

An Introduction to untyped λ -calculus

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1 Introduction

Across the different definitions of an *algorithm* the execution of a computation is in some way a process of allowed transformations to an expression which either continues indefinitely or terminates after a finite number of steps. For example,

$$1 + 2 + 3 \tag{1}$$

is not *computed* yet, as it may be transformed to

$$3 + 3$$

which can then be transformed to 6. Of course, there is another route of computation which could have been taken, performing the second addition of (1) first obtains $1 + 5$, which then yields 6. The property that there exists the term 6 which both computation paths $1 + 2 + 3 \rightarrow 3 + 3$ and $1 + 2 + 3 \rightarrow 1 + 5$ can be computed to is the property of *confluence* of natural number addition.

The goal of this note is to introduce a system of computation, the *untyped λ -calculus*, and prove the Church-Rosser theorem which states that the untyped λ -calculus is *confluent*.

2 The Untyped λ -Calculus

The untyped λ -calculus sits among a collection of *type theories* which have been used as a foundation for mathematics [7], a foundation for logic [1], (although it was later found to be inconsistent [2]), and a foundation of certain programming languages such as AGDA, Lisp, Haskell, Coq, COC, etc. The untyped λ -calculus is the simplest of these theories, and although is rarely used in its original form, is a good entry point to many of the important ideas concerning the more modern type theories.

The main reference for this section is [4, §3.3].

Definition 1. Let \mathcal{V} be a (countably) infinite set of variables, and let \mathcal{L} be the language consisting of \mathcal{V} along with the special symbols

$$\lambda \quad . \quad (\quad)$$

Let \mathcal{L}^* be the set of words of \mathcal{L} , more precisely, an element $w \in \mathcal{L}^*$ is a finite sequence (w_1, \dots, w_n) where each w_i is in \mathcal{L} , for convenience, such an element will be written as $w_1 \dots w_n$. Now let Λ_p denote the smallest subset of \mathcal{L}^* such that

- if $x \in \mathcal{V}$ then $x \in \Lambda_p$,
- if $M, N \in \Lambda_p$ then $(MN) \in \Lambda_p$,
- if $x \in \mathcal{V}$ and $M \in \Lambda_p$ then $(\lambda x.M) \in \Lambda_p$

Λ_p is the set of **preterms**. A preterm M such that $M \in \mathcal{V}$ is a **variable**, if $M = (M_1 M_2)$ for some preterms M_1, M_2 , then M is an **application**, and if $M = (\lambda x, M')$ for some $x \in \mathcal{V}$ and $M' \in \Lambda_p$ then M is an **abstraction**.

In practice, it becomes unwieldy to use this notation for the preterms exactly, and so the following notation is adopted:

Definition 2. • For preterms M_1, M_2, M_3 , the preterm $M_1 M_2 M_3$ means $((M_1 M_2) M_3)$,
 • For variables x, y and a preterm M , the preterm $\lambda x y.M$ means $(\lambda x.(\lambda y.M))$.

The variables x which appear in the subpreterm M of a preterm $\lambda x.M$ are viewed as “markers for substitution”, (see Remark 3). For this reason, a distinction is made between the variable x and the variable y in, for example, the preterm $\lambda x.x y$:

Definition 3. Given a preterm M , let $\text{FV}(M)$ be the following set of variables, defined recursively

- if $M = x$ where x is a variable then $\text{FV}(M) = \{x\}$,
- if $M = M_1 M_2$ then $\text{FV}(M) = \text{FV}(M_1) \cup \text{FV}(M_2)$,
- if $M = \lambda x.M'$ then $\text{FV}(M) = \text{FV}(M') \setminus \{x\}$.

A variable $x \in \text{FV}(M)$ is a **free variable** of M , a variable x which appears in M but is not a free variable is a **bound variable**.

As mentioned, bound variables will be viewed as “markers for substitution”, so we define the following equivalence relation on Λ_p which relates a preterm M to M' if M can be obtained by replacing every bound occurrence of a variable x in M' with another variable y :

Definition 4. For any term M , let $M[x := y]$ be the preterm given by replacing every bound occurrence of x in M with y . Define the following equivalence relation on Λ_p : $M \sim_\alpha M'$ if there exists $x, y \in \mathcal{V}$ such that $M[x := y] = M'$, where no free variable of M becomes bound in $M[x := y]$. In such a case, we say that M is **α -equivalent** to M' .

Remark 1. The reason why we need to let x and y be such that no free variable of M becomes bound in $M[x := y]$ is so that a preterm such as $\lambda x.y$ does not get identified with the preterm $\lambda y.y$.

We are now in a position to define the underlying language of λ -calculus:

Definition 5. Let $\Lambda = \Lambda_p / \sim_\alpha$ be the set of **λ -terms**. The set of **free variables** of a λ -term $[M]$ is $\text{FV}(M)$, which can be shown to be well defined. For convenience, M will be written instead of $[M]$.

Now the dynamics of the computation of λ -terms will be defined.

Definition 6. Single step β -reduction \rightarrow_β is the smallest relation on Λ satisfying:

- the **reduction axiom**:
 - for all variables x and λ -terms M, M' , $(\lambda x.M)M' \rightarrow_\beta M[x := M']$, where $M[x := M']$ is the term given by replacing every free occurrence of x in M with M' ,
- the following **compatibility axioms**:
 - if $M \rightarrow_\beta M'$ then $(MN) \rightarrow_\beta (M'N)$ and $(NM) \rightarrow_\beta (NM')$,
 - if $M \rightarrow_\beta M'$ then for any variable x , $\lambda x.M \rightarrow_\beta \lambda x.M'$.

A subterm of the form $(\lambda x.M)M'$ is a **β -redex**, and $(\lambda x.M)M'$ **single step β -reduces** to $M[x := M']$.

Remark 2. Strictly, single step β reduction should be defined on preterms and then shown that a well defined relation is induced on terms, but this level of detail has been omitted for the sake of clarity.

Remark 3. The reduction axiom shows precisely in what sense a bound variable is a “marker for substitution”. For example, $(\lambda x.x)M \rightarrow_\beta M$ and $(\lambda y.y)M \rightarrow_\beta M$, which is why $\lambda x.x$ is identified with $\lambda y.y$.

It is through single step β -reduction that computation may be performed. In fact, λ -calculus is capable of performing natural number addition:

Example 1. Define the following λ -terms:

- $\text{ONE} := \lambda f x. f x$,

- $TWO := \lambda fx.f fx,$
- $THREE := \lambda fx.f f fx,$
- $PLUS := \lambda mnfx.mf(\underline{nf}x)$

then

$$\begin{aligned}
PLUS\ ONE\ TWO &= (\lambda mnfx.\underline{mf}(\underline{nf}x))(\underline{\lambda fx.fx})(\lambda fx.f fx) \\
&\rightarrow_{\beta} (\lambda nfx.(\lambda fx.\underline{f}x)\underline{f}(\underline{nf}x))(\lambda fx.f fx) \\
&\rightarrow_{\beta} (\lambda nfx.(\lambda x.\underline{f}x)(\underline{nf}x))(\lambda fx.f fx) \\
&\rightarrow_{\beta} (\lambda nfx.f\underline{nf}x)(\underline{\lambda fx.f fx}) \\
&\rightarrow_{\beta} (\lambda fx.f(\lambda fx.\underline{f}fx)\underline{fx}) \\
&\rightarrow_{\beta} (\lambda fx.f(\lambda x.f\underline{fx})\underline{x}) \\
&\rightarrow_{\beta} (\lambda fx.f f fx) = THREE
\end{aligned}$$

where each step is obtained by substituting the right most underlined λ -term in place of the left most underlined variable.

Historically, is this how Church first defined computable functions.

3 The Church-Rosser Theorem

Example 1 shows one possible sequence of β -reductions which reduces PLUS ONE TWO to THREE, however, different valid sequences exist. Moreover, no matter what path is taken, one can always find a path to THREE. The following theorem, which is the main point of this note, states that such a term always exists:

Definition 7. Multi step β -reduction \rightarrow (or simply **β -reduction**) is the smallest relation on Λ satisfying

- the **reduction axiom**:
 - if $M \rightarrow_{\beta} M'$ then $M \rightarrow M'$,
- **reflexivity**:
 - if $M = M'$ then $M \rightarrow M'$,
- **transitivity**:
 - if $M_1 \rightarrow M_2$ and $M_2 \rightarrow M_3$ then $M_1 \rightarrow M_3$

If $M \rightarrow M'$, then M **multistep β -reduces** to M' .

Theorem 1 (The Church Rosser Theorem). *If $M_1 \rightarrow M_2$ and $M_1 \rightarrow M_3$ then there exists a term M_4 such that the diagram*

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \\ M_3 & \longrightarrow & M_4 \end{array}$$

*commutes. That is, multi step β reduction is **confluent**.*

Proof. The proof will proceed by introducing a new relation \Rightarrow on Λ which satisfies the following:

- if $M \rightarrow_\beta M'$ then $M \Rightarrow M'$,
- if $M \Rightarrow M'$ then $M \rightarrow M'$,
- if $M_1 \Rightarrow M_2$ and $M_1 \Rightarrow M_3$ then there exists $M_4 \in \Lambda$ which makes the following diagram commute

$$\begin{array}{ccc} M_1 & \Longrightarrow & M_2 \\ \Downarrow & & \Downarrow \\ M_3 & \Longrightarrow & M_4 \end{array}$$

This is sufficient as if $M_1 = M^{11}, \dots, M^{1m}$ and $M_1 = M^{11}, \dots, M^{n1}$ are sequences of λ -terms such that

$$M^{11} \rightarrow_\beta M^{12} \rightarrow_\beta \dots \rightarrow_\beta M^{1m}$$

and

$$M^{11} \rightarrow_\beta M^{21} \rightarrow_\beta \dots \rightarrow_\beta M^{n1}$$

then the diagram

$$\begin{array}{c} M_1 = M^{11} \Longrightarrow M^{12} \Longrightarrow \dots \Longrightarrow M^{1m} = M_2 \\ \Downarrow \\ M^{21} \\ \Downarrow \\ \vdots \\ \Downarrow \\ M_3 = M^{n1} \end{array}$$

can be completed to the following commuting diagram

$$\begin{array}{ccccccc}
M_1 = M^{11} & \Longrightarrow & M^{12} & \Longrightarrow & \dots & \Longrightarrow & M^{1m} = M_2 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
M^{21} & \Longrightarrow & M^{22} & \Longrightarrow & \dots & \Longrightarrow & M^{2m} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\vdots & \Longrightarrow & \vdots & \Longrightarrow & \ddots & \Longrightarrow & \vdots \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
M_3 = M^{n1} & \Longrightarrow & M^{n2} & \Longrightarrow & \dots & \Longrightarrow & M^{nm}
\end{array}$$

from which it follows that M^{nm} satisfies the required properties of M_4 .

Towards this end, define the following relation on Λ :

Definition 8. Parallel β reduction \Rightarrow is the smallest relation on Λ satisfying

- the **reduction axiom**:
 - if $M \Rightarrow M'$ and $N \Rightarrow N'$ then $(\lambda x.M)N \Rightarrow M'[x := N']$,
- **reflexivity**:
 - if $M = M'$ then $M \Rightarrow M'$,
- the following **compatibility axioms**:
 - if $M \Rightarrow M'$ and $N \Rightarrow N'$ then $(MN) \Rightarrow (M'N')$,
 - if $M \Rightarrow M'$ then $\lambda x.M \Rightarrow \lambda x.M'$.

Remark 4. β -reduction might introduce new β -redexes which are not “visible” in the original term. For example

$$(\lambda x.xxx)(\lambda x.x) \rightarrow (\lambda x.x)(\lambda x.x)(\lambda x.x)$$

By transitivity, $(\lambda x.xxx)(\lambda x.x) \rightarrow \lambda x.x$. However, parallel β -reduction is not transitive, so $(\lambda x.xxx)(\lambda x.x) \not\Rightarrow \lambda x.x$. So $M \Rightarrow N$ only if N is obtained from M by reducing a collection of the β redexes in M and not ones which are introduced by this reduction process.

Clearly, if $M \rightarrow_\beta M'$ then $M \Rightarrow M'$ and if $M \Rightarrow M'$ then $M \rightarrow M'$. It remains to show that parallel β reduction is confluent.

First, we claim that if $M_1 \Rightarrow M_2$ and $N_1 \Rightarrow N_2$ then $M_1[x := N_1] \Rightarrow M_2[x := N_2]$. To

prove this claim, we proceed by inducting on the “minimum number of usages of the axioms of parallel β reduction required to prove that $M_1 \Rightarrow M_2$ ”. More precisely, let

$$S_0 := \{(M, M) \mid M \in \Lambda\}$$

and for $i > 0$, let S_i be the smallest set such that

- $S_{i-1} \subseteq S_i$,
- if $(M_1, M_2), (N_1, N_2) \in S_{i-1}$ then $((M_1 N_1), (M_2 N_2)) \in S_i$,
- if $(M, N) \in S_{i-1}$ then $(\lambda x.M, \lambda x.N) \in S_i$,
- if $(M_1, M_2), (N_1, N_2) \in S_{i-1}$ then $((\lambda x.M_1)N_1, N_2[x := M_2]) \in S_i$

Clearly, $M \Rightarrow N$ if and only if $(M, N) \in S := \cup_{i=0}^{\infty} S_i$. Define the following function:

$$\begin{aligned} \varphi : S &\rightarrow \mathbb{N} \\ (M, N) &\mapsto \min\{i \in \mathbb{N} \mid (M, N) \in S_i\} \end{aligned}$$

we proceed by (strong) induction on $\varphi(M_1, M_2)$. If $\varphi(M_1, M_2) = 0$ then $M_1 = M_2$ from which it follows that $M_1[x := N_1] \Rightarrow M_2[x := N_2]$. Say the result holds true for $\varphi(M_1, M_2) < k$. Then there are three cases, corresponding to M_1 being a variable, an application, or an abstraction (see Definition 1). If M_1 is a variable, then $\varphi(M_1, M_2) = 0$ and we have reduced to the base case. If $M_1 = \lambda y.M'_1$ then $M_1 \Rightarrow M_2$ implies that $M_2 = \lambda x.M'_2$. By the inductive hypothesis $M'_1[x := N_1] \Rightarrow M'_2[x := N_2]$ which implies

$$\begin{aligned} &\lambda y.(M'_1[x := N_1]) \Rightarrow \lambda y.(M'_2[x := N_2]) \\ \text{so, } &(\lambda y.M'_1)[x := N_1] \Rightarrow (\lambda y.M'_2)[x := N_2] \\ \text{so, } &M_1[x := N_1] \Rightarrow M_2[x := N_2] \end{aligned}$$

Lastly, say $M_1 = (M_1^1 M_1^2)$. Then either M_1^1 is an abstraction or it is not. If it is not then the proof is similar to the case where M_1 is an abstraction. Say $M_1^1 = (\lambda x.M_1^{1'})$. Now, either $M_2 = (\lambda x.M_2^{1'})M_2^2$, in which case the proof is similar to the case when M_1 is an abstraction, or $M_2 = M_2^{1'}[x := M_2^2]$. In this case, by the inductive hypothesis we have

$$M_1^{1'}[x = N_1] \Rightarrow M_2^{1'}[x = N_2]$$

and

$$M_1^2[x = N_1] \Rightarrow M_2^2[x = N_2]$$

from which it follows that

$$(\lambda x.M_1^{1'}[x := N_1])(M_1^2[x := N_1]) \Rightarrow (\lambda x.M_2^{1'}[x := N_2])(M_2^2[x := N_2])$$

which implies

$$M_1[x := N_1] = ((\lambda x.M_1^{1'})M_1^2)[x := N_1] \Rightarrow ((\lambda x.M_2^{1'})M_2^2)[x := N_2] = M_2[x := N_2]$$

which establishes the claim.

To finish the proof, say $M_1 \Rightarrow M_2$ and $M_1 \Rightarrow M_3$, we will show that there exists an appropriate term M_4 by induction on $l(M_1)$, the length of M_1 . This is broken up into cases in a similar way to the proof of the claim above, the only non-trivial case is when

$$M_1 = (\lambda x.M_1^{1'})M_1^2, \quad M_2 = M_2^{1'}[x := M_2^2], \quad M_3 = M_3^{1'}[x := M_3^2]$$

By the inductive hypothesis, there exists $M_4^{1'}$ and M_4^2 such that the diagrams

$$\begin{array}{ccc} M_1^{1'} & \Longrightarrow & M_2^{1'} \\ \Downarrow & & \Downarrow \\ M_3^{1'} & \Longrightarrow & M_4^{1'} \end{array} \quad \begin{array}{ccc} M_1^2 & \Longrightarrow & M_2^2 \\ \Downarrow & & \Downarrow \\ M_3^2 & \Longrightarrow & M_4^2 \end{array}$$

both commute. Now, by the claim proved above,

$$M_2^{1'}[x := M_2^2] \Rightarrow M_4^{1'}[x := M_4^2] \quad M_3^{1'}[x := M_3^2] \Rightarrow M_4^{1'}[x := M_4^2]$$

and so,

$$(\lambda x.M_2^{1'})M_2^2 \Rightarrow (\lambda x.M_4^{1'})M_4^2 \quad (\lambda x.M_3^{1'})M_3^2 \Rightarrow (\lambda x.M_4^{1'})M_4^2$$

ie, the diagram

$$\begin{array}{ccc} M_1 & \Longrightarrow & M_2 \\ \Downarrow & & \Downarrow \\ M_3 & \Longrightarrow & M_4 \end{array}$$

commutes, as required. □

References

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