# Ax-Grothendieck via model theory

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**Definition 0.0.1.** We define $\mathcal{F}$, the first order theory of fields, beginning with the first order language of fields. Let $\Sigma$ be a signature consisting of a single sort $A$. We introduce 5 function symbols.

- $0, 1 : A$,
- $- : A \longrightarrow A$,
- $+, \cdot : A \times A \longrightarrow A$.

The first order language of fields has no relation symbols.

The axioms are given as follows.

$$(x + y) + z = x + (y + z) \tag{1}$$
$$x + y = y + x \tag{2}$$
$$x + 0 = x \tag{3}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{4}$$
$$x \cdot 1 = 1 \cdot x = x \tag{5}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{6}$$
$$x + (-x) = 0 \tag{7}$$
$$x \neq 0 \Rightarrow \exists y, xy = 1 \tag{8}$$

This set of formulas forms the axioms of $\mathcal{F}$.

**Definition 0.0.2.** For each $d \geq 1$ define the following formula.

$$P_d := \forall a_0 \ldots \forall a_d \exists x, a_d \neq 0 \wedge a_0 + a_1 x + \ldots + a_{d-1} x^{d-1} + a_d x^d = 0 \tag{9}$$

For each prime number $p$ define the following formula.

$$S_d := 1 + \ldots + 1 = 0 \tag{10}$$

where there are $d$ instances of 1 in (10).

**Definition 0.0.3.** Let $\mathcal{ACF}$ denote the **first order theory of algebrically closed fields** which is over the same language as $\mathcal{F}$ and consists of all the axioms of Definition 0.0.1 along with $P_d$ for each $d \geq 1$.

The **first order theory of algebraically closed fields of characteristic** $p$ is denoted $\mathcal{ACF}_p$ and consists of all the axioms of $\mathcal{ACF}$ along with $S_p$.

Lastly, the **first order theory of algebraically closed fields of characteristic** 0 is denoted $\mathcal{ACF}_0$ and consists of all the axioms of $\mathcal{ACF}$ along with the formula $\neg S_p$ for each prime number $p$.

**Lemma 0.0.4.** *Let $f : \overline{\mathbb{F}_p}^n \longrightarrow \overline{\mathbb{F}_p}^n$ be a polynomial. If $f$ is injective then it is surjective.*

*Proof.* Let $\underline{y} = (y_1, \ldots, y_n) \in \overline{\mathbb{F}_p}^n$ be arbitrary. Consider the field extension $K \supseteq \mathbb{F}_p$ generated by $y_1, \ldots, y_n$ as well as the coefficients of $f$. Since every element of $\overline{\mathbb{F}_p}$ is algebraic over $\mathbb{F}_p$ (by the definition of an algebraic closure) we have $K$ is an algebraic extension and thus finite of $\mathbb{F}_p$. Since $\mathbb{F}_p$ is finite, this implies $K$ is finite. Lastly, we notice that fields are closed under polynomial expressions, and so $f(K^n) \subseteq K^n$, which by injectivity and finiteness implies surjectivity. $\square$

**Corollary 0.0.5.** *Let $k$ be an algebraically closed field and $f : k^n \longrightarrow k^n$ a polynomial. If $f$ is injective then it is surjective.*

*Proof.* There is a small slieght of hand involved, we need to turn the statement of the corollary into a first order formula, but we cannot do that if try to work with a polynomial of arbitrary degree. So instead we will consider the statement "If $f$ is injective and has degree at most $d$ then it is surjective". The idea is to write out the following statement

$$\forall a_0 \ldots \forall a_d (\forall x \forall y, f(x) = f(y) \Rightarrow x = y) \tag{11}$$
$$\Longrightarrow \forall y \exists x, y = f(x) \tag{12}$$

however we need to write out $f$ explicitly. This is where we use the fact that $f$ is a polynomial. Our first order statement is:

$$\begin{aligned}\forall a_0 \ldots \forall a_d (\forall x \forall y, a_0 + a_1 x + \ldots + a_{d-1} x^{d-1} + a_d x^d \\ = a_0 + a_1 y + \ldots + a_{d-1} y^{d-1} + a_d y^d \Rightarrow x = y) \\ \Longrightarrow \forall y \exists x, y = a_0 + a_1 x + \ldots + a_{d-1} x^{d-1} + a_d x^d\end{aligned}$$

Denote this formula $B_d$.

Since $\mathcal{ACF}_0$ is complete, this statement is either proveable or its negation is proveable. That is, either $\mathcal{ACF}_0 \vdash B_d$ or $\mathcal{ACF}_0 \vdash \neg B_d$. Suppose $\mathcal{ACF}_0 \vdash \neg B_d$ and let $\pi$ be such a proof. Since $\pi$ is finite, only finitely many axioms of $\mathcal{ACF}_0$ appear amongst its premises. This, there exists some prime $q$ such that $\neg S_q$ does *not* appear amongst the premises of $\pi$. That is, $\pi$ is a valid proof in $\mathcal{ACF}_q$! By soundness of $\mathcal{ACF}_q$ we derive a contradiction to Lemma 0.0.4, and so we must have $\mathcal{ACF}_0 \vdash B_d$. The result then follows by soundness of $\mathcal{ACF}_0$. □

**Lemma 0.0.6.** *Every field $F$ can be embedded into an algebraically closed field $\bar{F}$.*

*Proof.* Let $\Lambda$ be the collection of monic, irreducible polynomials with coefficients in $F$. For each $f \in F$, let $u_{f,0}, ..., u_{f,d}$ be formal indeterminants, where $d$ is the degree of $f$. Let $F[\{U\}]$ be the polynomial ring over $F$ where $U$ is the collection of all $u_{f,i}$. Write

$$f - \prod_{i=0}^{d} (x - u_{f,i}) = \sum_{i=0}^{d-1} \alpha_{f,i} x^i \in F[\{U\}][x]$$

Let $I$ be the ideal generated by $\alpha_{f,i}$. $I$ is not all of $F[\{U\}]$ so there exists a maximal ideal $M$ containing $I$. Let $F_1 = F[\{U\}]/M$. Repeat this process to define $f_i$ for all $i > 0$. Then $\cup_{i=1}^{\infty} F_i$ is algebraically closed which $F$ embeds into, and moreover is an algebraic extension of $F$. □

**Corollary 0.0.7.** *If $F$ is infinite, then the cardinality of $F$ is equal to the cardinality of $\overline{F}$.*

*If $F$ is finite, then the cardinality of $\overline{F}$ is countably infinite.*

*Proof.* Using the notation of the proof of Lemma 0.0.6, we first observe that the $|\{U\}| = |F|$ □

**Lemma 0.0.8.** *Let $p$ be either a prime number or 0 and let $\kappa \geq \aleph_1$ be an uncountable cardinal. There exists an algebraically closed field of characteristic $p$ whose cardinality is $\kappa$. Moreover, this field is unique up to isomorphism.*

*Proof.* Define $F$ to be

$$p = \begin{cases} \mathbb{Q}, & p = 0 \\ \mathbb{F}_p, & p \neq 0 \end{cases} \tag{13}$$

Let $X$ be any set of cardinality $\kappa$ (eg, $X = \mathbb{R}$) and consider the polynomial ring $F[\{X\}]$. The ideal $I \subseteq F[\{X\}]$ generated by $X$ is not all of $F[\{X\}]$ and so is contained in some maximal ideal $\mathfrak{m}$ (using Zorn's Lemma). The field $F[\{X\}]/\mathfrak{m}$ has cardinality $\aleph_1$, as there are countably many polynomials over a single indeterminant, and we claim that the algebraic closure $\overline{F[\{X\}]/\mathfrak{m}}$ of $F[\{X\}]/\mathfrak{m}$ also has cardinality $\kappa$.

The argument above show that in the notation of Lemma 0.0.6, that $F_i$ has cardinality $\kappa$ for all $i \geq 0$. Since $\overline{F[\{X\}]/\mathfrak{m}}$ is the countable union of all of these fields, it follows that the cardinality of $\overline{F[\{X\}]/\mathfrak{m}}$ is $\kappa$.

The uniqueness claim follows easily by considering a transcendental basis of $\overline{F[\{X\}]/\mathfrak{m}}$ and observing that this basis has cardinality $\kappa$. The rest follows from the universal property of the algebraic closure. □

**Corollary 0.0.9.** *There is only one model (up to isomorphism) of $\mathcal{ALG}_0$ and of $\mathcal{ALG}_p$ for each cardinal $\kappa \geq \aleph_1$.*