Part II Electrodynamics and Optics

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Preface

There are, to the best of our knowledge, four forces at play in the Universe. At the very largest scales – those of planets or stars or galaxies – the force of gravity dominates. At the very smallest distances, the two nuclear forces hold sway. For everything in between, it is force of electromagnetism that rules.

At the atomic scale, electromagnetism (admittedly in conjunction with some basic quantum effects) governs the interactions between atoms and molecules. It is the force that underlies the periodic table of elements, giving rise to all of chemistry and, through this, much of biology. It is the force which binds atoms together into solids and liquids. And it is the force which is responsible for the incredible range of properties that different materials exhibit.

At the macroscopic scale, electromagnetism manifests itself in the familiar phenomena that give the force its name. In the case of electricity, this means everything from rubbing a balloon on your head and sticking it on the wall, through to the fact that you can plug any appliance into the wall and be pretty confident that it will work. For magnetism, this means everything from the shopping list stuck to your fridge door, through to trains in Japan which levitate above the rail. Harnessing these powers through the invention of the electric dynamo and motor has transformed the planet and our lives on it.

As if this wasn't enough, there is much more to the force of electromagnetism for it is, quite literally, responsible for everything you've ever seen. It is the force that gives rise to light itself.

Rather remarkably, a full description of the force of electromagnetism is contained in four simple and elegant equations. These are known as the *Maxwell equations*. There are few places in physics, or indeed in any other subject, where such a richly diverse set of phenomena flows from so little. The purpose of this course is to introduce the Maxwell equations and to extract some of the many stories they contain.

However, there is also a second theme that runs through this course. The force of electromagnetism turns out to be a blueprint for all the other forces. There are various mathematical symmetries and structures lurking within the Maxwell equations, structures which Nature then repeats in other contexts. Understanding the mathematical beauty of the equations will allow us to see some of the principles that underly the laws of physics, laying the groundwork for future study of the other forces.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Charge and Current

Each particle in the Universe carries with it a number of properties. These determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called *electric charge*.

For the purposes of this course, we can think of electric charge as a real number, $q \in \mathbb{R}$. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

The SI unit of charge is the *Coulomb*, denoted by C. It is, like all SI units, a parochial measure, convenient for human activity rather than informed by the underlying laws of the physics. (We'll learn more about how the Coulomb is defined in Section 3.1). At a fundamental level, Nature provides us with a better unit of charge. This follows from the fact that charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron which we denoted as $-e$, with

$$e = 1.602176634^{-19}\mathrm{C}. \tag{1.1}$$

A much more natural unit would be to simply count charge as $q = ne$ with $n \in \mathbb{Z}$. Then electrons have charge -1 while protons have charge $+1$ and neutrons have charge 0. Nonetheless, in this course, we will bow to convention and stick with SI units.[1]

One of the key goals of this course is to move beyond the dynamics of point particles and onto the dynamics of continuous objects known as fields. To aid in this, it's useful to consider the *charge density*,

$$\rho(\mathbf{x}, t) \tag{1.2}$$

defined as charge per unit volume. The total charge Q in a given region $\mathcal{V}$ is simply $Q = \int_{\mathcal{V}} \mathrm{d}^3x\, \rho(\mathbf{x}, t)$. In most situations, we will consider smooth charge densities, which can be thought of as arising from averaging over many point-like particles. But, on occasion, we will return to the idea of a single particle of charge q, moving on some trajectory $\mathbf{r}(t)$, by writing $\rho = q\delta(\mathbf{x} - \mathbf{r}(t))$ where the delta-function ensures that all the charge sits at a point.

More generally, we will need to describe the movement of charge from one place to another. This is captured by a quantity known as the *current density* $\mathbf{J}(x, t)$, defined as follows: for every surface $\mathcal{S}$, the integral

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \tag{1.3}$$

[1](An aside: the charge of quarks is actually $q = -e/3$ and $q = 2e/3$. This doesn't change the spirit of the above discussion since we could just change the basic unit. But, apart from in extreme circumstances, quarks are confined inside protons and neutrons so we rarely have to worry about this).

Fig. 1.1: Current flux

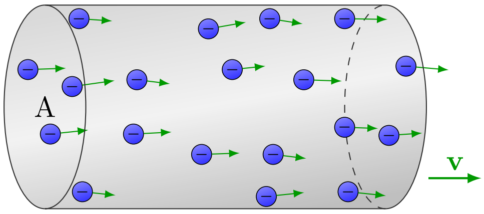

Fig. 1.2: Caption

counts the charge per unit time passing through $\mathcal{S}$. (Here $\mathrm{d}\mathbf{S}$ is the unit normal to $\mathcal{S}$). The quantity I is called the *current*. In this sense, the current density is the current-per-unit-area

The above is a rather indirect definition of the current density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point $\mathbf{x}$, is given by $\mathbf{v}(\mathbf{x}, t)$. Then, neglecting relativistic effects, the current density is

$$\mathbf{J} = \rho \mathbf{v}. \tag{1.4}$$

In particular, if a single particle is moving with velocity $\mathbf{v} = \dot{\mathbf{r}}(t)$, the current density will be $\mathbf{J} = q\mathbf{v}\delta^3(\mathbf{x} - \mathbf{r}(t))$. This is illustrated in Fig. 1.1, where the underlying charged particles are shown as red balls, moving through the blue surface $\mathcal{S}$.

As a simple example, consider electrons moving along a wire (see Fig. 1.2). We model the wire as a long cylinder of cross-sectional area A as shown below. The electrons move with velocity $\mathbf{v}$, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take $\mathbf{v}$ to be their average velocity). If there are n electrons per unit volume, each with charge q, then the charge density is $\rho = nq$ and the current density is $\mathbf{J} = nq\mathbf{v}$. The current itself is $I = |\mathbf{J}|A$..

Throughout this course, the current density $\mathbf{J}$ plays a much more prominent role than the current I. For this reason, we will often refer to $\mathbf{J}$ simply as the "current" although we'll be more careful with the terminology when there is any possibility for confusion.

1.1.1 The Conservation Law

The most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change. But it means much more than that because electric charge is conserved *locally*. An electric charge can't just vanish from one part of the Universe and turn up somewhere else. It can only leave one point in space by moving to a neighbouring point.

The property of local conservation means that ρ can change in time only if there is a compensating current flowing into or out of that region. We express this in the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{1.5}$$

This is an important equation. It arises in any situation where there is some quantity that is locally conserved.

To see why the continuity equation captures the right physics, it's best to consider the change in the total charge Q contained in some region $\mathcal{V}$.

$$\frac{dQ}{dt} = \int_{\mathcal{V}} d^3x \, \frac{\partial \rho}{\partial t} = -\int_{\mathcal{V}} d^3x \, \nabla \cdot \mathbf{J} = -\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{S} \, . \tag{1.6}$$

From our previous discussion, $\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{S}$ is the total current flowing out through the boundary $\mathcal{S}$ of the region $\mathcal{V}$. (It is the total charge flowing *out*, rather than in, because $d\mathbf{S}$ is the outward normal to the region $\mathcal{V}$). The minus sign is there to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $dQ/dt = 0$. This is the statement of (global) conservation of charge. In many applications we will take $\mathcal{V}$ to be all of space, $\mathbb{R}^3$, with both charges and currents localised in some compact region. This ensures that the total charge remains constant.

1.2 Forces and Fields

Any particle that carries electric charge experiences the force of electromagnetism. But the force does not act directly between particles. Instead, Nature chose to introduce intermediaries. These are *fields*.

In physics, a "field" is a dynamical quantity which takes a value at every point in space and time. To describe the force of electromagnetism, we need to introduce two fields, each of which is a three-dimensional vector. They are called the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{B}$,

$$\mathbf{E}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t). \tag{1.7}$$

When we talk about a "force" in modern physics, we really mean an intricate interplay between particles and fields. There are two aspects to this. First, the charged particles create both electric and magnetic fields. Second, the electric and magnetic fields guide the charged particles, telling them how to move. This motion, in turn, changes the fields that the particles create. We're left with a beautiful dance with the particles and fields as two partners, each dictating the moves of the other.

This dance between particles and fields provides a paradigm which all other forces in Nature follow. It feels like there should be a deep reason that Nature chose to introduce fields associated to all the forces. And, indeed, this approach does provide one overriding advantage: all interactions are local. Any object – whether particle or field – affects things only in its immediate neighbourhood. This influence can then propagate through the field to reach another point in space, but it does not do so instantaneously. It takes time for a particle in one part of space to influence a particle elsewhere. This lack of instantaneous interaction allows us to introduce forces which are compatible with the theory of special relativity, something that we will explore in more detail in Section 5.

The purpose of this course is to provide a mathematical description of the interplay between particles and electromagnetic fields. In fact, you've already met one side of this dance: the position $\mathbf{r}(t)$ of a particle of charge q is dictated by the electric and magnetic fields through the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}). \tag{1.8}$$

The motion of the particle can then be determined through Newton's equation $\mathbf{F} = m\ddot{\mathbf{r}}$. Roughly speaking, an electric field accelerates a particle in the direction $\mathbf{E}$, while a magnetic field causes a particle to move in circles in the plane perpendicular to $\mathbf{B}$.

We can also write the Lorentz force law in terms of the charge distribution $\rho(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$. Now we talk in terms of the *force density* $\mathbf{f}(\mathbf{x}, t)$, which is the force acting on a small volume at point $\mathbf{x}$. Now the Lorentz force law reads

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{1.9}$$

1.2.1 The Maxwell Equations

In this course, most of our attention will focus on the other side of the dance: the way in which electric and magnetic fields are created by charged particles. This is described by a set of four equations, known collectively as the *Maxwell equations*. They are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1.10}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.11}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.12}$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}. \tag{1.13}$$

The equations involve two constants. The first is the *electric constant* (known also, in slightly old-fashioned terminology, as the *permittivity of free space*),

$$\epsilon_0 \approx 8.85 \times 10^{-12}\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2. \tag{1.14}$$

It can be thought of as characterising the strength of the electric interactions. The other is the *magnetic constant* (or *permeability of free space*),

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2} \\ &\approx 1.25 \times 10^{-6}\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2}. \end{aligned} \tag{1.15}$$

The presence of 4π in this formula isn't telling us anything deep about Nature, but simply reflects a rather outdated way in which this constant was first defined. (We will explain this in more detail in Section 3.1). Nonetheless, this can be thought of as characterising the strength of magnetic interactions (in units of Coulombs).

The Maxwell equations (1.10-1.13) will occupy us for the rest of the course. Rather than trying to understand all the equations at once, we'll proceed bit by bit, looking at situations where only some of the equations are important. By the end of the lectures, we will understand the physics captured by each of these equations and how they fit together.

However, equally importantly, we will also explore the mathematical structure of the Maxwell equations. At first glance, they look just like four random equations from vector calculus. Yet this couldn't be further from the truth. The Maxwell equations are special and, when viewed in the right way, are the essentially unique equations that can describe the force of electromagnetism. The full story of why these are the unique equations involves both quantum mechanics and relativity and will only be told in later courses. But we will start that journey here. The goal is that by the end of these lectures you will be convinced of the importance of the Maxwell equations on both experimental and aesthetic grounds.

CHAPTER 2

Electrostatics

In this section, we will be interested in electric charges at rest. This means that there exists a frame of reference in which there are no currents; only stationary charges. Of course, there will be forces between these charges but we will assume that the charges are pinned in place and cannot move. The question that we want to answer is: what is the electric field generated by these charges?

Since nothing moves, we are looking for time independent solutions to Maxwell's equations with $\mathbf{J} = \mathbf{0}$. This means that we can consistently set $\mathbf{B} = \mathbf{0}$ and we're left with two of Maxwell's equations to solve. They are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

and

$$\nabla \times \mathbf{E} = 0. \tag{2.2}$$

If you fix the charge distribution ρ, Eqs. (2.1) and (2.2) have a unique solution. Our goal in this section is to find it.

2.1 Gauss' Law

Before we proceed, let's first present equation (2.1) in a slightly different form that will shed some light on its meaning. Consider some closed region $\mathcal{V} \subset \mathbb{R}^3$ of space. We'll denote the boundary of $\mathcal{V}$ by $\mathcal{S} = \partial\mathcal{V}$. We now integrate both sides of (2.1) over $\mathcal{V}$. Since the left-hand side is a total derivative, we can use the divergence theorem to convert this to an integral over the surface $\mathcal{S}$. We have

$$\int_{\mathcal{V}} \mathrm{d}^3x\, \nabla \cdot \mathbf{E} = \int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3x\, \rho. \tag{2.3}$$

The integral of the charge density over $\mathcal{V}$ is simply the total charge contained in the region. We'll call it $Q = \int \mathrm{d}^3x\, \rho$. Meanwhile, the integral of the electric field over $\mathcal{S}$ is called the *flux* through $\mathcal{S}$. We learn that the two are related by

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} \tag{2.4}$$

This is *Gauss's law.* However, because the two are entirely equivalent, we also refer to the original (2.1) as Gauss's law.

Notice that it doesn't matter what shape the surface $\mathcal{S}$ takes. As long as it surrounds a total charge Q, the flux through the surface will always be Q/ϵ_0. This is shown, for example, in the left-hand figure above. A fancy way of saying this is that the integral of the flux doesn't depend on the geometry of the surface, but does depend on its topology

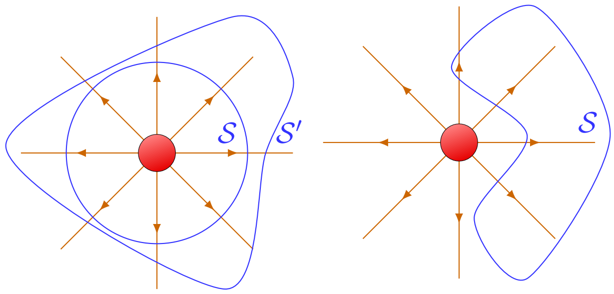

Fig. 2.1: Left: The flux through $\mathcal{S}$ and $\mathcal{S}'$ is the same. Right: The flux through $\mathcal{S}$ vanishes

since it must surround the charge Q. The choice of $\mathcal{S}$ is called the *Gaussian surface*; often there's a smart choice that makes a particular problem simple.

Only charges that lie inside $\mathcal{V}$ contribute to the flux. Any charges that lie outside will produce an electric field that penetrates through $\mathcal{S}$ at some point, giving negative flux, but leaves through the other side of $\mathcal{S}$, depositing positive flux. The total contribution from these charges that lie outside of $\mathcal{V}$ is zero, as illustrated in the right-hand figure above.

For a general charge distribution, we'll need to use both Gauss' law (2.1) and the extra equation (2.2). However, for rather special charge distributions – typically those with lots of symmetry – it turns out to be sufficient to solve the integral form of Gauss' law (2.4) alone, with the symmetry ensuring that (2.2) is automatically satisfied. We start by describing these rather simple solutions. We'll then return to the general case in Section (2.2).

2.1.1 The Coulomb Force

We'll start by showing that Gauss' law (2.4) reproduces the more familiar Coulomb force law that we all know and love. To do this, take a spherically symmetric charge distribution, centered at the origin, contained within some radius R. This will be our model for a particle. We won't need to make any assumption about the nature of the distribution other than its symmetry and the fact that the total charge is Q.

We want to know the electric field at some radius $r > R$. We take our Gaussian surface $\mathcal{S}$ to be a sphere of radius r as shown in Fig. 2.2. Gauss' law states

$$\int_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}. \tag{2.5}$$

At this point we make use of the spherical symmetry of the problem. This tells us that the electric field must point radially outwards: $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$. And, since the integral is only over the angular coordinates of the sphere, we can pull the function $E(r)$ outside. We have

$$\int_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{S} = E(r)\int_{\mathcal{S}} \hat{\mathbf{r}}\cdot d\mathbf{S} = E(r)4\pi r^2 = \frac{Q}{\epsilon_0}, \tag{2.6}$$

where the factor of $4\pi r^2$ has arisen simply because it's the area of the Gaussian sphere.

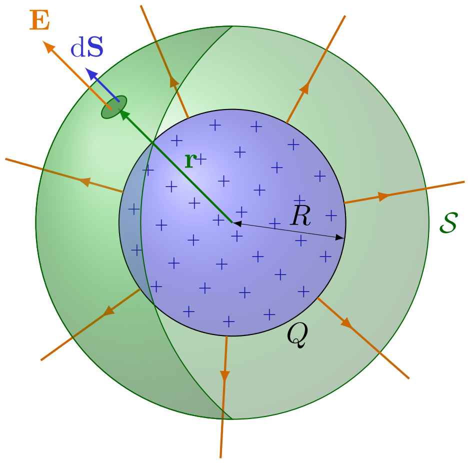

Fig. 2.2: Electric field produced by a spherically symmetric charge distribution, centered at the origin, contained within some radius R.

We learn that the electric field outside a spherically symmetric distribution of charge Q is

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.7}$$

That's nice. This is the familiar result that we've seen before. The Lorentz force law (1.8) then tells us that a test charge q moving in the region $r > R$ experiences a force

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.8}$$

This, of course, is the *Coulomb force* between two static charged particles. Notice that, as promised, $1/\epsilon_0$ characterises the strength of the force. If the two charges have the same sign, so that $Qq > 0$, the force is repulsive, pushing the test charge away from the origin. If the charges have opposite signs, $Qq < 0$, the force is attractive, pointing towards the origin. We see that Gauss's law (2.1) reproduces this simple result that we know about charges.

Finally, note that the assumption of symmetry was crucial in our above analysis. Without it, the electric field $\mathbf{E}(\mathbf{x})$ would have depended on the angular coordinates of the sphere $\mathcal{S}$ and so been stuck inside the integral. In situations without symmetry, Gauss' law alone is not enough to determine the electric field and we need to also use $\nabla \times \mathbf{E} = 0$. We'll see how to do this in Section 2.2. If you're worried, however, it's simple to check that our final expression for the electric field (2.7) does indeed solve $\nabla \times \mathbf{E} = 0$.

2.1.1.1 Coulomb vs Newton

The inverse-square form of the force is common to both electrostatics and gravity. It's worth comparing the relative strengths of the two forces. For example, we can look at the relative strengths of Newtonian attraction and Coulomb repulsion between two electrons. These are point particles with mass me and charge $-e$ given by

$$e \approx 1.6 \times 10^{-19}\text{C} \quad \text{and} \quad m_e \approx 0.1 \times 10^{-31}\text{kg}. \tag{2.9}$$

Regardless of the separation, we have

$$\frac{F_{\text{Coulomb}}}{F_{\text{Newton}}} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{Gm_e^2}. \tag{2.10}$$

The strength of gravity is determined by Newton's constant $G \approx 6.7 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^2$. Plugging in the numbers reveals something extraordinary:

$$\frac{F_{\text{Coulomb}}}{F_{\text{Newton}}} \approx 10^{42} \tag{2.11}$$

Gravity is puny. Electromagnetism rules. In fact you knew this already. The mere act of lifting up you arm is pitching a few electrical impulses up against the gravitational might of the entire Earth. Yet the electrical impulses win.

However, gravity has a trick up its sleeve. While electric charges come with both positive and negative signs, mass is only positive. It means that by the time we get to macroscopically large objects – stars, planets, cats – the mass accumulates while the charges cancel to good approximation. This compensates the factor of 10^{-42} suppression until, at large distance scales, gravity wins after all.

The fact that the force of gravity is so ridiculously tiny at the level of fundamental particles has consequence. It means that we can neglect gravity whenever we talk about the very small. (And indeed, we shall neglect gravity for the rest of this course). However, it also means that if we would like to understand gravity better on these very tiny distances – for example, to develop a quantum theory of gravity – then it's going to be tricky to get much guidance from experiment.

2.1.2 A Uniform Sphere

The electric field outside a spherically symmetric charge distribution is always given by (2.7). What about inside? This depends on the distribution in question. The simplest is a sphere of radius R with uniform charge distribution ρ. The total charge is

$$Q = \frac{4\pi}{3}R^3\rho. \tag{2.12}$$

Let's pick our Gaussian surface to be a sphere, centered at the origin, of radius$r < R$. The charge contained within this sphere is $4\pi\rho r^3/3 = Qr^3/R^3$, so Gauss' law gives

2.2 The Electrostatic Potential

Chapter 3

Magnetostatics

3.1 Units of Electromagnetism

Chapter 4

Electrodynamics

Chapter 5

Electromagnetism and Relativity

Chapter 6

Light Scattering

Chapter 7

Special Relativity

Chapter 8

Radiation and Relativistic Electrodynamics

Appendix A

Code