Theory of Computation Notes

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Contents

1 September 9 **2**

2 September 12 **4**

3 September 19 **5**

4 September 26 **6**
- 4.1 Generalized NFA 6
- 4.2 Non-Regular Languages (Pumping Lemma) 8

5 October 3 **10**
- 5.1 Turing Machines 10
  - 5.1.1 David Hilbert's Descision Problem 10

6 October 7 **12**
- 6.1 Building a Turing Machine 12

7 October 10 **13**
- 7.1 Language of a TM 13
- 7.2 Specifiying a Turing Machine 13
- 7.3 Beyond Boolean Functions 14

8 October 14 - Turing Machine Variants **15**

9 October 21 **17**
- 9.1 Universal Turing Machine 17
- 9.2 Non-deterministic TM 17

10 October 28 **18**
- 10.1 Comparing Infinities 18
- 10.2 Uncountablility 18
- 10.3 Undecidability 18
- 10.4 The TM Self-Acceptance Problem 19
- 10.5 Undecidability as Diagonalization 19
- 10.6 Reductions in Undecidability 19
  - 10.6.1 Halting Problem 19

11 November 7 **21**
11.1 Proof Systems 21
11.2 Gödel's Sentence 21

12 Exam 2 Prep November 13 **22**

1 September 9

Definition 1. *A function $f : D \to R$ has domain D and range R. Each input $x \in D$ is mapped to exactly one output $f(x) \in R$.*

Example 1. *The function $add : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined by*

$$add(x, y) = x + y.$$

Goal of Computation

We focus on computing functions $f : \Sigma^* \to \{\text{accept}, \text{reject}\}$.

- **Domain:** strings over alphabet Σ.
- **Range:** Boolean $\{0, 1\}$ or $\{\text{accept}, \text{reject}\}$.

Why strings? Any input can be encoded as a string. Why booleans? Simplicity, while still capturing many interesting functions.

Functions as Languages

A language L over Σ is a subset of Σ^*. Example: $L = \{w \in \{0, 1\}^* : w \text{ ends with } 1\} = \{1, 01, 11, 001, 101, \dots\}$.

Equivalence between functions and languages:

$$f \leftrightarrow L \quad \text{where} \quad L = \{w : f(w) = \text{accept}\}.$$

Observation

Languages may be finite or infinite, but a "program" is always a finite description.

Finite Automata

A **deterministic finite automaton (DFA)** consists of:

- States (nodes).
- Transitions labeled by alphabet symbols.
- Unique start state q_0.
- Accept states (double circles).

Definition 2. *A DFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where:*

- Q = *finite set of states*
- Σ = *alphabet*
- $\delta : Q \times \Sigma \to Q$ = *transition function*
- $q_0 \in Q$ = *start state*
- $F \subseteq Q$ = *accepting states*

Definition 3. *The extended transition function* $\delta^* : Q \times \Sigma^* \to Q$ *is defined by:*

$$\delta^*(q, \epsilon) = q, \quad \delta^*(q, wa) = \delta(\delta^*(q, w), a).$$

2 September 12

Theorem 1. *If A is regular, then so is its complement A^c.*

Definition 4. *A nondeterministic finite automaton (NFA) is a tuple $M = (Q, \Sigma, \gamma, q_{start}, F)$ where transitions may be nondeterministic or labeled with ϵ.*

An NFA accepts w if there exists some computation path leading to an accept state.

3 September 19

Theorem 2. *If A, B are regular languages, then so is $A \cup B$.*

Theorem 3. *If A is a regular language, then A^* is regular.*

Regular Expressions

Definition 5. *A regular expression (RE) over Σ is defined inductively:*

- *Atomic: $\emptyset$, ϵ, or $a \in \Sigma$.*
- *If R_1, R_2 are REs, then so are:*

$$(R_1 \cup R_2), \quad (R_1 R_2), \quad (R_1^*).$$

Given regular expressions R and S, the following operations over them are defined to produce regular expressions:

- <u>Concatenation</u> (RS): denotes the set of strings that can be obtained by concatenating a string accepted by R and a string accepted by S (in that order). For example, let R denote {"ab", "c"} and S denote {"d", "ef"}. Then, (RS) denotes {"abd", "abef", "cd", "cef"}.
- <u>Alternation</u> (R|S) denotes the set union of sets described by R and S. For example, if R describes {"ab", "c"} and S describes {"ab", "d", "ef"}, expression (R|S) describes {"ab", "c", "d", "ef"}.
- <u>Kleene Star</u> (R*) denotes the smallest superset of the set described by R that contains ε and is closed under string concatenation. This is the set of all strings that can be made by concatenating any finite number (including zero) of strings from the set described by R. For example, if R denotes {"0", "1"}, (R*) denotes the set of all finite binary strings (including the empty string). If R denotes {"ab", "c"}, (R*) denotes {ε, "ab", "c", "abab", "abc", "cab", "cc", "ababab", "abcab", ...}.

Definition 6. *The semantics of a RE R are given by its language $L(R)$:*

$$L(\emptyset) = \emptyset, \quad L(\epsilon) = \{\epsilon\}, \quad L(a) = \{a\}.$$

$$L(R_1 \cup R_2) = L(R_1) \cup L(R_2), \quad L(R_1 R_2) = L(R_1) L(R_2), \quad L(R^*) = (L(R))^*.$$

Theorem 4. *A language A is regular $\iff$ there exists a DFA, NFA, or regular expression R such that $A = L(R)$.*

4 September 26

4.1 Generalized NFA

Definition 7. *A Generalized NFA (GNFA) has transitions labeled with regular expressions. Can follow a transition by reading a matching block of input symbols.*

We can convert any GNFA into a "simple form":

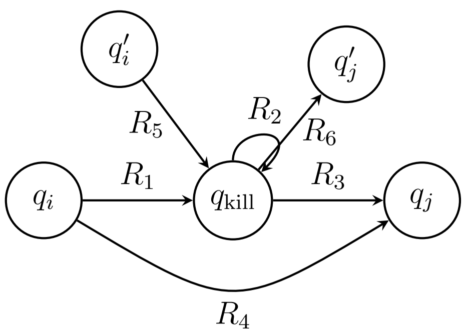
1. Accept state is unique. No transitions leaving accept state. Add new accept state, ε-transitions from old accept states to new one.

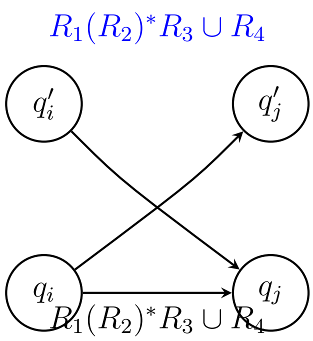
2. No transitions entering the start state. Add new start state, ε-transitions from new start states to old one.

3. At most 1 transition between each pair of states. If multiple transitions between two nodes labeled $R_1, R_2, ...$, replace with transition $R_1 \cup R_2 \cup ...$

Theorem 5. *A is Regular $\Rightarrow$ $\exists$ reg. ex. $R : A = L(R)$.*

- Start with DFA D that recognizes A.
- Convert it to GNFA G that has "simple form".
- **Remove intermediate states of G one at a time without changing which language it recognizes.**
  - **Remove state q_{kill}**
  - **For each pair of states q_i, q_j connected through q_{kill} as shown, add a transition from q_i to q_j labeled with RE .**
  - **Continue to combine edges with $\cup$ until the GNFA is just a single RE**





Regular Expressions In Practice:

- Used widely for pattern-matching, searching.
  - Specify the format of a string such as a phone number, address, credit card number, license plate ...
  - Search a document to see if it contains some credit card number.
- What algorithm is used:
  regex $\to$ NFA $\to$ DFA

4.2 Non-Regular Languages (Pumping Lemma)

Not all languages are regular. Consider: $L = \{0^n 1^n : n \geq 0\}$ (e.g. $0011 \in L$ but $001 \notin L$)

Theorem 6. *Intuition: A DFA for L would need to remember how many 0's it has seen. Not enough memory*

Our Plan:

- Direct proof that $L = \{0^n 1^n : n \geq 0\}$ is not regular
- Generalize the above ideas to "pumping lemma"
- Use "Pumping Lemma" to prove several other languages are not regular.

Proof. 6 Assume by contradiction there is a DFA M that recognized L. Let number of states in M $= p$.
Let $r_k \in Q$ be the state M reaches after reading 0^k.
Then for some $0 \leq i \leq j \leq q$ we have $r_i = r_j$.
M must accept $0^i 1^i \in L$.
$\Rightarrow$ M accepts $0^j 1^i \notin L$. □

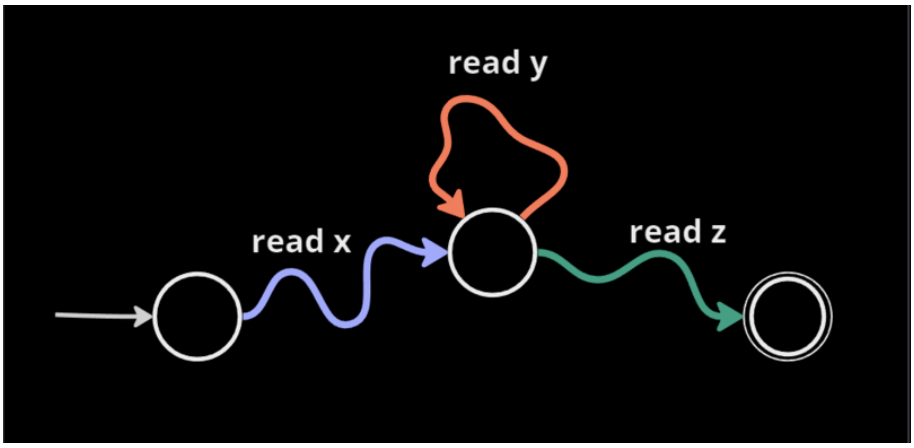
Lemma 7. *Pumping Lemma:*
If L is a regular language, then:
$\exists p \in \mathbb{Z}, p \geq 0$ *(pumping length)*
$\forall$*strings* $w \in L$ *of length* $|w| \geq P$
$\exists$ *strings* $x,y,z : w = xyz, |y| > 0,\ |xy| \leq p$
$\forall i \in \mathbb{N}, xy^i z \in L.$

To prove L is not regular, use contrapositive:

Definition 8. *Contrapositive:* $A \Rightarrow B = \neg B \Rightarrow \neg A$

Recall De Morgan's law:

Definition 9. $\neg \forall x \phi(x)$ *is the same as*



Recall that the pumping lemma requires you to give a strategy for winning the following game:

- The adversary chooses some integer p.
- You choose a string $w \in L$ such that $|w| \geq p$.
- The adversary chooses strings x, y, z such that $w = xyz$ and $|xy| \leq p, |y| \geq 1$.
- You choose an integer i and you win if xy^iz is not in L.

Therefore, for each language you ONLY need to answer the following questions (be concise):

1. What's your strategy for choosing the string w in the above game?
2. What's your strategy for choosing the integer i in the above game?
3. How do you know that xy^iz is not in L?

To show L is not regular, show:

1. $\forall p \in \mathbb{N}$
2. $\exists$ strings $w \in L$ of length $|w| \geq P$
3. $\forall$ strings $x, y, z \,:\, w = xyz, |y| > 0, |xy| \leq p$
4. $\exists i \in \mathbb{N}, xy^iz \notin L$

5 October 3

5.1 Turing Machines

"If you want to learn anything about automata you can just ask chatGPT"
A Turing machine is a General Model of Computation

- **Algorithms have been around since dawn of time.**
  - Long addition, multiplication, division.
  - Compass and straightedge constructions
  - Euclid's greatest common divisor algorithm
  - Quadratic formula: finding roots of polynomials
- Traditionally, algorithms were understood as a human construct. No precise mathematical definition.

Already saw a limited notion of algorithms (DFA). Using the pumping lemma, we proved that there are some problems that are not computable in this model.

5.1.1 David Hilbert's Descision Problem

In 1928, David Hilbert asked for an "algorithm" that takes as input a mathemattical statement and decides whether the statement is true or false.
During the years 1931-1936, a series of works showed there is no algorithm for the decision problem.
Each of these works included a different definition of a "general algorithm".

- Kurt Godel relied on recursive functions.
- Alonzo Church developed λ-calculus.
- Alan Turing developed the Turing Machine.

All of these definitions turn out to be equivalent.
Turing Machines are perhaps the most intuitive. They provided inspiration for a general computer, the Von Neumann Architecture

Turing Machines cont.

Our Plan

- Define Turing Machines (TM). See how they work.
- Convince ourselves that TMs are powerful enough to implement any "reasonable algorithm".

A TM is like a DFA with infinite memory tape. Information can be saved and accessed using the tape instead of the DFA's state space.

- Initially, tape contains the input, followed by "blanks". The tape head is at the left-most position.
- In each step, the machine can overwrite the symbol under the tape-head and move the tape left or right.
  - The tape head cannot move left of the start.
  - TMs can use additional symbols to write to tape.
- At any point in time, the machine can halt the computation and accept or reject. (If there is no decision edge at the state the head is on, also reject)
- This is implemented via states and transitions like a DFA

Definition 10. *A Turing Machine consists of a tuple:*
$M = (Q, \Sigma, \Gamma, \delta, q_{start}, q_{accept}, q_{reject})$ *Where*

- Q *is a finite set called the states.*
- Σ *is an input alphabet.*
- Γ *is the tape alphabet such that* $\Sigma \subseteq \Gamma$ *and* Γ *contains a special blank symbol '-' that is not in* Σ*.*
- $q_{start} \in Q$ *is the start state.*
- $q_{accept} \in Q$ *is the accept state,* q_{reject} *is the reject state.*
- $\delta : Q' \times \Gamma \to Q \times \Gamma \times \{L, R\}$ *is the transition function.*
  Where $Q' = Q \setminus \{q_{accept}, q_{reject}\}$

6 October 7

6.1 Building a Turing Machine

Definition 11. ***Configuration*** *encodes all information about a particular step in the computation of a turing machine.*
All information:

- *Current state*
- *Content on the tape*
- *Tape-head position*

Let $M = (Q, \Sigma, \Gamma, \delta, q_{start}, q_{accept}, q_{reject}$ *be a Turing Machine.*

- *A* ***configuration*** *of* M *is a tuple* $C = (u, q, v)$ *such that* $u, v \in \Gamma^*$ *and* $q \in Q$*. Can write* $C = uqv$ *without commas.*
- *A configuration* C ***yields*** C' *if* M *goes from* C *to* C' *in 1 step.*
  - $C = (u, q, bw)$ *yields* $C' = (ub', q', w)$ *if* $\delta(q, b) = (q', b', R)$
  - $C = (ua, q, bw)$ *yields* $C' = (u, q', ab'w)$ *if* $\delta(q, b) = (q', b', L))$
  - $C = (q, bw)$ *yields* $C' = (q', b'w)$ *if* $(q, b) = (q', b', L)$ *(don't fall off)*
- *A start configuration of* M *on input* W *is* $q_{start}W$
- *An accepting (resp. rejecting) configuration is one where the state is* q_{accept} *(resp.* q_{reject}*).*

M accepts (resp. rejects) w if there is a sequence of configurations $C_1, C_2, ..., C_n$ *such that:*

- C_1 *is the start configuration of* M *on input* w*.*
- C_i *yields* C_{i+1} *for* $i = 1, ..., n-1$*.*
- C_n *is an accepting (resp. rejecting) configuration.*

This is a way to save your current state for later.
If $C = (ua, q, bv)$, $\delta(a, b) = (q', c, L)$, then $C' = (u, q', acv)$
If $\delta(a, b) = (q', C, R)$ then $C' = (uac, q', v)$

7 October 10

7.1 Language of a TM

- A TM M on input w can either accept, reject, or loop
- For a TM M, we define $L(M) = \{w|M \text{ accepts } w\}$
- If TM M and a language L satisfies "for any $x \in L$, M accepts x", we cannot say "M recognizes L". We must prove that $L(M) = \{w|M \text{ accepts } w\}$
- Do this using $\subseteq$ (M accepts any string in L) and $\supseteq$ (If a string is accepted by M, the string is in L).
- We say the M **decides** L if
  - M accepts $w \in L$ and M rejects $w \notin L$.
  - equivalently: M recognizes L and M always halts.
- A language L is **recognizable** (resp. **decidable**) if there is some TM that recognizes (resp. decides) L.
- The set of all languages decided by turing machines is a subset of all languages recognized by turing machines.

7.2 Specifiying a Turing Machine

Instead of drawing a state diagram, we give a "tape-head" level description that abstracts out the states/transitions via pseudocode.

- Imagine tape-head has small local memort which is "fixed" and cannott grow with input size (states of TM).
- Describe how the tape-head should **walk across the tape** and **what it should write**.

Example 2. $L = \{a^{2n}|n \geq 0\}$ *all powers of 2.*
Walk tape-head from left to right and cross out any other a.

- *If tape contained a single 0, accept.*
- *Else if number of 0s was odd, reject.*
- *Else return to the left-hand end of tape, repeat.*

Example 3. $L = \{w\#w : w \in (0,1)^*\}$

- *Check input is of form* $\{0,1\}^*\#\{0,1\}^*$ *and reject otherwise.*
-

7.3 Beyond Boolean Functions

We can also consider TM's that output more than just "accept/reject"
Idea: define the output of a TM as the contents of its tape when it enters a halt state.
A TM M computes a function $f : \Sigma^* \to \Sigma^*$ if on every input $w \in \Sigma^*$ the TM **halts** and its tape contains f(w).

Definition 12. *We say that f is **computable** if some TM M computes it.*

Example 4. $f : \{0,1\}^* \to \{0,1\}^*$*: f(binary rep of n) = binary rep of n + 1. f is the binary successor function.*
Show that this is computable.
Given any input string such as '01101101111':
Move to the end of the string and continue left until the beginning. Flip all 1's until the first 0, flip the first 0.
To prevent crashing off the beginning of the tape if all ones: convert first character to a # if it is a 1 and add a zero on the end if it is reached.

8 October 14 - Turing Machine Variants

Multi-Tape TM

A TM with one input tape and multiple work tapes. Transition function is defined by $\delta = Q' \times T \to Q \times T \times \{L, R\}$. $\delta(g, w) = (g', w, L/R)$.

- You can concatenate multiple tapes to one tape and separate their contents by #.
- Remember tape-head positions by storing an <u>underlined</u> version of tape symbols.
- Each step of multi-tape TM is simulated by scanning entire tape of single-tape TM.
- If you run out of space on the tapes, shift all elements to the right. (halting problem)

Tape-Head Level Description:
f(a, b) = a + b. Input (binary interpretations of two integers separated by a #)

- Reverse each input and copy each one to a different tape and clear main tape.
- Return all tape heads to the left. Store 1-bit carry as 0
- Add two bits under each tape head (using 0 if one head is empty) and carry bit:
  - Write result mod 2 to the main tape.
  - Move all tape heads 1 right.
  - Repeat until both heads are empty.
- Reverse main tape.

Random Access TM

Can read/write to arbitrary locations in memory without scanning a tape. Memory modeled as infinite array R.

- In addition to the standard tape that contains the input the TM has location and value tapes.
- There is a special write transition which sets R[location] = value using the content of the tapes.
- There is a read transition which sets the contents of the values tape to R[location]

Compliling to normal TM:
We use a multi-tape machine (which can be converted to a single-tape)

- Store contents of array R on a tape as tuples (location$_n$, value$_n$).
- To simulate a read, scan R until find a location that matches content of location tape. Write the value on the value tape. Put a blank if no such value is found.
- To simulate a write, scan R until you find location that matches content of location tape. Update value. If none found, append (location, value) to end of R.

Turing Completeness

Theorem 8. *Church-Turing Thesis: Any algorithm (in an informal sense) can be computed by a TM.*

Proof Outline:
Design a compiler that converts Java program into a TM.

- All programming languages are already compiled to "assembly code" for modern CPUs
- Assembly code instructions can be implemented on a Random-Access TM.

9 October 21

For each object O, let $< O >$ be a string that encodes O.

- If i is an integer $< i >$ can be its representation
- If O is a string, then $< O >$ is just O itself.
- If G is a graph $< G >$ can be defined in many ways; such as vertex lists, etc
- If M is a TM, we can define $< M >$ by writing down the formal definition. $M = (Q, \Sigma, \Gamma, \delta, q_{start}, q_{accept}, q_{reject})$

Now we will consider more complex languages, i.e. primes, graphs, etc.

9.1 Universal Turing Machine

- There is a Turing Machine M_{UNIV} that can run any other TM.
- $M_{\text{UNIV}}(< M >, w)$. Takes as input a description of any TM M and any string W. Runs M on w.
  - If M accepts w, then $M_{\text{UNIV}}(< M >, w)$ accepts.
  - If M rejects w, then $M_{\text{UNIV}}(< M >, w)$ rejects.
  - If M loops on w, then $M_{\text{UNIV}}(< M >, w)$ also loops.
- TMs are algorithms and a universal TM is a general purpose computer.

9.2 Non-deterministic TM

- Multiple possible actions the TM can take at any point in time.
- Transition function is $\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$
- TM accepts if there ecists some way to run the computation that ends in an accept state.
- TM rejects if all computations reject.

Compiler: NDTM to TMs

- THink of computation of a non-deterministic TM as a tree of TM configutations.
- Using a deterministic TM, explore this tree using a breadth-first seach. If it finds an accepting configurations, accept. If all branches reject, reject.
- If the NDTM runs for N steps, what's the run-time of the above algorithm.

10 October 28

Goal: Show some languages are not decidable.

10.1 Comparing Infinities

There are many infinite sets:

- $\mathbb{N}$ - Natural numbers
- Even numbers
- $\mathbb{Q}$ - Rationals

Some infinite sets are bigger than others. We can show that one set A is larger than another B if a one-to-one map exists between B and A.
If there is a one-to-one function in both directions, B is the same size as A.
Natural numbers are the "smallest" infinite set, if A is infinite: $|\mathbb{N}| \leq |A|$. An infinite set A is countable if $|A| = |\mathbb{N}|$. We can also show $|A| \leq |\mathbb{N}|$ as there is no set $|S| < |\mathbb{N}|$.

10.2 Uncountablility

To show A is uncountable, it is enough to show some set $|B| \leq |A|$ where B is uncountable.
Proof: Because of the transitivity of the $\leq$ operator,

- The real numbers $\mathbb{R}$ are uncountable: $|\mathbb{S}| \leq |\mathbb{R}|$. The one-to-one function $f : \mathbb{S} \to \mathbb{R}$ defined by: $f(s) = .a_1a_2.a_3...$ (in decimal) where $s = a_1, a_2, a_3...$
- The set $\mathbb{P}$ PowerSet($\mathbb{N}$) is uncountable: $|\mathbb{S}| \leq |\mathbb{P}|$. One-to-one function $f : \mathbb{S} \to \mathbb{P}$ is defined by $f(s) = \{i : a_i = 1\}$ where $s = a_1, a_2, a_3...$

10.3 Undecidability

We know:

- The set $\mathbb{L}$ of all languages is uncountable.
- The set $\mathbb{M}$ of all TM's are countably infinite.

Because of the previous proof, we know that $|\mathbb{M}| \leq |\mathbb{L}|$
We will show $A_{TM} = \{\langle M, w\rangle :$ M is a TM that accepts w$\}$ is undecidable.
Given a description of a TM M and a string w:

1. Decide if M accepts w. (this is A_{TM})
2. Decide if M halts on the input w.
3. Decide if M halts on empty input ε
4. Decide if $L(M) = \emptyset$

10.4 The TM Self-Acceptance Problem

Take the turing machine $SA_{TM} = \{\langle M \rangle : \text{M is a TM that accepts } \langle M \rangle\}$
The complement of this: $SU_{TM} = \{\langle M \rangle : \text{M is a TM that does not accept} \langle M \rangle\}$

1. A TM is "self-accepting" if it accepts the string $\langle M \rangle$ denoting its own description
2. To decide the language SA_{TM}, you need to design an algorithm that gets $\langle M \rangle$ and decides M is self-accepting.
3. SA_{TM}, SU_{TM} are complements of each other (assume every string denotes some TM). One is decidable $\iff$ the other is.

Theorem 9. ***Claim:*** *SU_{TM} is an undecidable language.*
Proof: *By contradiction, Assume we have a decider D (a TM that always halts for SU_{TM}). D accepts $\langle M \rangle \iff$ M does not accept $\langle M \rangle$.*
This can be rewritten as D accepts $\langle D \rangle \iff$ D does not accept $\langle D \rangle$, hence we arrive at a contradiction.

10.5 Undecidability as Diagonalization

Because the set of all TMs are countable, we can create the matrix $[M_i \times \langle M_i \rangle]$ where the diagonals of the row show when the TM accepts (resp. rejects) itself.
We can use this to directly prove undecidability of self-acceptance.
Suppose by contradiction there exists a decider D for $SA_{TM} = \{\langle M \rangle | \text{ M accepts} \langle M \rangle\}$.
We can construct $M^*(\langle M \rangle)$: Outputs $\neg D(\langle M \rangle)$
Consider $M^*(\langle M^* \rangle)$ (the diagonal position in our matrix). This would result in $M^*(\langle M^* \rangle)$accepts $\iff$ $D(\langle M^* \rangle)$rejects $\iff \neg(M^*(\langle M^* \rangle)$accepts). This results in a contradiction and therefore a counterexample.

10.6 Reductions in Undecidability

The TM Acceptance problem: $A_{TM} = \{\langle M, w \rangle : \text{M is a TM that accepts w}\}$.
We previously showed that $SA_{TM} = \{\langle M \rangle | \text{ M accepts} \langle M \rangle\}$ is undecidable. If we had a decider D_A for A_{TM}, we could construct a decider D_S for SA_{TM}.
$D_S(\langle M \rangle)\{\text{Output } D_A(\langle M, M \rangle)\}$
Use reduction to solve problems. Reduce problem A to B - show how to solve A given a way to solve B.
If A and B are languages, than we reduce A to B by constructing decider D_B as a subroutine.
By reducing A to B, we show:

- If B is decidable then A is decidable. (algorithms)
- If A is undecidable then B is undecidable. (this course)

10.6.1 Halting Problem

Consider the TM $H = \{\langle M, w \rangle : \text{M is a TM and M halts on w}\}$

Theorem 10. ***Claim:*** *H is undecidable.*
Proof: *Reduce A_{TM} to H.*
Construct decider D_{ATM} using decider D_H as a subroutine.
$D_{ATM}(\langle M, w\rangle)\{$
   Run $D_H(\langle M, w\rangle)$ and if it rejects, output reject;
   Else run M(w) and output whatever it outputs;
$\}$
Because we know that D_H is a decider D_{ATM} always halts.
$\forall\langle M, w\rangle, \langle M, w\rangle \in A_{TM} \iff D_{ATM}(\langle M, w\rangle) accepts \iff M\ accepts\ w \iff$

11 November 7

11.1 Proof Systems

By Russel's Paradox, sets "cannot" be non-wellfounded.

Corollary 1. *There is no "proof system" in Number Theory*

- *Soundness: No false statement ϕ has a valid proof π.*
- *Completeness: Every trye statement ϕ has a valid proof π.*
- *Decidability: Can decide if π is a valid proof for ϕ*

We generally give up soundness for completeness.

Proof. If there was a proof system for number theory, then we could decide number theory.

- Given ϕ try all valid strings π one-by-one:
  - Check if π is a valid proof of ϕ. If so, accept.
  - Check if π is a valid proof of $\neg\phi$. If so, reject.

□

11.2 Gödel's Sentence

We want to construct a statement in a proof system that is true and unprovable.
Take proposition p(x) = "p(x) is unprovable."

1. Proving p(x) is true: $\neg p(x) \implies p(x)$ is provable $\implies p(x)$ by the property of soundedness in proof systems. This is clearly a contradiction.
2. If p(x) is true, it must also be unprovable.

Showing that a proposition can describe itself.
$\Phi^*(n)$:

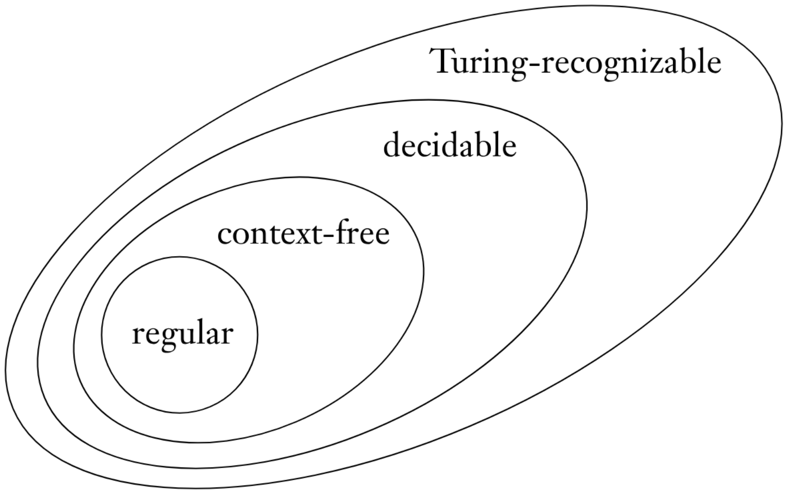
1. parse n as $\langle\phi\rangle$
2. True, if $\nexists$ proof for $\phi(n)$

$\phi^*(n) := \nexists$ proof for $\phi(n)$ where $\langle\phi\rangle = n$.
Consider $\phi^*(\langle\phi^*\rangle)$

1. Proving $\phi^*(\langle\phi^*\rangle)$ is true: Suppose not, $\implies \exists$ proof for $\phi^*(\langle\phi^*\rangle)$. By soundedness, we then know that $\phi^*(\langle\phi^*\rangle)$ is true.
2. Proving $\phi^*(\langle\phi^*\rangle)$ is unprovable: Suppose not, $\implies$ $\phi^*(\langle\phi^*\rangle)$ is true which results in a contradiction.

We can use diagonalization to build a matrix where each row represents the mathematical equation *phi* and each column represents a natural number n. Each cell represents if there exists a proof for $\phi(n)$.

12 Exam 2 Prep November 13



- The set $\mathbb{S}$ of all infinite binary sequences is uncountable. We can show a set is uncountable by proving it is larger than $\mathbb{S}$
- The TM Acceptance problem: $A_{TM} = \{\langle M, w \rangle$: M is a TM that accepts w$\}$.
- A_{TM} is recognizable but not decidable. $\overline{A_{TM}}$ is not recognizable.
- A language is decidable $\iff$ it is Turing-recognizable and co-Turing-recognizable.
- If A is reducible to B and B is decidable, A also is decidable. Equivalently, if A is undecidable and reducible to B, B is undecidable.
- By reducing A to B, we show if A is undecidable then B is undecidable.
- If $A \leq_m B$ and B is decidable, then A is decidable.
- If $A \leq_m B$ and A is decidable, then B is undecidable.
- The set of all Turing machines is countable, because every Turing machine has a finite, unique description.
- Any Turing-undecidable language is also non-regular, because its contrapositive is true: any regular language is decidable by Turing machines.
- Rice's theorem. Let P be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given Turing machine's language has property P is undecidable.