Operator exponential

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1 Introduction

The (completely fulfilling) motivation of this note is to solve a differential equation using the exponential of an operator.

2 Defining the exponential of an operator

Throughout, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. All vector spaces will be over $\mathbb{F}$.

First we recall the definition of a norm.

Definition 2.0.1. A **norm** on V is a function $\|\cdot\| : V \longrightarrow \mathbb{F}$ satisfying the following axioms.

- $\|x\| = 0 \iff x = 0$.
- $\forall x \in V, \|x\| \geq 0$.
- $\forall x, y \in V, \|x + y\| \leq \|x\| + \|y\|$.
- $\forall x \in V, \forall \lambda \in \mathbb{F}, \|\lambda x\| = |\lambda|\|x\|.$, where $|\lambda|$ denotes the magnitude (ie, the standard norm) on $\mathbb{F}$.

We now work towards Proposition 2.0.5 below which states that all norms on the same finite dimensional vector space are equivalent. This assists greatly with calculations.

Definition 2.0.2. Two norms $\|\cdot\|_1, \|\cdot\|_2$ are **Lipschitz equivalent** if there exists $c, C \in \mathbb{F}$ subject to the following.

$$\forall x \in V, c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \tag{1}$$

Lemma 2.0.3. *Let $\|\cdot\|_1$ denote the L_1 norm on $\mathbb{F}^n$, that is,*

$$\forall x = (x_1, \dots, x_n) \in \mathbb{F}^n, \|x\| = \sum_{i=1}^{n} |x_i| \tag{2}$$

Let $\|\cdot\|$ denote the standard norm, given by the following equation.

$$\forall x = (x_1, \dots, x_n) \in \mathbb{F}^n, \|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{3}$$

Then $\mathbb{F}^n$ endowed with the topology induced by $\|\cdot\|_1$ is homeomorphic to $\mathbb{F}^n$ endowed with the topology induced by $\|\cdot\|$.

Proof. We let $(\mathbb{F}, \|\cdot\|_1)$ and $(\mathbb{F}, \|\cdot\|)$ respectively denote these two topological spaces. We claim the identity function $\mathrm{id} : (\mathbb{F}, \|\cdot\|_1) \longrightarrow (\mathbb{F}, \|\cdot\|)$ is a homeomorphism. The only check to do is that both identity functions

$$(\mathbb{F}, \|\cdot\|_1) \longrightarrow (\mathbb{F}, \|\cdot\|) \tag{4}$$
$$(\mathbb{F}, \|\cdot\|) \longrightarrow (\mathbb{F}, \|\cdot\|_1) \tag{5}$$

are continuous.

First, consider (4). We use the simple fact that $\sqrt{x_1 + \ldots + x_n} \leq \sqrt{x_1} + \ldots + \sqrt{x_n}$ for all $x_1, \ldots, x_n \in \mathbb{F}$ to show that for all $x = (x_1, \ldots, x_n)$ and all $x' = (x'_1, \ldots, x'_n)$ we have

$$\|x - x'\| = \sqrt{\sum_{i=1}^{n} |x_i = x'_i|^2} \leq \sum_{i=1}^{n} \sqrt{|x_i - x'_i|^2} = \sum_{i=1}^{n} |x_i - x'_i| = \|x - x'\|_1 \tag{6}$$

This proves continuity of (4).

On the other hand, for all $x = (x_1, \ldots, x_n)$ and all $x' = (x'_1, \ldots, x'_n)$ the following holds.

$$\|x - x'\|_1 = \sum_{i=1}^{n} |x_i - x'_i| \leq n \sup_{i=1,\ldots,n}\{|x_i - x'_i|\} \leq n\sqrt{\sum_{i=1}^{n} |x_i - x'_i|^2} \leq \|x - x'\| n \tag{7}$$

It follows that (5) is continuous. □

Remark 2.0.4. We remark that the proof of Lemma 2.0.3 shows that in fact the functions (4), (5) are *uniformly* continuous.

Proposition 2.0.5. *Any two norms on a finite dimensional vector space V are Lipschitz equivalent.*

Proof. First, we claim that Lipschitz equivalence is an equivalence relation on the set of norms on V. Reflexivity is immediate, one takes $c = C = 1$. For symmetric, say $\|\cdot\|_1$ and $\|\cdot\|_2$ are Lipschitz equivalent. Then there exists $c, C \in \mathbb{F}$ satisfying (1) and we notice that

$$\forall x \in V, \frac{1}{C}\|x\|_2 \leq \|x\|_1 \leq \frac{1}{c}\|x\| \tag{8}$$

establishing symmetry. Moreover, if there was a third norm $\|\cdot\|_3$ for which there exists $k, K \in \mathbb{F}$ subject to

$$\forall x \in V, k\|x\|_2 \leq \|x\|_3 \leq K\|x\|_2 \tag{9}$$

then we have the following chain of inequalities.

$$\forall x \in V, ck\|x\|_1 \leq k\|x\|_2 \leq \|x\|_3 \leq K\|x\|_2 \leq CK\|x\|_1 \tag{10}$$

which proves transitivity.

Thus, it suffices to show that any norm $\|\cdot\|$ is Lipschitz equivalent to the following norm. Pick a basis $\{v_1, \ldots, v_n\}$ for V and define

$$\|\cdot\|_1 : V \longrightarrow \mathbb{F} \tag{11}$$
$$x = \sum_{i=1}^{n} \alpha_i v_i \longmapsto \sum_{i=1}^{n} |\alpha_i| \tag{12}$$

It is easy to check that this is a norm, indeed it is the norm induced by the normed space isomorphism $V \cong \mathbb{F}^n$ where $\mathbb{F}^n$ is endowed with the L_1 norm.

Our next claim is that this norm $\|\cdot\|$ is uniformly continuous, when V is endowed with the $\|\cdot\|_1$ norm. We must show

$$\forall x, x' \in V, \forall \epsilon > 0, \exists \delta > 0, \|x - x'\|_1 < \delta \Longrightarrow |\|x\| - \|x'\|| < \epsilon \tag{13}$$

Towards this end, let $x, x' \in V, \epsilon > 0$ be arbitrary and define

$$M := \sup_{i=1,\dots,n}\{\|v_i\|\} \tag{14}$$

Then we have the following calculation, where we write $x = \sum_{i=1}^n \alpha_i v_i, x' = \sum_{i=1}^n \alpha_i' v_i$.

$$\begin{aligned}\|x - x'\| &= \|\sum_{i=1}^n (\alpha_i - \alpha_i')v_i\| \\ &\le |\sum_{i=1}^n (\alpha_i - \alpha_i')|\|v\| \\ &\le M|\sum_{i=1}^n (\alpha_i - \alpha_i')| \\ &= M\|x - x'\|_1\end{aligned}$$

So $\delta = \epsilon/M$ is an appropriate choice for δ. This establishes the claim.

Now we consider the unit ball $\{v \in V \mid \|v\|\} \subseteq V$ which is compact by Lemma 2.0.3; the image of this set under the homeomorphism $V \cong \mathbb{F}^n$ is closed and bounded, thus compact by the Heine-Borel Theorem.

Thus, by the extreme value Theorem, we have that $\|\cdot\|$ admits its maximum and minimum on this set.

$$\begin{aligned}c &:= \inf\{\|v\| \mid \|v\|_1 = 1\} \\ C &:= \sup\{\|v\| \mid \|v\|_1 = 1\}\end{aligned}$$

Thus, if $v \in V$ is such that $\|v\|_1 = 1$, we have

$$c\|v\|_1 \le \|v\| \le C\|v\|_1 \tag{15}$$

If $\|v\|_1 = 0$ then $v = 0$ and (15) clearly holds. If $\|v\|_1 \neq 1, 0$ then we divide by $\|v\|_1$ and reduce to the case of (15). $\square$