1 Sample Spaces

Definition 1. *An experiment is any process that can be repeated with a well-defined set of outcomes. Example: roll a die, toss a coin.*

Sets: discrete/finite, discrete/infinite, continuous

Definition 2. *A sample space is the set of all possible outcomes of an experiment. Example:* $S = \{1, 2, 3, 4, 5, 6\}$*,* $S = \{H, T\}$

Definition 3. *The union of* A *and* B*,* $A \cup B$*, is all elements in* A *or* B*.*

Definition 4. *The intersection of* A *and* B*,* $A \cap B$*, is all elements in both* A *and* B*.*

Definition 5. A *and* B *are mutually exclusive (disjoint) if* $A \cap B = \emptyset$*.*

Definition 6. *The complement of* A*,* A^c*, is the set of elements not in* A*:* $S - A$*.*

2 Probability

Definition 7. *The function* P *is a probability function if it satisfies:*

1. $P(A) \in [0, 1]$ *for all* $A \subseteq S$
2. $P(S) = 1$
3. *If* A *and* B *are disjoint, then* $P(A \cup B) = P(A) + P(B)$
4. *If* $A_1, A_2, \ldots$ *are disjoint, then* $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Lemma 1. $P(A^c) = 1 - P(A)$

Proof: $A \cup A^c = S$, $A \cap A^c = \emptyset$, so $P(S) = P(A \cup A^c) = P(A) + P(A^c) = 1$. Therefore, $P(A^c) = 1 - P(A)$.

2.1 Fair Dice Example

If we roll a fair die, $P(i) = \frac{1}{6}$ for $i = 1, 2, \ldots, 6$.
$P(\{1, 2\} \cup \{3, 4\} \cup \{5, 6\}) = P(\{1\}) + P(\{2\}) + \cdots + P(\{6\}) = 6 \cdot P(\{1\}) \Rightarrow P(\{1\}) = \frac{1}{6}$
Note: If all elements in S are equally likely, $P(A) = \frac{\#\text{ in } A}{\#\text{ in } S}$

2.2 Card Example

Roll a fair dice, find $P(\text{sum} = 5)$.

A	B	Total
1	4	5
2	3	5
3	2	5
4	1	5

There are 26 people, find $P(\text{at least 2 same birthday})$.
$1 - P(\text{0 different})$

Lemma 2 (Addition Rule)**.** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If disjoint: $P(A \cup B) = P(A) + P(B) - 0 = P(A) + P(B)$
Proof: $P(A \cup B^c) + P(A \cap B^c) + P(B \cap A^c) = P(A \cup B)$
Prove: If $A \subseteq B$, then $P(A) \leq P(B)$
If $A \subseteq B$, $B^c \cap A = \emptyset$
$B = A \cup (A^c \cap B)$ (disjoint union), so $P(B) = P(A) + P(A^c \cap B)$
Because $P(\cdot)$ is non-negative, $P(B) \geq P(A)$

3 Counting

3.1 Example 2.8

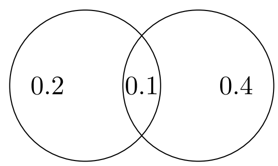
Given A, B, C:

a) none occur: $A^c \cap B^c \cap C^c$ (only A occurs: $A \cap B^c \cap C^c$)

b) all three occur: $A \cap B \cap C$ (exactly one occurs: $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$)

c) at least one occurs: $A \cup B \cup C$ $((A^c \cap B \cap C) \cup (A \cap B^c \cap C))$

$P(A) = 0.3$, $P(B) = 0.7$, $P(A \cap B) = 0.1$, WANTED
$P(A \cup B) = 0.3 + 0.7 - P(A \cap B) = 0.9 - 0.5 = 0.4$
Blue/green/red
$P(A \cup B) = 0.5 + 0.7 = P(A \cap B) = 0.9$
$P(A \cap B) = 0.3$



3.2 Counting Example

Choose a card at random: $P(A) = \frac{3}{7}$, $P(A \cap \text{red}) = \frac{15}{70} = \frac{3}{14}$, $P(A \cup \text{red}) = \frac{4}{7}$

Choose two at random without replacement (conditional probability): if the first card has an A, find $P(\text{second} = A) = \frac{2}{69}$, $P(\{2^{\text{nd}} = A\} \mid \{1^{\text{st}} = A\}) = \frac{1}{51} = \frac{1}{17}$

One card: $P(\text{red} \mid A) = \frac{15}{30} = 0.5$

$P(\text{red} \mid A) = \frac{P(\text{red} \cap A)}{P(A)} = \frac{3/7}{1} = \frac{3}{7}$

4 Conditional Probability

Definition 8. $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Choose two without replacement: $P(\text{both blue}) = P(\{1^{\text{st}} \text{ blue} \cap 2^{\text{nd}} \text{ blue}\}) = P(\text{blue})P(2^{\text{nd}} \text{ blue} \mid 1^{\text{st}} \text{ blue})$
$= \frac{14}{70} \cdot \frac{13}{69} = \frac{182}{4830} = \frac{14}{70} = \frac{1}{5}$

Choose 5 without replacement: $P(\text{at least one is blue}) = 1 - P(\text{no blue}) = 1 - \frac{56 \cdot 55 \cdots}{70 \cdot 69 \cdots}$

Example: Two buckets - 11 blue/1 green, 2 blue/3 green. Choose one from bucket one and move to bucket 2, choose 1 from bucket 2:

$P(1^{\text{st}} \text{ blue} \cap 2^{\text{nd}} \text{ blue}) = \text{WORK}$
$P(1^{\text{st}} \text{ green} \cap 2^{\text{nd}} \text{ blue}) = \frac{1}{4}$
$P(2^{\text{nd}} \text{ blue}) = \frac{6}{12}$

4.1 Total Probability - Given Events

Total probability: Given $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $\bigcup A_i = S$,

then $P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$

5 Conditional Probability (continued)

Find $P(1^{\text{st}} \text{ blue} \mid 2^{\text{nd}} \text{ blue}) = \frac{P(1^{\text{st}} \text{ blue} \cap 2^{\text{nd}} \text{ blue})}{P(2^{\text{nd}} \text{ blue})} = \frac{4/4}{5/12} = 0.6$

See setup for total probability.

Theorem 1 (Bayes' Theorem)**.**

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{P(B \cap A)P(A)}{\sum_{all\ k} P(B \mid A_k)P(A_k)}$$

5.1 Medical Test Example

A test is 95% accurate: $P(\{+\} \mid \text{disease}) = 0.95$, $P(\{-\} \mid \text{not disease}) = 0.95$

If $P(\text{disease}) = 0.01$, find $P(\text{test}+)$:

$$0.01 \cdot 0.95 = (0.01)(0.95) + (1.00)(0.05) = 0.059$$

$P(\text{disease} \mid +) = \frac{P(\text{is} + \cap)}{P(+)} = \frac{(0.01)(0.95)}{0.059} = 0.16$

Roll a die until you get a 6:

$P(\{3\}) + P(\{4\}) + P(\{5\}) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right)$

Geometric sequence: $a + ar + ar^2 + \cdots = \frac{a}{1-r}$ if $|r| < 1$

$a = \frac{1}{6}$, $r = \frac{5}{6}$, so $\left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) = \frac{a \cdot r^2}{1-r} = \frac{(1/6)(5/6)^2}{1/6} = \frac{25}{36}$

Question 2 if $r \geq 1$

P if $A \subseteq B$, then $P(A) \leq P(B)$

6 Homework

6.1 Roll 2 dice

Find $P(\text{sum} = 6)$: $S' = \{2, 3, 4, \ldots, 12\}$, $S_Y = \{0, 1, 2, \ldots\}$

Roll 2 dice: Find $P(\text{exactly 4 green})$

$n = 10$, $k = 4$, $p = \frac{10}{50} = 0.2$, $P(4 \text{ green}) = \binom{10}{4}(0.2)^4(0.8)^6$

Binomial pdf (n, p, k): $n\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Choose 5 without replacement:

6.2 Discrete Random Variables

Example 1. *Roll 2 dice: X = sum of two rolls, $Y = \#$ of 6s*

$S_X = \{2, 3, 4, \ldots, 12\}$, $S_Y = \{0, 1, 2\}$

Example: Roll 1 dice, let $X = (1^{st}$ roll$)$, $Y = \#$ of $\{$

$S_X = \{2, 3, 4, \ldots, 12\}$, $S_Y = \{0, 1, 2, 3\}$

Theorem 2. *A random variable X is a function from a sample set to the real numbers (with an associated probability distribution).*

Example: Roll two dice: X = sum of two rolls, $Y = \#$ of $\{$

$S_X = \{2, 3, 4, \ldots, 12\}$, $S_Y = \{0, 1, 2, 3\}$

Theorem 3. *Associated with a random variable is a probability density function (pdf) which gives the probability of all elements.*

Theorem 4. *The cumulative distribution function (cdf) for X is: $F_X(x) = P(X \leq x)$*

7 Binomial Distribution

Definition 9. $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,2,\ldots,n$

where $n=\#$ of trials, $k=\#$ of successes, $p=P(\text{success in one trial})$, $\binom{n}{k}$ = number of ways choosing k out of n

Example 2. *50 red, 40 red, 10 green*

Choose 10 with replacement: Find $P(\text{exactly 4 green})$

$n=10$, $k=4$, $p=\frac{10}{50}=0.2$, $P(4\ \text{green})=\binom{10}{4}(0.2)^4(0.8)^6$

Binompdf (n,p,k): $n\binom{n}{k}=\frac{n!}{k!(n-k)!}$

Choose 5 without

8 Discrete Random Variables

Example 3. *Roll 2 dice, find $P(sum=5)$*

Choose 10 with replacement: Find $P(\text{exactly 4 green})$

$n=10$, $k=4$, $p=\frac{10}{50}$, $p=0.2$

$P(4\ \text{green})=\binom{10}{4}(0.2)^4(0.8)^6$

9 Continuous Random Variables

Example 4. *Choose a number at random from the interval $[0,100]$ in $\mathbb{R}$:*

$P(X\le 50)=0.5$, $P(X\le 10)=\frac{1}{10}$, $P(X\le 1)=\frac{1}{100}$, $P(X=0)=0$

$P(X\le V_1)=\int_0^{V_1}p_{2-x}dx=\frac{x^2}{2}\Big|_0^{V_1}=\frac{V_1}{4}$

Example 5. $P_X(x)=2x$, $0\le x\le 1$

$P(X\le V_1)=P(X\le V_1)=\int_0^{V_1}$

Definition 10. *For $P_X(x)$ to be a pdf, we need: $f_X(x)\ge 0$, $\int_{-\infty}^{\infty}f_X(x)dx=1$*

Find the cdf $F_X(x)=P(X\le x)=\int_{-\infty}^{x}2xdx=x^2\Big|_0^x=x^2$ for $0\le x\le 1$

Example 6. $f_X(x)=2e^{-2x}$, $x\ge 0$

$P(X\ge 1)\int_1^{\infty}2e^{-2x}dx=-e^{-2x}\Big|_1^{\infty}=0+e^{-2}$

$f_X(x)=cx^2e^{x^3}$, $x\ge 0$

Find c so this is a pdf:

$1=\int_0^{\infty}cx^2e^{x^3}dx=c\left(-\frac{1}{3}e^{-x^3}\right)\Big|_0^{\infty}=\frac{c}{3}(0-(-1))=\frac{c}{3}$

$c=3$

$P_X(x)=3x^2e^{-x^3}$

Find c for pdf: $1=\int_0^{\infty}cxe^{-2x}dx$

$u=-\frac{1}{2}e^{-2x}$, $dv=xdx$

$dv=1$, $dv=e^{-x}$

$c\left[-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\right]\Big|_0^{\infty}=c\cdot\frac{1}{4}=1$, $c=4$

$f_X(x)=2x$, $0\le x\le 1$

If $y=2x+1$, find $P_Y(y)$

Go through: $F_Y(y)=P(Y\le y)=P(2x+1\le y)=P\left(x\le\frac{y-1}{2}\right)$

$=\int_0^{\frac{y-1}{2}}2xdx=x^2\Big|_0^{\frac{y-1}{2}}=\left(\frac{y-1}{2}\right)^2=\frac{(y-1)^2}{4}$, $1\le y\le 3$

$f_Y(y)=\frac{d}{dy}(y-1)^2=\frac{(y-1)}{2}$ pdf

10 Expected Value

Example 7. *We have: 3, 2.5, 4, 3.5, 2, 5.5. Mean:* $\frac{7.5}{20} = 3.75$
$2\left(\frac{3}{20}\right) + 3\left(\frac{8}{20}\right) + 5\left(\frac{2}{20}\right) = \sum_i n_i P(n_i)$

Definition 11. *The expected value of a random variable X is:*

$$E(X) = \begin{cases} \sum_{x \in S} x \cdot P(X = x) & \textit{discrete} \\ \int_{-\infty}^{\infty} x \cdot f_X(x)dx & \textit{continuous} \end{cases}$$

Example 8. $P(X = x) = \frac{1}{10}$, $x = 1, 2, 3, 4$
$E(X) = \frac{1}{10}[1 + 2 + 3 + 4] = 3$
$P(X = 1) = \frac{1}{10}$

Roll a die until the first 6, with # of rolls: Find $E(X)$
$P(X = 1) = \frac{1}{6}$, $P(X = 2) = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$, $P(X = k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$, $k = 1, 2, \ldots$
$E(X) = \sum_{k=1}^{\infty} k \cdot \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$
$\sum ar^k = \frac{a}{1-r}$, $a + 2ar + 3ar^2 + \cdots = \frac{a}{(1-r)^2}$, $E(X) = \frac{1/6}{(1-5/6)^2} = 6$
Binomial: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$
$E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$

10.1 Continuous Example

$f_X(x) = 2x$, $0 \leq x \leq 1$
$E(X) = \int_0^1 x(2x^2)dx = \frac{2}{3}$
$f_X(x) = 2e^{-2x}$, $x \geq 0$
$E(X) = \int_0^{\infty} x \cdot 2e^{-2x}dx \Rightarrow \hat{\lambda} = \frac{1}{2}$
Toss a coin until the first T, if it's on 1st win 1
$1 \cdot P(T \text{ on } 1^{\text{st}}) + 2 \cdot P(T \text{ on } 2^{\text{nd}}) + \cdots = 2^n E(W_n) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + \cdots + 2^n\left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} 1 = \infty$
$f_X(x) = \frac{1}{2}$, $x \geq 1$
Show this is a pdf:
$1 = \int_1^{\infty} \frac{c}{x^2}dx = -\frac{c}{x}\Big|_1^{\infty} = 0 + 1 = 1 \checkmark$
$E(X) = \int_1^{\infty} x\left(\frac{1}{x^2}\right)dx = \ln(x)\Big|_1^{\infty} = \infty$

11 Median

Definition 12. *The median m is the number so that $P(X \leq m^*) \geq 0.5$ and $P(X \geq m^*) \geq 0.5$.*

Example 9. $P_X(x) = 3x^2$, $0 \leq x \leq 1$
$0.5 = \int_0^{m^*} 3x^2dx$, *so* $0.5 = x^3\Big|_0^{m^*} = (m^*)^3$, *thus* $m^* = \sqrt[3]{0.5}$

Notation: $E(X) = \mu$, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ (population mean vs sample mean)

Note 1. $E(g(x)) = \sum g(x)P(X = x)$ *or* $\int_{-\infty}^{\infty} g(x)P_X(x)dx$

Example 10. $x = [-1, 1, 3]$

$$E(X) = -0.3 + 0.4 + 0.9 = 1$$

$E(\cos(x)) = \cos(-1)(0.3) + \cos(1)(0.2) + \cos(3)(0.3)$
$E(X) = \int_0^1 3x^3dx$, $E(x^2) = \int_0^1 x^2(3x^2)dx$

12 Variance

Definition 13.

$$Var(X) = E[(X-\mu)^2] = \sigma^2 \qquad 1,2,5 \Rightarrow \sigma^2 = 2 \qquad Var = 4$$

Theorem 5. $E[(X-\mu)^2] = Var(X) = E(X^2) - \mu^2$

Note 2. $E(aX+b) = aE(X)+b$

13 Variance Examples

Example 11 (Finite discrete dist.)**.** $x = 1,2,3,4,\ E(X) = \frac{1}{4}(1+2+3+4) = 2.5$
$Var(X) = \frac{1}{4}[(1-2.5)^2 + (2-2.5)^2 + (3-2.5)^2 + (4-2.5)^2] = 1.25$

Example 12 (Roll die)**.** $E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$
$E(X^2) = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$
$Var(X) = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$

Example 13 (Binomial)**.** $E(X) = np,\ Var(X) = np(1-p)$
Roll 600, k $\neq$ *of 6's:* $E(X) = 100,\ Var(X) = 600 \cdot \frac{1}{6} \cdot \frac{5}{6}$

14 Further Properties of Mean and Variance

Theorem 6 (Lemma)**.** $E(aX+bY) = aE(X) + bE(Y)$

Useless: $\sigma = S(+) + EY + \sqrt{VAR}$

Definition 14. $\bar{X} = \frac{1}{n}\sum X_i$ *where* X_i *are from a list with* $E(X_i) = \mu$
$E(\bar{X}) = \frac{1}{n}\sum E(X_i) = \mu$

Note 3. $Var(\bar{X}) = E[(\bar{X}-\mu)^2] = E\left[\left(\frac{1}{n}\sum X_i - \mu\right)^2\right] = E\left[X^2\right] - \mu^2$

Note 4. *If* X *and* Y *are independent,* $Var(X) + Var(Y)$

$\mathrm{Var}(aX+bY)$ if X and Y are independent: $a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$
$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$ if X and Y are independent
$\mathrm{Var}(\bar{X}) = \sigma^2$ where $\sigma^2 = \mathrm{Var}(X_i) = \frac{\sigma^2}{n}$
Notation (if points equally likely): $\sigma^2 = \frac{1}{n}\sum(X_i - \mu)^2$ = population variance
$s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$ = sample variance

Y	1	2	3	4	5	6
X	1	1	1	1	1	1
1	$\times$					
2	$\times$	$\times$				
3	$\times$	$\times$	$\times$			
4	$\times$	$\times$	$\times$	$\times$		
5					$\times$	
6						$\times$

PMF with respect to Y

15 The k-th Moment

Definition 15. *The k-th moment of* X *is* $E(X^k)$

Notation (if points equally likely): $\sigma^2 = \frac{1}{n}\sum(X_i - \mu)^2$ = population variance = standard
$s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$ = sample var

16 Joint Densities

Example 14. *$f_{X,Y}(x,y) = \frac{2}{3}xy$, $0 \le x \le 1$, $0 \le y \le 2$, $x + y > 1$*
Find $E(XY) = \int\int (xy)\left[\frac{2}{3}xy\right] dydx = \frac{2}{3}\int_0^1 \int_{1-x}^2 x^2y^2 dydx$
Find $F(X) = \int_0^2 \frac{2}{3}xydy = \frac{2}{3}xy\Big|_{y=0}^{2} = \frac{2}{3}x(2) - \frac{2}{3}x(0) = \frac{4}{3}x$, $0 \le x \le 1$
Expected $f(x)$: $P(Y > X) = 1 - P(Y \le X) = 1 - \int_0^1 \int_0^x \frac{2}{3}xydydx$
$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$
Find $E(XY) = \sum\sum(xy)P(X,Y) = (.5)(2)(.5) + 1.5$
$E(Y) = 1(.3) + 1(.6) + 2(.1) = .5$
Find $E(XY) = (.1)(.1) + (.2) + 2(.2) + 2(.3) + 2(.2) + 2(.3) = .6$

Definition 16. *The covariance is the $Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - E(X)E(Y)$*

Are the sets $\{x = 13\}$ and $\{y = 13\}$ independent? $P(\{x = 13\} \cap \{y = 13\}) = P(\{x = 13\})P(\{y = 13\})$
$P(\{x = x\})$

Theorem 7. *The random variables X and Y are independent if $P(X \in A \cap Y \in B) = P(X \in A)P(Y \in B)$ for all sets $A \in S_X$, $B \in S_Y$*

If X and Y are independent, $E(XY) = E(X)E(Y) \Rightarrow \mathrm{Cov}(X,Y) = 0$
Lemma: The continuous random variables X and Y are independent if $f_{XY}(x,y) = g(x)h(y)$ where $g(x) = kf_X(x)$ and $h(y) = \ell f_Y(y)$ for all $x \in S_X$, $y \in S_Y$

Example 15. *$f_{XY}(x,y) = \frac{1}{33}xy$, $0 \le x \le 1$, $0 \le y \le 2$, $x + y > 1$*
Are X and Y independent? NO - need 5 to be rectangle

Example 16. *$f_{XY}(x,y) = d(x \cdot y)$, $0 \le x \le 2$, $0 \le y \le 5$*
$f_{XY}(x,y) = \frac{1}{(x+y)}p \ne h(x)g(y)$ NO

Example 17. *$f_{XY}(x,y) = e^{-(x+y)}$, $x \ge 0$, $y \ge 0$*
$= e^{-x} \cdot e^{-y} = g(x)h(y)$ YES

Example 18. *If Var $f_X(x) = 4x^3$, $0 \le x \le 1$, $f_Y(y) = 3y^2$, $0 \le y \le 1$*
Find $P(Y > X)$: If Y and Y are independent
$f_{XY}(x,y) = 4x^3(3y^2) \Rightarrow$
$\int_0^1 \int_x^1 (2x^3y^2)dydx = \int_0^1 4x^3y^3\Big|_x^1 dx$
...

17 Further Properties of Mean and Variance

Lemma 3. *$E(aX + bY) = aE(X) + bE(Y)$*

Definition 17. *$\bar{X} = \frac{1}{n}\sum X_i$ where X_i are from a list with $E(X_i) = \mu$*
$E(\bar{X}) = \frac{1}{n}\sum E(X_i) = \mu$
Note: $Var(\bar{X}) = E[(\bar{X} - \mu)^2] = E[(X_i - \mu)]^2 = E[X^2] - \mu^2$

Note 5. *If X and Y are independent, $Var(X + Y) = Var(X) + Var(Y)$*

$\mathrm{Var}(aX + bY)$ if X and Y are independent $= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$
$\mathrm{Var}(\bar{X})$ for X and Y independent: $\sigma^2 = \mathrm{Var}(X_i) = \frac{1}{n^2}[\sigma^2 + \cdots + \sigma^2] = \frac{\sigma^2}{n}$
$\mathrm{Var}(N) = 100\sigma^2(X) - 36\mathrm{Var}(Y) - 110\sigma^2(X) + 36\mathrm{Var}(Y)$ if X and Y are *ind*

Definition 18. *The correlation coefficient is: $\rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$, $-1 \le \rho \le 1$*

PPP:

1. $-1 \leq \rho \leq 1$
2. $\rho = \pm 1 \Leftrightarrow y = a + bx$
3. $\rho = 0$ no linear association
4. ρ gives the percent of the variation in y due to the change in x

Example 19 (20)**.** *$X \sim Binom(n, p_x)$, $Y \sim Binom(m, p_y)$, $U = X + Y$*
Find $E(U)$, $Var(U)$
$E(X) = np_x$, $E(Y) = mp_y$, $Var(X) = np_x(1-p_x)$, $Var(Y) = mp_y(1-p_y)$
$E(U) = 4E(X) + 6E(Y) = 4np_x + 6mp_y$
$Var(N) = 10\sigma^2(X) - 36\,Var(Y) = 110p_x(1-p_x) + 36mp_y(1-p_y)$ if X and Y are ind

18 The Poisson Distribution

Definition 19. *The Poisson distribution is: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k \in \mathbb{N}$ (or $k = 0, 1, \ldots$)*
$E(X) = \lambda = Var(X)$

This usually gives the number of occurrences in a time interval. Something is Poisson if:

1. The occurrences in Δt happen simultaneously
2. For a short enough time frame, there are 0 or 1 occurrences
3. Occurrences are independent
4. The rate of occurrences is constant (only matters well if a Δt-age unit 2 small)

Example 20. *A TV has 6 million pixels, where the probability that one is defective is $\frac{1}{2}$ mill. λ = mean = 5 A mill/2 mill = 2.5*
$P(3 \text{ are defective}) \approx e^{-2.5}\frac{(2.5)^3}{3!} = P(Poissonp(\lambda = 2.5, k)) = 0.2136$

Find $P(X \geq 3)$: WORK = Poisson $p(\lambda = 2.5, k)$

Example 21. *The number of people who walk into a store is Poisson with a mean of $\frac{50}{1\ hour}$. Find $P(\text{at least in one hour with } Poissonp(\lambda = 2.5, k))$ or $P(3 \text{ in } 10 \text{ mins})$*
$\lambda = 3$, Poisson $p(\lambda = 5, k) = 0.104$

Note: If the number of occurrences is Poisson, then the time between occurrences is exponential: $P(X = k) = \lambda e^{-\lambda k}$ (ECN $\Rightarrow X$)

Example 22. *The mean is 492 for two hours: $\lambda = 492$ (for 2 hours)*
Find λ (in 2 min): $\lambda = \frac{492}{60} \approx 8.0333$, $\lambda^{492}/60 = 8.0333$
Therefore Poisson $p(\lambda = 492/60, k) = 0.029$ or $P(E3;\ \text{in } 2\ \text{min}) =$ Poisson $p(\lambda = 492/60, k) = 0.014$

Find $P(E3;\ \text{in 2 min}) = \text{Poisson } p(\lambda = 492/60, k)$

Example 23. *20000 People play, if $P(win) = \frac{1}{5000}$. Find $P(2 > \text{People win})$*
$\lambda = \frac{20000}{5000} = 4$, $1 - P(X \leq 3) = 1 - PoissonCDF(4, 2) = \overline{7}(2)$

19 The Normal/Gaussian Distribution

Definition 20. *Normal/Gaussian distribution:* $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$

Standard: $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, *human*

Example 24. *Find* $P(2 \leq (1.2)) = \textit{Normal CDF}(2, 100000, 1.2) \cap \mu = 0 \Rightarrow P = 0.8849$

Lemma: *If* X *is Normal with* $\mu = 100$, $\sigma = 10$

Find $P(2 \leq (1.2))$ *If we take a sample of 100* $P(\bar{X} > 52) = \textit{Normal CDF}(52, 100000, 50)$

Find a so: $P(\mu = k \leq \mu) \Rightarrow \textit{inv Normal CDF}(.75, 100, 10, \textit{center})$

HW 3

Bernoulli: $P(X = 1) = p$, $P(X = 0) = 1 - p$

a) Find $E(X)$ and $\text{Var}(X)$

$E(X) = \sum xP(X = x)$ and $\text{Var}(X)$

b) Let $Y = X_1 + X_2 + \cdots + X_n$ when X_i = Bernoulli with $P(X_i = 1) = p$ and

$E(Y) \cdot \sum xP(X = x)$

$\text{Var}(Y) = \text{Var}(X_1 + \cdots) = np(1 - p)$

$\text{Var}(Y) = \text{Var}(2X_i) \Rightarrow \text{Var}(\bar{X}_i)^2$

20 Central Limit Theorem

Given a sample $X_1, X_2, X_3, \ldots X_n$ all from the same population taken independently, then $\bar{X} = \frac{1}{n}\sum X_i$ and the sum $\sum X_i$ are normal

No outliers or strong skewed normal converges under sample $E(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}) \Rightarrow$ so $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}}$, $\text{Var}(\bar{X}) = n\sigma^2$

For X we know $\mu = 50$, $\sigma = 8$

Find $P(\bar{X} > 52) =$? If we take a sample of 100 $P(\bar{X} > 52) = \text{Normal CDF}(52, 10000, 50, 30)$

Find $P(X > 52) =$? Find a $P(\mu \leq k \leq \mu) \Rightarrow a$ with $P(X \geq 52) = \text{Normal CDF}(52, 10000, 50, 30) = 0.0062$

Find a so that $P(\mu \leq k \leq \mu)$: inv Normal $(1.75, 100, 10, \text{center})$

Example 25. X *is normal with* $\mu = 12$, $\sigma = 42$

Find $P(X \leq 150) = \textit{Normal CDF}(10000, 130, 12, 41) = 0.0655$

With a sample of 400: Find $P(10 \leq \bar{X} \leq 1(2)) = \textit{Normal CDF}(112, 110, 112, 41/\sqrt{400})$

Example 26. *Roll a dice 600 times, estimate* $P(90 \leq \#\ \textit{of}\ 6\textit{'s} \leq 115)$*: Mean* $= np = 106$

variance $= np(1 - p) = \frac{500}{6} \approx 71.27$

continuity correction: $P(\bar{X} < 70) \Rightarrow P(\bar{X} \leq 70.5) \Rightarrow$

$P(10 \leq k \leq 115) = P(90.52k \leq 115) = \textit{Normal CDF}(115, 90.5, 115.5, 70.5, 100.5)$

$k < 100 \Rightarrow X \leq 100.5$, $X \geq 100 \Rightarrow X \geq 99.5$