In the category of simplicial sets

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1 In the topos of simplicial sets

As a hands on example of the methodology presented in Section [1, §5] we consider the particular topos <u>sSet</u> of simplicial sets (Definition 1.0.4 below). Recall that associated to every simplicial set S is its *Geometric Realisation* |S| [20]. Although this notion will not be required for this Section, awareness of it will help with guiding intuition.

Definition 1.0.1. The simplex category Δ is the category whose objects are sets of the form $\{0, 1, ..., n\}$ for some n, these will be denoted [n]. The morphisms of this category are order preserving functions. For any positive integer k, let $\Delta_{\leq k}$ be the full subcategory of Δ with objects $\{[0], ..., [k]\}$.

There is a canonical way of factorising morphisms in the simplex category:

Definition 1.0.2. Define

$$\begin{split} \epsilon_n^i : [n-1] \to [n] \\ j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases} \end{split}$$

and

$$\eta_n^i: [n+1] \to [n]$$

$$j \mapsto \begin{cases} j & j \le i \\ j-1 & j > i \end{cases}$$

Theorem 1.0.3. Any morphism $[n] \to [m]$ in Δ can be written uniquely as

$$\epsilon_m^{i_1} \epsilon_{m-1}^{i_2} ... \epsilon_{m-k+1}^{i_l} \eta_{m-k}^{j_1} \eta_{m-k+1}^{j_2} ... \eta_{m-1}^{j_{k-1}} \eta_m^{j_k}$$

with $m \ge i_1 \ge i_2 \ge ... \ge i_l \ge 0$, and $0 \le j_1 \le j_2 \le ... \le j_k \le n$.

Definition 1.0.4. A simplicial set is a functor $\Delta^{op} \to \underline{Set}$, where \underline{Set} is the category of sets. The collection of these, along with the collection of natural transformations between them, forms a category \underline{sSet} , the category of simplicial sets.

Example 1.0.5. Consider the simplicial set S given by the colimit of the following diagram, the geometric realisation of which is the interval.

$$\begin{bmatrix}
0 \\
\downarrow^{\epsilon_0^1} \\
[1] \\
\uparrow^{\epsilon_1^1} \\
[0]$$

We construct the diagram (??) in this setting. There are 5 morphisms in this diagram, including the identity morphisms. These induce two morphisms:

$$\Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[1]} \xrightarrow{s_0} \Omega^{[0]} \coprod \Omega^{[1]} \coprod \Omega^{[0]} \tag{1}$$

We now give the description of these morphisms s_1, s_2 as terms t_1, t_2 . Let $z_1, z_2, z_3, z_4, z_5 : \Omega^{[0]}$, and let $z_6 : \Omega^{[1]}$. Here, the idea is that z_1 corresponds to $[0]^{\epsilon_0^1}$, z_2 corresponds to $[1]^{\mathrm{id}}$, z_3 to $[0]^{\mathrm{id}}$, z_4 to $[0]^{\epsilon_1^1}$, and z_5 to $[1]^{\mathrm{id}}$.

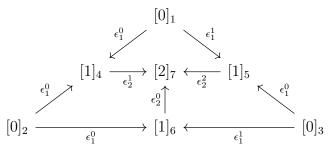
$$t_0 = \langle \langle z_1 \cup z_2, z_3 \cup z_4 \rangle, z_5 \rangle \tag{2}$$

$$t_1 = \langle \langle z_2, z_3 \rangle, \epsilon_0^1(z_1) \cup \epsilon_1^1(z_5) \cup z_6 \rangle \tag{3}$$

The interesting term here is t_1 . Reading t_1 from left to right, we first read $\langle z_2, z_3 \rangle$, indicating that the two copies of [0] are not glued together, and the next component $\epsilon_0^1(z_1) \cup \epsilon_1^1(z_5) \cup z_6$ which describes how the 0 dimensional components are glued to the 1 dimensional component. This fits our intuition of how the geometric realisation of the simplicial set S is constructed.

A more complicated example is given by the following.

Example 1.0.6. Let S be the simplicial set given by the colimit of the following diagram, the geometric realisation of which is a triangle. We have artificially added labellings to the copies of objects in this diagram for clarity.



We define the following variables.

$$\begin{array}{lll} z_1^1, z_2^1: \Omega^{[0]_1} & z_1^2, z_2^2: \Omega^{[0]_2} & z_1^3, z_2^3: \Omega^{[0]_3} \\ x_4: \Omega^{[1]_4} & x_5: \Omega^{[1]_5} & x_6: \Omega^{[1]_6} \\ y_6: \Omega^{[2]_7} & \end{array}$$

The term of interest is the following, we ignore the bracketting.

$$t_1 = \langle z_1^1 \cup z_2^1, z_1^2 \cup z_2^2, z_1^3 \cup z_2^3, \epsilon_1^0(z_2^1) \cup \epsilon_1^0(z_1^2), \epsilon_1^1(z_2^1) \cup \epsilon_1^0(z_1^3), \epsilon_1^0(z_2^2) \cup \epsilon_1^1(z_1^3), \epsilon_2^0(x_6) \cup \epsilon_2^1(x_4) \cup \epsilon_2^2(x_5), y_7 \rangle$$
(4)

Which, as in Example 1.0.5, agrees with the glueing instructions corresponding to the geometric realisation of S.

References

- [1] W. Troiani, Finite Colimits in the Internal Language of Topos
- [2] J. May, A Crash Course in Algebraic Topology, University of Chicago Press, Chicago, 1999.