Part II Electrodynamics and Optics

William Royce

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Part II Physics, The University of Cambridge

Preface

There are, to the best of our knowledge, four forces at play in the Universe. At the very largest scales – those of planets or stars or galaxies – the force of gravity dominates. At the very smallest distances, the two nuclear forces hold sway. For everything in between, it is force of electromagnetism that rules.

At the atomic scale, electromagnetism (admittedly in conjunction with some basic quantum effects) governs the interactions between atoms and molecules. It is the force that underlies the periodic table of elements, giving rise to all of chemistry and, through this, much of biology. It is the force which binds atoms together into solids and liquids. And it is the force which is responsible for the incredible range of properties that different materials exhibit.

At the macroscopic scale, electromagnetism manifests itself in the familiar phenomena that give the force its name. In the case of electricity, this means everything from rubbing a balloon on your head and sticking it on the wall, through to the fact that you can plug any appliance into the wall and be pretty confident that it will work. For magnetism, this means everything from the shopping list stuck to your fridge door, through to trains in Japan which levitate above the rail. Harnessing these powers through the invention of the electric dynamo and motor has transformed the planet and our lives on it.

As if this wasn't enough, there is much more to the force of electromagnetism for it is, quite literally, responsible for everything you've ever seen. It is the force that gives rise to light itself.

Rather remarkably, a full description of the force of electromagnetism is contained in four simple and elegant equations. These are known as the *Maxwell equations*. There are few places in physics, or indeed in any other subject, where such a richly diverse set of phenomena flows from so little. The purpose of this course is to introduce the Maxwell equations and to extract some of the many stories they contain.

However, there is also a second theme that runs through this course. The force of electromagnetism turns out to be a blueprint for all the other forces. There are various mathematical symmetries and structures lurking within the Maxwell equations, structures which Nature then repeats in other contexts. Understanding the mathematical beauty of the equations will allow us to see some of the principles that underly the laws of physics, laying the groundwork for future study of the other forces.

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Introduction

1.1 Charge and Current

Each particle in the Universe carries with it a number of properties. These determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called *electric charge*.

For the purposes of this course, we can think of electric charge as a real number, $q \in \mathbb{R}$. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

The SI unit of charge is the *Coulomb*, denoted by C. It is, like all SI units, a parochial measure, convenient for human activity rather than informed by the underlying laws of the physics. (We'll learn more about how the Coulomb is defined in Section 3.1). At a fundamental level, Nature provides us with a better unit of charge. This follows from the fact that charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron which we denoted as $-e$, with

$$e = 1.602176634^{-19} C. \tag{1.1}$$

A much more natural unit would be to simply count charge as $q = ne$ with $n \in \mathbb{Z}$. Then electrons have charge -1 while protons have charge $+1$ and neutrons have charge 0. Nonetheless, in this course, we will bow to convention and stick with SI units.[1]

One of the key goals of this course is to move beyond the dynamics of point particles and onto the dynamics of continuous objects known as fields. To aid in this, it's useful to consider the *charge density*,

$$\rho(\mathbf{x}, t) \tag{1.2}$$

defined as charge per unit volume. The total charge Q in a given region $\mathcal{V}$ is simply $Q = \int_{\mathcal{V}} \mathrm{d}^3 x \, \rho(\mathbf{x}, t)$. In most situations, we will consider smooth charge densities, which can be thought of as arising from averaging over many point-like particles. But, on occasion, we will return to the idea of a single particle of charge q, moving on some trajectory $\mathbf{r}(t)$, by writing $\rho = q\delta(\mathbf{x} - \mathbf{r}(t))$ where the delta-function ensures that all the charge sits at a point.

More generally, we will need to describe the movement of charge from one place to another. This is captured by a quantity known as the *current density* $\mathbf{J}(x, t)$, defined as follows: for every surface $\mathcal{S}$, the integral

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \tag{1.3}$$

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[1](An aside: the charge of quarks is actually $q = -e/3$ and $q = 2e/3$. This doesn't change the spirit of the above discussion since we could just change the basic unit. But, apart from in extreme circumstances, quarks are confined inside protons and neutrons so we rarely have to worry about this).

Fig. 1.1: Current flux
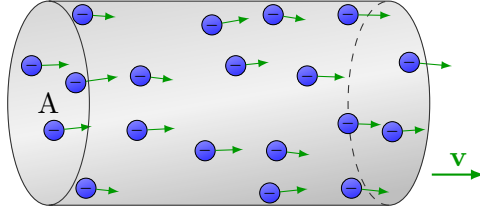
Fig. 1.2: Caption

counts the charge per unit time passing through $\mathcal{S}$. (Here $d\mathbf{S}$ is the unit normal to $\mathcal{S}$). The quantity I is called the *current*. In this sense, the current density is the current-per-unit-area

The above is a rather indirect definition of the current density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point $\mathbf{x}$, is given by $\mathbf{v}(\mathbf{x}, t)$. Then, neglecting relativistic effects, the current density is

$$\mathbf{J} = \rho\mathbf{v}. \tag{1.4}$$

In particular, if a single particle is moving with velocity $\mathbf{v} = \dot{\mathbf{r}}(t)$, the current density will be $\mathbf{J} = q\mathbf{v}\delta^3(\mathbf{x} - \mathbf{r}(t))$. This is illustrated in Fig. 1.1, where the underlying charged particles are shown as red balls, moving through the blue surface $\mathcal{S}$.

As a simple example, consider electrons moving along a wire (see Fig. 1.2). We model the wire as a long cylinder of cross-sectional area A as shown below. The electrons move with velocity $\mathbf{v}$, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take $\mathbf{v}$ to be their average velocity). If there are n electrons per unit volume, each with charge q, then the charge density is $\rho = nq$ and the current density is $\mathbf{J} = nq\mathbf{v}$. The current itself is $I = |\mathbf{J}|A$..

Throughout this course, the current density $\mathbf{J}$ plays a much more prominent role than the current I. For this reason, we will often refer to $\mathbf{J}$ simply as the "current" although we'll be more careful with the terminology when there is any possibility for confusion.

1.1.1 The Conservation Law

The most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change. But it means much more than that because electric charge is conserved *locally*. An electric charge can't just vanish from one part of the Universe and turn up somewhere else. It can only leave one point in space by moving to a neighbouring point.

The property of local conservation means that ρ can change in time only if there is a compensating current flowing into or out of that region. We express this in the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot \mathbf{J} = 0. \tag{1.5}$$

This is an important equation. It arises in any situation where there is some quantity that is locally conserved.

To see why the continuity equation captures the right physics, it's best to consider the change in the total charge Q contained in some region $\mathcal{V}$.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}^3x \, \frac{\partial \rho}{\partial t} = -\int_{\mathcal{V}} \mathrm{d}^3x \, \boldsymbol{\nabla} \cdot \mathbf{J} = -\int_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \,. \tag{1.6}$$

From our previous discussion, $\int_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ is the total current flowing out through the boundary $\mathcal{S}$ of the region $\mathcal{V}$. (It is the total charge flowing *out*, rather than in, because $\mathrm{d}\mathbf{S}$ is the outward normal to the region $\mathcal{V}$). The minus sign is there to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $\mathrm{d}Q/\mathrm{d}t = 0$. This is the statement of (global) conservation of charge. In many applications we will take $\mathcal{V}$ to be all of space, $\mathbb{R}^3$, with both charges and currents localised in some compact region. This ensures that the total charge remains constant.

1.2   Forces and Fields

Any particle that carries electric charge experiences the force of electromagnetism. But the force does not act directly between particles. Instead, Nature chose to introduce intermediaries. These are *fields*.

In physics, a "field" is a dynamical quantity which takes a value at every point in space and time. To describe the force of electromagnetism, we need to introduce two fields, each of which is a three-dimensional vector. They are called the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{B}$,

$$\mathbf{E}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t). \tag{1.7}$$

When we talk about a "force" in modern physics, we really mean an intricate interplay between particles and fields. There are two aspects to this. First, the charged particles create both electric and magnetic fields. Second, the electric and magnetic fields guide the charged particles, telling them how to move. This motion, in turn, changes the fields that the particles create. We're left with a beautiful dance with the particles and fields as two partners, each dictating the moves of the other.

This dance between particles and fields provides a paradigm which all other forces in Nature follow. It feels like there should be a deep reason that Nature chose to introduce fields associated to all the forces. And, indeed, this approach does provide one overriding advantage: all interactions are local. Any object – whether particle or field – affects things only in its immediate neighbourhood. This influence can then propagate through the field to reach another point in space, but it does not do so instantaneously. It takes time for a particle in one part of space to influence a particle elsewhere. This lack of instantaneous interaction allows us to introduce forces which are compatible with the theory of special relativity, something that we will explore in more detail in Section 5.

The purpose of this course is to provide a mathematical description of the interplay between particles and electromagnetic fields. In fact, you've already met one side of this dance: the position $\mathbf{r}(t)$ of a particle of charge q is dictated by the electric and magnetic fields through the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}). \tag{1.8}$$

The motion of the particle can then be determined through Newton's equation $\mathbf{F} = m\ddot{\mathbf{r}}$. Roughly speaking, an electric field accelerates a particle in the direction $\mathbf{E}$, while a magnetic field causes a particle to move in circles in the plane perpendicular to $\mathbf{B}$.

We can also write the Lorentz force law in terms of the charge distribution $\rho(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$. Now we talk in terms of the *force density* $\mathbf{f}(\mathbf{x}, t)$, which is the force acting on a small volume at point $\mathbf{x}$. Now the Lorentz force law reads

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{1.9}$$

1.2.1   The Maxwell Equations

In this course, most of our attention will focus on the other side of the dance: the way in which electric and magnetic fields are created by charged particles. This is described by a set of four equations, known collectively as the *Maxwell equations*. They are:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1.10}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{1.11}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.12}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{J}. \tag{1.13}$$

The equations involve two constants. The first is the *electric constant* (known also, in slightly old-fashioned terminology, as the *permittivity of free space*),

$$\epsilon_0 \approx 8.85 \times 10^{-12} \mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2. \tag{1.14}$$

It can be thought of as characterising the strength of the electric interactions. The other is the *magnetic constant* (or *permeability of free space*),

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2} \\ &\approx 1.25 \times 10^{-6}\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2}. \end{aligned} \tag{1.15}$$

The presence of 4π in this formula isn't telling us anything deep about Nature, but simply reflects a rather outdated way in which this constant was first defined. (We will explain this in more detail in Section 3.1). Nonetheless, this can be thought of as characterising the strength of magnetic interactions (in units of Coulombs).

The Maxwell equations (1.10-1.13) will occupy us for the rest of the course. Rather than trying to understand all the equations at once, we'll proceed bit by bit, looking at situations where only some of the equations are important. By the end of the lectures, we will understand the physics captured by each of these equations and how they fit together.

However, equally importantly, we will also explore the mathematical structure of the Maxwell equations. At first glance, they look just like four random equations from vector calculus. Yet this couldn't be further from the truth. The Maxwell equations are special and, when viewed in the right way, are the essentially unique equations that can describe the force of electromagnetism. The full story of why these are the unique equations involves both quantum mechanics and relativity and will only be told in later courses. But we will start that journey here. The goal is that by the end of these lectures you will be convinced of the importance of the Maxwell equations on both experimental and aesthetic grounds.

CHAPTER **2**

Electrostatics

In this section, we will be interested in electric charges at rest. This means that there exists a frame of reference in which there are no currents; only stationary charges. Of course, there will be forces between these charges but we will assume that the charges are pinned in place and cannot move. The question that we want to answer is: what is the electric field generated by these charges?

Since nothing moves, we are looking for time independent solutions to Maxwell's equations with $\mathbf{J} = \mathbf{0}$. This means that we can consistently set $\mathbf{B} = \mathbf{0}$ and we're left with two of Maxwell's equations to solve. They are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

and

$$\boldsymbol{\nabla} \times \mathbf{E} = 0. \tag{2.2}$$

If you fix the charge distribution ρ, Eqs. (2.1) and (2.2) have a unique solution. Our goal in this section is to find it.

2.1 Gauss' Law

Before we proceed, let's first present equation (2.1) in a slightly different form that will shed some light on its meaning. Consider some closed region $\mathcal{V} \subset \mathbb{R}^3$ of space. We'll denote the boundary of $\mathcal{V}$ by $\mathcal{S} = \partial\mathcal{V}$. We now integrate both sides of (2.1) over $\mathcal{V}$. Since the left-hand side is a total derivative, we can use the divergence theorem to convert this to an integral over the surface $\mathcal{S}$. We have

$$\int_{\mathcal{V}} \mathrm{d}^3 x \, \boldsymbol{\nabla} \cdot \mathbf{E} = \int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 x \, \rho. \tag{2.3}$$

The integral of the charge density over $\mathcal{V}$ is simply the total charge contained in the region. We'll call it $Q = \int \mathrm{d}^3 x \, \rho$. Meanwhile, the integral of the electric field over $\mathcal{S}$ is called the *flux* through $\mathcal{S}$. We learn that the two are related by

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0} \tag{2.4}$$

This is *Gauss's law.* However, because the two are entirely equivalent, we also refer to the original (2.1) as Gauss's law.

Notice that it doesn't matter what shape the surface $\mathcal{S}$ takes. As long as it surrounds a total charge Q, the flux through the surface will always be Q/ϵ_0. This is shown, for example, in Fig 2.1. A fancy way of saying this is that the integral of the flux doesn't depend on the geometry of the surface, but does depend on its topology since it must
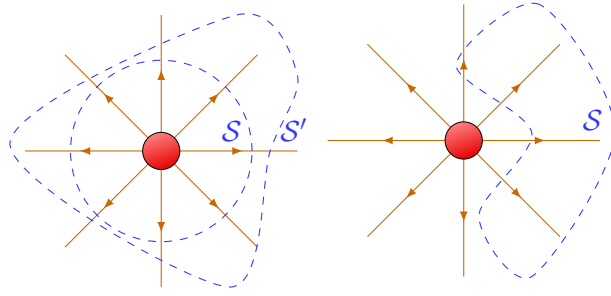
Fig. 2.1: Left: The flux through $\mathcal{S}$ and $\mathcal{S}'$ is the same. Right: The flux through $\mathcal{S}$ vanishes

surround the charge Q. The choice of $\mathcal{S}$ is called the *Gaussian surface*; often there's a smart choice that makes a particular problem simple.

Only charges that lie inside $\mathcal{V}$ contribute to the flux. Any charges that lie outside will produce an electric field that penetrates through $\mathcal{S}$ at some point, giving negative flux, but leaves through the other side of $\mathcal{S}$, depositing positive flux. The total contribution from these charges that lie outside of $\mathcal{V}$ is zero, as illustrated in the right-hand figure above.

For a general charge distribution, we'll need to use both Gauss' law (2.1) and the extra equation (2.2). However, for rather special charge distributions – typically those with lots of symmetry – it turns out to be sufficient to solve the integral form of Gauss' law (2.4) alone, with the symmetry ensuring that (2.2) is automatically satisfied. We start by describing these rather simple solutions. We'll then return to the general case in Section (2.2).

2.1.1 The Coulomb Force

We'll start by showing that Gauss' law (2.4) reproduces the more familiar Coulomb force law that we all know and love. To do this, take a spherically symmetric charge distribution, centered at the origin, contained within some radius R. This will be our model for a particle. We won't need to make any assumption about the nature of the distribution other than its symmetry and the fact that the total charge is Q.

We want to know the electric field at some radius $r > R$. We take our Gaussian surface $\mathcal{S}$ to be a sphere of radius r as shown in Fig. 2.2. Gauss' law states

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}. \tag{2.5}$$

At this point we make use of the spherical symmetry of the problem. This tells us that the electric field must point radially outwards: $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$. And, since the integral is only over the angular coordinates of the sphere, we can pull the function $E(r)$ outside. We have

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} = E(r) \int_{\mathcal{S}} \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r) 4\pi r^2 = \frac{Q}{\epsilon_0}, \tag{2.6}$$

where the factor of $4\pi r^2$ has arisen simply because it's the area of the Gaussian sphere.
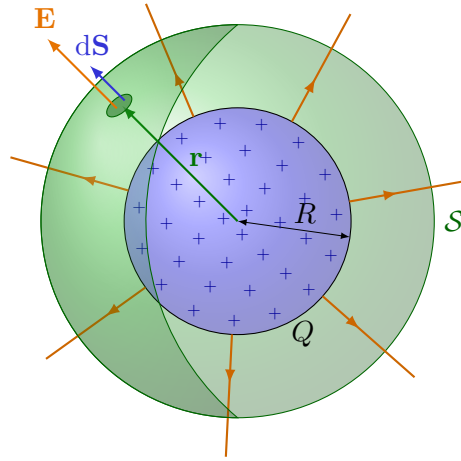
Fig. 2.2: Electric field produced by a spherically symmetric charge distribution, centered at the origin, contained within some radius R.

We learn that the electric field outside a spherically symmetric distribution of charge Q is

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.7}$$

That's nice. This is the familiar result that we've seen before. The Lorentz force law (1.8) then tells us that a test charge q moving in the region $r > R$ experiences a force

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.8}$$

This, of course, is the *Coulomb force* between two static charged particles. Notice that, as promised, $1/\epsilon_0$ characterises the strength of the force. If the two charges have the same sign, so that $Qq > 0$, the force is repulsive, pushing the test charge away from the origin. If the charges have opposite signs, $Qq < 0$, the force is attractive, pointing towards the origin. We see that Gauss's law (2.1) reproduces this simple result that we know about charges.

Finally, note that the assumption of symmetry was crucial in our above analysis. Without it, the electric field $\mathbf{E}(\mathbf{x})$ would have depended on the angular coordinates of the sphere $\mathcal{S}$ and so been stuck inside the integral. In situations without symmetry, Gauss' law alone is not enough to determine the electric field and we need to also use $\boldsymbol{\nabla} \times \mathbf{E} = 0$. We'll see how to do this in Section 2.2. If you're worried, however, it's simple to check that our final expression for the electric field (2.7) does indeed solve $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

2.1.1.1 Coulomb vs Newton

The inverse-square form of the force is common to both electrostatics and gravity. It's worth comparing the relative strengths of the two forces. For example, we can look at the relative strengths of Newtonian attraction and Coulomb repulsion between two electrons. These are point particles with mass me and charge $-e$ given by

$$e \approx 1.6 \times 10^{-19}\text{C} \quad \text{and} \quad m_e \approx 0.1 \times 10^{-31}\text{kg}. \tag{2.9}$$
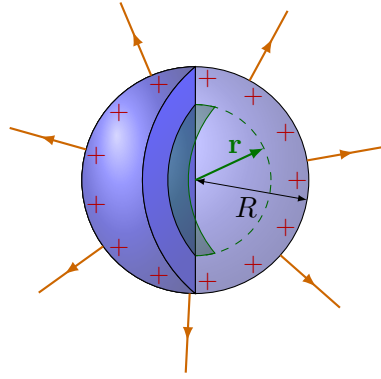
Fig. 2.3: Electric field inside a spherically symmetric charge distribution.

Regardless of the separation, we have

$$\frac{F_{\text{Coulomb}}}{F_{\text{Newton}}} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{Gm_e^2}. \tag{2.10}$$

The strength of gravity is determined by Newton's constant $G \approx 6.7 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^2$. Plugging in the numbers reveals something extraordinary:

$$\frac{F_{\text{Coulomb}}}{F_{\text{Newton}}} \approx 10^{42} \tag{2.11}$$

Gravity is puny. Electromagnetism rules. In fact you knew this already. The mere act of lifting up you arm is pitching a few electrical impulses up against the gravitational might of the entire Earth. Yet the electrical impulses win.

However, gravity has a trick up its sleeve. While electric charges come with both positive and negative signs, mass is only positive. It means that by the time we get to macroscopically large objects – stars, planets, cats – the mass accumulates while the charges cancel to good approximation. This compensates the factor of 10^{-42} suppression until, at large distance scales, gravity wins after all.

The fact that the force of gravity is so ridiculously tiny at the level of fundamental particles has consequence. It means that we can neglect gravity whenever we talk about the very small. (And indeed, we shall neglect gravity for the rest of this course). However, it also means that if we would like to understand gravity better on these very tiny distances – for example, to develop a quantum theory of gravity – then it's going to be tricky to get much guidance from experiment.

2.1.2 A Uniform Sphere

The electric field outside a spherically symmetric charge distribution is always given by (2.7). What about inside? This depends on the distribution in question. The simplest is a sphere of radius R with uniform charge distribution ρ (see Fig. 2.3). The total charge is

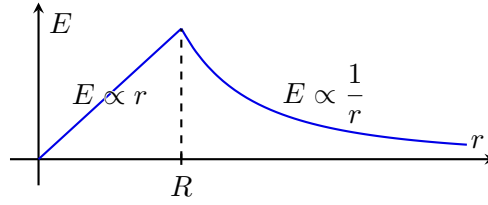$$Q = \frac{4\pi}{3}R^3\rho. \tag{2.12}$$

Fig. 2.4: Radial dependence of electric field strength inside and outside a spherically symmetric charge distribution of radius R.

Let's pick our Gaussian surface to be a sphere, centered at the origin, of radius $r < R$. The charge contained within this sphere is $4\pi\rho r^3/3 = Qr^3/R^3$, so Gauss' law gives

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Qr^3}{\epsilon_0 R^3}. \tag{2.13}$$

Again, using the symmetry argument we can write $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$ and compute

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(r) \int_{\mathcal{S}} \hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S} = E(r)4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3}. \tag{2.14}$$

This tells us that the electric field grows linearly inside the sphere

$$\mathbf{E}(\mathbf{x}) = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}}, \quad r < R. \tag{2.15}$$

Outside the sphere we revert to the inverse-square R r E Figure 7: form (2.7). At the surface of the sphere, $r = R$, the electric field is continuous but the derivative, $\mathrm{d}E/\mathrm{d}r$, is not. This is shown in Fig. 2.4.

2.1.3   Line Charges

Consider, next, a charge smeared out along a line which we'll take to be the z-axis. We'll take uniform charge density η per unit length. (If you like you could consider a solid cylinder with uniform charge density and then send the radius to zero). We want to know the electric field due to this line of charge.

Our set-up now has cylindrical symmetry. We take the Gaussian surface to be a cylinder of length L and radius r (See Fig. 2.5). We have

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{\eta L}{\epsilon_0}. \tag{2.16}$$

Again, by symmetry, the electric field points in the radial direction, away from the line. We'll denote this vector in cylindrical polar coordinates as $\hat{\mathbf{r}}$ so that $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. The symmetry means that the two end caps of the Gaussian surface, $\mathcal{S}_1$ and $\mathcal{S}_2$ in Fig. 2.5, don't contribute to the integral because their normal points parallel to the $\hat{\mathbf{z}}$ direction and $\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$. We're left only with a contribution from the curved side of the cylinder, $\mathcal{S}_3$,

$$\int_{\mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(r)2\pi r L = \frac{\eta L}{\epsilon_0}. \tag{2.17}$$
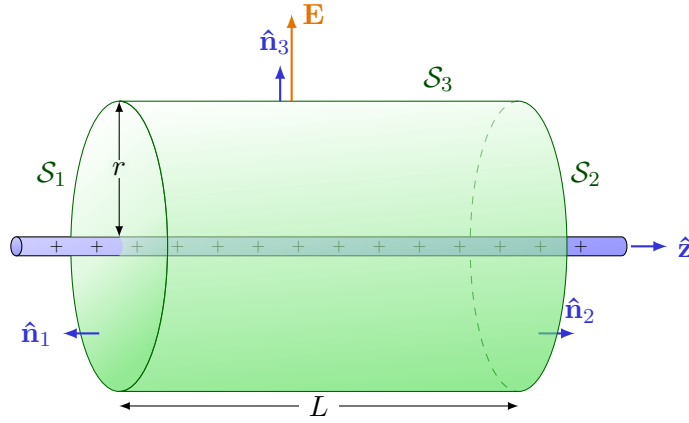
Fig. 2.5: The electric field $\mathbf{E}$ of an infinite line charge with a uniform linear charge density η. Considering a Gaussian surface in the form of a cylinder at radius r, the electric field has the same magnitude at every point on the surface $\mathcal{S}_3$ and is directed outward, parallel to $\hat{\mathbf{n}}_3$.

So that the electric field is

$$\mathbf{E}(\mathbf{x}) = \frac{\eta}{2\pi\epsilon_0}\hat{\mathbf{r}}. \tag{2.18}$$

Note that, while the electric field for a point charge drops off as $1/r^2$ (with r the radial distance), the electric field for a line charge drops off more slowly as $1/r$. (Of course, the radial distance r means slightly different things in the two cases: it is $r = \sqrt{x^2 + y^2 + z^2}$ for the point particle, but is $r = \sqrt{x^2 + y^2}$ for the line).

2.1.4   Surface Charges and Discontinuities

Now consider an infinite plane, which we take to be $z = 0$, carrying uniform charge per unit area, σ. We again take our Gaussian surface to be a cylinder, this time with its axis perpendicular to the plane as shown in the figure. In this context, the cylinder is sometimes referred to as a Gaussian "pillbox" (on account of Gauss' well known fondness for aspirin). On symmetry grounds, we have

$$\mathbf{E}(\mathbf{r}) = E(z)\hat{\mathbf{z}}. \tag{2.19}$$

Moreover, the electric field in the upper plane, $z > 0$, must point in the opposite direction from the lower plane, $z < 0$, so that $E(z) = -E(-z)$.

The surface integral now vanishes over the curved side of the cylinder and we only get contributions from the end caps, which we take to have area A. This gives

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} = E(z)A - E(-z)A = 2E(z) = \frac{\sigma A}{\epsilon_0}. \tag{2.20}$$

The electric field above an infinite plane of charge is therefore

$$\mathbf{E}(\mathbf{x}) = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}. \tag{2.21}$$

Note that the electric field is independent of the distance from the plane! This is because the plane is infinite in extent: the further you move from it, the more comes into view.
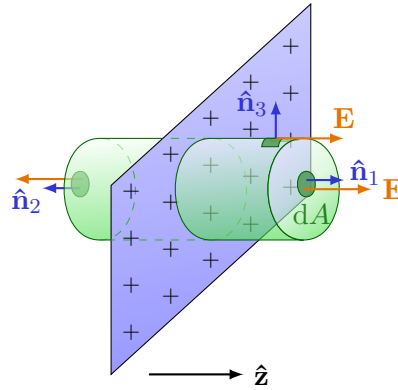
Fig. 2.6: For an infinite sheet of charge with charge density σ, the electric field **E** will be perpendicular to the surface. Therefore only the ends of a cylindrical Gaussian surface will contribute to the electric flux. In this case a cylindrical Gaussian surface perpendicular to the charge sheet is used. The resulting field is half that of a conductor at equilibrium with this surface charge density.

Fig. 2.7: Caption

There is another important point to take away from this analysis. The electric field is not continuous on either side of a surface of constant charge density. We have

$$E(z \to 0^+) - E(z \to 0^-) = \frac{\sigma}{\epsilon_0}. \tag{2.22}$$

For this to hold, it is not important that the plane stretches to infinity. It's simple to redo the above analysis for any arbitrary surface with charge density σ. There is no need for σ to be uniform and, correspondingly, there is no need for **E** at a given point to be parallel to the normal to the surface $\hat{\mathbf{n}}$. At any point of the surface, we can take a Gaussian cylinder, as shown in the left-hand figure above, whose axis is normal to the surface at that point. Its cross-sectional area A can be arbitrarily small (since, as we saw, it drops out of the final answer). If $\mathbf{E}_\pm$ denotes the electric field on either side of the surface, then

$$\hat{\mathbf{n}} \cdot \mathbf{E}_+ - \hat{\mathbf{n}} \cdot \mathbf{E}_- = \frac{\sigma}{\epsilon_0}. \tag{2.23}$$

In contrast, the electric field tangent to the surface is continuous. To see this, we need to do a slightly different calculation. Consider, again, an arbitrary surface with surface charge. Now we consider a loop $\mathcal{C}$ with a length L which lies parallel to the surface and a length a which is perpendicular to the surface. We've drawn this loop in the right-hand figure above, where the surface is now shown side-on. We integrate **E** around the loop. Using Stoke's theorem, we have

$$\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = \int_{\mathcal{S}} \times \mathbf{E} \cdot d\mathbf{S}, \tag{2.24}$$

where $\mathcal{S}$ is the surface bounded by $\mathcal{C}$. In the limit $a \to 0$, the surface $\mathcal{S}$ shrinks to zero size so this integral gives zero. This means that the contribution to line integral must also vanish, leaving us with

$$\hat{\mathbf{n}} \times \mathbf{E}_+ - \hat{\mathbf{n}} \times \mathbf{E}_- = 0. \tag{2.25}$$

This is the statement that the electric field tangential to the surface is continuous.

Fig. 2.8: A pair of infinite planes at $z = 0$ and $z = a$, carrying uniform surface charge density $\pm\sigma$
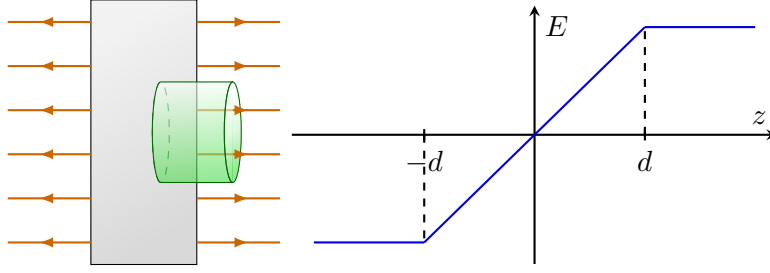
Fig. 2.9: Left: The Gaussian surface for a plane slab. Right: The resulting electric field.

2.1.4.1 A Pair of Planes

As a simple generalisation, consider a pair of infinite planes at $z = 0$ and $z = a$, carrying uniform surface charge density $\pm\sigma$ respectively as shown in Fig. 2.8. To compute the electric field we need only add the fields for arising from two planes, each of which takes the form (2.21). We find that the electric field between the two planes is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}, \qquad 0 < z < a, \tag{2.26}$$

while $\mathbf{E} = 0$ outside the planes.

2.1.4.2 A Plane Slab

We can rederive the discontinuity (2.23) in the electric field by considering an infinite slab of thickness $2d$ and charge density per unit volume ρ. When our Gaussian pillbox lies inside the slab, with $z < d$, we have

$$2AE(z) = \frac{2zA\rho}{\epsilon_0} \quad \implies \quad E(z) = \frac{\rho z}{\epsilon_0}. \tag{2.27}$$

Meanwhile, for $z > d$ we get our earlier result (2.21). The electric field is now continuous as shown in the figure. Taking the limit $d \to 0$ and $\rho \to \infty$ such that the surface charge $\sigma = \rho d$ remains constant reproduces the discontinuity (2.22).

2.1.4.3 A Spherical Shell

Let's give one last example that involves surface charge and the associated discontinuity of the electric field. We'll consider a spherical shell of radius R, centered at the origin, with uniform surface charge density σ. The total charge is

$$Q = 4\pi R^2 \sigma \tag{2.28}$$

We already know that outside the shell, $r > R$, the electric field takes the standard inverse-square form (2.7). What about inside? Well, since any surface with $r < R$ doesn't
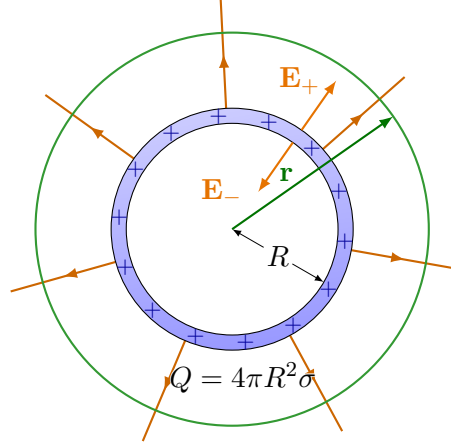
Fig. 2.10: A spherical shell of radius R, centered at the origin, with uniform surface charge density σ.

surround a charge, Gauss' law tells us that we necessarily have $E = 0$ inside. That means that there is a discontinuity at the surface $r = R$,

$$\mathbf{E} \cdot \hat{\mathbf{r}}_+ - \mathbf{E} \cdot \hat{\mathbf{r}}_- = \frac{Q}{4\pi R^2 \epsilon_0} = \frac{\sigma}{\epsilon_0}, \tag{2.29}$$

in accord with the expectation (2.23).

2.2 The Electrostatic Potential

For all the examples in the last section, symmetry considerations meant that we only needed to consider Gauss' law. However, for general charge distributions Gauss' law is not sufficient. We also need to invoke the second equation, $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

In fact, this second equation is easily dispatched since $\boldsymbol{\nabla} \times \mathbf{E} = 0$ implies that the electric field can be written as the gradient of some function,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{2.30}$$

The scalar ϕ is called the *electrostatic potential* or *scalar potential* (or, sometimes, just the *potential*). To proceed, we revert to the original differential form of Gauss' law (2.1). This now takes the form of the *Poisson equation*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \Longrightarrow \quad \nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \tag{2.31}$$

In regions of space where the charge density vanishes, we're left solving the Laplace equation

$$\nabla^2 \phi = 0. \tag{2.32}$$

Solutions to the Laplace equation are said to be *harmonic* functions.

A few comments:

- The potential ϕ is only defined up to the addition of some constant. This seemingly trivial point is actually the beginning of a long and deep story in theoretical physics known as *gauge invariance.* We'll come back to it in Subsection 5.0.1. For now, we'll eliminate this redundancy by requiring that $\phi(r) \to 0$ as $r \to \infty$.

- We know from our study of Newtonian mechanics that the electrostatic potential is proportional to the potential energy experienced by a test particle. Specifically, a test particle of mass m, position $\mathbf{r}(t)$ and charge q moving in a background electric field has conserved energy

$$E = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\phi(\mathbf{r}). \tag{2.33}$$

- The Poisson equation is linear in both ϕ and ρ. This means that if we know the potential ϕ_1 for some charge distribution ρ_1 and the potential ϕ_1 for another charge distribution ρ_2, then the potential for $\rho_1 + \rho_2$ is simply $\phi_1 + \rho_2$. What this really means is that the electric field for a bunch of charges is just the sum of the fields generated by each charge. This is called the *principle of superposition* for charges. This linearity of the equations is what makes electromagnetism easy compared to other forces of Nature.

- We stated above that $\boldsymbol{\nabla} \times \mathbf{E} = 0$ is equivalent to writing $\mathbf{E} = -\boldsymbol{\nabla}\phi$. This is true when space is $\mathbb{R}^3$ or, in fact, if we take space to be any open ball in $\mathbb{R}^3$. But if our background space has a suitably complicated topology then there are solutions to $\boldsymbol{\nabla} \times \mathbf{E} = 0$ which cannot be written in the form $\mathbf{E} = -\boldsymbol{\nabla}\phi$. This is tied ultimately to the beautiful mathematical theory of de Rham cohomology. Needless to say, in this starter course we're not going to worry about these issues. We'll always take spacetime to have topology $\mathbb{R}^4$ and, correspondingly, any spatial hypersurface to be $\mathbb{R}^3$.

2.2.1 The Point Charge

Let's start by deriving the Coulomb force law yet again. We'll take a particle of charge Q and place it at the origin. This time, however, we'll assume that the particle really is a point charge. This means that the charge density takes the form of a delta-function, $\rho(\mathbf{x}) = Q\delta^3(\mathbf{x})$. We need to solve the equation

$$\nabla^2\phi = -\frac{Q}{\epsilon_0}\delta^3(\mathbf{x}). \tag{2.34}$$

The solution is essentially the Green's function (See Appendix A.1) for the Laplacian ∇^2, an interpretation that we'll return to in Subsection 2.2.3. Let's recall how we find this solution. We first look away from the origin, $r \neq 0$, where there's no funny business going on with delta-function. Here, we're looking for the spherically symmetric solution to the Laplace equation. This is

$$\phi = \frac{\alpha}{r}, \tag{2.35}$$

or some constant α. To see why this solves the Laplace equation, we need to use the result

$$\boldsymbol{\nabla}r = \hat{\mathbf{r}} \tag{2.36}$$

where $\hat{\mathbf{r}}$ is the unit radial vector in spherical polar coordinates, so $\mathbf{x} = r\hat{\mathbf{r}}$. Using the chain rule, this means that $\boldsymbol{\nabla}(1/r) = -\hat{\mathbf{r}}/r^2 = -\mathbf{x}/r^3$. This gives us

$$\boldsymbol{\nabla}\phi = -\frac{\alpha}{r^3}\mathbf{x} \quad \Longrightarrow \quad \nabla^2\phi = -\alpha\left(\frac{\boldsymbol{\nabla}\cdot\mathbf{x}}{r^3} - \frac{3\mathbf{x}\cdot\mathbf{x}}{r^5}\right). \tag{2.37}$$

But $\boldsymbol{\nabla}\cdot\mathbf{x} = 3$ and we find that $\nabla^2\phi = 0$ as required.

It remains to figure out what to do at the origin where the delta-function lives. This is what determines the overall normalisation α of the solution. At this point, it's simplest to use the integral form of Gauss' law to transfer the problem from the origin to the far flung reaches of space. To do this, we integrate (2.34) over some region $\mathcal{V}$ which includes the origin. Integrating the charge density gives

$$\rho(\mathbf{x}) = Q\delta^3(\mathbf{x}) \quad \Longrightarrow \quad \int_{\mathcal{V}} \mathrm{d}^3x\,\rho = Q. \tag{2.38}$$

So, using Gauss' law (2.4), we require

$$\int_{\mathcal{S}} \boldsymbol{\nabla}\phi\cdot\mathrm{d}\mathbf{S} = -\frac{Q}{\epsilon_0}. \tag{2.39}$$

But this is exactly the kind of surface integral that we were doing in the last section. Substituting $\phi = \alpha/r$ into the above equation, and choosing $\mathcal{S}$ to be a sphere of radius r, tells us that we must have $\alpha = Q/4\pi\epsilon_0$, or

$$\phi = \frac{Q}{4\pi\epsilon_0 r}. \tag{2.40}$$

Taking the gradient of this using (2.36) gives us Coulomb's law

$$\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\phi = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.41}$$

The derivation of Coulomb's law using the potential was somewhat more involved than the technique using Gauss' law alone that we saw in the last section. However, as we'll now see, introducing the potential allows us to write down the solution to essentially any problem.

A Note on Notation Throughout these lectures, we will use $\mathbf{x}$ and $\mathbf{r}$ interchangeably to denote position in space. For example, sometimes we'll write integration over a volume as $\int \mathrm{d}^3x$ and sometimes as $\int \mathrm{d}^3r$. The advantage of the r notation is that it looks more natural when working in spherical polar coordinates. For example, we have $|\mathbf{r}| = r$ which is nice. The disadvantage is that it can lead to confusion when working in other coordinate systems, in particular cylindrical polar. For this reason, we'll alternate between the two notations, adopting the attitude that clarity is more important than consistency.

2.2.2 The Dipole

A *dipole* consists of two point charges, Q and $-Q$, a distance d apart. We place the first charge at the origin and the second at $\mathbf{r} = -\mathbf{d}$. The potential is simply the sum of the potential for each charge,

$$\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\mathbf{r}+\mathbf{d}|}\right). \tag{2.42}$$

Similarly, the electric field is just the sum of the electric fields made by the two point charges. This follows from the linearity of the equations and is a simple application of the principle of superposition that we mentioned earlier.

It will prove fruitful to ask what the dipole looks like far from the two point charges, at a distance $r \gg |\mathbf{d}|$. We need to Taylor expand the second term above. The vector version of the Taylor expansion for a general function $f(\mathbf{r})$ is given by

$$f(\mathbf{r} + \mathbf{d}) \approx f(\mathbf{r}) + \mathbf{d} \cdot \boldsymbol{\nabla} f(\mathbf{r}) + \frac{1}{2}(\mathbf{d} \cdot \boldsymbol{\nabla})^2 f(\mathbf{r}) + \cdots . \tag{2.43}$$

Applying this to the function $1/|\mathbf{r} + \mathbf{d}|$ gives

$$\begin{aligned}
\frac{1}{|\mathbf{r} + \mathbf{d}|} &\approx \frac{1}{r} + \mathbf{d} \cdot \boldsymbol{\nabla} \frac{1}{r} + \frac{1}{2}(\mathbf{d} \cdot \boldsymbol{\nabla})^2 \frac{1}{r} + \cdots \\
&= \frac{1}{r} - \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} - \frac{1}{2}\left(\frac{\mathbf{d} \cdot \mathbf{d}}{r^3} - \frac{3(\mathbf{d} \cdot \mathbf{r})^2}{r^5}\right) + \cdots
\end{aligned} \tag{2.44}$$

(To derive the last term, it might be easiest to use index notation for $\mathbf{d} \cdot \boldsymbol{\nabla} = d_i \partial_i$). For our dipole, we'll only need the first two terms in this expansion. They give the potential

$$\phi \approx \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r} + \mathbf{d} \cdot \boldsymbol{\nabla} \frac{1}{r} + \cdots\right) = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{d} \cdot \mathbf{r}}{r^3} + \cdots . \tag{2.45}$$

We see that the potential for a dipole falls off as $1/r^2$. Correspondingly, the electric field drops off as $1/r^3$; both are one power higher than the fields for a point charge.

The electric field is not spherically symmetric. The leading order contribution is governed by the combination

$$\mathbf{p} = Q\mathbf{d}. \tag{2.46}$$

This is called the electric *dipole moment*. By convention, it points from the negative charge to the positive. The dipole electric field is

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\right) + \cdots . \tag{2.47}$$

Notice that the sign of the electric field depends on where you sit in space. In some parts, the force will be attractive; in other parts repulsive.

It's sometimes useful to consider the limit $d \to 0$ and $Q \to \infty$ such that $\mathbf{p} = Q\mathbf{d}$ remains fixed. In this limit, all the $\cdots$ terms in (2.45) and (2.47) disappear since they contain higher powers of d. Often when people talk about the "dipole", they implicitly mean taking this limit.

2.2.3   General Charge Distributions

Our derivation of the potential due to a point charge (2.40), together with the principle of superposition, is actually enough to solve – at least formally – the potential due to any charge distribution. This is because the solution for a point charge is nothing other than

the Green's function for the Laplacian. The Green's function is defined to be the solution to the equation

$$\phi(\mathbf{r}) = -\frac{1}{\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 r'\, G(\mathbf{r}; \mathbf{r}') \rho \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 r'\, \frac{\rho \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{2.48}$$

(To check this, you just have to keep your head and remember whether the operators are hitting $\mathbf{r}$ or $\mathbf{r}'$. The Laplacian acts on $\mathbf{r}$ so, if we compute $\nabla^2 \phi$, it passes through the integral in the above expression and hits $G(\mathbf{r}; \mathbf{r}')$, leaving behind a delta-function which subsequently kills the integral).

Similarly, the electric field arising from a general charge distribution is

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 r'\, \rho(\mathbf{r}') \boldsymbol{\nabla} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 r'\, \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2.49}$$

Given a very complicated charge distribution $\rho(\mathbf{r})$, this equation will give back an equally complicated electric field $\mathbf{E}(\mathbf{r})$. But if we sit a long way from the charge distribution, there's a rather nice simplification that happens.

2.2.3.1   Long Distance Behaviour

Suppose now that you want to know what the electric field looks like far from the region $\mathcal{V}$. This means that we're interested in the electric field at $\mathbf{r}$ with $|\mathbf{r}| \gg |\mathbf{r}'|$ for all $\mathbf{r}' \in \mathcal{V}$. We can apply the same Taylor expansion (2.43), now replacing $\mathbf{d}$ with $-\mathbf{r}'$ for each $\mathbf{r}'$ in the charged region. This means we can write

$$frac1{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} - \mathbf{r}' \cdot \boldsymbol{\nabla} \frac{1}{r} + \frac{1}{2}(\mathbf{r} \cdot \boldsymbol{\nabla})^2 \frac{1}{r} + \cdots$$

$$= \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \frac{1}{2}\left(\frac{3(\mathbf{r} \cdot \mathbf{r}')^2}{r^5} - \frac{\mathbf{r}' \cdot \mathbf{r}'}{r^3}\right) + \cdots, \tag{2.50}$$

and our potential becomes

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathrm{d}^3 r'\, \rho(\mathbf{r}')\left(\frac{1}{r} + \frac{\mathbf{r}' \cdot \mathbf{r}'}{r^3} + \cdots\right). \tag{2.51}$$

The leading term is just

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} + \cdots, \tag{2.52}$$

where $Q = \int_{\mathcal{V}} \mathrm{d}^3 r'\, rho(\mathbf{r}')$ is the total charge contained within $\mathcal{V}$. So, to leading order, if you're far enough away then you can't distinguish a general charge distribution from a point charge localised at the origin. But if you're careful with experiments, you can tell the difference. The first correction takes the form of a dipole,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \cdots\right), \tag{2.53}$$

where

$$\mathbf{p} = \int_{\mathcal{V}} \mathrm{d}^3 r' \, \mathbf{r}' \rho(\mathbf{r}') \tag{2.54}$$

is the dipole moment of the distribution. One particularly important situation is when we have a neutral object with $Q = 0$. In this case, the dipole is the dominant contribution to the potential.

We see that an arbitrarily complicated, localised charge distribution can be characterised by a few simple quantities, of decreasing importance. First comes the total charge Q. Next the dipole moment $\mathbf{p}$ which contains some basic information about how the charges are distributed. But we can keep going. The next correction is called the quadrupole and is given by

$$\Delta \phi = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{r_i r_j \mathbb{Q}_{ij}}{r^5}, \tag{2.55}$$

where $\mathbb{Q}_{ij}$ is a symmetric traceless tensor known as the quadrupole moment, given by

$$\mathbb{Q}_{ij} = \int_{\mathcal{V}} \mathrm{d}^3 r' \, \rho(\mathbf{r}') \left(3 r_i' r_j' - \delta_{ij} r'^2 \right) \tag{2.56}$$

It contains some more refined information about how the charges are distributed. After this comes the octopole and so on. The general name given to this approach is the *multipole expansion*. It involves expanding the function ϕ in terms of spherical harmonics. A systematic treatment can be found, for example, in the book by Jackson.

2.2.3.2   A Comment on Infinite Charge Distribution

In the above, we assumed for simplicity that the charge distribution was restricted to some compact region of space, $\mathcal{V}$. The Green's function approach still works if the charge distribution stretches to infinity. However, for such distributions it's not always possible to pick $\phi(\mathbf{r}) \to 0$ as $r \to \infty$. In fact, we saw an example of this earlier. For an infinite line charge of density η, we computed the electric field in (**??**). It goes as

$$E(\mathbf{x}) = \frac{\eta}{2\pi\epsilon_0} \hat{\mathbf{r}}, \tag{2.57}$$

where now $r^2 = x^2 + y^2$ is the cylindrical radial coordinate perpendicular to the line. The potential ϕ which gives rise to this is

$$\phi(r) = -\frac{\eta}{2\pi\epsilon_0} \log\left(\frac{r}{r_0}\right). \tag{2.58}$$

Because of the log function, we necessarily have $\phi(r) \to 0$ as $r \to \infty$. Instead, we need to pick an arbitrary, but finite distance, r_0 at which the potential vanishes.

2.2.4   Field Lines

The usual way of depicting a vector is to draw an arrow whose length is proportional to the magnitude. For the electric field, there's a slightly different, more useful way to

show what's going on. We draw continuous lines, tangent to the electric field **E**, with the density of lines proportional to the magnitude of **E**. This innovation, due to Faraday, is called the *field line*. (They are what we have been secretly drawing throughout these notes).

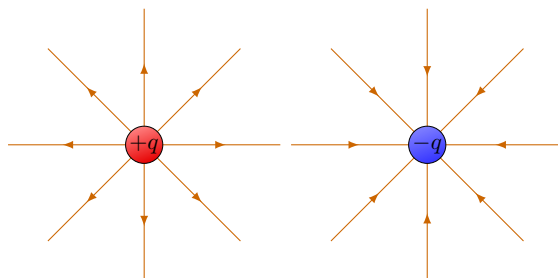Field lines are continuous. They begin and end only at charges. They can never cross.

Fig. 2.12: Field lines for positive and negative point charges

By convention, the positive charges act as sources for the lines, with the arrows emerging. The negative charges act as sinks, with the arrows approaching.

It's also easy to draw the equipotentials – surfaces of constant ϕ – on this same figure. These are the surfaces along which you can move a charge without doing any work. The relationship $\mathbf{E} = -\boldsymbol{\nabla}\phi$ ensures that the equipotentials cut the field lines at right angles. We usually draw them as dotted lines.
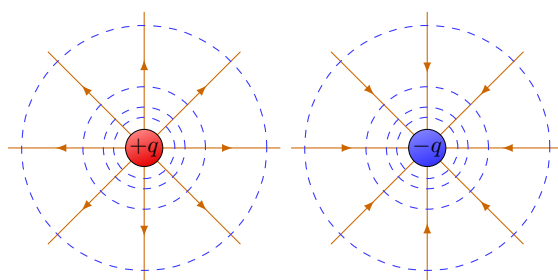

Fig. 2.13: Equipotentials for positive and negative point charges

2.2.5 Electrostatic Equilibrium

Here's a simple question: can you trap an electric charge using only other charges? In other words, can you find some arrangements of charges such that a test charge sits in stable equilibrium, trapped by the fields of the others.

There's a trivial way to do this: just allow a negative charge to sit directly on top of a positive charge. But let's throw out this possibility. We'll ask that the equilibrium point lies away from all the other charges.

There are some simple set-ups that spring to mind that might achieve this. Maybe
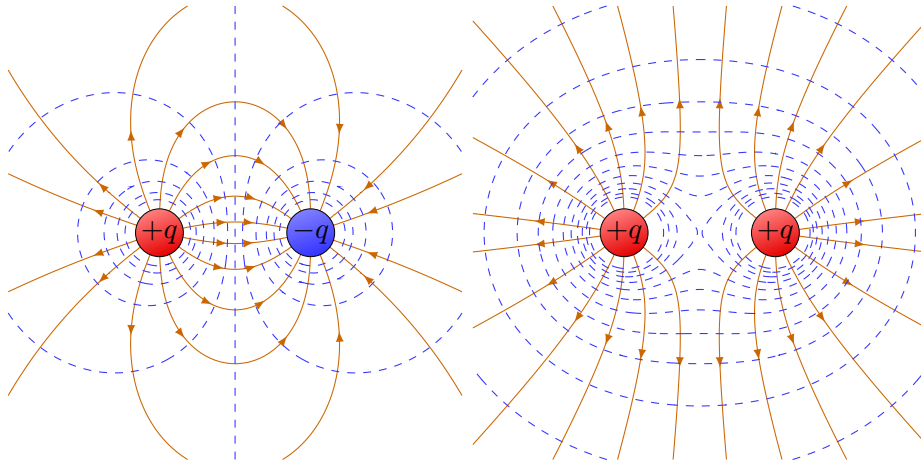
Fig. 2.14: Field lines and equipotentials for the dipole (left) and for a pair of charges of the same sign (on the right).

you could place four positive charges at the vertices of a pyramid; or perhaps 8 positive charges at the corners of a cube. Is it possible that a test positive charge trapped in the middle will be stable? It's certainly repelled from all the corners, so it might seem plausible.

The answer, however, is no. There is no electrostatic equilibrium. You cannot trap an electric charge using only other stationary electric charges, at least not in a stable manner. Since the potential energy of the particle is proportional to ϕ, mathematically, this is the statement that a harmonic function, obeying $\nabla^2 \phi = 0$, can have no minimum or maximum.

To prove that there can be no electrostatic equilibrium, let's suppose the opposite: that there is some point in empty space $\mathbf{r}_*$ that is stable for a particle of charge $q > 0$. By "empty space", we mean that $\rho(\mathbf{r}) = 0$ in a neighbourhood of $\mathbf{r}_*$. Because the point is stable, if the particle moves away from this point then it must always be pushed back. This, in turn, means that the electric field must always point inwards towards the point $\mathbf{r}_*$; never away. We could then surround $\mathbf{r}_*$ by a small surface $\mathcal{S}$ and compute

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} < 0. \tag{2.59}$$

But, by Gauss' law, the right-hand side must be the charge contained within $\mathcal{S}$ which, by assumption, is zero. This is our contradiction: electrostatic equilibrium does not exist.

Of course, if you're willing to use something other than electrostatic forces then you can construct equilibrium situations. For example, if you restrict the test particle to lie on a plane then it's simple to check that equal charges placed at the corners of a polygon will result in a stable equilibrium point in the middle. But to do this you need to use other forces to keep the particle in the plane in the first place.

2.3   Electrostatic Energy

There is energy stored in the electric field. In this section, we calculate how much.

Suppose we have some test charge q moving in a background electrostatic potential ϕ. We'll denote the potential energy of the particle as $U(\mathbf{r})$. The potential $U(\mathbf{r})$ of the particle can be thought of as the work done bringing the particle in from infinity;

$$U(\mathbf{r}) = -\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = +q \int_{\infty}^{r} \boldsymbol{\nabla}\phi \cdot d\mathbf{r} = q\phi(\mathbf{r}), \qquad (2.60)$$

where we've assumed our standard normalisation of $\phi(\mathbf{r}) \to 0$ as $r \to \infty$.

Consider a distribution of charges which, for now, we'll take to be made of point charges q_i at positions $\mathbf{r}_i$. The electrostatic potential energy stored in this configuration is the same as the work required to assemble the configuration in the first place. (This is because if you let the charges go, this is how much kinetic energy they will pick up). So how much work does it take to assemble a collection of charges?

Well, the first charge is free. In the absence of any electric field, you can just put it where you like - say, $\mathbf{r}_1$. The work required is $W_1 = 0$.

To place the second charge at $\mathbf{r}_2$ takes work

$$W_2 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}. \qquad (2.61)$$

Note that if the two charges have the same sign, so $q_1 q_2 > 0$, then $W_2 > 0$ which is telling us that we need to put work in to make them approach. If $q_1 q_2 < 0$ then $W_2 < 0$ where the negative work means that the particles wanted to be drawn closer by their mutual attraction.

The third charge has to battle against the electric field due to both q_1 and q_2. The work required is

$$W_3 = \frac{q_3}{4\pi\epsilon_0} \left(\frac{q_2}{|\mathbf{r}_2 - \mathbf{r}_2|} + \frac{q_1}{|\mathbf{r}_1 - \mathbf{r}_3|} \right), \qquad (2.62)$$

and so on. The total work needed to assemble all the charges is the potential energy stored in the configuration,

$$U = \sum_{i=1}^{N} W_i = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \qquad (2.63)$$

where $\sum_{i<j}$ means that we sum over each pair of particles once. In fact, you probably could have just written down (2.63) as the potential energy stored in the configuration. The whole purpose of the above argument was really just to nail down a factor of 1/2: do we sum over all pairs of particles $\sum_{i<j}$ or all particles $\sum_{i\neq j}$? The answer, as we have seen, is all pairs.

We can make that factor of 1/2 even more explicit by writing

$$U = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i} \sum_{j\neq i} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \qquad (2.64)$$

where now we sum over each pair twice.

There is a slicker way of writing (2.64). The potential at $\mathbf{r}_i$ due to all the other charges q_j, $j \neq i$ is

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0} \sum_{i \neq j} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{2.65}$$

which means that we can write the potential energy as

$$U = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\mathbf{r}_i). \tag{2.66}$$

This is the potential energy for a set of point charges. But there is an obvious generalisation to charge distributions $\rho(\mathbf{r})$. We'll again assume that $\rho(\mathbf{r})$ has compact support so that the charge is localised in some region of space. The potential energy associated to such a charge distribution should be

$$U = \frac{1}{2} \int \mathrm{d}^3 r \, \rho(\mathbf{r}) \phi(\mathbf{r}), \tag{2.67}$$

where we can quite happily take the integral over all of $\mathbb{R}^3$, safe in the knowledge that anywhere that doesn't contain charge has $\rho(\mathbf{r}) = 0$ and so won't contribute.

Now this is in a form that we can start to play with. We use Gauss' law to rewrite it as

$$U = \frac{\epsilon_0}{2} \int \mathrm{d}^3 r \, (\boldsymbol{\nabla} \cdot \mathbf{E}) \phi = \frac{\epsilon_0}{2} \int \mathrm{d}^3 r \, [\boldsymbol{\nabla} \cdot (\mathbf{E}\phi) - \mathbf{E} \cdot \boldsymbol{\nabla}\phi]. \tag{2.68}$$

But the first term is a total derivative. And since we're taking the integral over all of space and $\phi(\mathbf{r}) \to \infty$ as $r \to \infty$, this term just vanishes. In the second term we can replace $\boldsymbol{\nabla}\phi = -\mathbf{E}$. We find that the potential energy stored in a charge distribution has an elegant expression solely in terms of the electric field that it creates,

$$U = \frac{\epsilon_0}{2} \int \mathrm{d}^3 r \, \mathbf{E} \cdot \mathbf{E}. \tag{2.69}$$

Isn't that nice!

2.3.1 The Energy of a Point Particle

There is a subtlety in the above derivation. In fact, I totally tried to pull the wool over your eyes. Here it's time to own up.

First, let me say that the final result (2.69) is right: this is the energy stored in the electric field. But the derivation above was dodgy. One reason to be dissatisfied is that we computed the energy in the electric field by equating it to the potential energy stored in a charge distribution that creates this electric field. But the end result doesn't depend on the charge distribution. This suggests that there should be a more direct way to arrive at (2.69) that only talks about fields and doesn't need charges. And there is. We will see it later.

But there is also another, more worrying problem with the derivation above. To illustrate this, let's just look at the simplest situation of a point particle. This has electric field

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2.70}$$

So, by (2.69), the associated electric field should carry energy. But we started our derivation above by assuming that a single particle didn't carry any energy since it didn't take any work to put the particle there in the first place. What's going on?

Well, there was something of a sleight of hand in the derivation above. This occurs when we went from the expression $q\phi$ in (2.66) to $\rho\phi$ in (2.67). The former omits the "self-energy" terms; there is no contribution arising from $q_i\phi(\mathbf{r}_i)$. However, the latter includes them. The two expressions are not quite the same. This is also the reason that our final expression for the energy (2.69) is manifestly positive, while $q\phi$ can be positive or negative.

So which is right? Well, which form of the energy you use rather depends on the context. It is true that (2.69) is the correct expression for the energy stored in the electric field. But it is also true that you don't have to do any work to put the first charge in place since we're obviously not fighting against anything. Instead, the "self-energy" contribution coming from $\mathbf{E} \cdot \mathbf{E}$ in (2.70) should simply be thought of – using $E = mc^2$ – as a contribution to the mass of the particle.

We can easily compute this contribution for, say, an electron with charge $q = -e$. Let's call the radius of the electron a. Then the energy stored in its electric field is

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APPENDIX **A**

Code

A.1   Green's Functions