1 Introduction

Many links between fundamental mathematical concepts and elements of musicology have been found. Sometimes these links offer instructive ways to think about mathematical objects. As a novel example of this, consider the lexicographical ordering of the group $\mathbb{Z}_4 \oplus \mathbb{Z}_3$,

$$(0,0) < (0,1) < (0,2) < (1,0) < (1,1) < (1,2) < (2,0) < (2,1) < (2,2) < (3,0) < (3,1) < (3,2)$$

which induces the following order on $\mathbb{Z}_{12}$:

$$0 < 4 < 8 < 9 < 1 < 5 < 6 < 10 < 2 < 3 < 7 < 11 \tag{1}$$

This order is perhaps more easily conceptualised as the order of notes in the arpeggio-like scale depicted in Figure 1. This scale itself can be conceptualised as the C augmented appegiated triad, followed by the $C\natural$ augmented appegiated triad second inversion, followed by the D augmented appegiated triad in first inversion, followed by the $D\natural$ augmented appegiated traid. The pattern is thus to appegiate the C augmented chord, repeat this three more times where each repition the tonic note moves up a semi-tone, and the inversion of the chord moves down by one. In this way, the elements of $\mathbb{Z}_{12}$ are interpreted as the 12 tones of an oc-

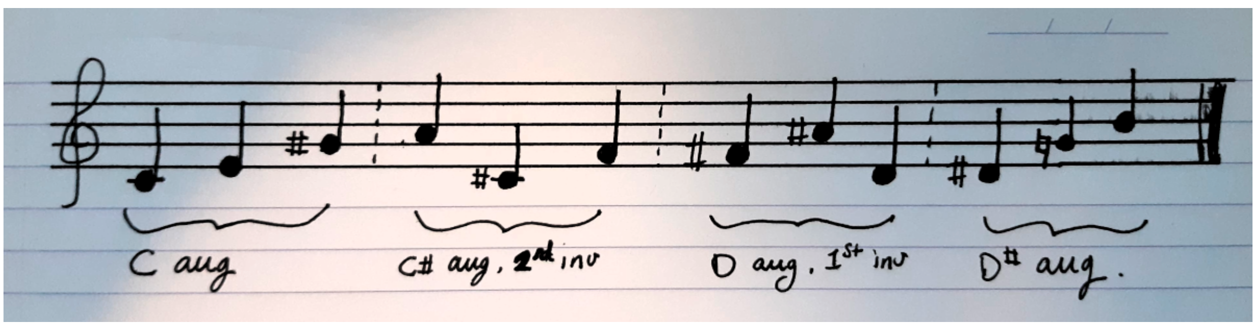

Figure 1: Appegio-like scale representing an order on $\mathbb{Z}_{12}$

tave (under standard tuning), and $n < m$ means n comes before m in the scale. See [1, §6.8.1] for more details.

The motivating question of this project is the following:

Question 1.0.1. *Are there any links between fundamental* computational *concepts and music?*

The first investigation will be on computation and *musical composition*. The formal objects on the side of musical composition will be *global compositions*, due to Mazzola [1]. In short, a global composition consists of a collection of *local compositions*, ie, small musical snippets, along with *glueing instructions* describing how these snippets fit together. The guiding intuition which will relate this to computation is that just as a musical composer begins with a collection of motifs and organises them into a cohesive whole, a program consists of a collection of smaller programs which are slotted together. In other words, once a musical structure of a particular piece (ie, a global composition) has been written, appropriate local compositions can be *substituted* in to *realise* a complete piece. Since the language of substitution naturally arise here, we adopt the λ-calculus as our formalisation of a *program*. Indeed, the ultimate goal is an appropriate category of *global compositions* lying on the musical side, and an equivalence of categories between this and $\mathscr{L}_Q$ [2], an appropriate category of λ-terms.

2 Forms, local compositions, global compositions

The following Definitions are particular instances of the extremely general formulations of those in [1] which carry the same name. The full generality is avoided here due to our underlying agenda: to relate musical composition to computation. The indications of such a relationship described in the Introduction encourage us to look toward *global compositions*, which consist of a collection of particular *local compositions*, satisfying suitable compatibility conditions. Our approach avoids the full generality of *forms* and avoids *denotators* completely, which greatly reduces the work needed to arrive at *global compositions*.

First we Define *forms*, we differ from Mazzola's recursive Definition and provide an inductive one instead, throughout rings are assumed to be commutative with unit.

Definition 2.0.1. *The set of* ***simple forms*** $\mathscr{F}$ *consists of tuples* $(N(F), T(F), C(F), I(F))$ *where*

- *the* ***name*** $N(F)$ *is a word in* ASCII^*,
- *the* ***type*** $T(F)$ *is the word* $\mathrm{Simple} \in \mathrm{ASCII}^*$,
- *the* ***coordinator*** *is a ring* R *along with an* R*-module* M,
- *the* ***identifier*** $I(F)$ *is a presheaf* $S : \underline{Modd}^{op} \to \underline{Set}$ *along with a monic* $S \rightarrowtail \underline{Modd}(\_, M)$.

We now define ***compound forms***, *let* $i > 0$ *and* $N(F) \in \mathrm{ASCII}^*$,

- *if* $(F' = N(F'), T(F'), C(F'), I(F')) \in \mathscr{F}_{i-1}$, *then*
  - *if* $I(F) : X \to \mathrm{Dom}\, I(F')$ *is some monic of functors, then* $F = (N(F), \mathrm{Syn}, F', I(F)) \in \mathscr{F}_i$, *we say* F *has type* ***synonym***,
  - *if* $I(F) : X \to \Omega^{\mathrm{Dom}\, I(F')}$ *is some monic of functors, then* $F = (N(F), \mathrm{power}, F', I(F)) \in \mathscr{F}_i$, *we say* F *has type* ***Power***
- *given a diagram* $D : \mathscr{J} \to \underline{Set}^{\underline{Modd}^{op}}$ *and a* $\mathscr{J}$*-indexed collection* $\{F_j : (N(F_j), T(F_j), C(F_j), I(F_j))\}_{j \in J}$ *where* $F_j \in \mathscr{F}_{i-1}$ *and* $D(j) = \mathrm{Dom}\, I(F_j)$ *then*
  - $F = (N(F), \mathrm{Limit}, D, \mathrm{Limit}_D\, \mathscr{J}) \in \mathscr{F}_j$, *we say* F *has type* ***limit***, *and*
  - $F = (N(F), \mathrm{Colimit}, D, \mathrm{Colimit}_D\, \mathscr{J}) \in \mathscr{F}_j$, *we say* F *has type* ***colimit***.

As with simple forms, the ***name*** *of a compound form* $F = (N(F), T(F), C(F), I(F))$ *is* $N(F)$, *etc.*

Example 2.0.1. *An extremely bare bones way of thinking about a musical phrase played by a musician is as a sequence of notes. Assuming these notes belong to the standard* 12*-tone equal temprament tuning, and are played on a monophonic instrument, this can be formalised as a sequence of elements of the* $\mathbb{Z}$*-module* $\mathbb{Z}_m \oplus \mathbb{Z}_{12}$, *where* m *is the length of the sequence. An element* (a, b) *means the* a^{th} *note in the music phrase is that which is* b *semitones above* C*. That is,* $\mathbb{Z}_m$ *determines the* onset *of the note, and* $\mathbb{Z}_{12}$ *determins the* pitch*. This data is captured by the following form:*

$$(\mathrm{OnsetPitch}_{(m,12)}, \mathrm{Simple}, \mathbb{Z}_m \oplus \mathbb{Z}_{12}, \mathrm{Id}_{y(\mathbb{Z}_m \oplus \mathbb{Z}_{12})})$$

where $y : \underline{Modd} \to \underline{Set}^{\underline{Modd}^{op}}$ *is the Yoneda embedding.*

Remark 2.0.1. *At this point, the roll of the identifier of a form may seem like an illusive concept, especially since the only Example given here is so trivial. Indeed the full weight of the abstract setting will not be needed for our current purposes, in fact the identifier can safely be completely ignored for the remainder of this note. It was included in order to align with [1]. See [1, p.56] for the motivation for having it as part of the Definition of a form.*

Definition 2.0.2. *Let* R *be a ring and* M *and* R*-module. An* M ***addressed local composition*** $\mathcal{C}$ *is a tuple* $(N(\mathcal{C}), F, X)$ *where*

- $N(\mathcal{C})$ *is a name, that is, a word in* ASCII^*,

- *F is a form,*
- *X is a subset of* $\mathrm{Dom}\, I(F)(M)$

Remark 2.0.2. *The corresponding notion to 2.0.2 in [1] is* objective local compositions*, see [1, §7, p90].*

Definition 2.0.3. *A* ***morphism*** *of M addressed local compositions $f : (N, F_N, X) \to (P, F_P, Y)$ is a function $f : X \to Y$ such that there exists a natural transformation $\eta : \mathrm{Dom}\, I(F_N) \to \mathrm{Dom}\, I(F_P)$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "f"'] & \mathrm{Dom}\, I(F_N)(M) \arrow[d, "h_M"] \\ Y \arrow[r, tail] & \mathrm{Dom}\, I(F_P)(M) \end{tikzcd}$$

Local compositions can be *glued together* to form *global compositions*:

Definition 2.0.4. *Let R be a ring and M an R-module. A M* ***addressed pre-global composition*** *consists of the following data:*

- *a set G,*
- *a non-empty covering $\{I_j\}_{j \in J}$ of G,*
- *an M* ***addressed atlas*** *for the covering I of G, which consists of*
  - *a family $\{\mathcal{C}_t = (N(\mathcal{C}_t), F_t, X_t)\}_{t \in T}$ of M addressed local compositions,*
  - *a surjection $s : T \twoheadrightarrow J$, we write I_t for $I_{s(t)}$,*
  - *for each $t \in T$, a bijection $\phi_t : X_t \xrightarrow{\sim} I_t$*

  such that if $I_t \cap I_{t'} \neq \varnothing$, we have that $\phi_t : \phi_t^{-1}(I_t \cap I_{t'}) \to \phi_{t'}^{-1}(I_t \cap I_{t'})$ is part of an isomorphism of local compositions $\mathcal{C}_t \xrightarrow{\sim} \mathcal{C}_{t'}$.

Two M addressed atlases for the covering I of G are ***equivalent*** *if their disjoint union is again an M addressed atlas for the covering I of G. Two M addressed pre-global compositions are* ***equivalent*** *if their M addressed atlases for the covering I of G are equivalent. An M* ***addressed global composition*** *is an equivalence class of M addressed pre-global compositions.*

Example 2.0.2. *See Figure 2.0.2 for a musical phrase consisting of 12 notes. The set G is this collection of notes, and the covering are the labelled squares. The address is the zero-module, and all 15 local compositions have associated form given by* $\mathrm{OnsetPitch}_{(0,12)}$*. This is included here only to provide rough intuition. See [1, §13.2] for details.*

Definition 2.0.5. *The* ***nerve*** *of an M addressed global composition is the categorical nerve of the covering I of G which is a poset thought of as a category.*

Remark 2.0.3. *Definition 2.0.5 differs from the notion with the same name is [1], there, the nerve is taken as a simplicial complex rather than a simplicial set.*

Lemma 2.0.1. *The Geometric realisation of the nerve of a global composition gives the nerve as defined in [1, §13.2.1].*

Remark 2.0.4. *There exists an appropriate notion of* morphism *of global compositions which together with global compositions themselves form a category, and moreover the nerve can be extended to a functor from this category to the homotopy category of simplicial sets. See [1, §13,§14] for details. This provides a way to inject homotopy theory into musicology, and provides a way to derive topological and therefor algebraic invariants of global compositions. This is not the direction this note will take though.*

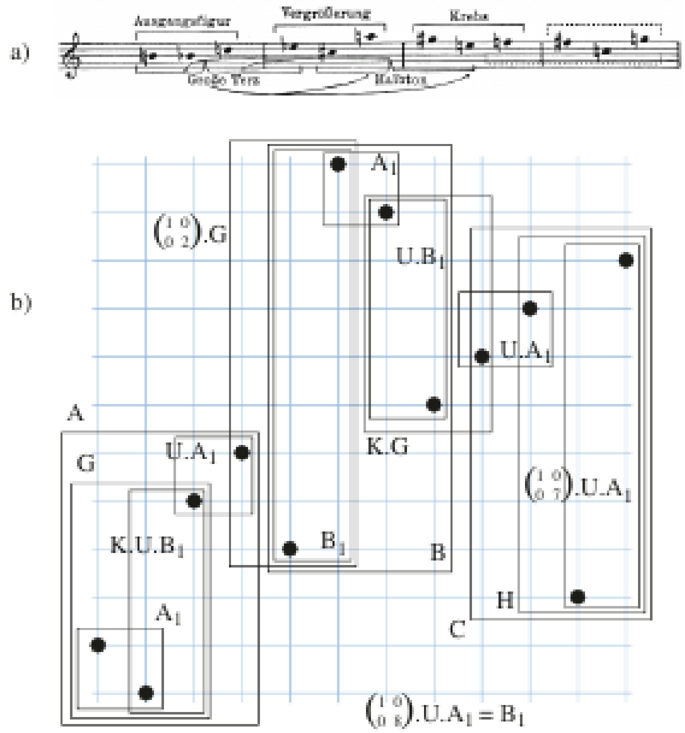

Figure 2: A global composition

This leads us to the following question:

Question 2.0.1. *If global compositions are simplicial sets, then how can they be related to something "computational", or "algorithmic"?*

Another unfinished project, that (finite) simplicial sets are algorithms [3] may be relevant here.

Also, music plays from start to finish, so the local compositions "pan from left to right" in some sense, that is, they play out in *order* from left to right, but with overlap due to that of the charts of the global composition. These charts when thought of as *transitions* from one *musical element* τ to another σ appear strikingly similar to a λ-term of type $\tau \to \sigma$. However, as we have just seen, global compositions are not λ-terms, they are simplicial sets. So a mathematical question arises:

Question 2.0.2. *Is there any circumstance in which a simplicial set can be described by a typed λ-term?*

A positive answer to this question would be "yes, when that simplicial set arrises from a global composition". Fleshing out the details of such an answer would then lead to a result similar to the *curry-howard correspondence* [2] which extends the "types as propositions, terms as proofs" paradigm to "types as propositions as musical elements, terms as proofs as compositions" paradigm. A holy grail in this search would be a new perspective which renders mathematical musicology relevant to computer science, in the same way that insights of the 1950s and 1960s rendered proof theory relevant to computer science.

Notice also that with careful construction of the notion of "musical element", there could be collections of "musical elements" of particular sorts, for example, a local composition $\mathcal{C}$ which starts on a C and a local composition $\mathcal{C}$ which starts on G are both "musical elements" of sort "belong to the C-major scale". So perhaps even System F is the appropriate computational model for global compositions.

References

[1] G. Mazzola, *The Topos of Music I, Theory*, Springer International Publishing AG, part of Springer Nature 2002,2017.

[2] D. Murfet, W. Troiani, *The Curry-Howard Correspondence*

[3] W. Troiani. *Finite Simplicial Sets are Algorithms*, `https://williamtroiani.github.io/2019/06/01/Will-Troiani-Finite-Simplicial-Sets.html`