Normal funct~~ors~~[ions], [the irrelevance of] power series, and [a new model of] λ-calculus.

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Where did Linear Logic come from?

Girard was considering a categorical model of the untyped λ-calculus where each term t in context $\{x_1, \ldots, x_n\}$ is interpretted as a normal functor:

$$[\![x_1, \ldots, x_n \mid t]\!] : \left(\underline{\mathrm{Set}}^A\right)^n \longrightarrow \underline{\mathrm{Set}}^A$$

where A is any countably infinite set.

Definition

A functor $F : \underline{\mathrm{Set}}^A \longrightarrow \underline{\mathrm{Set}}$ is **normal** if it

- preserves wide pullbacks,
- preserves filtered colimits.

Girard's Normal Functor Theorem

Theorem

Let $\mathscr{F} : \mathrm{Set}^A \longrightarrow \mathrm{Set}$ *be a functor. Then the following are equivalent.*

- *The functor* $\mathscr{F}$ *is normal.*
- *The functor* $\mathscr{F}$ *is isomorphic to an analytic functor.*
- *The functor* $\mathscr{F}$ *satisfies the finite normal form property.*

Definition

A functor $\mathscr{F} : \mathrm{Set}^A \longrightarrow \mathrm{Set}$ is **analytic** if there exists a family of sets $\{C_G\}_{G \in \mathrm{Set}^A}$ such that for all objects $F \in \mathrm{Set}^A$ and all morphisms $\mu : F \longrightarrow G$ we have

$$\mathscr{F}(F) = \coprod_{G \in \mathrm{Int}(A)} (C_G \times \mathrm{Set}^A(G, F))$$

Is all of this machinery necessary?

A critical definition of Girard's is the following.

Definition

Let A be a set. Define

$$\mathrm{Int}(A) \subseteq \mathrm{Set}^A$$

to be the set of integral functors G. That is

$$|\bigcup_{a \in A} G(a)| < \infty$$

and each $G(a) \in \mathbb{N}$. Ie, $G(a)$ is one of the following sets

$$0 = \varnothing, \quad 1 = \{0\} = \{\varnothing\}, \quad n = \{0, \ldots, n-1\}, \ldots$$

So... why not replace $\mathrm{I}(A)$ with the set of finite multisets of A?

Our new model

Girard's setup seems categorically unnatural. So we "decategorified" his model and came up with the following.

Girard	**Us**
Set^A	$\mathcal{Q}(A) := \{f : A \longrightarrow \mathbb{N} \cup \{\infty\}\}$
$\mathrm{Int}(A)$	$\mathcal{I}(A) := \{f : A \longrightarrow \mathbb{N} \mid f$ has finite support$\}$
Normal	Preserves supremums

We have the following important observation: let $f : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A)$ be an order preserving function which preserves supremums, and let $\underline{a} \in \mathcal{Q}(A)$ be arbitrary.

$$\begin{aligned} f(\underline{a}) &= f(\sup_{\underline{a}' \in \mathcal{I}(A)} \underline{a}' \cdot \delta_{\underline{a}' \leq \underline{a}}) \\ &= \sup_{\underline{a}' \in \mathcal{I}(A)} f(\underline{a}' \cdot \delta_{\underline{a}' \leq \underline{a}}) \\ &= \sup_{\underline{a}' \leq \underline{a}} f(\underline{a}') \end{aligned}$$

Thus f is determined by its restriction to $\mathcal{I}(A)$.

In fact we have a pair of functions

$$\begin{tikzcd} \mathrm{Norm}\Big(\mathcal{Q}(A)^n \times \mathcal{Q}(A), \mathcal{Q}(A)\Big) \arrow[d, shift left, "(-)^+"] \\ \mathrm{Norm}\Big(\mathcal{Q}(A)^n, \mathcal{Q}(\mathcal{I}(A) \times A)\Big) \arrow[u, shift left, "(-)^-"] \end{tikzcd}$$

defined as follows, where $\alpha \in \mathcal{Q}(A)^n, (\underline{a}, a) \in \mathcal{I}(A) \times A$.

$$\begin{aligned} f^+(\alpha)(\underline{a}, a) &= f(\alpha, \underline{a})(a) \\ g^-(\underline{\alpha}, \underline{a})(a) &= \sup_{\underline{a}' \in \mathcal{I}(A)} g(\alpha)(\underline{a}', a) \cdot \delta_{\underline{a}' \leq \underline{a}} \end{aligned}$$

We think of this as currying.

Lemma

We have that $(f^+)^- = f$, but in general $(g^-)^+ \neq g$.

Terms

Definition

Let $\underline{x} = \{x_1, \ldots, x_n\}$ be a set of variables and let t be a λ-term for which $\underline{x}$ is a valid context.

- **The term t is a variable** $x_i \in \underline{x}$. We define

$$\begin{aligned} [\![\underline{x} \mid x_i]\!] : \mathcal{Q}(A)^n &\longrightarrow \mathcal{Q}(A) \\ (\underline{a}_1, \ldots, \underline{a}_n) &\longmapsto \underline{a}_i \end{aligned}$$

  to be the projection map.

Application and abstraction

Since A is countably infinite, so is $\mathcal{I}(A) \times A$. We fix a bijection $q : \mathcal{I}(A) \times A \longrightarrow A$ which induces a bijection $\overline{q} : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(\mathcal{I}(A) \times A)$.

Definition

The term t is an application $t = t_1 t_2$.

$$(\overline{q}[\![\underline{x} \mid t_1]\!])^{-} \circ [\![\underline{x} \mid t_2]\!] : \mathcal{Q}(A)^n \longrightarrow \mathcal{Q}(A)$$

The term t is an abstraction $t = \lambda y.t'$.

$$[\![\underline{x}, y \mid t']\!] : \mathcal{Q}(A)^{n+1} \longrightarrow \mathcal{Q}(A)$$

We assume that this function is normal. We define

$$[\![\underline{x} \mid t]\!] := \overline{q}^{-1}[\![\underline{x}, y \mid t']\!]^{+} : \mathcal{Q}(A)^n \longrightarrow \mathcal{Q}(A)$$

Substitution Lemma, and denotation model Theorem

Lemma

Let t, s be λ-terms and $\underline{x} = \{x_1, \ldots, x_n\}$ and y be such that $\underline{x} \cup \{y\}$ is a valid context for t and $\underline{x}$ is a valid context for s. Then for any $\alpha \in \mathcal{Q}(A)^n$ we have

$$[\![\underline{x} \mid t[y := s]]\!](\alpha) = [\![\underline{x}, y \mid t]\!](\alpha, [\![\underline{x} \mid s]\!](\alpha))$$

Theorem

This is a denotational model of the λ-calculus. That is, if t is a λ-term and $\underline{x}$ a valid context for t and for s, then we have the following equality.

$$[\![\underline{x} \mid (\lambda y.t)s]\!] = [\![\underline{x} \mid t[y := s]]\!]$$

Extending to Linear Logic

- $\mathcal{I}(A)$ looks a lot like $!A$
- Recall that a normal function $f : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A)$ is equivalent to an order preserving function $\mathcal{I}(A) \longrightarrow \mathcal{Q}(A)$.
- Is there a property which f may satisfy which means it is determined by its restriction to A? Yes! Assume f is linear:

Let $\underline{a} \in \mathcal{Q}(A)$

$$\begin{aligned} f(\underline{a}) &= f(\sum_{a \in A} \underline{a}(a) \cdot \delta_a) \\ &= \sum_{a \in A} \underline{a}(a) \cdot f(\delta_a) \end{aligned}$$

So this model has a concept of linearity: *linearity*.

A genuine bijection

In fact we have a pair of bijections.

$$\begin{array}{c} \mathrm{Add}\big(\prod_{i=1}^n \mathcal{Q}(A_i) \times \mathcal{Q}(A), \mathcal{Q}(B)\big) \\ {\scriptstyle (-)^{\div}}\big\uparrow\big\downarrow{\scriptstyle (-)^{\times}} \\ \mathrm{Add}\big(\prod_{i=1}^n \mathcal{Q}(A_i), \mathcal{Q}(A \times B)\big) \end{array}$$

Defined as follows, for $\alpha \in \prod_{i=1}^n \mathcal{Q}(A_i), \underline{a} \in \mathcal{Q}(A), (a,b) \in A \times B$.

$$\begin{aligned} f^{\times}(\alpha)(a,b) &= f(\alpha, \delta_a)(b) \\ g^{\div}(\alpha, \underline{a}) &= \sum_{a \in A} \underline{a}(a) \cdot g(\alpha)(a,b) \end{aligned}$$

We use this to define a model of multiplicative, exponential linear logic.

A taste

Say the last rule of π is given by $(\mathrm{R} \multimap)$.

$$\dfrac{\begin{array}{c}\pi' \\ \vdots \\ \Gamma, A, \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \multimap B}\ (\mathrm{R} \multimap)$$

We define

$$[\![\pi]\!] := [\![\pi']\!]^{\times}$$

$$\text{Say } \Gamma = A_1, \ldots, A_n, \Delta = B_1, \ldots, B_m$$

$$\dfrac{[\![\pi']\!] : \prod_{i=1}^{n} \mathcal{Q}(A_i) \times \mathcal{Q}(A) \times \prod_{i=1}^{m} \mathcal{Q}(B_i) \longrightarrow \mathcal{Q}(B)}{[\![\pi]\!] = [\![\pi']\!]^{\times} : \prod_{i=1}^{n} \mathcal{Q}(A_i) \times \prod_{i=1}^{m} \mathcal{Q}(B_i) \longrightarrow \mathcal{Q}(A \times B)}\ \times$$

A taste

$$\dfrac{\begin{array}{c}\pi' \\ \vdots \\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi'' \\ \vdots \\ B, \Delta \vdash C\end{array}}{A \multimap B, \Gamma, \Delta \vdash C} \; (\mathrm{L} \multimap)$$

$$\llbracket \pi' \rrbracket : \prod_{i=1}^{n} \mathcal{Q}(A_i) \longrightarrow \mathcal{Q}(A)$$

$$\llbracket \pi'' \rrbracket : \prod_{i=1}^{m} \mathcal{Q}(B) \times \mathcal{Q}(B_i) \longrightarrow \mathcal{Q}(C)$$

We define

$$\begin{aligned} \llbracket \pi \rrbracket : &\mathcal{Q}(A \times B) \times \prod_{i=1}^{n} \mathcal{Q}(A_i) \times \prod_{i=1}^{m} \mathcal{Q}(B_i) \longrightarrow \mathcal{Q}(C) \\ &(f, \underline{\alpha}, \beta) \longmapsto \llbracket \pi'' \rrbracket \Big(\beta, \sum_{a \in A} \llbracket \pi' \rrbracket(\alpha)(a) \cdot f(a, (-) \Big) \end{aligned}$$

(cut)-reduction invariance

In the special case where $f = [\![\zeta]\!]^{\times}(\gamma)$ for some $\gamma \in \prod_{i=1}^{k} \mathcal{Q}(C_i)$, we obtain

$$
\begin{aligned}
(\alpha, \beta, \gamma) \longmapsto\ & [\![\pi'']\!]\Big(\beta, \sum_{a \in A} [\![\pi']\!](\alpha)(a) \cdot [\![\zeta]\!](\gamma)(a, -)\Big) \\
&= [\![\pi'']\!]\Big(\beta, ([\![\zeta]\!]^{\times})^{\div}(\gamma, [\![\pi']\!])(\alpha)\Big) \\
&= [\![\pi'']\!]\Big(\beta, [\![\zeta]\!](\gamma, [\![\pi']\!])(\alpha)\Big)
\end{aligned}
$$

This calculation proves equality of the interpretations of the two proofs:

$$(\alpha, \beta, \gamma) \longmapsto [\![\pi'']\!]\Big(\beta, \sum_{a \in A} [\![\pi']\!](\alpha)(a) \cdot [\![\zeta]\!](\gamma)(a, -)\Big)$$

$$(\alpha, \beta, \gamma) \longmapsto [\![\pi'']\!]\Big(\beta, [\![\zeta]\!](\gamma, [\![\pi']\!](\alpha))\Big)$$

$$\dfrac{\dfrac{\begin{array}{c}\zeta\\ \vdots\\ \Theta, A \vdash B\end{array}}{\Theta \vdash A \multimap B}\,(\mathrm{R} \multimap) \qquad \dfrac{\begin{array}{c}\pi'\\ \vdots\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi''\\ \vdots\\ \Delta, B \vdash C\end{array}}{A \multimap B, \Gamma, \Delta \vdash C}\,(\mathrm{L} \multimap)}{\Theta, \Gamma, \Delta \vdash C}\,(\mathrm{cut})$$

$$\longrightarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi'\\ \vdots\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\zeta\\ \vdots\\ \Theta, A \vdash B\end{array}}{\Gamma, \Theta \vdash B}\,(\mathrm{cut}) \qquad \begin{array}{c}\pi''\\ \vdots\\ \Delta, B \vdash C\end{array}}{\Gamma, \Theta, \Delta \vdash C}\,(\mathrm{cut})$$

Can we go further?

- Now that we have decategorified Girard's model, can we re-categorify it? At the moment, we are *not* anticipating our model to be an instance of a $\ast$-autonomous category with a comonad satisfying the relevant conditions, it seems as though we have something else.
- Once a more general framework is established, can we recover the famous relational model? Taking $\mathcal{Q} = \mathcal{P}$ and relaxing the requirement that our functions be order preserving is a start...
- Once we have recovered the relational model, can we transfer the differential structure across to obtain a model of the differential λ-calculus?

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