Quantum error correction and cut-elimination

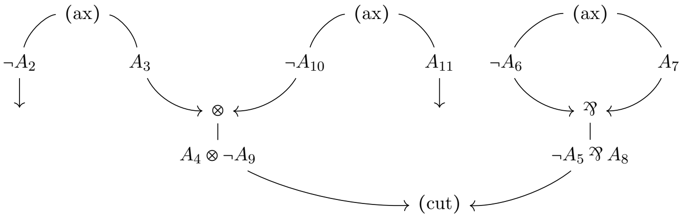
Daniel Murfet, William Troiani

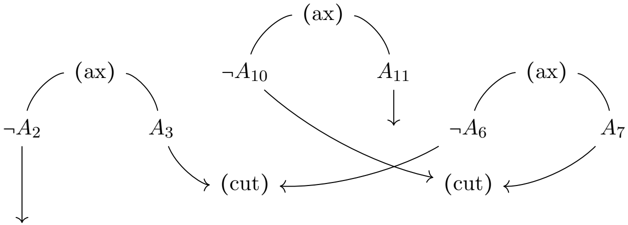
$$\begin{tikzcd}
\mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] \arrow[d, "g'"] & \mathcal{H}_{\pi}^{C_\pi} \arrow[d, "g"'] \\
\mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] & \mathcal{H}_{\pi}^{C_\pi}
\end{tikzcd}$$

"But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry that what is perfectly accurate is called geometrical; what is less so, is called mechanical. However, the errors are not in the art, but in the artificers." I. Newton, *Principia*

Geometry of Interaction

Proofs as codes, reduction as renormalisation.





Qubits

Dirac notation: $|0\rangle : \mathbb{C} \longrightarrow \mathcal{H}$ denotes the linear map $1 \longmapsto (1,0)$, and $|1\rangle$ denotes the linear map $1 \longmapsto (0,1)$.

- A **qubit** is a copy of the $\mathbb{C}$-Hilbert space $\mathbb{C}^2$.
- The **state** of a qubit $\mathbb{C}^2$ is a vector $|\psi\rangle \in \mathbb{C}^2$ of norm 1.
- A **measurement** on a state space $\mathcal{H}$ is a finite family of linear operators $\{M_m : \mathcal{H} \longrightarrow \mathcal{H}\}_{m \in \mathcal{M}}$ satisfying the **completeness condition**.

$$\sum_{m \in \mathcal{M}} M_m^{\dagger} M_m = I \tag{1}$$

- An element $m \in \mathcal{M}$ is an **outcome** (simply a set of labels).
- Associated to every measurement and state vector $|\psi\rangle$ there is a value, the **probability of outcome** m

$$p(m) := \langle\psi| M_m^{\dagger} M_m |\psi\rangle = \| M_m |\psi\rangle \|^2$$

- The **resulting state** after measurement $\{M_m\}_{m \in \mathcal{M}}$ and outcome m is:

$$\frac{M_m |\psi\rangle}{\sqrt{p(m)}} \tag{2}$$

Quantum Error Correction

$$Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, |\psi\rangle \in (\mathbb{C}^2)^{\otimes 3}$$

1. perform the following measurements:

   $$\langle\psi| Z_1 Z_2 |\psi\rangle \text{ with resulting state } |\psi'\rangle,$$

   followed by

   $$\langle\psi'| Z_2 Z_3 |\psi'\rangle$$

   let (r_1, r_2) be the values given by these measurements.
2. Now retrieve $|\varphi\rangle$ based on the values of r_1, r_2:
   - if $(r_1, r_2) = (1, 1)$, return $|\psi\rangle$,
   - if $(r_1, r_2) = (-1, 1)$, return $X_1 |\psi\rangle$,
   - if $(r_1, r_2) = (1, -1)$, return $X_3 |\psi\rangle$,
   - if $(r_1, r_2) = (-1, -1)$, return $X_2 |\psi\rangle$

$$Z_1 Z_2 |000\rangle = |000\rangle \qquad Z_1 Z_2 |001\rangle = |001\rangle$$
$$Z_1 Z_2 |010\rangle = -|010\rangle \qquad Z_1 Z_2 |011\rangle = -|011\rangle$$
$$Z_1 Z_2 |100\rangle = -|100\rangle \qquad Z_1 Z_2 |101\rangle = -|101\rangle$$
$$Z_1 Z_2 |110\rangle = |110\rangle \qquad Z_1 Z_2 |111\rangle = |111\rangle$$

Let $|\psi\rangle := a|010\rangle + b|101\rangle$ be a state, ie, an element of $\mathbb{H}^{\otimes 3}$. We perform the measurement $Z_1 Z_2$ followed by $Z_2 Z_3$:

$$\begin{aligned}
\langle\psi| Z_1 Z_2 |\psi\rangle &= (a\langle 010| + b\langle 101|) Z_1 Z_2 (a|010\rangle + b|101\rangle) \\
&= (a\langle 010| + b\langle 101|)(-a|010\rangle - b|101\rangle) \\
&= -a^2 - b^2 = -1
\end{aligned}$$

and

$$\begin{aligned}
\langle\psi| Z_2 Z_3 |\psi\rangle &= (a\langle 010| + b\langle 101|) Z_1 Z_2 (a|010\rangle + b|101\rangle) \\
&= (a\langle 010| + b\langle 101|)(-a|010\rangle - b|101\rangle) \\
&= -a^2 - b^2 = -1
\end{aligned}$$

We can infer from the fact that both of these came out as -1 that it was the second bit which was flipped, and so we can correct this. However, what is the impact of this measurement on the state? Again we calculate:

$$\begin{aligned} Z_1 Z_2(a\,|010\rangle + b\,|101\rangle) &= Z_1(-a\,|010\rangle + b\,|101\rangle) \\ &= -a\,|010\rangle - b\,|101\rangle \end{aligned}$$

and

$$\begin{aligned} Z_2 Z_3(-a\,|010\rangle - b\,|101\rangle) &= Z_2(-a\,|010\rangle + b\,|101\rangle) \\ &= a\,|010\rangle + b\,|101\rangle \end{aligned}$$

and so the measurements (in the end) did not impact our state.

Definition

A **quantum error correcting code (QECC)** is a pair $\mathcal{Q} = (\mathcal{H}, S)$ consisting of a state space $\mathcal{H}$ along with a set of operators S on $\mathcal{H}$. The elements of S are the **stabilisers**. The **codespace** $\mathcal{H}^S$ of $\mathcal{Q}$ is the maximal subspace of $\mathcal{H}$ invariant under all the operators in S.

In the previous example, $S = \{Z_1 Z_2, Z_2 Z_3\}$ and $\mathcal{H}^S = \mathrm{Span}\{|000\rangle, |111\rangle\}$.

Proof nets

Definition

There is an infinite set of **unoriented atoms** $X, Y, Z, \ldots$ and an **oriented atom** (or **atomic proposition**) is a pair $(X, +)$ or $(X, -)$ where X is an unoriented atom. The set of **pre-formulas** is defined as follows.

- Any atomic proposition is a pre-formula.
- If A, B are pre-formulas then so are $A \otimes B$, $A \parr B$.
- If A is a pre-formula then so is $\neg A$.

The set of **formulas** is the quotient of the set of pre-formulas by the equivalence relation $\sim$ generated by, for arbitrary formulas A, B and unoriented atom X, the following.

$$\neg(A \otimes B) \sim \neg A \parr \neg B, \qquad \neg(A \parr B) \sim \neg A \otimes \neg B$$
$$\neg(X, +) \sim (X, -), \qquad \neg(X, -) \sim (X, +)$$

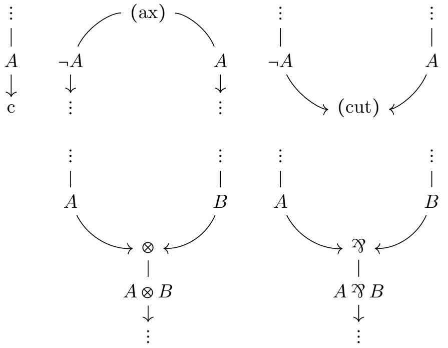
Proof structures

A **proof structure** is a directed multigraph. Edges: formulas. Nodes: $\{(\mathrm{ax}), (\mathrm{cut}), \otimes, \parr, \mathrm{c}\}$. Incoming edges: **premises**, outgoing edges: **conclusions**.

- Each node labelled (ax) has exactly two conclusions $\neg A, A$ and no premise.
- Each node labelled (cut) has exactly two premises $A, \neg A$ and no conclusion.
- Each node labelled $\otimes$ has exactly two premises A, B and one conclusion $A \otimes B$. These two premises are ordered. Smallest one: *left* premise A. Biggest one: *right* premise B.
- Each node labelled $\parr$ has exactly two ordered premises and one conclusion.
- Each node labelled c has exactly one premise and no conclusion. **Conclusions** of the proof structure.

Links

Let π be a proof structure. A **conclusion link** consists of a node labelled c along with its premise. An **axiom link** of π is a subgraph consisting of a node labelled (ax) along with its conclusions. A (cut) link consists of a node labelled (cut) along with its premises. A **tensor link** of π consists of a node labelled $\otimes$ along with its premises and conclusion. A **par link** consists of a node labelled $\parr$ along with its premises and conclusion.



Unoriented atoms of a *link*

Let π be a proof structure. To each link l in π we associate a set of unoriented atoms, denoted $[l]$. This definition depends on what type of link l is.

$$\text{Conclusion link } l: \quad \begin{array}{c} \vdots \\ | \\ A \\ \downarrow \\ \mathrm{c} \end{array}$$

We define $[l]$ to be the empty set.

$$[l] := \varnothing \tag{3}$$

Axiom/Cut links

For Axiom/Cut links:

Axiom/Cut link l:

$$\begin{tikzcd} & (\mathrm{ax}) \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] & \\ \neg A \arrow[d] & & A \arrow[d] \\ \vdots & & \vdots \end{tikzcd} \qquad \begin{tikzcd} \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\ \neg A \arrow[dr, bend right] & & A \arrow[dl, bend left] \\ & (\mathrm{cut}) & \end{tikzcd}$$

If A has set of unoriented axioms given by $\{X_1, ..., X_n\}$ then so does $\neg A$, and we define:

$$[l] := \{X_1, ..., X_n\} \tag{4}$$

Tensor/Par links

Tensor/Par link l:

$$\begin{tikzcd} \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\ A \arrow[dr, bend right] & & B \arrow[dl, bend left] \\ & \otimes \arrow[d, dash] & \\ & A \otimes B \arrow[d] & \\ & \vdots & \end{tikzcd} \qquad \begin{tikzcd} \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\ A \arrow[dr, bend right] & & B \arrow[dl, bend left] \\ & \parr \arrow[d, dash] & \\ & A \parr B \arrow[d] & \\ & \vdots & \end{tikzcd}$$

If A, B respectively have sets of unoriented atoms $\{X_1, ..., X_n\}, \{Y_1, ..., Y_m\}$ then the set of unoriented atoms of $A \otimes B$ and of $A \parr B$ is $\{X_1, ..., X_n, Y_1, ..., Y_m\}$, we define $[l]$ to be this set:

$$[l] := \{X_1, ..., X_n, Y_1, ..., Y_m\} \tag{5}$$

Total space

Let π be a proof structure with associated set of links $\mathcal{L}$. The exterior algebra of the complex Hilbert space freely generated by the set $[l]$ of unoriented atoms of l. ψ_X^l is the basis element corresponding to $X \in [l]$.

$$\mathcal{H}_l := \bigwedge \bigoplus_{X \in [l]} \mathbb{C}\psi_X^l$$

$$\mathcal{H}_\pi := \bigwedge \bigoplus_{l \in \mathcal{L}} \bigoplus_{X \in [l]} \mathbb{C}\psi_X^l \overset{*}{\cong} \otimes_{l \in \mathcal{L}, X \in [l]} \mathbb{C}\psi_X^l$$

The *set of qubits* $[\pi]$ of π is the following disjoint union, where $\mathcal{L}$ is the set of links of π.

$$[\pi] := \coprod_{l \in \mathcal{L}} [l] \qquad (6)$$

Notice that there are two copies of the atomic axioms coming from premises to cut links in $[\pi]$.
$(*)$ A *qubit ordering* of π is a bijection between $[\pi]$ and $\{1, \ldots, r\}$ where r is the number of elements of $[\pi]$.

Annihilation and creation operators

Given a generator ψ_i:

$$\psi_i : \bigwedge^d \mathbb{C}\underline{\psi} \longrightarrow \bigwedge^{d+1} \mathbb{C}\underline{\psi}$$

which behaves as follows on the basis vectors:

$$\psi_{i_1} \wedge \cdots \wedge \psi_{i_d} \longmapsto \psi_i \wedge \psi_{i_1} \wedge \cdots \wedge \psi_{i_d}$$

Associated to any element η of the vector space $(\bigoplus_{i=1}^n \mathbb{C}\psi_i)^*$ dual to the vector space $\bigoplus_{i=1}^n \mathbb{C}\psi_i$ there is a linear map:

$$\eta_\lrcorner : \bigwedge^d \mathbb{C}\underline{\psi} \longrightarrow \bigwedge^{d-1} \mathbb{C}\underline{\psi}$$

behaving as follows on the basis vectors:

$$\psi_{i_1} \wedge \cdots \wedge \psi_{i_d} \longrightarrow \sum_{j=1}^d (-1)^{j-1} \eta(\psi_{i_j}) \psi_{i_1} \wedge \cdots \wedge \hat{\psi}_{i_j} \wedge \cdots \wedge \psi_{i_d}$$

Bit operators

Lemma

Let $B_i : \{0,1\}^n \longrightarrow \{0,1\}^n$ *send* $a_1 \ldots a_n$ *to* $a_1 \ldots \overline{a_i} \ldots a_n$ *where* $\overline{0} = 1, \overline{1} = 0$. *Then*

$$(\psi_i + \psi_i^*)\psi^{\underline{a}} = (-1)^{a_1+\ldots+a_{i-1}}\psi^{B_i(\underline{a})}$$
$$(\psi_i - \psi_i^*)\psi^{\underline{a}} = (-1)^{a_1+\ldots+a_i}\psi^{B_i(\underline{a})}$$

we define the following linear functions on $\bigwedge \mathbb{C}\psi_{U_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{U_r}$, for $i = 1, \ldots, r$, determined by linearity along with the following equations.

$$X_i(\psi^{\underline{a}}) = \psi^{B_i(\underline{a})} \qquad Z_i(\psi^{\underline{a}}) = \begin{cases} \psi^{\underline{a}}, & a_i = 0 \\ -\psi^{\underline{a}}, & a_i = 1 \end{cases}$$

Edges

Let π be a proof structure and v, v' vertices respectively corresponding to links l, l' which are not conclusion links. Let $e : v \longrightarrow v'$ be an edge and let A be the formula labelling e. For every oriented atom (U, y_u) of A we have a corresponding generator $\psi_U \in \mathcal{H}_l$ and $\psi'_U \in \mathcal{H}_{l'}$. The *edge operator* associated to e and U is:

$$\Theta_U^{l \longrightarrow l'} := y_u(\psi'_U - y_u \psi'^*_U)(\psi_U + y_u \psi^*_U) : \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi$$

Ranging over all edges $e : v \longrightarrow v' \in E$ of π, where the vertices v, v' respectively correspond to links l, l' and every unoriented atom $U \in [A]$ of the formula A labelling e, we obtain the *stabilisers* of π.

$$S_\pi := \{\Theta_U^{l \longrightarrow l'}\}_{e \in E, U \in [A]}$$

Lemma

Choose a qubit ordering $U_1 < \cdots < U_r$ of π. Choose an edge $e : v \longrightarrow v'$, where v, v' respectively correspond to links l, l' connecting non-conclusion links. Let (U, y_u) be an oriented atom of the formula A labelling e and suppose the corresponding unoriented atoms of the links are $U_i \in [l], U_j \in [l']$ as in the diagram below.

$$\overset{l}{\bullet} \quad \xrightarrow[\mathcal{H}_i]{\quad U_i \quad} \quad A \quad \xrightarrow[\mathcal{H}_j]{\quad U_j \quad} \quad \overset{l'}{\bullet}$$

Let Θ_U be the corresponding edge operator on $\mathcal{H}_\pi$.

1. *If $y_u = +$ and $j < i$ the following diagram commutes, in what follows the morphism Q.*

$$\begin{tikzcd}
\bigwedge \mathbb{C}\psi_{U_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{U_r} \arrow[r, "Q"] \arrow[d, "X_j Z_{j+1} \cdots Z_{i-1} X_i"'] & \mathcal{H}_\pi \arrow[d, "\Theta_U"] \\
\bigwedge \mathbb{C}\psi_{U_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{U_r} \arrow[r, "Q"'] & \mathcal{H}_\pi
\end{tikzcd}$$

The QECC corresponding to a proof net

The *Quantum Error Correcting Code* $[\![\pi]\!]$ associated to a proof structure π is the pair consisting of the Hilbert space $\mathcal{H}_\pi$ and the stabiliser code S_π.

$$[\![\pi]\!] := (\mathcal{H}_\pi, S_\pi)$$

The *codespace* of π is the invariant subspace

$$\mathcal{H}_\pi^{S_\pi} = \{|\varphi\rangle \in \mathcal{H}_\pi \mid \forall X \in S_\pi, X|\psi\rangle = |\psi\rangle\}$$

Dynamics

Theorem (The Reduction Theorem)

For each reduction $\gamma : \pi \longrightarrow \pi'$ there exists a subset $C_\pi \subseteq S_\pi$ and an isomorphism:

$$\hat{\gamma} : \mathcal{H}_{\pi'} \longrightarrow \mathcal{H}_\pi^{C_\pi} \tag{7}$$

such that for every $g \in S_\pi \setminus C_\pi$ there is a unique $g' \in S_{\pi'}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] \arrow[d, "g'"] & \mathcal{H}_\pi^{C_\pi} \arrow[d, "g"'] \\ \mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] & \mathcal{H}_\pi^{C_\pi} \end{tikzcd} \tag{8}$$

and this map $g \longmapsto g'$ is a bijection $S_\pi \setminus C_\pi \longrightarrow S_{\pi'}$.

We label the relevant links of π, π' according to the following diagram.

$$
\begin{tikzcd}
 & \overset{l_{(\mathrm{ax})}}{(\mathrm{ax})} \arrow[dl, dash, bend right] \arrow[dr, dash, bend left] & & & \overset{l}{\bullet} \arrow[d, dash] & \overset{l}{\bullet} \arrow[d, dash] \\
A \arrow[d] & & \neg A \arrow[dr, bend right] & & A \arrow[dl, bend left] & A \arrow[d] \\
\overset{m}{\bullet} & & & \overset{l_{(\mathrm{cut})}}{(\mathrm{cut})} & & \overset{m}{\bullet}
\end{tikzcd}
\tag{9}
$$

For each oriented atom (U, y) of A we define a $\mathbb{Z}_2$-degree zero map for $y = +$ by:

$$
\begin{aligned}
\gamma_U : \bigwedge \mathbb{C}\psi_U^{l} &\longrightarrow \bigwedge \mathbb{C}\psi_U^{l} \otimes \bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{cut})}} \otimes \bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{ax})}} && (10)\\
|j\rangle &\longmapsto \frac{1}{\sqrt{2}}\big(|+++\rangle + (-1)^j\,|---\rangle\big) && (11)
\end{aligned}
$$

If $y = -$ then γ_U has the same domain and formula, but its codomain is:

$$
\bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{ax})}} \otimes \bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{cut})}} \otimes \bigwedge \mathbb{C}\psi_U^{l} \tag{12}
$$

Making the dynamics of the model precise...

The remaining question:

$$\mathcal{H} \longmapsto \mathcal{H}^{C_\pi}$$

In fact, Quantum Error Correction can be recast in the framework of *normalisation*, which is a deep idea coming from physics, which allows us to talk about the same quantum system but at different *scales*. It is more natural to think of the process of transforming $\mathcal{H}_\pi$ to $\mathcal{H}_\pi^{C_\pi}$ in the language of renormalisation, and indeed that is what we are currently making precise.

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