1 Introduction

Many links between fundamental mathematical concepts and elements of musicology have been found. Sometimes these links offer instructive ways to think about mathematical objects. As a novel example of this, consider the lexicographical ordering of the group $\mathbb{Z}_4 \oplus \mathbb{Z}_3$,

$$(0,0) < (0,1) < (0,2) < (1,0) < (1,1) < (1,2) < (2,0) < (2,1) < (2,2) < (3,0) < (3,1) < (3,2)$$

which inudces the following order on $\mathbb{Z}_{12}$:

$$0 < 4 < 8 < 9 < 1 < 5 < 6 < 10 < 2 < 3 < 7 < 11 \tag{1}$$

This order is perhaps more easily conceptualised as the order of notes in the arpeggio-like scale depicted in Figure 1. This scale itself can be conceptualised as the C augmented appegiated triad, followed by the $C\sharp$ augmented appegiated triad second inversion, followed by the D augmented appegiated triad in first inversion, followed by the $D\sharp$ augmented appegiated traid. The pattern is thus to appegiate the C augmented chord, repeat this three more times where each repition the tonic note moves up a semi-tone, and the inversion of the chord moves down by one. In this way, the elements of $\mathbb{Z}_{12}$ are interpreted as the 12 tones of an oc-

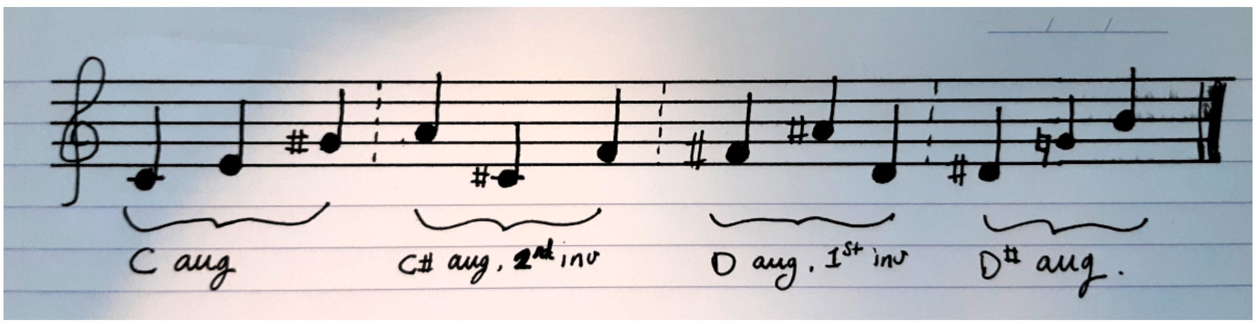

Figure 1: Appegio-like scale representing an order on $\mathbb{Z}_{12}$

tave (under standard tuning), and $n < m$ means n comes before m in the scale. See [1, §6.8.1] for more details.

The motivating question of this project is the following:

Question 1.0.1. *Are there any links between fundamental* computational *concepts and music?*

The first investigation will be on computation and *musical composition*. The formal objects on the side of musical composition will be *global compositions*, due to Mazzola need citation to original paper. In short, a global composition consists of a collection of *local compositions*, ie, small musical snippets, along with *glueing instructions* describing how these snippets fit together. The guiding intuition which will relate this to computation is that just as a musical composer begins with a collection of motifs and organises them into a cohesive whole, a program consists of a collection of smaller programs which are slotted together. In other words, once a musical structure of a particular piece (ie, a global composition) has been written, appropriate local compositions can be *substituted* in to *realise* a complete piece. Since the language of substitution naturally arise here, we adopt the λ-calculus as our formalisation of a *program*. Indeed, the ultimate goal is an appropriate category of *global compositions* lying on the musical side, and an equivalence of categories between this and $\mathscr{L}_Q$ [2], an appropriate category of λ-terms.

2 Forms, local compositions, global compositions

The following Definitions are particular instances of the extremely general formulations of those in [1] which carry the same name. The full generality is avoided here due to our underlying agenda: to relate musical composition to computation. The indications of such a relationship described in the Introduction encourage us to look toward *global compositions*, which consist of a collection of particular *local compositions*, satisfying suitable compatibility conditions. Our approach avoids the full generality of *forms* and avoids *denotators* completely, which greatly reduces the work needed to arrive at *global compositions*.

First we Define *forms*, we differ from Mazzola's confusing presentation "recursive Definition" and provide an inductive one instead, throughout rings are assumed to be commutative with unit.

Definition 2.0.1. *The set of* ***simple forms*** $\mathscr{F}$ *consists of tuples* $(N(F), T(F), C(F), I(F))$ *where*

- *the* ***name*** $N(F)$ *is a word in* ASCII^**,*
- *the* ***type*** $T(F)$ *is the word* $\mathrm{Simple} \in \mathrm{ASCII}^*$*,*
- *the* ***coordinator*** *is a ring* R *along with an* R*-module* M*,*
- *the* ***identifier*** $I(F)$ *is a presheaf* $S : \underline{Modd}^{op} \to \underline{Set}$ *along with a monic* $S \rightarrowtail \underline{Modd}(\_, M)$*.*

We now define ***compound forms****, let* $i > 0$ *and* $N(F) \in \mathrm{ASCII}^*$*,*

- *if* $(F' = N(F'), T(F'), C(F'), I(F')) \in \mathscr{F}_{i-1}$*, then*
  - *if* $I(F) : X \to \mathrm{Dom}\, I(F')$ *is some monic of functors, then* $F = (N(F), \mathrm{Syn}, F', I(F)) \in \mathscr{F}_i$*, we say* F *has type* ***synonym****,*
  - *if* $I(F) : X \to \Omega^{\mathrm{Dom}\, I(F')}$ *is some monic of functors, then* $F = (N(F), \mathrm{power}, F', I(F)) \in \mathscr{F}_i$*, we say* F *has type* ***Power***
- *given a diagram* $D : \mathscr{J} \to \underline{Set}^{\underline{Modd}^{op}}$ *and a* $\mathscr{J}$*-indexed collection* $\{F_j : (N(F_j), T(F_j), C(F_j), I(F_j))\}_{j \in J}$ *where* $F_j \in \mathscr{F}_{i-1}$ *and* $D(j) = \mathrm{Dom}\, I(F_j)$ *then*
  - $F = (N(F), \mathrm{Limit}, D, \mathrm{Limit}_D\, \mathscr{J}) \in \mathscr{F}_j$*, we say* F *has type* ***limit****, and*
  - $F = (N(F), \mathrm{Colimit}, D, \mathrm{Colimit}_D\, \mathscr{J}) \in \mathscr{F}_j$*, we say* F *has type* ***colimit****.*

As with simple forms, the ***name*** *of a compound form* $F = (N(F), T(F), C(F), I(F))$ *is* $N(F)$*, etc.*

Example 2.0.1. *Give the OP example*

Note: there is an order on the local compositions corresponding to a global composition. It is "left first, followed by inner first", this order pescribes a simplicial set whose geometric realisation is the nerve that Mazzola defines.

3 λ-terms and global compositions

Our motivating example is Figure 3, which consists of a small excerpt of music (see [1, §13.1], Figure 13.1 for a reference). Our first goal is to realise this as a *global composition* in the formal sense, and then to derive a λ-term which corresponds to it as a composition. From this example, we hope a general Theory will be formed.

References

[1] G. Mazzola, *The Topos of Music I, Theory*, Springer International Publishing AG, part of Springer Nature 2002,2017.

[2] D. Murfet, W. Troiani, *The Curry-Howard Correspondence*

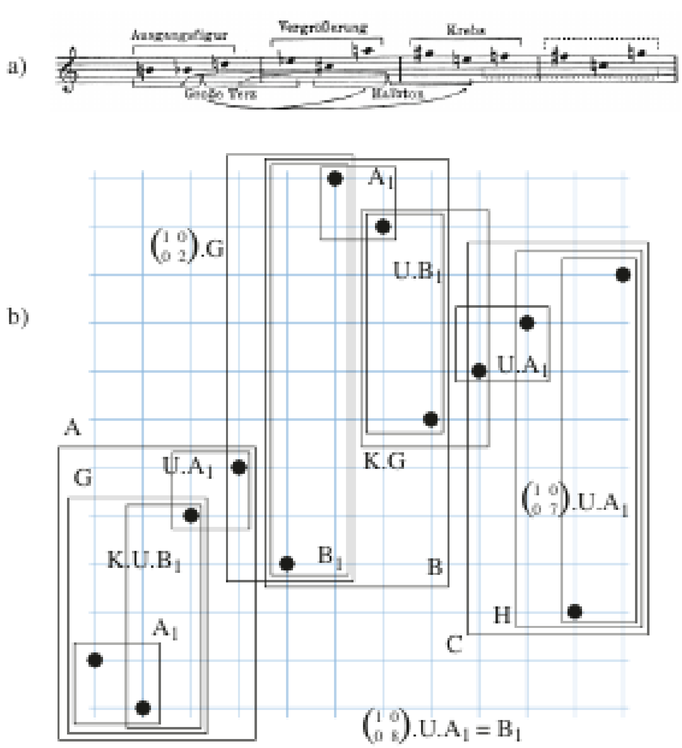

Figure 2: A global composition