Tutorial Week 2 - CSCB63 -Complexity

Draw a 2D table with column headers and row headers as follows:

	$\ln(n)$	$\lg(n)$	$\lg(n^2)$	$(\lg n)^2$	n	$n*\lg(n)$	2^n	$2^{(3n)}$
$\ln(n)$								
$\lg(n)$								
$\lg(n^2)$								
$(\lg n)^2$								
n								
$n*\lg(n)$								
2^n								
$2^{(3n)}$								

(Remember that lg means log base 2.)

In each cell, fill in "Y" iff (its row function)$\in O$(its column function).

The filled table should go like:

	$\ln(n)$	$\lg(n)$	$\lg(n^2)$	$(\lg n)^2$	n	$n*\lg(n)$	2^n	$2^{(3n)}$
$\ln n$	Y	Y	Y	Y	Y	Y	Y	Y
$\lg n$	Y	Y	Y	Y	Y	Y	Y	Y
$\lg(n^2)$	Y	Y	Y	Y	Y	Y	Y	Y
$(\lg n)^2$				Y	Y	Y	Y	Y
n					Y	Y	Y	Y
$n\lg n$						Y	Y	Y
2^n							Y	Y
$2^{(3n)}$								Y

How much proof/argument is required before filling in each cell? Answer: For most cells very straightforward.

We haven't talked about transitivity (if $f \in O(g)$ and $g \in O(h)$, then simply deduce $f \in O(h)$ and be done), but you may prove on your own and use it.

Observation. Even though $3n \in O(n)$, we cannot "exponentiate both sides" to infer $2^{3n} \in O(2^n)$.

Lets look 3 of the more interesting cells to show proofs for.

Fill in proofs for selected big-O cells:

1. $n \in O(n\lg(n))$, using the definition of big-O:

   Choose $c = 1, n_0 = 2$. We need $2 = n_0$ because we plan to do:

   $$2 \leq n,$$

   take lg on both sides, get

   $$1 \leq \lg(n)$$

So we can do: for all $n \geq n_0$

$$\begin{aligned} &= n \\ &= n \cdot 1 \\ &\leq n \cdot \lg(n) \\ &= c \cdot n \cdot \lg(n) \end{aligned}$$

And also $0 \leq n$ of course. Overall: for all $n \geq n_0, 0 \leq n \leq c \cdot n \cdot \lg(n)$.

2. $n\lg(n) \notin O(n)$ using the definition of big-O: (Note: This can be explained as a proof by contradiction...other ways possible too). (Not the only way to explain, but it seems to make sense to most students.)

   Suppose
   $$\exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, 0 \leq n\lg(n) \leq cn$$
   is true.

   Then changing "$\forall n, n \geq n_0$" to "$\forall n, n \geq max(n_0, 1)$" still gives a true statement. $max(n_0, 1)$ is equal to or larger than n_0.

   We do this because we want $n \geq 1$, because I will be dividing by n. Here:

   So for all $n \geq max(n_0, 1)$,

$$\begin{aligned} n \cdot \lg(n) &\leq c \cdot n, \quad \text{since } n \geq 1, \text{can divide both sides by } n && (1) \\ \lg(n) &\leq c && (2) \\ n &\leq 2^c && (3) \end{aligned}$$

   So for all $n \geq max(n_0, 1), n \leq 2^c$. If we now choose $n = max(n_0, 1, 2^c + 1)$, then $n \geq max(n_0, 1)$, so we should get $n \leq 2^c$, but this $n > 2^c$. We get a contradiction.

3. $2^{(3n)} \notin O(2^n)$, using a limit theorem from lecture (you may not have seen the limit theorem if your tutorial is before the Wed. class).

   The limit theorem says:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \notin O(g(n))$$

   Below we will for brevity omit "$n \to \infty$".

$$\begin{aligned} & \lim \frac{2^{(3n)}}{2^n} && (4) \\ = & \lim 2^{(3n-n)} && (5) \\ = & \, lim 2^{(2n)} && (6) \\ = & \, \infty && (7) \end{aligned}$$

   So $2^{(3n)} \notin O(2^n)$.

Some practice questions:

1. $6n^5 + n^2 - n^3 \in \Theta(n^5)$
2. $3n^2 - 4n \in \Omega(n^2)$

For these the intentions are to use the **definitions** of Theta and Omega not use the limit laws.

So for example for 1, for the $O(n^5)$ part one could do:

$$6n^5 + n^2 - n^3 \leq 6n^5 + n^2 \leq 6n^5 + n^5 = 7n^5 \text{ for } n \geq 1$$

For $\Omega(n^5)$ can do:

$$6n^5 + n^2 - n^3 >= 6n^5 - n^3 >= 6n^5 - n^5 = 5n^5 \text{ so } b = 5 \text{ for } n \geq 1.$$

For 2.

$$\begin{aligned}
3n^2 - 4n &\geq bn^2 & (8)\\
3n - 4 &\geq bn & (9)\\
n(3-b) &\geq 4 & (10)\\
n &\geq 4/(3-b) & (11)\\
& & (12)
\end{aligned}$$

if $b = 2$ then $n \geq 4$ or $b = 1$ then $n \geq 2$.