Part II Electrodynamics and Optics

William Royce

July 26, 2024

Part II Physics, The University of Cambridge

Preface

Contents

List of Tables

List of Figures

Chapter 1

Electromagnetic Waves

1.1 Revision of Part IB

Electrostatic and magnetostatic experiments lead to the concepts of the electric field $\mathbf{E}$ and the magnetic induction (or magnetic flux density) $\mathbf{B}$ in vacuum. Experimentally the Lorentz force on a point charge q moving with velocity $\mathbf{v}$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1.1}$$

where $\mathbf{E}$ is the "force per unit charge," and $\mathbf{B}$ is the "force per unit current." More generally, the charge is distributed, with $\rho_T(\mathbf{r})$ and $\mathbf{J}_T(\mathbf{r})$ the total charge and current density distributions. We additionally have the continuity equation:

$$\iint_{\mathcal{S}} \mathbf{J}_T \cdot \mathrm{d}\mathbf{S} = -\iiint_{\mathcal{V}} \frac{\partial \rho_T}{\partial t} \,\mathrm{d}V \tag{1.2}$$

In statics and in vacuum summarise experimental observations:

$$\text{Gauss (ES):} \quad \iint_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \frac{\rho_T}{\epsilon_0} \,\mathrm{d}V\,, \tag{1.3}$$

$$\text{Gauss (MS):} \quad \iint_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0, \tag{1.4}$$

$$\oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0, \tag{1.5}$$

$$\text{Ampére} \quad \oint_{\mathcal{C}} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \iint_{\mathcal{S}} \mu_0 \mathbf{J}_T \cdot \mathrm{d}\mathbf{S}\,. \tag{1.6}$$

The surface $\mathcal{S}$ bounds the volume $\mathcal{V}$ and contour $\mathcal{C}$ bounds the surface $\mathcal{S}$.

Differential form (using Gauss's and Stokes's theorems):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_T}{\epsilon_0}, \tag{1.7}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{1.8}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{1.9}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}_T, \tag{1.10}$$

$\rho_T(\mathbf{r})$ and $\mathbf{J}_T(\mathbf{r})$ are sources for $\mathbf{E}$ and $\mathbf{B}$ respectively.

Eqs. (1.5) and (1.9) imply that in electrostatics $\mathbf{E}$ is a conservative field, and can be written in terms of a scalar potential:

$$\mathbf{E} = -\boldsymbol{\nabla} \phi(\mathbf{r}). \tag{1.11}$$

Eqs. (1.6) and (1.10) imply that in magnetostatics $\mathbf{B}$ is a conservative field in regions where $\mathbf{J}_T = 0$, and can be written in terms of a magnetic scalar potential:

$$\mathbf{B} = -\mu_0 \boldsymbol{\nabla} \phi_m(\mathbf{r}) \tag{1.12}$$

1.1.1 Maxwell's Equations

This leads to Maxwell's Equations for EM fields in vacuum:

$$\nabla \cdot \mathbf{E} = \frac{\rho_T}{\epsilon_0}, \tag{1.13}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.14}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_T + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.16}$$

The equations for $\mathbf{E}$ and $\mathbf{B}$ are now interlinked by the time-dependent inductive and displacement current terms - Electromagnetism.

Eqs. (1.7) and (1.11) give Poisson's Equation:

$$\nabla^2 \phi = -\frac{\rho_T}{\epsilon_0} \tag{1.17}$$

Similarly Eqs. (1.12) and (1.8) give

$$\nabla^2 \phi_m = 0, \tag{1.18}$$

for regions where $\mathbf{J}_T = 0$.

The equations for $\mathbf{E}$ and $\mathbf{B}$ are independent, electrostatics and magnetostatics are separate theories. Faraday's experimental results on electromagnetic induction , and Maxwell's insights allow electricity and magnetism to be unified. The unified theory of Electromagnetism is fully relativistic as we will see.

1.1.1.1 Media

In the presence of a medium

$$\left.\begin{matrix}\mathbf{E}\\ \mathbf{B}\end{matrix}\right\} \to \left\{\begin{matrix}\text{electric}\\ \text{magnetic}\end{matrix}\right\} \text{dipole moments per unit volume} \left\{\begin{matrix}\mathbf{P}\\ \mathbf{M}\end{matrix}\right. . \tag{1.19}$$

The electric polarization $\mathbf{P}$:

$$\mathbf{P} \to \begin{cases}\text{volume charge density} & \rho_P = -\nabla \cdot \mathbf{P}\\ \text{surface charge density} & \sigma_P = \mathbf{P} \cdot \hat{\mathbf{n}}\end{cases}, \tag{1.20}$$

where $\hat{\mathbf{n}}$ is the surface normal unit vector.

If $\mathbf{P}$ varies with time, so does ρ_P, and there is an associated volume current density $\mathbf{J}_P$ given by the continuity equation for the polarization charge:

$$\nabla \cdot \mathbf{J}_P = -\dot{\rho}_P = \nabla \cdot \dot{\mathbf{P}} \quad \Longrightarrow \quad \mathbf{J}_P = \dot{\mathbf{P}}. \tag{1.21}$$

The magnetization $\mathbf{M}$:

$$\mathbf{P} \to \begin{cases} \text{volume charge density} & \mathbf{J}_M = \boldsymbol{\nabla} \times \mathbf{M} \\ \text{surface charge density} & \mathbf{j}_M = \mathbf{M} \times \hat{\mathbf{n}} \end{cases}. \tag{1.22}$$

Eqs. (1.13-1.16) involve the total charge and current volume distributions.

$$\rho_T = \rho + \rho_P = \rho - \boldsymbol{\nabla} \cdot \mathbf{P} \tag{1.23}$$

$$\mathbf{J}_T = \mathbf{J} + \mathbf{P} + \mathbf{J}_M = \mathbf{J} + \dot{\mathbf{P}} + \boldsymbol{\nabla} \times \mathbf{M}, \tag{1.24}$$

where ρ and $\mathbf{J}$ refer to the "real", free charges and currents present.

Substituting for ρ_T and $\mathbf{J}_T$ in Eqs. (1.13-1.16):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho - \boldsymbol{\nabla} \cdot \mathbf{P}}{\epsilon_0}, \tag{1.25}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{1.26}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}}, \tag{1.27}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\Big(\mathbf{J} + \dot{\mathbf{P}} + \boldsymbol{\nabla} \times \mathbf{M}\Big) + \mu_0\epsilon_0\dot{\mathbf{E}}. \tag{1.28}$$

Define the auxiliary fields, $\mathbf{D}$ the electric displacement and $\mathbf{H}$ the magnetic intensity:

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \tag{1.29}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \tag{1.30}$$

Eqs. (1.25-1.28) then become the standard form for Maxwell's Equations:

$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D} &= \rho, & (1.31)\\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, & (1.32)\\ \boldsymbol{\nabla} \times \mathbf{E} &= -\dot{\mathbf{B}}, & (1.33)\\ \boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J} + \dot{\mathbf{D}}. & (1.34) \end{aligned}}$$

ρ and $\mathbf{J}$ are free charge and current densities, the response of the medium is included via the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$. In integral form:

$$\boxed{\begin{aligned} &\text{Gauss (ES):} && \iint_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \rho \,\mathrm{d}V\,, & (1.35)\\ &\text{Gauss (MS):} && \iint_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0, & (1.36)\\ &\text{Faraday, Lenz:} && \oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial}{\partial t}\iint_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}\,, & (1.37)\\ &\text{Ampére:} && \oint_{\mathcal{C}} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \iint_{\mathcal{S}} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathrm{d}\mathbf{S}\,. & (1.38) \end{aligned}}$$

1.1.1.2 Constitutive Relations

For linear, isotropic media:

$$\mathbf{P} = \epsilon_0\chi\mathbf{E} \quad \text{and} \quad \mathbf{M} = \chi_m\mathbf{H}, \tag{1.39}$$

where χ is the dielectric susceptibility and χ_m is the magnetic susceptibility. Defining:

$$\text{the relative permittivity (or dielectric “constant”)}\epsilon = 1 + \chi \tag{1.40}$$
$$\text{the relative magnetic permeability}\mu = 1 + \chi_m \tag{1.41}$$

gives the constitutive relations:

$$\boxed{\mathbf{D} = \epsilon\epsilon_0\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu\mu_0\mathbf{H}.} \tag{1.42}$$

ϵ and μ encapsulate the responses (polarization and magnetisation) of the medium to the EM fields.

1.2 Electromagnetic Energy

The energy U of the EM fields in a volume $\mathcal{V}$ bounded by the surface $\mathcal{S}$ is:

$$\iiint_{\mathcal{V}} \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right) \mathrm{d}V \tag{1.43}$$

and from the MEs it can be shown that:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\iint_{\mathcal{S}} \mathbf{E}\times\mathbf{H}\cdot\mathrm{d}\mathbf{S} - \iiint_{\mathcal{V}} \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V\,, \tag{1.44}$$

the sum of a power flux and a dissipative term. So the energy density at a point is

$$\boxed{u = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H},} \tag{1.45}$$

and the energy flux is given by the Poynting Vector

$$\boxed{\mathbf{N} = \mathbf{E}\times\mathbf{H}.} \tag{1.46}$$

1.3 The Vector Potential

Note from Eq. (1.37) that when there is a time-dependent magnetic induction $\mathbf{B}$, $\mathbf{E}$ is no longer in general a conservative field and $\mathbf{E} \neq -\boldsymbol{\nabla}\phi$. Now $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$ for any vector field $\mathbf{F}$, so Eq. (1.32) implies that there is a vector field $\mathbf{A}$ – the magnetic vector potential – of which the curl is $\mathbf{B}$,

$$\boxed{\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}.} \tag{1.47}$$

From Eq. (1.37):

$$\boldsymbol{\nabla}\times\mathbf{E} = -\dot{\mathbf{B}} = -\boldsymbol{\nabla}\times\dot{\mathbf{A}} \tag{1.48}$$

and integrating:

$$\mathbf{E} = -\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi. \tag{1.49}$$

1.4 Boundary Conditions

From the integral forms of Maxwell's Eqs. (1.35-1.38) it is easy to derive conditions for the continuity of components of the EM fields at interfaces between two media

- $\mathbf{D}_\perp$ is conserved in the absence of any free charges at the interface,
- $\mathbf{B}_\perp$ is conserved,
- $\mathbf{E}_{||}$ is conserved,
- $\mathbf{H}_{||}$ is conserved in the absence of any free current at the interface.

1.5 Electromagnetic Waves

For any vector field $\mathbf{F}$, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2\mathbf{F}$. In regions with $\rho = 0$, $\mathbf{J} = \mathbf{J}$, taking the curl of Eq. (1.33) and substituting $\boldsymbol{\nabla} \times \mathbf{H}$ from Eq. (1.34):

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times E) = -\boldsymbol{\nabla} \times \dot{\mathbf{B}} \tag{1.50}$$

Chapter 2

Optics

Chapter 3

Electrodynamics

Chapter 4

Dipole Radiation

Chapter 5

Antennas

Chapter 6

Light Scattering

Chapter 7

Special Relativity

Chapter 8

Radiation and Relativistic Electrodynamics

Appendix A

Code