Math Reasoning Notes

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Contents

1 Number Sets

A few famous sets are:

- $\varnothing$: the *empty set* containing no elements;
- $\mathbb{N}$: the set of *natural numbers* (that is, nonnegative integers);
- $\mathbb{Z}$: the set of *integers*;
- $\mathbb{Q}$: the set of *rational numbers*;
- $\mathbb{R}$: the set of *real numbers*;
- $\mathbb{C}$: the set of *complex numbers*.

2 Definitions

- Given universal set U and proposition schema P(x)
  - $\neg(\forall x \in U, P(x)) = \exists x \in U, \neg P(x)$
  - $\neg(\exists x \in U, P(x)) = \forall x \in U, \neg P(x)$
- A **relation** from set X to set Y is a subset $R \subseteq X \times Y$.
- A **function** with domain X and codomain Y is a relation such that:
  - $\forall x \in X, \exists y \in Y$
- A **Model of the second-order formulation of peano arithmetic** is a 3-tuple (N, O, S) where N is a set, $O \in N$, and $S : N \to N$ which all satisfy the second-order peano axioms. All models like this are isomorphic.

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3.1 Quantification

Notation: Given some schema P(x), U a universal set, then we _universally quantify P(x) by forming the prop: "for all x $\in$ U, P(x) is true" and write $\forall x \in U, P(x)$ or $\forall x \in U : P(x)$.
Similarly, we existentially quantify P(x) by forming the proposition: "there exists an x $\in$ U such that P(x) is true" and write $\exists x \in U, P(x)$ or $\exists x \in U : P(x)$.

Note 1. *The name of a free variable is irrelevant. For example,* $\forall x \in U, P(x)$ *and* $\forall y \in U, P(y)$ *are the same proposition.*

Proposition 1. *Let U be a universal set, and let P(x) be a schema with free variable x, then:*

- $\neg(\forall x \in U, P(x)) \equiv \exists x \in U, \neg P(x)$
- $\neg(\exists x \in U, P(x)) \equiv \forall x \in U, \neg P(x)$

Proof. Let us assume $(\forall x \in U, P(x))$ is true, then for any $x \in U, P(x)$ is true. Thus, there does not exist an $x \in U$ such that $P(x)$ is false, i.e. $\neg P(x)$ is true. Thus, $\exists x \in U, \neg P(x)$ is false.
Conversely, if we assume $\exists x \in U, \neg P(x)$ is true, then there exists an $x \in U$ such that $P(x)$ is false. Combining these two paragraphs, we have shown that $\exists x \in U, \neg P(x)$ is true if and only if $\forall x \in U, P(x)$ is false. Thus, $\neg(\forall x \in U, P(x)) \equiv \exists x \in U, \neg P(x)$. ∎

Example 1. *Negate* $\forall x[(x \in A) \implies (x \in B)]$
$\neg(\forall x[(x \in A) \implies (x \in B)]) \equiv \exists x \neg[(x \in A) \implies (x \in B)] \equiv \exists x[(x \in A) \wedge \neg(x \in B)] \equiv \exists x[(x \in A) \wedge (x \notin B)]$
In english: "There is an element of A which is not and element in B". i.e. $A \not\subseteq B$.

Example 2. $U = \mathbb{Z}$ *and consider the propositions:* $[\forall a \in U, \exists b \in U, a + b = 1]$ *and* $[\exists a \in U, \forall b \in U, a + b \neq 1]$.
The first proposition is true, since for any integer a, we can choose b = 1 - a, which is also an integer, and a + b = 1.
The second proposition is false, since for any integer a, we can choose b = 1 - a, and a + b = 1.
Note that the order of quantifiers matters!

Proposition 2. *Let U be a universal set, and let P(x) and Q(x) be schemas with free variable x, then:*

- $\forall x \in U, [P(x) \wedge Q(x)] \implies [\forall x \in U, P(x)] \wedge [\forall x \in U, Q(x)]$
- $\exists x \in U, [P(x) \vee Q(x)] \implies [\exists x \in U, P(x)] \vee [\exists x \in U, Q(x)]$
- $\forall x \in U, [P(x) \vee Q(x)] \implies [(\forall x \in U, P(x)) \vee (\forall x \in U, Q(x))]$
- $\exists x \in U, [P(x) \wedge Q(x)] \implies [(\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))]$

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4.1 The Natural Numbers

Definition 1. A <u>relation</u> R from a set A to a set B is a subset of $A \times B$.

Definition 2. A <u>function</u> f with domain X and codomain Y is a relation $f \subseteq X \times Y$ such that $\forall x \in X, \exists y \in Y, (x, y) \in f$
And $((x, y) \in f) \wedge ((x, z) \in f) \implies (y = z)$ We write $f : A \to B$ and $f(a) = b$.

The Peano Axioms:

1. $0 \in \mathbb{N}$
2. $\forall x \in \mathbb{N}, x = x$
3. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, (x = y) \implies (y = x)$
4. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, [(x = y) \wedge (y = z)] \implies (x = z)$
5. $\forall a, \forall b, [(b \in \mathbb{N}) \wedge (a = b)] \implies (a \in \mathbb{N})$
6. $\exists S \subseteq \mathbb{N} \times \mathbb{N}$ called the successor function, such that $\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$
7. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, [S(x) = S(y)] \implies (x = y)$ (i.e. S is injective)
8. $\forall n \in \mathbb{N}, S(n) \neq 0$
9. (Axiom of Induction) If k is a set such that:
   (a) $0 \in k$
   (b) $\forall x \in \mathbb{N}, (x \in k) \Rightarrow (S(n) \in k)$

   Then $\mathbb{N} \subseteq k$.

Axiom 9': If P(x) is a proposition schema with one free variable x such that:

1. P(0) is true
2. $\forall n \in \mathbb{N}, (P(n)) \Rightarrow (P(S(n)))$

Then P(n) is true for all $n \in \mathbb{N}$.
Using these axioms, we can define addition and multiplication recursively for all $a, b \in \mathbb{N}$:

- $a + 0 = a$
- $a + S(b) = S(a + b)$
- $a \cdot 0 = 0$
- $a \cdot S(b) = a \cdot b + a$

Theorem 1. *Addition is associative:* $\forall a, b, c \in \mathbb{N}, (a + b) + c = a + (b + c)$

Proof. Fix $a, b \in \mathbb{N}$ and let P(c) be the proposition $(a+b)+c = a+(b+c)$. We will prove that P(c) is true for all $c \in \mathbb{N}$ using Axiom 9'.
Base Case: P(0) is true since $(a+b)+0 = a+b$ and $a+(b+0) = a+b$.
Inductive Step: Assume P(c) is true for some $c \in \mathbb{N}$, i.e. $(a+b)+c = a+(b+c)$. We want to show that P(S(c)) is true, i.e. $(a+b)+S(c) = a+(b+S(c))$.

$$\begin{aligned}
(a+b)+S(c) &= S((a+b)+c) && \text{(by definition of addition)} \\
&= S(a+(b+c)) && \text{(by inductive hypothesis)} \\
&= a+S(b+c) && \text{(by definition of addition)} \\
&= a+(b+S(c)) && \text{(by definition of addition)}
\end{aligned}$$

Thus, by Axiom 9', P(c) is true for all $c \in \mathbb{N}$. ■

Definition 3. Inequality on $\mathbb{N}$: For $a, b \in \mathbb{N}, a \leq b \iff \exists c \in \mathbb{N}, a+c = b$.
We say $a < b \iff a \leq b$ and $a \neq b$.

Using $\leq$, we can restate Axiom 9' as follows:
Axiom 9" (the strong induction axiom): If P(x) is a proposition with one free variable and P(0) is true, for every $n \in \mathbb{N}$ if P(k) is true for all $k \in \mathbb{N}$ where $k \leq n$, P(n) is true.
All axioms 1-8 are phrased in first-order logic, which is what have called predicate logic with quantification <u>of elements of universal sets</u>. Axioms 9, 9' and 9" all require <u>second-order logic</u> where we are allowed to quantify over proposition schemata.