

Part II Relativity

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Abstract

General relativity is the theory of space and time and gravity. The essence of the theory is simple: gravity is geometry. The effects that we attribute to the force of gravity are due to the bending and warping of spacetime, from falling cats, to orbiting spinning planets, to the motion of the cosmos on the grandest scale. The purpose of these lectures is to explain this.

Before we jump into a description of curved spacetime, we should first explain why Newton's theory of gravity, a theory which served us well for 250 years, needs replacing. The problems arise when we think about disturbances in the gravitational field. Suppose, for example, that the Sun was to explode. What would we see? Well, for 8 glorious minutes – the time that it takes light to reach us from the Sun – we would continue to bathe in the Sun's light, completely oblivious to the fate that awaits us. But what about the motion of the Earth? If the Sun's mass distribution changed dramatically, one might think that the Earth would start to deviate from its elliptic orbit. But when does this happen? Does it occur immediately, or does the Earth continue in its orbit for 8 minutes before it notices the change?

Of course, the theory of special relativity tells us the answer. Since no signal can propagate faster than the speed of light, the Earth must continue on its orbit for 8 minutes. But how is the information that the Sun has exploded then transmitted? Does the information also travel at the speed of light? What is the medium that carries this information? As we will see throughout these lectures, the answers to these questions forces us to revisit some of our most basic notions about the meaning of space and time and opens the door to some of the greatest ideas in modern physics such as cosmology and black holes.

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CHAPTER 1

Introduction

1.1 Newtonian Gravity

There is a well trodden path in physics when trying to understand how objects can influence other objects far away. We introduce the concept of a field. This is a physical quantity which exists everywhere in space and time; the most familiar examples are the electric and magnetic fields. When a charge moves, it creates a disturbance in the electromagnetic field, ripples of which propagate through space until they reach other charges. To develop a causal theory of gravity, we must introduce a gravitational field that responds to mass in some way.

It's a simple matter to cast Newtonian gravity in terms of a field theory. A particle of mass m_G experiences a force that can be written as

$$\mathbf{F} = -m_G \nabla \Phi. \quad (1.1)$$

The quantity m_G is the *passive gravitational mass*, and it determines the gravitational force on the particle. The gravitational field $\Phi(\mathbf{r}, t)$ is determined by the surrounding matter distribution which is described by the mass density $\rho(\mathbf{r}, t)$. If the matter density is static, so that $\rho(\mathbf{r})$ is independent of time, then the gravitational field obeys

$$\nabla^2 \Phi = 4\pi G \rho, \quad (1.2)$$

with Newton's constant G given by

$$G \approx 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}. \quad (1.3)$$

This equation is simply a rewriting of the usual inverse square law of Newton. For example, if a mass M is concentrated at a single point we have

$$\rho(\mathbf{r}) = M \delta^{(3)}(\mathbf{r}) \implies \Phi = -\frac{GM}{r}, \quad (1.4)$$

which is the familiar gravitational field for a point mass.

The question that we would like to answer is: how should we modify (1.2) when the mass distribution $\rho(\mathbf{r})$ changes with time? Of course, we could simply postulate that (1.2) continues to hold even in this case. A change in ρ would then immediately result in a change of Φ throughout all of space. Such a theory clearly is not consistent with the requirement that no signal can travel faster than light. Our goal is to figure out how to generalise (1.2) in a manner that is compatible with the postulates of special relativity. The end result of this goal will be a theory of gravity that is compatible with special relativity: this is the general theory of relativity.

Fixing this incompatibility will ultimately require a radical modification of how we think about gravity and, indeed, spacetime itself. Sticking with Newtonian gravity for

the moment, it is not immediately obvious that the mass density appearing in Poisson's equation should refer to the density of the passive gravitational mass. Rather, let us also introduce the active gravitational mass m_A , so that the relevant mass density for a point particle at position $\mathbf{r}'(t)$ at time t is

$$\rho(\mathbf{r}, t) = m_A \delta^{(3)}(\mathbf{r} - \mathbf{r}'(t)). \quad (1.5)$$

For the point particle, the relevant solution of Poisson's equation is

$$\Phi(\mathbf{r}, t) = -\frac{Gm_A}{|\mathbf{r} - \mathbf{r}'(t)|}. \quad (1.6)$$

It follows that the force on a test particle of passive gravitational mass $m_{G,1}$ at position \mathbf{r}_1 at time t due to a particle of active gravitational mass $m_{A,2}$ at position \mathbf{r}_2 *at the same time* t is

$$\mathbf{F}_{2 \text{ on } 1} = -Gm_{G,1}m_{A,2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \quad (1.7)$$

Similarly, the force on the second particle due to the first is

$$\mathbf{F}_{1 \text{ on } 2} = -Gm_{G,2}m_{A,1} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \quad (1.8)$$

If momentum is to be conserved, i.e., $\mathbf{F}_{2 \text{ on } 1} = \mathbf{F}_{1 \text{ on } 2}$, we must have

$$m_{G,1}m_{A,2} = m_{G,2}m_{A,1}. \quad (1.9)$$

Since this must hold for arbitrary masses, we must have that the ratio of passive to active gravitational mass is the same for all particles. Thus, we can take these masses to be equal, $m_G = m_A$, for all matter (absorbing their universal ratio in the gravitational constant).

This sort of universality is not unusual in physics – a similar thing happens in electromagnetism, for example, where the passive and active electric charges are equal. However, there is a further equality of masses in Newtonian gravity that is rather more surprising: the equality of gravitational and inertial masses. A particle acted on by a force \mathbf{F} experiences an acceleration such that

$$\mathbf{F} = m_I \frac{d^2\mathbf{r}}{dt^2}, \quad (1.10)$$

where m_I is the *inertial mass*. For the gravitational force, the acceleration is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{m_G}{m_I} \nabla \Phi. \quad (1.11)$$

It is an experimental fact¹ (known since Galileo's time) that the ratio m_G/m_I is the same for all particles, so we can always take $m_G = m_I$ (further absorbing their universal ratio in the gravitational constant). This means that if two particles of different composition fall freely in a gravitational field, they have the same acceleration. This is often rephrased as the *weak equivalence principle*:

Freely-falling particles with negligible gravitational self-interaction follow the same path through space and time if they have the same initial position and velocity, independent of their composition.

This property of gravity is in striking contrast to other forces; for example, in electromagnetism the acceleration of a point particle in a given electric field depends on the ratio of the electric charge to inertial mass, which is definitely not universal.

¹The equality of gravitational and inertial masses is now verified to the level of one part in 10^{13} .

1.2 Implications of the Equivalence Principle

Consider an observer in a free-falling, non-rotating elevator in a uniform gravitational field. Relative to this observer, free-falling particles move on straight lines at constant velocity – the effects of the uniform gravitational field have been removed and the observer perceives that the usual laws of special relativistic kinematics hold. This idea motivates an extension of the weak equivalence principle to what is known as the *strong equivalence principle*:

In an arbitrary gravitational field, *all* the laws of physics in a free-falling, non-rotating laboratory occupying a sufficiently small region of spacetime look locally like special relativity (with no gravity).

Note how the strong equivalence principle is supposed to apply to all laws of physics, not just the dynamics of free-falling particles. Why the qualification of observations over a sufficiently small region of spacetime?

Consider the same elevator falling freely in the non-uniform gravitational field of the earth. Free particles initially at rest in the elevator will move together over time as they follow radial trajectories towards the centre of the earth. It is these tidal effects that are the physical manifestation of the gravitational field, and that cannot be removed by passing to the free-falling frame. However, for sufficiently local measurements in space and time, these tidal effects are undetectable, and physics relative to the free-falling elevator looks just like special relativistic physics in an inertial frame of reference in the absence of gravity.

The strong equivalence principle implies the local equivalence of a gravitational field and acceleration. In particular, it implies that a constant gravitational field is unobservable – observations in a reference frame at rest in such a field would be indistinguishable from those in a uniformly-accelerating reference frame in the absence of gravity. In special relativity, physics looks simple when referred to an inertial frame, one defined by comoving, unaccelerated observers with synchronised clocks. However, with gravity, the equivalence principle tells us that physics looks equally simple *locally* in a free-falling reference frame, suggesting that we should *define* inertial reference frames locally by free-falling observers. Acceleration should be defined relative to such local inertial frames, so that a particle acted on by no other force (and so free-falling) should be regarded as unaccelerated.

1.2.1 Gravity as Spacetime Curvature

The universality of free fall suggested to Einstein that the trajectories of free-falling particles should be determined by the local structure of spacetime, rather than by the action of a gravitational force with a mysterious universal coupling to matter.

Local inertial reference frames correspond to local systems of coordinates over spacetime so that the geometry over a small region looks like that of the spacetime of special relativity. Gravity manifests itself through our inability to extend such coordinates globally, reflecting the *curvature of spacetime*.

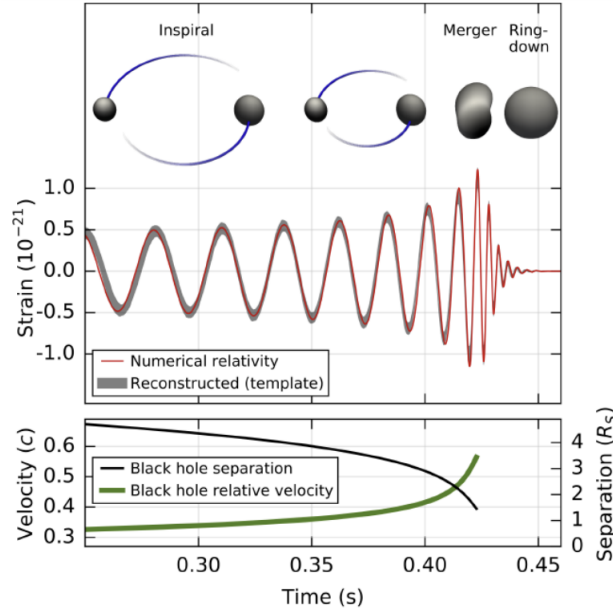


Fig. 1.1: Top: Estimated gravitational wave strain amplitude inferred from the LIGO data for their discovery event. The signal is generated from the inspiral, merger and ring-down of two massive black holes. The properties of the source can be estimated by comparing the measured waveform with detailed calculations in general relativity. Bottom: the relative speed and separation (in units of the Schwarzschild radius, $R_s = 2GM/c^2$) of the blackholes during the event. For reference, the Newtonian potential at R_s away from a mass M is $|\Phi|/c^2 = 1/2$. Figure taken from Abbot et al., Phys. Rev. Lett. 116, 061102 (2016).

General relativity abandons the idea of gravity as a force defined on the fixed space-time of special relativity, replacing it with a geometric theory in which the geometry of spacetime determines the trajectories of free-falling particles, the geometry itself being curved by the presence of matter.

1.3 Further Motivation: Extreme Gravity

Newtonian gravity is recovered from general relativity in the limit of low relative speeds of particles, $v \ll c$, and weak gravitational fields, typically $|\Phi| \ll c^2$. Note that in situations where speeds are determined by gravity, these two regimes are generally equivalent.

To see this, consider a particle in a circular orbit of radius R around a mass M in Newtonian gravity: the speed is determined by

$$\frac{v^2}{R} = \frac{GM}{R^2}, \quad (1.12)$$

and so

$$\frac{v^2}{c^2} = \frac{GM}{Rc^2} = \frac{|\Phi|}{c^2}. \quad (1.13)$$

However, increasingly we are observing phenomena where Newtonian gravity is a very poor approximation. A striking example is the recent first detection of gravitational waves by the LIGO interferometer; see Fig. 1.1.

Gravitational waves are wavelike disturbances in the geometry of spacetime, which can be detected by looking for their characteristic quadrupole distortion (i.e., a shortening in one direction and stretching in an orthogonal direction) of the two arms of a laser interferometer. Gravitational waves propagate at the speed of light and are a natural prediction of general relativity; they do not arise in Newtonian gravity where the potential responds instantly to distant rearrangements of mass.

The first LIGO signal was generated by a truly extreme astrophysical source: two merging black holes each with a mass around 30 times that of the Sun at a distance from us of around 2 Gly. As the blackholes orbited their common centre of mass, the system radiated gravitational waves causing the blackholes to spiral inwards and increase their speed until they merged to form a single black hole. Such sources probe the strong-field regime of general relativity during the merger phase and involve highly relativistic speeds (see Fig. 1.1). At its peak, the source was losing energy to gravitational waves at a rate of $3.6 \times 10^{49} \text{W}$, which is equivalent to 200 times the rest mass energy of the Sun per second!

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