DRP Notes

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1 Functions

1.1 Definitions

Definition 1. *A **Function** is defined by a subset of $A \times B$:*

$$\Gamma_f := \{(a, b) \in A \times B | b = f(a)\} \subseteq A \times B$$

This set Γ_f is the graph of f; a function is fully represented by its graph.
Functions are required to follow $(\forall a \in A)(\exists! b \in B)\ f(a) = b$

Identity function: $id_A : A \to A$ or $(\forall a \in A)\ id_A(a) = a$

1.2 Indexed Sets

An indexed set $\{a_i\}_{i \in I}$ is informaly defined as a_i for i ranging over some set of indicies I. The more formal definition is a function $I \to A$ where A is some set from which we draw the elements a_i.

1.3 Composition of functions

Functions may be composed if $f : A \to B$ and $g : B \to C$ are functions, then so is the operation $g \circ f$ defined by:

$$(\forall a \in A)\ (g \circ f)(a) := g(f(a))$$

Composition is commutative and associative.

1.4 Injections, surjections, bijections

- A function $f : A \to B$ is *injective* if $(\forall a' \in A)(\forall a'' \in A)\, a' \neq a'' \implies f(a') \neq f(a'')$. That is, if f sends different elements to different elements.
- A function $f : A \to B$ is *surjective* if $(\forall b \in B)(\exists a \in A)\, b = f(a)$. That is, if f 'covers the whole of B' (im f = b)

Injections are often drawn $\hookrightarrow$; surjections are often drawn $\twoheadrightarrow$.
If f is both injective and surjective, we say it is *bijective* or and *isomorphism of sets.* Where we write $\cong$

1.5 Injections, surjections, bijections: Second viewpoint

If $f : A \to B$ is a bijection, than we can 'flip its graph' to define a function $g : B \to A$.
Assume $A \neq \emptyset$, and let $f : A \to B$ be a function:

1. f has a left-inverse if and only if it is injective.
2. f has a right-inverse if and only if it is surjective.

This implies a function $f : A \to B$ if a bijection if and only if it has a two-sided inverse. ??.

1.6 Monomorphisms and epimorphisms

There is another way to express injectivity and surjectivity.
A function $f : A \to B$ is a *monomorphism* if the following holds:
for all sets Z and all functions $\alpha', \alpha'' : Z \to A$
$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''$

1.7 Excercises

Excercise 1. *Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.*
First to prove the inverse of a bijective function f is injective:
By Proposition 2.1, b

2 Section 3: Categories

2.1 Definition

A category consists of a collection of 'objects' and of 'morphisms" between objects, satisying a list of conditions.
Categories are explicitly not sets, as we would like to create a category of all sets and a set cannot contain all sets. [1] While a collection doesn't really have a formal definitions, *class* is used to deal with collections of sets. In some cases a class is a set (and is called small).

Definition 2. *A* <u>*category*</u> *C consists of:*

- *a class Obj(**C**) of objects of the category*
- *for every two objects A, B of **C**, a set Hom$_{\boldsymbol{C}}$(A,B) of morphisms, with the properties listed below*

Think of objects as sets and morphisms as functions. Morphisms have these properties:

- For every object A of C, there exists at least one morphism $1_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$, the 'identity' on A.
- One can compose morphisms: two morphisms $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$ determine a morphism $gf \in \mathrm{Hom}_C(A, C)$. That is, for every triple of objects A, B, C of **C** there is a function where:

$$\mathrm{Hom}_{\mathbf{C}}(A, B) \times \mathrm{Hom}_{\mathbf{C}}(B, C) \to \mathrm{Hom}_C(A, C)$$

[1] This might be because a set of all sets must include itself which violates ZF axiom of foundation. The text says this is because of Russel's Paradox which seems to be an alternative to ZF.