MAST20026 Assignment 2 - Solutions
Semester 2 2022

PART A

PART B

(2) (*7 marks*) In this question we prove the following theorem about suprema.

Theorem. *Let $A \subseteq \mathbb{R}$ be bounded above and non-empty and let $\gamma \in \mathbb{R}$ be an upper bound of A in $\mathbb{R}$. We have $\sup A = \gamma$ if and only if for every $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$.*

Your task is to copy out the proof below, filling in the blanks. To aid the grader, please <u>underline</u> your filled in blanks. You will receive no marks for this question if you do not re-write out the proof in its entirety. (You are asked to re-write out the entire proof as a way to get you to practise writing sentences as part of your solutions.)

Proof. Let $A \subseteq \mathbb{R}$ be bounded above and let $\gamma \in \mathbb{R}$ be an upper bound of A in $\mathbb{R}$. By the Completeness Axiom, A has a supremum.

Assume $\sup A = \gamma$. Since γ is the supremum of A, we have $\gamma \geq$ _______ for every $\beta \in A$. We proceed by contradiction. Assume there exists $\epsilon > 0$ so that for every $\beta \in A$ we have $\beta \leq \gamma - \epsilon$. Therefore $\gamma - \epsilon$ is an __________ of A in $\mathbb{R}$. Notice $\gamma - \epsilon <$ ______. This contradicts that γ is the ________________ of A in $\mathbb{R}$. Therefore if $\sup A = \gamma$, then no element of A is greater than γ and for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$.

Assume for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$. We proceed by contradiction. Assume $\sup A \neq \gamma$. Since γ is an upper bound of A in $\mathbb{R}$ and $\sup A$ is the least such upper bound, we have $\sup A < \gamma$. Let $\delta = \gamma - \sup A$ and $\epsilon = \delta/2$. By hypothesis, there exists $r \in A$ so that $r > \gamma -$ _________. Recall $\delta = \gamma - \sup A$. Therefore

$$r > \gamma - \delta/2 = \sup A + \underline{\qquad} > \sup A.$$

Since $r \in A$ and $r > \sup A$, then $\sup A$ is not __________________________________, a contradiction. Therefore if for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$, then $\sup A = \gamma$. $\square$

Solution

Proof. Let $A \subseteq \mathbb{R}$ be bounded above and let $\gamma \in \mathbb{R}$ be an upper bound of A in $\mathbb{R}$. By the Completeness Axiom, A has a supremum.

Assume $\sup A = \gamma$. Since γ is the supremum of A, we have $\gamma \geq \underline{\beta}$ for every $\beta \in A$. We proceed by contradiction. Assume there exists $\epsilon > 0$ so that for every $\beta \in A$ we have $\beta \leq \gamma - \epsilon$. Therefore $\gamma - \epsilon$ is an <u>upper bound</u> of A in $\mathbb{R}$. Notice $\gamma - \epsilon < \underline{\gamma}$. This contradicts that γ is the <u>least upper bound</u> of A in $\mathbb{R}$. Therefore if $\sup A = \gamma$, then no element of A is greater than γ and for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$.

Assume for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$. We proceed by contradiction. Assume $\sup A \neq \gamma$. Since γ is an upper bound of A in $\mathbb{R}$ and $\sup A$ is the least such upper bound, we have $\sup A < \gamma$. Let $\delta = \gamma - \sup A$ and $\epsilon = \delta/2$. By hypothesis, there exists $r \in A$ so that $r > \gamma - \underline{\delta/2}$. Recall $\delta = \gamma - \sup A$. Therefore

$$r > \gamma - \delta/2 = \sup A + \underline{\delta/2} > \sup A.$$

Since $r \in A$ and $r > \sup A$, then $\sup A$ is not an upper bound of A in $\mathbb{R}$, a contradiction. Therefore if for every $\epsilon > 0$ there is an element of A greater than $\gamma - \epsilon$, then $\sup A = \gamma$. $\square$

(3) Let $A, B \subseteq \mathbb{R}$ such that A and B are non-empty and each bounded above in $\mathbb{R}$. Let $A+B = \{a+b \mid (a \in A) \wedge (b \in B)\}$. In this exercise we prove

$$\sup(A+B) = \sup A + \sup B.$$

Throughout this exercise you may assume that elements of $\mathbb{R}$ can be algebraically manipulated as you expect. You do not need to appeal to the Real Number Axioms in your solution.

(a) (*1 mark*) Write a sentence that explains how you know $\sup A$ and $\sup B$ both exist.

(b) (*3 marks*) Let $\sup A = \alpha$ and $\sup B = \beta$. Let $y \in A+B$. Using the definition of supremum and the definition of the set $A+B$, write a sentence or two that explains how you know $y \leq \alpha + \beta$.

(c) (*1 mark*) Let $\delta \in \mathbb{R}$ with $\delta > 0$ and let $\epsilon = \delta/2$. Write a sentence that explains how you know there exists $a \in A$ and $b \in B$ so that $a > \alpha - \epsilon$ and $b > \beta - \epsilon$.

(d) (*1 mark*) By algebraically manipulating inequalities from part (c), show $a + b > \alpha + \beta - \delta$.

(e) (*3 marks*) Prove

$$\sup(A+B) = \sup A + \sup B.$$

Refer your work from previous parts of this question as appropriate.

Solution

(a) Since A and B are both bounded above in $\mathbb{R}$, the Completeness Axiom implies $\sup A$ and $\sup B$ both exist.

(b) Since $y \in A+B$ there exists $a \in A$ and $b \in B$ so that $y = a+b$. By the definition of supremum we have $a \leq \alpha$ and $\beta \leq \beta$. Therefore $y < \alpha + \beta$.

(c) By the Theorem from Q3, since α and β are respectively suprema of A and B, for every $\epsilon > 0$ there exists $a \in A$ and $b \in B$ such that $a > \alpha - \epsilon$ and $b > \beta - \epsilon$.

(d)

$$a + b > \alpha + \beta - 2\epsilon = \alpha + \beta - \delta$$

(e) Let $\sup A = \alpha$ and $\sup B = \beta$. Let $y \in A+B$. By part (b), $\alpha + \beta$ is an upper bound of $A+B$ in $\mathbb{R}$. By part (d), for every $\delta > 0$ there exists an element of $A+B$ that is greater than $\alpha + \beta - \delta$. And so by the Theorem from Q3 we have $\sup A + B = \sup A + \sup B$.

(4) (*5 marks*) Let $q \in \mathbb{Q}$. Prove $(q)_{\mathbf{R}^\star}$ is a cut by proving it satisfies all parts of the definition of <u>cut</u>.

In your proof you may assume algebra and inequalities with rational numbers behave the way you expect. However, you may not assume there exists a rational number between any two rational numbers. If you want to use this fact, you must first give an algebraic proof of this fact.

Solution

To prove $(q)_{\mathbf{R}^\star}$ is a cut, we must prove it satisfies all three parts of the definition of <u>cut</u>:

1. Consider $x \in (q)_{\mathbf{R}^\star}$ and $y \in \mathbb{Q}$ so that $y < x$. Since $x \in (q)_{\mathbf{R}^\star}$, we have $x < q$. Since $y < x$ and $x < q$, we have $y < q$. And so by definition of the set $(q)_{\mathbf{R}^\star}$ it follows that $y \in (q)_{\mathbf{R}^\star}$

2. Consider $x \in x \in (q)_{\mathbf{R}^\star}$. Let $z = \frac{x+q}{2}$. We claim $z \in (q)_{\mathbf{R}^\star}$ and $x < z$.

Notice $\frac{x+q}{2} = \frac{x}{2} + \frac{q}{2}$. Since $x < q$ we have $\frac{x}{2} < \frac{q}{2}$. Therefore $z = \frac{x}{2} + \frac{q}{2} < \frac{q}{2} + \frac{q}{2} = q$.

Similarly, since $x < q$, we have $z = \frac{x}{2} + \frac{q}{2} > \frac{x}{2} + \frac{x}{2} = x$. Therefore

$$x < z < q$$

Since $z < q$ we have $z \in (q)_{\mathbf{R}^\star}$. Therefore there exists $z \in (q)_{\mathbf{R}^\star}$ so that $x < z$.

3. Since $q - 1 < q$ we have $q - 1 \in (q)_{\mathbf{R}^\star}$. Therefore $(q)_{\mathbf{R}^\star} \neq \{\}$.

Since $q + 1 > q$ we have $q - 1 \notin (q)_{\mathbf{R}^\star}$. Therefore $(q)_{\mathbf{R}^\star} \neq \mathbb{Q}$.

Since $(q)_{\mathbf{R}^\star}$ satisfies all three parts of the definition of cut, necessarily $(q)_{\mathbf{R}^\star}$ is a cut.