frac{n}{n - \frac{3}{2}} \right)^{n-3/2}} \quad (7.60)$$

Same functional form as in the isothermal case, but now with an additional coefficient related to the polytropic index. This is known as *Bondi Accretion*. We can recover the isothermal case by taking the limit $n \rightarrow \infty$.

The generalisation to the case of a star accreting from a medium that it is moving through is called *Bondi-Hoyle-Lyttleton Accretion*. The result is

$$\dot{M} \sim \frac{(GM)^2 \rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}}, \quad (7.61)$$

where v_∞ is the velocity of gas relative to the star at ∞ .

CHAPTER 8

Fluid Instabilities

Consider a fluid in a steady state ($\partial/\partial t = 0$). Thus it is in a state of equilibrium.

- If a small perturbation of this configuration grows with time, the configuration is *unstable* with respect to those perturbations;
- If a small perturbation decays with time or just oscillates around the equilibrium configuration, the configuration is *stable* with respect to those perturbations.

An awful lot of interesting astrophysics is due to the action of fluid instabilities!

- Convection in stars;
- Multiphase nature of the ISM;
- Mixing of fluids that have relative motion;
- Turbulence in accretion disks;
- Formation of stars and galaxies.

In this chapter, we discuss some of the most important instabilities.

8.1 Convective Instability

This concerns the stability of a hydrostatic equilibrium. We can gain insight without doing a full perturbation analysis.

Consider the following system:

- Ideal gas in hydrostatic equilibrium;
- Uniform gravitational field in $-\hat{\mathbf{z}}$ direction.

Now perturb a fluid element upwards, away from its equilibrium point.

We assume that any pressure imbalances are quickly removed by acoustic waves, but that heat exchange takes longer. This implies the displaced element evolves adiabatically with a pressure p' equal to the pressure at the new location in the atmosphere.

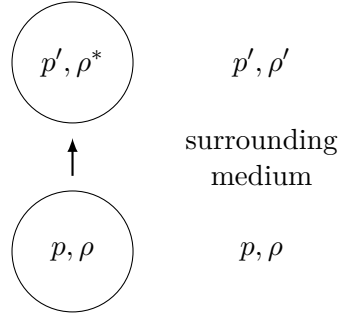


Fig. 8.1: Perturbing a fluid element upwards

Since we assume heat transfer is slow, initially perturbations will change adiabatically. Stability depends on the new density.

$$\begin{aligned}
 \rho^* < \rho' &\implies \text{perturbed element buoyant} \\
 &\implies \text{system unstable;} \\
 \rho^* > \rho' &\implies \text{perturbed element sinks back} \\
 &\implies \text{system stable.}
 \end{aligned}$$

For adiabatic change,

$$\left. \begin{aligned} p &= K\rho^\gamma \\ p' &= K\rho^{*\gamma} \end{aligned} \right\} \implies \rho^* = \rho \left(\frac{p'}{p} \right)^{1/\gamma}. \quad (8.1)$$

To first order,

$$p' = p + \frac{dp}{dz} \delta z, \quad (8.2)$$

thus,

$$\begin{aligned}
 \rho^* &= \rho \left(\frac{p + \frac{dp}{dz} \delta z}{p} \right)^{1/\gamma} \\
 &= \rho \left(1 + \frac{1}{p} \frac{dp}{dz} \delta z \right)^{1/\gamma} \\
 &\approx \rho + \frac{\rho}{p\gamma} \frac{dp}{dz} \delta z.
 \end{aligned} \quad (8.3)$$

In the surrounding medium,

$$\rho' = \rho + \frac{d\rho}{dz} \delta z, \quad (8.4)$$

and the system is unstable if $\rho^* < \rho'$. So instability needs

$$\begin{aligned}
 \rho + \frac{\rho}{p\gamma} \frac{dp}{dz} \delta z &< \rho + \frac{d\rho}{dz} \delta z \\
 \implies \frac{\rho}{p\gamma} \frac{dp}{dz} &< \frac{d\rho}{dz} \\
 \implies \frac{d}{dz} \ln p &< \gamma \frac{d}{dz} \ln \rho \\
 \implies \frac{d}{dz} (\ln p \rho^{-\gamma}) &< 0 \\
 \implies \frac{dK}{dz} &< 0. \quad (\text{instability})
 \end{aligned} \tag{8.5}$$

So, the system is unstable if the entropy of the atmosphere decreases with increasing height. This can also be related to temperature and pressure gradients.

$$\frac{dK}{dz} < 0 \implies \frac{d}{dz} \ln K < 0. \tag{8.6}$$

But,

$$K = p\rho^{-\gamma} = (\text{const.})p^{1-\gamma}T^\gamma, \quad (p = \mathcal{R}_*\rho T/\mu) \tag{8.7}$$

so,

$$\begin{aligned}
 \frac{d}{dz} \ln K &= (1 - \gamma) \frac{d}{dz} \ln p + \gamma \frac{d}{dz} \ln T < 0 \\
 \implies \frac{dT}{dz} &< \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{dp}{dz}. \quad (\text{instability})
 \end{aligned} \tag{8.8}$$

Hence, we have the *Schwarzschild stability criterion* which reads

$$\boxed{\frac{dT}{dz} > \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{dp}{dz}} \tag{8.9}$$

Since hydrostatic equilibrium requires $dp/dz < 0$, we see that (since $\gamma > 1$)

- Always stable to convection if $dT/dz > 0$;
- Otherwise, can tolerate a negative temperature gradient provide

$$\left| \frac{dT}{dz} \right| < \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \left| \frac{dp}{dz} \right| \tag{8.10}$$

So convective instability develops when T declines too steeply with increasing height.

Examples (Convectively unstable systems).

- Outer regions of low mass stars;
- Cores of high mass stars.

For stable configurations, we can examine atmosphere dynamics: equation of motion is

$$\begin{aligned}
 \rho^* \frac{d^2}{dt^2} \delta z &= -g(\rho^* - \rho') \\
 \Rightarrow (\rho + \underbrace{\delta \rho}_{\text{small}}) \frac{d^2}{dt^2} \delta z &= -g \left[\frac{\rho}{T} \frac{dT}{dz} - \left(1 - \frac{1}{\gamma} \right) \frac{\rho}{p} \frac{dp}{dz} \right] \delta z \\
 \Rightarrow \frac{d^2}{dt^2} \delta z &= -\frac{g}{T} \left[\frac{dT}{dz} - \left(1 - \frac{1}{\gamma} \right) \frac{T}{p} \frac{dp}{dz} \right] \delta z.
 \end{aligned} \tag{8.11}$$

So, it is a simple harmonic motion with angular frequency N where

$$N^2 = \frac{g}{T} \left[\frac{dT}{dz} - \left(1 - \frac{1}{\gamma} \right) \frac{T}{p} \frac{dp}{dz} \right], \tag{8.12}$$

which defines the *Brunt-Väisälä frequency*. These oscillations are internal gravity waves.

8.2 Jeans Instability

This concerns the stability of a self-gravitating fluid against gravitational collapse. Consider the following system:

- Uniform medium initially static;
- Barotropic EoS;
- Gravitational field generated by the medium itself.

So equilibrium is

$$\begin{aligned}
 p &= p_0, \quad (\text{const.}) \\
 \rho &= \rho_0, \quad (\text{const.}) \\
 \mathbf{u} &= \mathbf{0},
 \end{aligned} \tag{8.13}$$

and governing equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{8.14}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Psi \tag{8.15}$$

$$\nabla^2 \Psi = 4\pi G \rho. \tag{8.16}$$

Introduce a perturbation:

$$p = p_0 + \Delta p \tag{8.17}$$

$$\rho = \rho_0 + \Delta \rho \tag{8.18}$$

$$\mathbf{u} = \Delta \mathbf{u} \tag{8.19}$$

$$\Psi = \Psi_0 + \Delta \Psi \tag{8.20}$$

Note: There is an inconsistency between the assumption $\rho_0 = \text{constant} > 0$ and the assumption $\Psi_0 = \text{const.}$ We proceed anyways – this is the *Jeans swindle* (1902). This is closely tied to the fact that it is impossible to construct a model of a static infinite Universe. A more complete analysis of perturbations against a background of a (relativistic) homogenous expanding Universe recovers the same local instability as that found by Jeans, hence justifying the swindle.

Linearised equations are:

$$\frac{\partial \Delta \rho}{\partial t} + \rho_0 \nabla \cdot (\Delta \mathbf{u}) = 0 \quad (8.21)$$

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\frac{dp}{d\rho} \frac{1}{\rho_0} \nabla(\Delta \rho) - \nabla(\Delta \Psi) = -c_s^2 \frac{\nabla(\Delta \rho)}{\rho_0} - \nabla(\Delta \Psi) \quad (8.22)$$

$$\nabla^2(\Delta \Psi) = 4\pi G \Delta \rho. \quad (8.23)$$

Look for plane wave solutions

$$\Delta \rho = \rho_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (8.24)$$

$$\Delta \Psi = \Psi_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (8.25)$$

$$\Delta \mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}. \quad (8.26)$$

Substitution into the linear equations gives

$$(8.21) \implies -\omega \rho_1 + \rho_0 \mathbf{k} \cdot \mathbf{u}_1 = 0 \quad (8.27)$$

$$(8.22) \implies -\rho_0 \omega \mathbf{u}_1 = -c_s^2 \mathbf{k} \rho_1 - \rho_1 \mathbf{k} \Psi_1 \quad (8.28)$$

$$(8.23) \implies -k^2 \Psi_1 = 4\pi G \rho_1. \quad (8.29)$$

Eliminating \mathbf{u}_1 and Ψ_1 from these

$$(8.27) + (8.28) \implies \rho_1 \omega^2 = k^2 (\rho_1 c_s^2 + \rho_0 \Psi_1) \quad (8.30)$$

$$\begin{aligned} &= k^2 \rho_1 c_s^2 - 4\pi G \rho_0 \rho_1 \\ \implies \omega^2 &= c_s^2 \left(k^2 - \frac{4\pi G \rho_0}{c_s^2} \right). \end{aligned} \quad (8.31)$$

Introduce the Jeans wavenumber $k_J^2 = 4\pi G \rho_0 / c_s^2$ so we have the dispersion relation

$$\boxed{\omega^2 = c_s^2 (k^2 - k_J^2)}. \quad (8.32)$$

Notes:

- For $k \gg k_J$, we have normal sound waves $\omega^2 = c_s^2 k^2$.
- For $k \gtrsim k_J$, we have modified sound waves. Gravity leads to dispersion of the wave and a slower group velocity.
- For $k < k_J$, ω is purely imaginary (for $k \in \mathbb{R}$), giving

$$\omega = i\tilde{\omega}, \quad \tilde{\omega} \in \mathbb{R}, \quad (8.33)$$

and,

$$e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} = e^{\tilde{\omega} t} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (8.34)$$

leading to exponentially growing solution: *Gravitational Instability*.

The maximum stable wavelength is the *Jeans length*,

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}. \quad (8.35)$$

The associated mass is the *Jeans mass*,

$$M_J \sim \rho_0 \lambda_J^3. \quad (8.36)$$

These are central concepts in the theory of

- *Star formation* (instability of giant molecular clouds);
- *Cosmological structure formation* (instability of the homogeneous primordial gas).

8.3 Rayleigh-Taylor and Kelvin-Helmholtz Instability

This concerns the stability of an interface with a discontinuous change in tangential velocity and/or density.

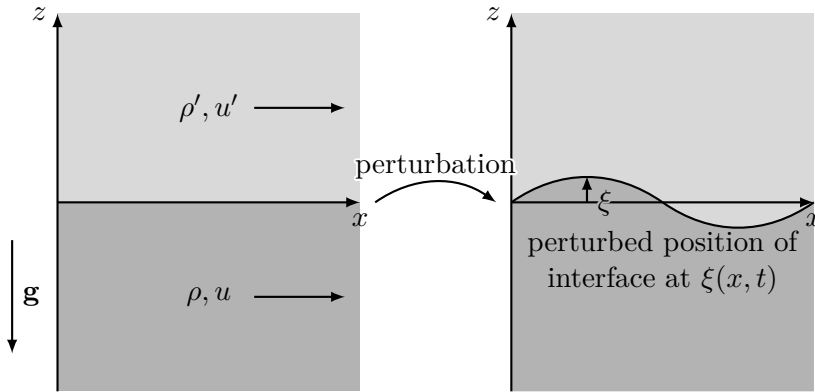


Fig. 8.2: Perturbation of interface of discontinuity

For convenience, let's assume:

- Constant gravity, ideal fluid;
- Pressure continuous across the interface;
- Incompressible flow $\nabla \cdot \mathbf{u} = 0$;
- Irrotational flow $\nabla \times \mathbf{u} = 0 \implies \mathbf{u} = -\nabla \Psi$;
- 2D problem (symmetry direction into the page of Fig. 8.2)

The momentum equation (for either upper or lower fluid) is

$$\begin{aligned}
 \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\rho} \nabla p + \mathbf{g} \\
 \Rightarrow -\nabla \frac{\partial \Psi}{\partial t} + \nabla \left(\frac{1}{2} u^2 \right) &= - \underbrace{\nabla \left(\frac{p}{\rho} \right)}_{\substack{\text{since } \rho = \text{const.} \\ \text{within each fluid}}} - \nabla \Psi \\
 \Rightarrow \nabla \left[-\frac{\partial \Psi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \Psi \right] &= 0 \\
 \Rightarrow -\frac{\partial \Psi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \Psi &= F(t).
 \end{aligned} \tag{8.37}$$

where $F(t)$ is a function that is constant in space but not in time.

Now consider a perturbation at the interface of these two fluids. Let us study the evolution of the perturbed position of the interface $\xi(x, t)$.

The velocity potential is $u = -\nabla \Psi$, so if the unperturbed velocities in the fluids are U and U' we have

$$\Phi_{\text{low}} = -Ux + \phi \tag{8.38}$$

$$\Phi_{\text{up}} = -U'x + \phi', \tag{8.39}$$

thus,

$$\nabla^2 \phi = \nabla^2 \phi' = 0. \quad (\text{since } \nabla \cdot \mathbf{u} = 0) \tag{8.40}$$

ϕ and ϕ' are sourced by displacements of the interface. Consider an element of the lower fluid that is at the interface. Then

$$u_z = \frac{D\xi}{Dt}, \tag{8.41}$$

giving

$$\left. \begin{aligned} -\frac{\partial \phi}{\partial z} &= \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \\ -\frac{\partial \phi'}{\partial z} &= \frac{\partial \xi}{\partial t} + U' \frac{\partial \xi}{\partial x} \end{aligned} \right\} \text{ to first order} \tag{8.42}$$

Now look for plane wave solutions,

$$\xi = Ae^{i(kx - \omega t)} \tag{8.43}$$

$$\phi = Ce^{i(kx - \omega t) + k_z z} \tag{8.44}$$

$$\phi' = C'e^{i(kx - \omega t) + k'_z z}, \tag{8.45}$$

where extra terms on the exponents $k_z z$ and $k'_z z$ are there to seek solutions where the perturbed potential decays at large $|z|$.

But we know that

$$\begin{aligned} \nabla^2 \phi = 0 &\implies -k^2 + k_z^2 = 0 \\ &\implies k_z = |k|, \end{aligned} \tag{8.46}$$

so $\phi \rightarrow 0$ as $z \rightarrow \infty$.

For now, let's stipulate $k > 0$. So

$$\phi = C e^{i(kx - \omega t) + kz} \quad (8.47)$$

$$\phi' = C' e^{i(kx - \omega t) - kz}. \quad (8.48)$$

From Eq. (8.42), we have

$$-kC = -i\omega A + iUkA = i(kU - \omega)A \quad (8.49)$$

$$kC' = i(kU' - \omega)A. \quad (8.50)$$

We need one more equation if we're to solve for A , C , C' . We get that from pressure balance across the interface.

$$p = -\rho \left(-\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + g\xi \right) + \rho F(t) \quad (8.51)$$

$$p' = -\rho' \left(-\frac{\partial \phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi \right) + \rho' F'(t), \quad (8.52)$$

and equality at $z = 0$:

$$\rho \left(-\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + g\xi \right) = \rho' \left(-\frac{\partial \phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi \right) + K(t), \quad (8.53)$$

where

$$K \equiv \rho F(t) - \rho' F'(t). \quad (8.54)$$

The perturbation vanishes for $z \rightarrow \pm\infty$ at all times, so we can look at equation Eq. (8.37) for each fluid in the limit $|z| \rightarrow \infty$, taking limit carefully so that Ψ terms cancel, to get

$$\rho F(t) - \rho' F'(t) = \underbrace{\frac{1}{2}U^2 \rho - \frac{1}{2}U'^2 \rho'}_{\substack{\text{conditions at } \infty \\ \text{and so a constant}}}. \quad (8.55)$$

Therefore, $K(t)$ is actually a constant.

Next in our attempt to use Eq. (8.53) to match across boundary, we need to determine u and u' . Now

$$\begin{aligned} \mathbf{u} &= -\nabla \Psi = -\nabla(-Ux + \phi) = U\hat{\mathbf{x}} - \nabla \phi \\ \implies u &= U^2 - 2U \frac{\partial \phi}{\partial x}, \quad (\text{dropping } 2^{\text{nd}} \text{ order terms}) \end{aligned} \quad (8.56)$$

and similarly

$$u'^2 = U'^2 - 2U' \frac{\partial \phi'}{\partial x}. \quad (8.57)$$

So, Eq. (8.53) reads

$$\begin{aligned}
 \rho \left(-\frac{\partial \phi}{\partial t} + \frac{1}{2} U^2 - U \frac{\partial \phi}{\partial x} + g\xi \right) &= \rho' \left(-\frac{\partial \phi'}{\partial t} + \frac{1}{2} U'^2 - U' \frac{\partial \phi'}{\partial x} + g\xi \right) + \underbrace{\frac{1}{2} U^2 \rho - \frac{1}{2} U'^2 \rho'}_K \\
 \implies \rho \left(-\frac{\partial \phi}{\partial t} - U \frac{\partial \phi}{\partial x} + g\xi \right) &= \rho' \left(-\frac{\partial \phi'}{\partial t} - U' \frac{\partial \phi'}{\partial x} + g\xi \right) \\
 \implies \rho i\omega C - \rho U i k C + \rho g A &= \rho' i\omega C' - \rho' U' i k C' + \rho' g A \\
 \implies \rho(kU - \omega)C + i\rho g A &= \rho'(kU' - \omega)C' + i\rho' g A.
 \end{aligned} \tag{8.58}$$

Now eliminate C and C' from Eqs. (8.49) and (8.50) to give

$$\boxed{\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho')}. \tag{8.59}$$

This is the dispersion relation for our system. Let's now look at some specific applications.

1. *Surface gravity waves*: two fluids at rest initially with $\rho' < \rho$ (i.e. denser fluid on bottom). The dispersion relation gives

$$\begin{aligned}
 \omega^2(\rho + \rho') &= kg(\rho - \rho') \\
 \implies \omega^2 &= k \frac{g(\rho - \rho')}{\rho + \rho'}.
 \end{aligned} \tag{8.60}$$

So, for $k \in \mathbb{R}$, we have that $\omega \in \mathbb{R}$ and hence the system displays oscillations/waves. Phase speed is

$$\frac{\omega}{k} = \pm \sqrt{\frac{g}{k} \frac{\rho - \rho'}{\rho + \rho'}} = \underbrace{f(k)}_{\text{waves are dispersive}}. \tag{8.61}$$

If $\rho' \ll \rho$, then $\omega/k = \pm \sqrt{g/k}$.

Example. Surface waves on the ocean.

2. *Static stratified fluid*: two fluids at rest initially with $\rho' > \rho$ (i.e. denser fluid on top). Then

$$\omega^2 = k \frac{g(\rho - \rho')}{\rho + \rho'}. \tag{8.62}$$

So, for $k \in \mathbb{R}$ we have $\omega^2 < 0$ and so ω is purely imaginary.

$$\frac{\omega}{k} = \pm i \sqrt{\frac{g}{k} \frac{\rho' - \rho}{\rho + \rho'}}. \tag{8.63}$$

The positive root of this gives us exponentially growing solutions. This is the *Rayleigh-Taylor Instability*.

3. *Fluids in motion*: two fluids with $\rho > \rho'$ (so stable to Rayleigh-Taylor) but different velocities, non-zero U and U' . Take full dispersion relation:

$$\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho'), \tag{8.64}$$

divide by k^2 and solve the quadratic in ω/k ,

$$\implies \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k} \frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho \rho' (U - U')^2}{(\rho + \rho')^2}}. \tag{8.65}$$

There is instability if

$$\frac{g}{k} \frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho \rho' (U - U')^2}{(\rho + \rho')^2} < 0. \quad (\text{instability}) \quad (8.66)$$

If $g = 0$, then any relative motion gives instability, i.e. *Kelvin-Helmholtz Instability*;

If $g \neq 0$, then unstable modes are those with

$$k > \frac{(\rho^2 - \rho'^2)g}{\rho \rho' (U - U')^2}, \quad (8.67)$$

i.e. gravity is a stabilising influence.

8.4 Thermal Instability

This concerns the stability of a medium in thermal equilibrium (heating = cooling) to perturbations in temperature. Consider the following system:

- No gravitational field;
- Static thermal equilibrium

$$\mathbf{u}_0 = \mathbf{0}, \dot{Q}_0 = 0, \underbrace{\nabla p_0 = \mathbf{0}, \nabla \rho_0 = \mathbf{0}}_{\nabla K_0 = 0} \quad \text{where} \quad p = K \rho^\gamma. \quad (8.68)$$

Let's start by deriving an alternative form of the energy equation that involves the entropy-like variable K ; this will be well suited to problems of thermal instability.

$$\begin{aligned} p = K \rho^\gamma \quad \implies \quad dp &= \rho^\gamma dK + K \gamma \rho^{\gamma-1} d\rho \\ &= \rho^\gamma dK + \frac{\gamma p}{\rho} d\rho, \end{aligned} \quad (8.69)$$

also,

$$\begin{aligned} p = \frac{\mathcal{R}_*}{\mu} \rho T \quad \implies \quad dp &= \frac{\mathcal{R}_*}{\mu} T d\rho + \frac{\mathcal{R}_*}{\mu} \rho dT \\ &= \frac{p}{\rho} d\rho + \frac{\mathcal{R}_*}{\mu} \rho dT. \end{aligned} \quad (8.70)$$

Equate Eqs. (8.69) and (8.70) to give

$$\begin{aligned} \rho^\gamma dK + \gamma \frac{p}{\rho} d\rho &= \frac{p}{\rho} d\rho + \frac{\mathcal{R}_*}{\mu} \rho dT \\ \implies \quad \rho^\gamma dK &= (1 - \gamma) \frac{p}{\rho} d\rho + \frac{\mathcal{R}_*}{\mu} \rho dT \\ \implies \quad dK &= \rho^{1-\gamma} (1 - \gamma) \underbrace{\left[\frac{p}{\rho^2} d\rho + \frac{\mathcal{R}_*}{\mu(1 - \gamma)} dT \right]}_{-dQ}. \end{aligned} \quad (8.71)$$

First law of thermodynamics:

$$\mathrm{d}Q = p \mathrm{d}V + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} \mathrm{d}T, \quad (\text{unit mass}) \quad (8.72)$$

and so

$$\begin{aligned} \mathrm{d}Q &= p \mathrm{d}(1/\rho) + C_V \mathrm{d}T \\ &= -\frac{p}{\rho^2} - \frac{\mathcal{R}_*}{\mu(1-\gamma)} \mathrm{d}T. \quad (\text{since we have } (\gamma-1)C_V = \mathcal{R}_*/\mu) \end{aligned} \quad (8.73)$$

Then we have

$$\mathrm{d}K = -(1-\gamma)\rho^{1-\gamma}\mathrm{d}Q, \quad (\text{for fluid element}) \quad (8.74)$$

Turn this into Lagrangian energy equation by noting that $\dot{Q} = -\mathrm{d}Q/\mathrm{d}t$,

$$\begin{aligned} \implies \quad \frac{\mathrm{D}K}{\mathrm{D}t} &= -(1-\gamma)\rho^{1-\gamma}\dot{Q} \\ \implies \quad \frac{1}{K} \frac{\mathrm{D}K}{\mathrm{D}t} &\equiv \frac{\mathrm{D}}{\mathrm{D}t}(\ln K) = -(\gamma-1)\frac{\rho}{p}\dot{Q}, \end{aligned} \quad (8.75)$$

and thus we derive the *Entropy form of the energy equation*,

$$\boxed{\frac{1}{K} \frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\frac{\rho\dot{Q}}{p}} \quad (8.76)$$

This joins our usual continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (8.77)$$

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p. \quad (8.78)$$

Now we look at thermal stability. Consider small perturbations to the equilibrium

$$\rho \rightarrow \rho_0 + \Delta\rho \quad (8.79)$$

$$p \rightarrow p_0 + \Delta p \quad (8.80)$$

$$\mathbf{u} \rightarrow \Delta \mathbf{u} \quad (8.81)$$

$$K \rightarrow K_0 + \Delta K. \quad (8.82)$$

Linearise the equations:

$$(8.77) \implies \frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla \cdot (\Delta \mathbf{u}) = 0 \quad (8.83)$$

$$(8.78) \implies \rho_0 \frac{\partial \Delta \mathbf{u}}{\partial t} = -\nabla(\Delta p) \quad (8.84)$$

$$(8.76) \implies \frac{\partial \Delta K}{\partial t} = -\frac{\gamma-1}{\rho_0^{\gamma-1}} \Delta \dot{Q}, \quad (8.85)$$

where we can write

$$\Delta \dot{Q} = \left. \frac{\partial \dot{Q}}{\partial p} \right|_{\rho} \Delta p + \left. \frac{\partial \dot{Q}}{\partial \rho} \right|_p \Delta \rho, \quad (8.86)$$

so that

$$\frac{\partial \Delta K}{\partial t} = -A^* \Delta p - B^* \Delta \rho, \quad (8.87)$$

with

$$A^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left. \frac{\partial \dot{Q}}{\partial p} \right|_\rho, \quad B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left. \frac{\partial \dot{Q}}{\partial \rho} \right|_p. \quad (8.88)$$

We also have

$$p = K \rho^\gamma \implies \Delta p = \rho_0^\gamma \Delta K + \gamma \frac{p_0}{\rho_0} \Delta \rho. \quad (8.89)$$

We seek solutions of the form

$$\Delta p = p_1 e^{i\mathbf{k} \cdot \mathbf{x} + qt} \quad (8.90)$$

$$\Delta \rho = \rho_1 e^{i\mathbf{k} \cdot \mathbf{x} + qt} \quad (8.91)$$

$$\Delta \mathbf{u} = \mathbf{u}_1 e^{i\mathbf{k} \cdot \mathbf{x} + qt} \quad (8.92)$$

$$\Delta K = K_1 e^{i\mathbf{k} \cdot \mathbf{x} + qt} \quad (8.93)$$

so, instability if $\text{Re}\{q\} > 0$. Substituting into linearised equations gives

$$(8.83) \implies q \rho_1 + \rho_0 i \mathbf{k} \cdot \mathbf{u}_1 = 0 \quad (8.94)$$

$$(8.84) \implies q \rho_0 \mathbf{u}_1 = -i \mathbf{k} p_1 \quad (8.95)$$

$$(8.87) \implies q K_1 = -A^* p_1 - B^* \rho_1 \quad (8.96)$$

$$(8.89) \implies p_1 = \rho_0^\gamma K_1 + \frac{\gamma p_0}{\rho_0} \rho_1. \quad (8.97)$$

We can combine these to obtain the dispersion relation:

$$\begin{aligned} \frac{A^* q}{k^2} - \frac{B^*}{q} &= - \left(\frac{q^2}{k^2} + \gamma \frac{p_0}{\rho_0} \right) \frac{1}{\rho_0^\gamma} \\ \implies \underbrace{q^3 + A^* \rho_0^\gamma q^2 + k^2 \gamma \frac{p_0}{\rho_0} q - B^* k^2 \rho_0^\gamma}_{\text{cubic in } q, \text{ call } E(q)} &= 0. \end{aligned} \quad (8.98)$$

This has at least one real root – system is unstable if that real root is positive, $q > 0$.

Now $E(\infty) = \infty$, $E(0) = -B^* k^2 \rho_0^\gamma$. So the system is unstable if $B^* > 0$.

$$B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left. \frac{\partial \dot{Q}}{\partial \rho} \right|_p > 0 \quad (\text{condition for stability}) \quad (8.99)$$

$$\begin{aligned} \implies \left. \frac{\partial \dot{Q}}{\partial \left(\frac{\mu p}{\mathcal{R}^* T} \right)} \right|_p &> 0 \\ \implies - \frac{T^2}{p} \left. \frac{\partial \dot{Q}}{\partial T} \right|_p &> 0, \end{aligned} \quad (8.100)$$

and thus we arrive at the *Field criterion*,

$$\boxed{\text{unstable if } \left. \frac{\partial \dot{Q}}{\partial T} \right|_p > 0.} \quad (8.101)$$

The system is always unstable if it's Field unstable (named after George Field who wrote the classic paper on thermal instability in 1965).

However, even a Field stable system can be unstable if $A^* < 0 \implies \partial \dot{Q} / \partial T \Big|_\rho < 0$. From the dispersion relation, we see that this can happen for long wavelength modes, i.e. k small. Then

$$q^2(q + A^* \rho_0^\gamma) \approx 0 \implies q \approx -A^* \rho_0^\gamma. \quad (8.102)$$

Interpretation:

- Short wavelength perturbations are readily brought into pressure equilibrium by the action of sound waves, therefore, thermal instability proceeds at fixed pressure;
- Long wavelength perturbations: there is insufficient time for sound waves to equalise pressure with surroundings, so they tend to develop at constant density.

Example. Let's assume a specific form for \dot{Q} ,

$$\begin{aligned} \dot{Q} &= A\rho T^\alpha - H \\ &= \frac{A\mu}{\mathcal{R}_*} p T^{\alpha-1} - H, \end{aligned} \quad (8.103)$$

so,

$$\left. \frac{\partial \dot{Q}}{\partial T} \right|_p = (\alpha - 1) \frac{A\mu p}{\mathcal{R}_*} T^{\alpha-2}. \quad (8.104)$$

This is Field unstable, $\partial \dot{Q} / \partial T \Big|_p < 0$ if $\alpha < 1$. Bremsstrahlung has $\alpha = 0.5 \implies$ Field unstable.

CHAPTER 9

Viscous Flows

Thus far, we have been assuming that changes in the momentum of a fluid element are due entirely to pressure forces (acting normal to the surface of the element) or gravity (acting on the bulk).

This assumption is justified in the limit $\lambda \rightarrow 0$, i.e. the particles composing the fluid have vanishingly small collisional mean-free-path.

For finite- λ , momentum can diffuse through the fluid. This brings us to a discussion of viscosity.

9.1 Basics of Viscosity

In a viscous flow, momentum can be transferred if there are velocity differences between fluid elements.

The continuity equation is unchanged

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0. \quad (9.1)$$

But the momentum equation needs to be changed

$$\frac{\partial}{\partial t}(\rho u_i) = \partial_j \sigma_{ij} + \rho g_i, \quad g_i = -\partial_i \Psi, \quad (9.2)$$

with

$$\sigma_{ij} = \rho u_i u_j + p \delta_{ij} - \underbrace{\sigma'_{ij}}_{\substack{\text{viscous} \\ \text{stress tensor}}}. \quad (9.3)$$

As we will see later, σ'_{ij} is related to velocity gradients.

The connection between the viscous stress tensor and the microphysics (i.e. the mean-free-path) is uncovered by considering a simple linear shear flow as in Fig. 9.1.

Microscopically, thermal/random motion of the particles can allow momentum to “diffuse” across streamlines. This becomes more important as gas gets less collisional.

Let’s analyse the microscopic behaviour: assume the typical (thermal) velocity in the j -direction is u_j . So the momentum flux associated with this is

$$\underbrace{\rho u_i u_j}_{\substack{i\text{-component of momentum} \\ \text{carried in } j\text{-direction}}}. \quad (9.4)$$

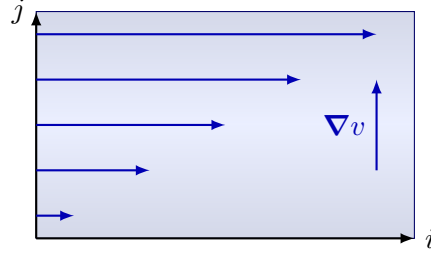


Fig. 9.1: Linear shear flow

The typical thermal velocity is $\sim \sqrt{kT/m}$. So, flux of the i -component of momentum in the upward j -direction is

$$\rho u_i \alpha \sqrt{\frac{kT}{m}}, \quad \alpha \sim 1. \quad (9.5)$$

For the element on the other side of the surface in the j -direction, the corresponding momentum flux across the surface is

$$-\rho u_i^* \alpha \sqrt{\frac{kT}{m}}, \quad (9.6)$$

where u_i^* is the i -velocity of that element. For a j -separation of δl we have

$$u_i^* = u_i + \delta l (\partial_j u_i). \quad (9.7)$$

So,

$$\text{net momentum flux} = -\rho (\partial_j u_i) \delta l \alpha \sqrt{\frac{kT}{m}}. \quad (9.8)$$

The relevant scale δl is the mean-free-path

$$\delta l \sim \lambda = \frac{1}{n\sigma}, \quad (9.9)$$

where σ is the collision cross section of the particles. If we treat the particles as hard spheres of radius a (decent approximation for neutral gas), then

$$\sigma = \pi a^2. \quad (9.10)$$

So,

$$\text{net momentum flux} = -\rho (\partial_j u_i) \frac{m}{\rho \pi a^2} \alpha \sqrt{\frac{kT}{m}}. \quad (9.11)$$

Putting this into the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p \delta_{ij}) + \partial_j \left[\underbrace{\frac{\alpha}{\pi a^2} \sqrt{mkT}}_{\equiv \eta, \text{ shear viscosity}} \partial_j u_i \right] + \rho g_i. \quad (9.12)$$

A rigorous derivation shows that, for this hard-sphere model, $\alpha = 5\sqrt{\pi}/64$.

Observations about the shear viscosity:

- η is independent of density (a denser gas has more particles to transport the momentum but the mean-free-path is shorter);
- η increases with T ;
- Isothermal system has $\eta = \text{const.}$

For a fully ionised plasma (e.g. the ICM), the mean-free-path is set by Coulomb collisions. Then

$$\lambda \propto T^2, \quad v_{\text{th}} \propto \sqrt{T} \quad \implies \quad \eta \propto T^{5/2}. \quad (9.13)$$

Thus the viscosity has a stronger temperature dependence than found for hard-sphere collisions.

9.2 Navier-Stokes Equation

The most general form of σ'_{ij} which is

- Galilean invariant;
- Linear in velocity components;
- Isotropic

is given by

$$\sigma'_{ij} = \eta \left(\partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \partial_k u_k \right) + \zeta \delta_{ij} \partial_k u_k, \quad (9.14)$$

with η and ζ independent of velocity. This is a *symmetric* tensor which ensures that there are no unbalanced torques on fluid elements.

The term associated with η relates to momentum transfer in *shear flows* (this term has zero trace). The term associated with ζ relates to momentum transfer due to bulk compression ($\partial_k u_k \equiv \nabla \cdot \mathbf{u}$).

Putting this into the momentum equation gives

$$\frac{\partial(\rho u_i)}{\partial t} = -\partial_j(\rho u_i u_j) - \partial_j p \delta_{ij} + \partial_j \left[\eta \left(\partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \partial_k u_k \right) + \zeta \delta_{ij} \partial_k u_k \right] + \rho g_i, \quad (9.15)$$

which we can combine with the continuity equation to give

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \partial_j u_i \right) = -\partial_j p \delta_{ij} + \partial_j \left[\eta \left(\partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \partial_k u_k \right) + \zeta \delta_{ij} \partial_k u_k \right] + \rho g_i. \quad (9.16)$$

This is the general form of the *Navier-Stokes equation*.

Outside of shocks ($\zeta \approx 0$) and for isothermal fluids ($\eta = \text{constant}$) we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Psi + \underbrace{\frac{\eta}{\rho}}_{\substack{\equiv \nu \\ \text{kinematic} \\ \text{viscosity}}} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right]. \quad (9.17)$$

9.3 Vorticity in Viscous Flows

Start with the Navier-Stokes equation with $\zeta = 0$ and $\eta = \text{const.}$, and take the curl of this, recalling the definition of the vorticity $\mathbf{w} = \nabla \times \mathbf{u}$:

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = \nabla \times \left(-\frac{1}{\rho} \nabla p - \nabla \Psi + \frac{\eta}{\rho} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right] \right). \quad (9.18)$$

To tidy up LHS, use the vector identity and definition of vorticity:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla u^2 - \mathbf{u} \times (\nabla \times \mathbf{u}) \quad (9.19)$$

$$\Rightarrow \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla \times (\mathbf{u} \times \mathbf{w}). \quad (9.20)$$

To tidy up RHS, assume a barotropic fluid, $p = p(\rho)$:

$$\begin{aligned} \Rightarrow \nabla \times \left(\frac{1}{\rho} \nabla p \right) &= \nabla \left(\frac{1}{\rho} \right) \times \nabla p + \frac{1}{\rho} \nabla \times \nabla p \\ &= -\frac{1}{\rho^2} \underbrace{\nabla \rho \times \nabla p}_{=0 \text{ since surfaces of constant } \rho \text{ and } p \text{ align}}. \end{aligned} \quad (9.21)$$

Putting the pieces together, we get

$$\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \nabla \times \left[\frac{\eta}{\rho} \nabla^2 \mathbf{u} \right], \quad (9.22)$$

and thus

$$\boxed{\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \frac{\eta}{\rho} \nabla^2 \mathbf{w}.} \quad (9.23)$$

where, in the last step, we have ignored gradients of $\nu = \eta/\rho$ (so strictly assumed uniform density). So, vorticity is carried with the flow but also diffuses through the flow due to action of vorticity.

9.4 Energy Dissipation in Incompressible Viscous Flows

Viscosity leads to dissipation of kinetic energy into heat – an irreversible process.

Let's analyse this in the case of an incompressible flow so that we don't need to worry about $p dV$ work. Then the total kinetic energy is

$$E_{\text{kin}} = \frac{1}{2} \int \rho u^2 dV. \quad (9.24)$$

Let's consider the rate of change of E_{kin} with time

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) &= u_i \frac{\partial}{\partial t} (\rho u_i) \\ &= -u_i \partial_j (\rho u_i u_j) - u_i \partial_j \delta_{ij} p + u_i \partial_j \sigma'_{ij} \\ &= -u_i \partial_j (\rho u_i u_j) - u_i \partial_i p + \partial_j (u_i \sigma'_{ij}) - \sigma'_{ij} \partial_j u_i. \end{aligned} \quad (9.25)$$

Look at the first term of RHS:

$$u_i \partial_j (\rho u_i u_j) = u_i \left(u_j \partial_j (\rho u_i) + \underbrace{\rho u_i \partial_j u_j}_{=0} \right), \quad (9.26)$$

where last term is zero due to the incompressible flow assumption,

$$\nabla \cdot \mathbf{u} \implies \partial_j u_j = 0. \quad (9.27)$$

Also note that

$$\begin{aligned} \partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i \right) &= \frac{1}{2} \rho u_i u_i \underbrace{\partial_j u_j}_{=0} + u_j \partial_j \left(\frac{1}{2} \rho u_i u_i \right) \\ &= u_j u_i \partial_j (\rho u_i), \end{aligned} \quad (9.28)$$

therefore

$$u_i \partial_j (\rho u_i u_j) = \partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i \right). \quad (9.29)$$

So,

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) &= -\partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i \right) - \partial_i (u_i p) + p \partial_i u_i + \partial_j (u_i \sigma'_{ij}) - \sigma'_{ij} \partial_j u_i \\ &= -\partial_i \left(\rho u_i \left[\frac{1}{2} u^2 + \frac{p}{\rho} \right] - u_j \sigma'_{ij} \right) - \sigma'_{ij} \partial_j u_i. \end{aligned} \quad (9.30)$$

Integrating over the volume,

$$\begin{aligned} \frac{\partial E_{\text{kin}}}{\partial t} &= \frac{\partial}{\partial t} \int_{\mathcal{V}} \frac{1}{2} \rho u^2 dV \\ &= - \int_{\mathcal{V}} \partial_i \left(\rho u_i \left[\frac{1}{2} u^2 + \frac{p}{\rho} \right] - u_j \sigma'_{ij} \right) dV - \int_{\mathcal{V}} \sigma'_{ij} \partial_j u_i dV \\ &= - \underbrace{\oint_S \left(\rho \mathbf{u} \left[\frac{1}{2} u^2 + \frac{p}{\rho} \right] - \mathbf{u} \cdot \underline{\underline{\sigma'}} \right) \cdot d\mathbf{S}}_{\text{Energy flux into volume including work done by viscous forces } \mathbf{u} \cdot \underline{\underline{\sigma'}}} - \underbrace{\int_{\mathcal{V}} \sigma'_{ij} \partial_j u_i dV}_{\text{Rate of change of } E_{\text{kin}} \text{ due to viscous dissipation}}. \end{aligned} \quad (9.31)$$

Let's take the volume \mathcal{V} to be the whole fluid so that the surface integral is zero (e.g. \mathbf{u} at bounding surface = 0, or \mathbf{u} at $\infty = 0$).

Then

$$\begin{aligned} \frac{\partial E_{\text{kin}}}{\partial t} &= - \int_{\mathcal{V}} \sigma'_{ij} \partial_j u_i dV \\ &= - \frac{1}{2} \int_{\mathcal{V}} \sigma'_{ij} (\partial_j u_i + \partial_i u_j) dV \quad \text{since } \sigma' \text{ is symmetric} \end{aligned} \quad (9.32)$$

But $\sigma'_{ij} = \eta (\partial_j u_i + \partial_i u_j)$ for an incompressible fluid. So,

$$\frac{\partial E_{\text{kin}}}{\partial t} = - \frac{1}{2} \int_{\mathcal{V}} \eta (\partial_j u_i + \partial_i u_j)^2 dV \quad (9.33)$$

We see that η needs to be positive in order for us to obey the 2nd law of thermodynamics.

9.5 Viscous Flow through a Pipe

Now consider flow through a long pipe with a constant circular cross-section (see Fig. 9.2).

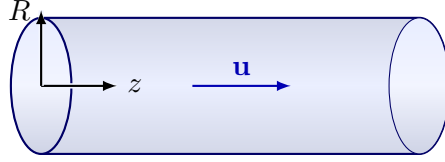


Fig. 9.2: A pipe

Assume

- Steady flow with $u_R = u_\phi = 0$, $u_z \neq 0$;
- Incompressible, uniform density fluid;
- Neglect gravity.

The Navier-Stokes equation reads

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{=0 \text{ since steady state}} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{u}}_{=0 \text{ due to symmetry}} = -\frac{1}{\rho} \nabla p + \nu \left[\nabla^2 \mathbf{u} + \underbrace{\frac{1}{3} \nabla (\nabla \cdot \mathbf{u})}_{=0 \text{ since incompressible}} \right]$$

$$\implies \nu \nabla^2 \mathbf{u} = \frac{1}{\rho} \nabla p. \quad (9.34)$$

By symmetry we have

$$u_R = u_\phi = 0 \implies \frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0. \quad (9.35)$$

For the z -component

$$\underbrace{\frac{1}{\rho} \frac{\partial p}{\partial z}}_{\text{function of } z \text{ only}} = \underbrace{\nu \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial u_z}{\partial R} \right)}_{\text{function of } R \text{ only}} = \underbrace{-\frac{1}{\rho} \frac{\Delta p}{l}}_{\text{constant, written in terms of global pressure gradient}}. \quad (9.36)$$

Integrating gives

$$u = -\frac{\Delta p}{4\rho\nu l} R^2 + a \ln R + b, \quad (9.37)$$

where a and b are constants. Apply boundary conditions:

- At $R = 0$, u finite $\implies a = 0$;



Fig. 9.3: Parabolic velocity profile for viscous flow through a pipe

- At $R = R_0$, $u = 0$ (no slip boundary condition at wall).

$$\Rightarrow u = \frac{\Delta p}{4\rho\nu l}(R_0^2 - R^2). \quad (9.38)$$

So the velocity profile is *parabolic* (see Fig. 9.3).

The mass flux passing through an annular element $2\pi R dR$ is $2\pi R \rho u dR$. So, the total mass flow rate is

$$Q = \int_0^{R_0} 2\pi \rho u R dR = \frac{\pi}{8} \frac{\Delta p}{\nu l} R_0^4. \quad (9.39)$$

As $\eta \rightarrow 0$, i.e. $\nu \rightarrow 0$, the flow rate $\rightarrow \infty$ (or, in other words, an inviscid flow cannot be in steady state in this pipe if there is a non-zero pressure gradient).

If Δp increases sufficiently, it becomes unstable and irregular, giving turbulent motions above a critical speed.

The actual transition to turbulence is usually phrased in terms of the *Reynolds number*

$$\text{Re} \equiv \frac{LV}{\nu}, \quad (9.40)$$

where L and V are “characteristic” length and velocity scales of the system. We have turbulence when

$$\text{Re} > \text{Re}_{\text{crit}} \quad (9.41)$$

9.6 Accretion Disks

Accretion disks are one of the most important applications of the Navier-Stokes equation in astrophysics.

Consider some gas flowing towards a central object (star, planet, black hole, ...). Almost always, the gas will have significant angular momentum about that object. If gravitationally bound to the object, the gas will settle into a plane defined by the mean angular momentum vector. Residual motions in other directions will be damped out on a free-fall timescale.

The gas will settle into circular orbits – the lowest energy configuration for a given angular momentum. In the vertical direction (parallel to the angular momentum vector) the system will come into hydrostatic equilibrium with an internal vertical pressure gradient

balancing the vertical component of gravity. In the radial direction (along the direction towards the central object), the system will achieve a state where the centripetal force is supplied by gravity and the radial pressure gradient.

A very important special case is when the disk is “thin”, meaning that the scale-height in the vertical direction h is much less than the radius r . Then, radial pressure gradients are negligible and we can just write

$$\Omega^2 R = \frac{GM}{R^2} \quad \Rightarrow \quad \Omega = \sqrt{\frac{GM}{R^3}}, \quad (9.42)$$

where Ω is the angular velocity of the flow around the central object. This means that

$$\frac{d\Omega}{dR} \neq 0 \quad \Rightarrow \quad \text{shear flow.} \quad (9.43)$$

Viscosity will allow angular momentum to be transferred from the fast moving inner regions to the more slowly moving outer regions. This means the inner disk fluid elements lose angular momentum. We have

$$J = R^2 \Omega = \sqrt{GM R}, \quad (\text{per unit mass}) \quad (9.44)$$

meaning that inner disk fluid elements drift inwards.

Ultimately, most of the mass flows inwards; a small amount of the mass carries all of the angular momentum out to large radii.

Let’s set up a simple model for a geometrically-thin accretion disk. We assume:

- Cylindrical polar coordinates (R, ϕ, z) ;
- Axisymmetric, $\partial/\partial\phi = 0$;
- Hydrostatic equilibrium in the z -direction, $u_z = 0$;
- u_ϕ close to Keplerian velocity (i.e. thin disk);
- u_R small and set by action of viscosity;
- Bulk viscosity zero.

The continuity equation in cylindrical polar coordinates is

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \rho u_R) + \underbrace{\frac{1}{R} \frac{\partial}{\partial \phi} (\rho u_\phi)}_{=0 \text{ due to axisymmetry}} + \underbrace{\frac{\partial}{\partial z} (\rho u_z)}_{=0 \text{ since hydrostatic}} &= 0 \\ \Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \rho u_R) &= 0. \end{aligned} \quad (9.45)$$

Define the surface density Σ by

$$\Sigma \equiv \int_{-\infty}^{\infty} \rho \, dz. \quad (9.46)$$

Then, integrating the above form of the continuity equation over z we have

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) = 0. \quad (9.47)$$

We can get the same result by thinking of the disk as a set of rings/annuli as shown in Fig. 9.4.

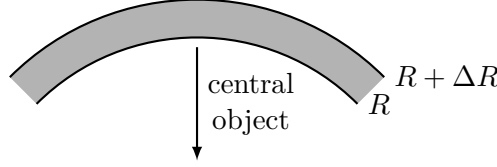


Fig. 9.4: Infinitesimal annulus element of disk

$$\begin{aligned} \text{rate of change} \\ \text{of mass in} \\ \text{the annulus} \end{aligned} = \begin{aligned} \text{flux into} \\ \text{annulus} \end{aligned} + \begin{aligned} \text{flux out} \\ \text{of annulus} \end{aligned} \quad (9.48)$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial t} (2\pi R \Delta R \Sigma) &= 2\pi R \Sigma(R) u_R(R) - 2\pi (R + \Delta R) \Sigma(R + \Delta R) u_R(R + \Delta R) \\ \Rightarrow R \frac{\partial \Sigma}{\partial t} &= - \left[\frac{(R + \Delta R) \Sigma(R + \Delta R) u_R(R + \Delta R) - R \Sigma(R) u_R(R)}{\Delta R} \right] \\ \Rightarrow R \frac{\partial \Sigma}{\partial t} &= - \frac{\partial}{\partial R} (R \Sigma u_R). \quad \text{taking } \Delta R \rightarrow 0. \end{aligned} \quad (9.49)$$

Now we look at the conservation of angular momentum. A derivation starting with the Navier-Stokes equation in cylindrical coordinates appears in Appendix A.1 at the end of this chapter. Here we use the ring/annulus approach:

$$\begin{aligned} \text{rate of change} \\ \text{of ang. mtm.} \end{aligned} = \begin{aligned} \text{ang. mtm.} \\ \text{of mass} \\ \text{entering ring} \end{aligned} - \begin{aligned} \text{ang. mtm.} \\ \text{of mass} \\ \text{leaving ring} \end{aligned} + \begin{aligned} \text{net torque on ring} \\ \text{(viscous, magnetic, etc.)} \end{aligned} \quad (9.50)$$

$$\Rightarrow \frac{\partial}{\partial t} (2\pi R \Delta R \Sigma R^2 \Omega) = f(R) - f(R + \Delta R) + G(R + \Delta R) - G(R), \quad (9.51)$$

where

$$f(R) \equiv 2\pi R \Sigma u_R \Omega R^2, \quad (9.52)$$

and $G(R)$ is the torque exerted by the disk outside of radius R on the disk inside of radius R :

$$G(R) = 2\pi R \nu \Sigma R \frac{d\Omega}{dR} = 2\pi R^3 \nu \Sigma \frac{d\Omega}{dR}, \quad (9.53)$$

therefore

$$\frac{\partial}{\partial t} (R \Sigma u_\phi) = - \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 u_\phi u_R) + \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{\partial \Omega}{\partial R} \right). \quad (9.54)$$

Now assume $\partial u_\phi / \partial t = 0$ since gas is on Keplerian orbits. Then

$$\begin{aligned}
 Ru_\phi \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 u_\phi u_R) &= \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right) \\
 \Rightarrow -u_\phi \frac{\partial}{\partial R} (R \Sigma u_R) + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 u_\phi u_R) &= \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right) \\
 \Rightarrow -u_\phi \frac{\partial}{\partial R} (R \Sigma u_R) + \frac{u_\phi R}{R} \frac{\partial}{\partial R} (R \Sigma u_R) + \Sigma u_R \frac{\partial}{\partial R} (u_\phi R) &= \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right) \\
 \Rightarrow R \Sigma u_R \frac{\partial}{\partial R} (R^2 \Omega) &= \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right) \\
 \Rightarrow u_R &= \frac{\frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right)}{R \Sigma \frac{\partial}{\partial R} (R^2 \Omega)}. \tag{9.55}
 \end{aligned}$$

Substitute this into Eq. (9.54) and specialise to the case of a Newtonian point source gravitational field $\Omega = \sqrt{GM/R^3}$, yielding

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left[R^{1/2} \frac{\partial}{\partial R} (\nu \Sigma R^{1/2}) \right]}. \tag{9.56}$$

So the surface density $\Sigma(R, t)$ obeys a diffusion equation.

Notes on accretion disks:

- In general, $\nu = \nu(R, \Sigma)$ and so this is a non-linear diffusion equation for Σ . It reduces to linear if $\nu = \nu(R)$;
- Solutions of this equation show that an initial ring of matter will broaden and then “slump” inwards towards the central object;

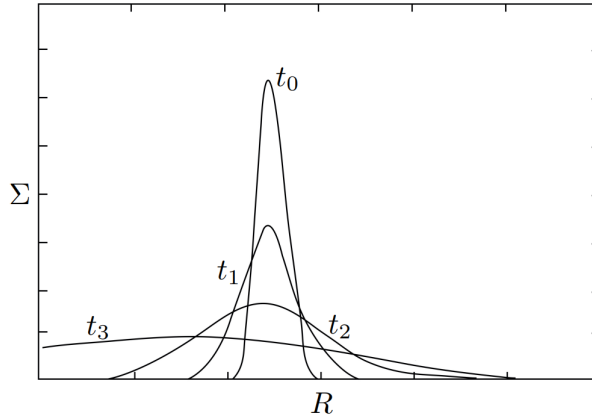


Fig. 9.5: The viscous evolution of a ring

- Timescale for evolution is t_ν where

$$\begin{aligned}
 \frac{\Sigma}{t_\nu} &\sim \frac{1}{R} \frac{1}{R} R^{1/2} \frac{1}{R} \nu \Sigma R^{1/2} \sim \frac{\nu \Sigma}{R^2} \\
 \Rightarrow t_\nu &\sim \frac{R^2}{\nu} = \frac{R}{u_\phi} \frac{R u_\phi}{\nu} = \Omega^{-1} \text{Re}, \tag{9.57}
 \end{aligned}$$

where Re is the Reynolds number;

- If viscosity is due to particle thermal motions, typical values would suggest that $\text{Re} \sim 10^{14}$! This means

$$t_\nu \gg \text{age of the Universe.} \quad (9.58)$$

There must be another source of effective viscosity: we now know that there is an *effective viscosity* due to magnetohydrodynamic turbulence driven by the *magnetorotational instability*.

9.7 Steady-State, Geometrically-Thin Disks

Consider a steady state such that $\partial/\partial t = 0$. Then

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma u_R) &= 0 \\ \implies R \Sigma u_R &= C_1 = -\frac{\dot{m}}{2\pi} \end{aligned} \quad (9.59)$$

where $\dot{m} = -2\pi R \Sigma u_R$ is the steady state mass inflow rate. Now recall from Eq. (9.55) that

$$u_R = \frac{\frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right)}{R \Sigma \frac{\partial}{\partial R} (R^2 \Omega)}, \quad (9.60)$$

therefore,

$$\begin{aligned} -\frac{\dot{m}}{2\pi R \Sigma} &= -\frac{3}{\Sigma R^{1/2}} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \quad \text{for } \Omega^2 = GM/R^3 \\ \implies \nu \Sigma &= \frac{\dot{m}}{3\pi} \left(1 - \sqrt{\frac{R_*}{R}} \right), \end{aligned} \quad (9.61)$$

where we have taken as a boundary condition that $\nu \Sigma = 0$ at $R = R_*$. This amounts to saying that there are no viscous torques at $R = R_*$. Physically R_* can be:

- Surface of an accreting star;
- Innermost circular orbit around a black hole.

Let's now calculate the viscous dissipation neglecting $p dV$ work and bulk viscosity. Specifically, we will calculate the viscous dissipation per unit surface area of the disk:

$$\begin{aligned} F_{\text{diss}} &= \int \sigma'_{ij} \partial_j u_i \frac{dV}{2\pi R dR d\phi} \\ &= \frac{1}{2} \int \eta (\partial_j u_i + \partial_i u_j)^2 dz \\ &= \int \eta R^2 \left(\frac{d\Omega}{dR} \right)^2 dz \\ &= \nu \Sigma R^2 \left(\frac{d\Omega}{dR} \right)^2. \end{aligned} \quad (9.62)$$

Combining with our previous result (9.61) for $\nu\Sigma$ and recalling that $\Omega^2 = GM/R^3$, we have

$$F_{\text{diss}} = \frac{3GM\dot{m}}{4\pi R^3} \left(1 - \sqrt{\frac{R_*}{R}} \right). \quad (9.63)$$

Notes on dissipation in a disk:

- Total energy emitted is

$$L = \int_{R_*}^{\infty} F_{\text{diss}} 2\pi R \, dR = \frac{GM\dot{m}}{2R_*}. \quad (9.64)$$

Here, $-GM/R_*$ is the gravitational potential at R_* . Therefore, $GM\dot{m}/R_*$ is the rate of gravitational energy loss of the flow. $GM\dot{m}/2R_*$ is radiated, the other half stays in the flow as kinetic energy and is dissipated in the boundary layer on an accreting star, or carried into the black hole;

- At a given location far from the inner edge ($R > R_*$) we have

$$F_{\text{diss}} \approx \frac{3GM\dot{m}}{4\pi R^3}. \quad (9.65)$$

But an elementary estimate based on the loss of gravitational potential energy would give

$$F_{\text{diss,est}} = \underbrace{\frac{1}{2\pi R \, dR}}_{\text{area of annulus}} \cdot \underbrace{\frac{\partial}{\partial R} \left(\frac{GM\dot{m}}{R} \right)}_{\text{change in grav. potential of } \dot{m} \text{ over annulus}} \cdot \underbrace{\frac{1}{2}}_{\text{half converts to radiation, rest to kinetic}} = \frac{GM\dot{m}}{4\pi R^3}. \quad (9.66)$$

The extra factor of “3” in the correct formula is due to the transport of energy through the disk by viscous torques.

9.7.1 Radiation from Steady-State Thin Disks

If a disk is optically-thick, all radiation is thermalised and it radiates locally as a black body

$$\underbrace{2}_{\text{top and bottom of disk}} \cdot \sigma_{\text{SB}} T_{\text{eff}}^4 = \frac{3GM\dot{m}}{4\pi R^3} \left(1 - \sqrt{\frac{R_*}{R}} \right) \\ \Rightarrow T_{\text{eff}} = \left[\frac{3GM\dot{m}}{8\pi\sigma_{\text{SB}}R^3} \right]^{1/4}. \quad (9.67)$$

So, for $R \gg R_*$, $T_{\text{eff}} \propto R^{-3/4}$.

The radiation emitted at a frequency f is

$$F_f = \int_{R_*}^{\infty} \frac{2h}{c^2} \frac{f^3}{e^{hf/kT_{\text{eff}}} - 1} 2\pi R \, dR. \quad (9.68)$$

So, we see that all of the observables from a steady-state disk are independent of the viscosity ν , provided it is large enough to supply the necessary angular momentum transport. To constrain ν , we need to study non-steady disks.

CHAPTER 10

Plasmas

Plasmas are fluids composed of charged particles. Thus, electromagnetic fields become important for both microphysics and large scale dynamics.

10.1 Magnetohydrodynamic (MHD) Equations

Consider a fully ionised hydrogen plasma, containing only protons (number density n^+ , bulk velocity \mathbf{u}^+) and electrons (n^- , \mathbf{u}^-). Mass conservation for each of the proton and electron fluids is

$$\frac{\partial n^+}{\partial t} + \nabla \cdot (n^+ \mathbf{u}^+) = 0 \quad (10.1)$$

$$\frac{\partial n^-}{\partial t} + \nabla \cdot (n^- \mathbf{u}^-) = 0. \quad (10.2)$$

The mass density is $\rho = m^+ n^+ + m^- n^-$ and the centre-of-mass velocity is

$$\mathbf{u} = \frac{m^+ n^+ \mathbf{u}^+ + m^- n^- \mathbf{u}^-}{m^+ n^+ + m^- n^-}. \quad (10.3)$$

So, we can combine these to give the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (10.4)$$

The continuity equation is the same as found before.

The charge density is $q = n^+ e^+ + n^- e^-$ and the current density is $\mathbf{j} = e^+ n^+ \mathbf{u}^+ + e^- n^- \mathbf{u}^-$. So, the above information also gives a *conservation of charge equation*

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (10.5)$$

When we formulate the momentum equation, we have to consider the Lorentz force on each particle

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (10.6)$$

So, for the two species of particles:

$$m^+ n^+ \left(\frac{\partial \mathbf{u}^+}{\partial t} + \mathbf{u}^+ \cdot \nabla \mathbf{u}^+ \right) = e^+ n^+ (\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}) - f^+ \nabla p \quad (10.7)$$

$$m^- n^- \left(\frac{\partial \mathbf{u}^-}{\partial t} + \mathbf{u}^- \cdot \nabla \mathbf{u}^- \right) = e^- n^- (\mathbf{E} + \mathbf{u}^- \times \mathbf{B}) - f^- \nabla p \quad (10.8)$$

where f^\pm is the fraction of the pressure gradient that accelerates the protons/electrons.

Summing these equations gives

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p. \quad (10.9)$$

Ohm's law lets us relate \mathbf{j} to \mathbf{E} and \mathbf{B} :

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \quad (10.10)$$

where σ is the electrical conductivity. This equation is needed to close the set of equations.

So, recapping the current set of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (10.11)$$

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (10.12)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p \quad (10.13)$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \quad (10.14)$$

We need to relate q , \mathbf{j} , \mathbf{E} and \mathbf{B} – Maxwell's equations!

$$\nabla \cdot \mathbf{B} = 0 \quad (10.15)$$

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0} \quad (10.16)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (10.17)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (10.18)$$

where we note $\epsilon_0 \mu_0 = 1/c^2$.

10.1.1 Simplifying MHD

Let us simplify in the case of a non-relativistic, highly conducting plasma. Suppose fields are varying over length scales l and timescales τ . Then

1.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \implies \frac{E}{B} \sim \frac{1}{\tau} \sim u. \quad (10.19)$$

2.

$$\left| \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right| / |\nabla \times \mathbf{B}| \sim \frac{1}{c^2} \left(\frac{l}{\tau} \right)^2 \sim \frac{u^2}{c^2} \ll 1, \quad (10.20)$$

for non-relativistic flows. Therefore, displacement current can be ignored in non-relativistic MHD;

3. Look at two terms from Eq. (10.13):

$$\frac{|q\mathbf{E}|}{|\mathbf{j} \times \mathbf{B}|} \sim \frac{qE}{jB} \sim \frac{\epsilon_0 E/l}{B/l\mu_0} \frac{E}{B} \sim u^2 \epsilon_0 \mu_0 \sim \frac{u^2}{c^2} \ll 1. \quad (10.21)$$

Therefore, charge neutrality is preserved to a high approximation due to the strength of electrostatic forces. If there is a charge imbalance, it will oscillate with a characteristic frequency, the *plasma frequency*

$$\omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}. \quad (10.22)$$

4. Neglecting displacement current in the relevant Maxwell equation, we get

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0 \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}). \quad (10.23)$$

Take curl:

$$\begin{aligned} \underbrace{\nabla \times (\nabla \times \mathbf{B})}_{= -\nabla^2 \mathbf{B} - \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}} &= \mu_0 \sigma \left(\underbrace{\nabla \times \mathbf{E}}_{-\frac{\partial \mathbf{B}}{\partial t}} + \nabla \times (\mathbf{u} \times \mathbf{B}) \right) \\ \Rightarrow \quad \frac{\partial \mathbf{B}}{\partial t} &= \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{advection of the field by the flow}} + \underbrace{\frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}}_{\text{dissipation of the field through the flow}}. \end{aligned} \quad (10.24)$$

If the fluid is a good conductor, i.e. σ is very large, then we can ignore the diffusion term and we have an equation that is analogous to the Helmholtz equation/Kelvin's theorem:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (10.25)$$

We talk about the “freezing” of the magnetic flux into the plasma. In the high σ limit we must also have

$$\begin{aligned} \mathbf{j} &= \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad \text{is finite} \\ \Rightarrow \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} &= 0 \quad \text{as } \sigma \rightarrow \infty \\ \Rightarrow \quad \mathbf{E} \cdot \mathbf{B} &= 0 \\ \text{i.e. } \mathbf{E} &\perp \mathbf{B}. \end{aligned} \quad (10.26)$$

So, the full set of *ideal MHD equations*, i.e. equations describing a non-relativistic, perfectly conducting, charge neutral plasma are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (10.27)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \mathbf{j} \times \mathbf{B} - \nabla p \quad (10.28)$$

$$\left. \begin{aligned} \mathbf{E} + \mathbf{u} \times \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B} / \partial t \\ p &= K \rho^\gamma \end{aligned} \right\} \Rightarrow \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \quad (10.29)$$

10.2 The Dynamical Effects of Magnetic Fields

The magnetic force density appearing in the above ideal MHD equations is

$$\mathbf{f}_{\text{mag}} = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (10.30)$$

So using vector identity this is

$$\mathbf{f}_{\text{mag}} = \left[\underbrace{-\nabla \left(\frac{B^2}{2} \right)}_{\substack{\text{magnetic pressure} \\ \text{term with} \\ p_{\text{mag}} = B^2/2\mu_0}} + \underbrace{(\mathbf{B} \cdot \nabla) \mathbf{B}}_{\substack{\text{magnetic tension} \\ \text{term (vanishes for} \\ \text{straight field lines)}}} \right] \quad (10.31)$$

Since there are new force terms in the momentum equation, this will change the nature of the waves that are possible.

10.3 Waves in Plasmas

We can repeat the perturbation analysis that we conducted for sound waves but now include the effects of a magnetic field. We will perturb about an equilibrium consisting of a static ($\mathbf{u} = \mathbf{0}$) plasma with uniform density ρ_0 , uniform pressure p_0 , and uniform magnetic field \mathbf{B}_0 .

We start by writing down the governing equations of ideal MHD, assuming a barotropic equation of state:

APPENDIX A

Appendix

A.1 Derivation of the Momentum Equation for Disk Evolution

The Navier–Stokes momentum equation, neglecting bulk viscosity, is

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \nabla \cdot \left[\eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \underline{\underline{\mathbf{I}}} \right) \right] + \rho \mathbf{g}. \quad (\text{A.1})$$

Let's look at the ϕ -component in cylindrical coordinates,

$$[(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \hat{\mathbf{e}}_\phi = u_R \frac{\partial u_\phi}{\partial R} + u_\phi \frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z} + \frac{u_R u_\phi}{R}, \quad (\text{A.2})$$

which simplifies with $u_z = 0$ and cylindrical symmetry to

$$[(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \hat{\mathbf{e}}_\phi = u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}. \quad (\text{A.3})$$

With cylindrical symmetry the pressure gradient in the ϕ direction is zero so we have

$$\rho \left(\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R} \right) = [\nabla \cdot \underline{\underline{\mathbf{T}}}] \cdot \hat{\mathbf{e}}_\phi, \quad (\text{A.4})$$

where we have defined the tensor

$$\underline{\underline{\mathbf{T}}} = \eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \underline{\underline{\mathbf{I}}} \right). \quad (\text{A.5})$$

We will use the vector gradient tensor in cylindrical coordinates

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_R}{\partial R} & \frac{1}{R} \frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} & \frac{1}{R} \frac{\partial u_\phi}{\partial \phi} + \frac{u_R}{R} & \frac{\partial u_\phi}{\partial z} \\ \frac{\partial u_z}{\partial R} & \frac{1}{R} \frac{\partial u_z}{\partial \phi} & \frac{\partial u_z}{\partial z} \end{bmatrix}, \quad (\text{A.6})$$

which simplifies with $u_z = 0$ and cylindrical symmetry to

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_R}{\partial R} & -\frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} & \frac{u_R}{R} & \frac{\partial u_\phi}{\partial z} \\ 0 & 0 & 0 \end{bmatrix}. \quad (\text{A.7})$$

Then, using $\nabla \cdot \mathbf{u} = \frac{1}{R} \frac{\partial}{\partial R} (R u_R)$ ($u_z = 0$, cylindrical symmetry), we get

$$\underline{\underline{\mathbf{T}}} = \eta \begin{bmatrix} 2 \frac{\partial u_R}{\partial R} - \frac{2}{3R} \frac{\partial}{\partial R} (R u_R) & \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} & \frac{\partial u_R}{\partial z} \\ \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} & 2 \frac{u_R}{R} - \frac{2}{3R} \frac{\partial}{\partial R} (R u_R) & \frac{\partial u_\phi}{\partial z} \\ \frac{\partial u_R}{\partial z} & \frac{\partial u_\phi}{\partial z} & -\frac{2}{3R} \frac{\partial}{\partial R} (R u_R) \end{bmatrix}, \quad (\text{A.8})$$

and the ϕ -component of the tensor divergence is

$$\left[\nabla \cdot \underline{\underline{\mathbf{T}}}\right] \cdot \hat{\mathbf{e}}_\phi = \frac{\partial}{\partial R} \left[\eta \left(\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} \right) \right] + \frac{\partial}{\partial z} \left(\eta \frac{\partial u_\phi}{\partial z} \right) + \frac{2}{R} \left[\eta \left(\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} \right) \right]. \quad (\text{A.9})$$

Putting all together, the ϕ -component of the momentum conservation is

$$\rho \left(\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R} \right) = \frac{\partial}{\partial R} \left(\eta \frac{\partial u_\phi}{\partial R} \right) + \frac{\partial}{\partial z} \left(\eta \frac{\partial u_\phi}{\partial z} \right) + \frac{1}{R} \frac{\partial}{\partial R} (\eta u_\phi) - 2 \frac{u_\phi}{R} \frac{\partial \eta}{\partial R} - \eta \frac{u_\phi}{R^2}. \quad (\text{A.10})$$

Integrating over z :

$$\Sigma \left(\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R} \right) = \frac{\partial}{\partial R} \left(\nu \Sigma \frac{\partial u_\phi}{\partial R} \right) + \frac{1}{R} \frac{\partial}{\partial R} (\nu \Sigma u_\phi) - 2 \frac{u_\phi}{R} \frac{\partial}{\partial R} (\nu \Sigma) - \frac{\nu \Sigma u_\phi}{R}. \quad (\text{A.11})$$

Adding the continuity equation (9.47) multiplied by Ru_ϕ with the momentum equation multiplied by R we get

$$\begin{aligned} \frac{\partial}{\partial t} (R \Sigma u_\phi) + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 u_\phi u_R) &= R \frac{\partial}{\partial R} \left(\nu \Sigma \frac{\partial u_\phi}{\partial R} \right) + \frac{\partial}{\partial R} (\nu \Sigma u_\phi) - 2 u_\phi \frac{\partial}{\partial R} (\nu \Sigma) - \frac{\nu \Sigma u_\phi}{R} \\ &= R \frac{\partial}{\partial R} \left(\nu \Sigma \frac{\partial u_\phi}{\partial R} \right) + 2 \nu \Sigma \left(\frac{1}{\nu \Sigma} \frac{\partial (\nu \Sigma u_\phi)}{\partial R} - \frac{u_\phi}{\nu \Sigma} \frac{\partial (\nu \Sigma)}{\partial R} \right) \\ &\quad - \frac{\partial (\nu \Sigma u_\phi)}{\partial R} - \frac{\nu \Sigma u_\phi}{R} \\ &= R \frac{\partial}{\partial R} \left(\nu \Sigma \frac{\partial u_\phi}{\partial R} \right) + 2 \nu \Sigma \frac{\partial u_\phi}{\partial R} - \frac{\partial (\nu \Sigma u_\phi)}{\partial R} - \frac{\nu \Sigma u_\phi}{R} \\ &= \frac{1}{R} \frac{\partial}{\partial R} \left[R^2 \nu \Sigma \frac{\partial u_\phi}{\partial R} - R \nu \Sigma u_\phi \right] \\ &= \frac{1}{R} \frac{\partial}{\partial R} \left[\nu \Sigma R^3 \left(\frac{1}{R} \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R^2} \right) \right]. \end{aligned} \quad (\text{A.12})$$

Using $u_\phi = \Omega R$ we get

$$\frac{\partial}{\partial t} (R \Sigma u_\phi) + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 u_\phi u_R) = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\Omega}{dR} \right), \quad (\text{A.13})$$

which is the same as Eq. (9.54).