Homological Algebra

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Contents

1 Abelian categories

We hold off for as long as possible from specialising to a particular category.

Definition 1.0.1. Let $\mathscr{C}$ be a category and $\{C_i\}_{i=1}^n$ a finite set of objects. A **direct sum** of $\{C_i\}_{i=1}^n$ is a tuple $(\bigoplus_{i=1}^n C_i, \{\pi_i : \bigoplus_{i=1}^n C_i \longrightarrow C_i,\}_{i=1}^n, \{\iota : C_i \longrightarrow \bigoplus_{i=1}^n\}_{i=1}^n)$ such that $(\bigoplus_{i=1}^n C_i, \{\pi_i\}_{i=1}^n)$ is a finite product and $(\bigoplus_{i=1}^n C_i, \{\iota_{i=1}^n\})$ is a finite coproduct.

The finite product of the empty set is defined by to be a zero object, that is, an object which is both initial and terminal.

Definition 1.0.2. Let $f : A \longrightarrow B$ be a morphism in a category which admits a 0 object. A **kernel for** f is a pair $(\ker f, \iota)$ consisting of an object ker and a monomorphism $\iota : \ker f \longrightarrow A$ satisfying:

if $g : C \longrightarrow A$ is such that fg factors through 0 then g factors through $\ker f$. Diagrammatically, if g is such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r] \arrow[d, "g"'] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{1}$$

then there exists a (necessarily unique) morphism $h : C \longrightarrow \ker f$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[d, "h"'] \arrow[dr, "h"] & \\ \ker f \arrow[r, tail, "\iota"'] & A \end{tikzcd} \tag{2}$$

A **cokernel** for f is defined analogously.

Definition 1.0.3. A category $\mathscr{A}$ is **abelian** if it satisfies the following properties:

- for every pair of objects $A, B \in \mathscr{A}$, the set $\hom(A, B)$ is an abelian group,
- composition is bilinear,
- $\mathscr{A}$ admits all finite direct sums (including the empty direct sum)
- every morphism admits a kernel and a cokernel,
- every monorphism f is the kernel of its cokernel,
- every epimorphism f is the cokernel of its kernel,
- every morphism factors as a epimorphism followed by a monomorphism.

Throughout, $\mathscr{A}$ denotes an abelian category.

Definition 1.0.4. The **image** of a morphism $f : A \longrightarrow B$ in $\mathscr{A}$ is the kernel of the cokernel of f.

Definition 1.0.5. A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3}$$

in $\mathscr{A}$ is **exact** is f is a monomorphism, g is an epimorphism, and $\ker g \cong \operatorname{im} f$.

Definition 1.0.6. Let $\mathscr{B}$ be another abelian category. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is **additive** if the induced map for all $A, B \in \mathscr{A}$

$$\hom(A, B) \longrightarrow \hom(FA, FB) \tag{4}$$

is a homomorphism.

An additive, covariant functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is **left exact** if given a short exact sequence in $\mathscr{A}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{5}$$

the resulting sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \tag{6}$$

is exact.

If F is *contravariant* then it is **left exact** if the resulting sequence

$$0 \longrightarrow FC \xrightarrow{Fg} FB \xrightarrow{Ff} FA \tag{7}$$

is exact.

We make the analogous definitions for **right exact** functors (covariant, or contravariant).

An additive functor F is **exact** if it is left and right exact.

Lemma 1.0.7. *Let $D \in \mathscr{A}$. The functor* $\hom(\mathrm{D}, \_)$ *is left exact.*

To prove Lemma 1.0.7 we first prove it in the special setting where the abelian category $\mathscr{A}$ is the category of left R-modules, where R is a commutative ring with unit.

Proof of special case. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{8}$$

be an exact sequence of R-modules.

To prove that $f \circ \_$ is injective amounts to proving that f is a monomorphism which is equivalent to f being injective.

That $(g \circ \_) \circ (f \circ \_) = 0$ follows immediately from the fact that $gf = 0$.

We now show that $\ker(g \circ \_) = \operatorname{im}(f \circ \_)$. Let $h : D \longrightarrow B$ be such that $gh = 0$. Notice that for every $d \in D$ that $gh(d) = 0 \Rightarrow \exists a_d \in A$ such that $f(a_d) = h(d)$, by exactness of (8). Moreover, by injectivity of f it follows that this a_d is unique. We define the following homomorphism

$$\begin{aligned} h' : D &\longrightarrow A \\ d &\longmapsto a_d \end{aligned}$$

This is indeed a homomorphism, to see this, notice that if $d_1, d_2 \in D$ are such that $f(a_{d_1}) = f(a_{d_2}) = h(d)$ then

$$a_{d_1+d_2} - a_{d_1} - a_{d_2} \in \ker f = 0 \tag{9}$$

similarly, if $d \in D$ with $f(a_d) = h(d)$ and $r \in R$ then $f(a_{rd}) = h(rd) = rh(d) = rf(a_d)$ and so

$$a_{rd} - ra_d \in \ker f = 0 \tag{10}$$

We now define

$$\begin{aligned} \Phi : \ker(g \circ \_) &\longrightarrow \operatorname{im}(f \circ \_) \\ h &\longmapsto f \circ h' \end{aligned}$$

which by commutativity of the following diagram:

$$\begin{tikzcd} D \arrow[d, "h'"'] \arrow[dr, "h"] & \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{11}$$

is clearly a bijection. □

The heart of the proof is given by the construction of h' given h. We now come up with a definition of this map which does not appeal to "elements of the objects D, A" which will generalise the proof to the setting of an arbitrary abelian category.

Proof of Lemma 1.0.7. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{12}$$

be an exact sequence in $\mathscr{A}$, which in particular means there is an isomorphism $\varphi : \ker g \cong \operatorname{im} f$.Now, say $h : D \longrightarrow B \in \ker(g \circ \_)$, that is, say $gh = 0$. We thus have that h factors through the kernel of g, that is, there exists a morphism $\hat{h} : D \longrightarrow \ker g$ such that the following diagram commutes

$$\begin{tikzcd} \ker g \arrow[dr, tail] & A \arrow[d, tail, "f"] \\ D \arrow[u, dashed, "\hat{h}"] \arrow[r, "h"'] & B \arrow[d, two heads, "g"] \\ & C \end{tikzcd} \tag{13}$$

Since $\mathscr{A}$ is abelian and f is monic, we have that it is the kernel of its cokernel, that is, there exists an isomorphism $A \cong \operatorname{coker} \ker f$ rendering the following diagram commutative

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\cong"'] & B \\ \ker(\operatorname{coker} f) \arrow[ur] & \end{tikzcd} \tag{14}$$

Moreover, by exactness of 12 we have $\ker g \cong \operatorname{im} f$ which implies $\ker(\operatorname{coker} f) \cong \ker g$. Let $\varphi : \ker g \longrightarrow A$ denote the resulting isomorphism. We now define the following homomorphism

$$\begin{aligned} \Phi : \ker(\_ \circ f) &\longrightarrow \operatorname{im}(\_ \circ g) \\ h &\longmapsto f\varphi\hat{h} \end{aligned}$$

It remains to show that this is an isomorphism, but this follows easily from the fact that f is a monomorphism and φ an isomorphism. □

Lemma 1.0.8. *Let $D \in \mathscr{A}$. The functor* $\hom(\_, D)$ *is left exact.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{15}$$

be short exact and consider

$$0 \longrightarrow \hom(C, D) \xrightarrow{-\circ g} \hom(B, D) \xrightarrow{-\circ f} \hom(A, D) \tag{16}$$

Let $h : D \longrightarrow C$ be a morphism such that $h \circ g = 0$. We have $h \circ g = 0 = 0 \circ g$ which implies $h = 0$ by way of g being an epimorphism.

Next we show $\ker(D \circ f) \cong \operatorname{im}(D \circ g)$. Let $h : B \longrightarrow D$ be such that $h \circ f = 0$. Then consider the following commutative diagram

$$\begin{tikzcd} A \arrow[d, tail, "f"] & \\ B \arrow[r, "h"] \arrow[d, two heads, "g"] \arrow[dr] & D \\ C & \operatorname{coker} f \arrow[u, dashed, "\hat{h}"'] \end{tikzcd} \tag{17}$$

where the dashed arrow $\hat{h}$ exists by the universal property of $\operatorname{coker} f$. Now, since $\mathscr{A}$ is an abelian category, g is the cokernel of its kernel, in other words, there is an isomorphism $C \cong \operatorname{coker}(\ker f)$ rendering the following diagram commutative

$$\begin{tikzcd} B \arrow[r, two heads, "g"] \arrow[dr] & C \arrow[d, "\cong"] \\ & \operatorname{coker}(\ker g) \end{tikzcd} \tag{18}$$

Moreover, by exactness of (15), we have $\ker g \cong \operatorname{im} f$ which implies $\operatorname{coker}(\ker g) \cong \operatorname{coker}(f)$. Let $\varphi : C \longrightarrow \operatorname{coker} f$ denote the resulting isomorphism. We now define the following function

$$\begin{aligned} \Phi : \ker(\_ \circ f) &\longrightarrow \operatorname{im}(\_ \circ g) \\ h &\longmapsto h'\varphi g \end{aligned}$$

It remains to show that this is an isomorphism, but this follows easily from the fact that g is an epimorphism and φ is an isomorphism. □

2 Resolutions

Throughout, A is a commutative ring with unit.

2.1 Short exact sequences

Definition 2.1.1. Given two short exact sequences

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{19}$$

and

$$0 \longrightarrow M' \longrightarrow N' \longrightarrow P' \longrightarrow 0 \tag{20}$$

which we denote by S_1, S_2 respectively, a **morphism of short exact sequences** $f : S_1 \longrightarrow S_2$ is a triple of module homomorphisms $f_1 : M \longrightarrow M'$, $f_2 : N \longrightarrow N'$, $f_3 : P \longrightarrow P'$ which render the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f_1"] & N \arrow[r] \arrow[d, "f_2"] & P \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0 \end{tikzcd} \tag{21}$$

Definition 2.1.2. A short exact sequence of A-modules

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \tag{22}$$

is **split** (or **splits**) if it is isomorphic to the short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus P \longrightarrow P \longrightarrow 0 \tag{23}$$

Lemma 2.1.3. *Given a short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \tag{24}$$

which we denote by S, the following are equivalent:

1. *S is split,*
2. *g admits a right inverse,*
3. *f admits a left inverse.*

Proof. First assume S is split. Then we have an isomorphism

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f_1"] & N \arrow[r, "g"] \arrow[d, "f_2"] & P \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & M \arrow[r, "f'"] & M \oplus P \arrow[r, "g'"] & P \arrow[r] & 0 \end{tikzcd} \tag{25}$$

The functions f', g' respectively admit left and right inverses given by $m \longmapsto (m, 0)$ and $p \longmapsto (0, p)$. Thus (1) implies (3) and (2).

Now say g admits a right inverse, $h : P \longrightarrow N$. We have $gh = \mathrm{id}_P$ and so h is injective. We thus have $P \cong \operatorname{im} h$ and similarly, $M \cong \operatorname{im} f$.

Moreover, there is a map $l : N \longrightarrow \operatorname{im} f \oplus \operatorname{im} h$ given by $n \longmapsto (n - hg(n), hg(n))$ rendering the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f"] & N \arrow[r, "g"] \arrow[d, "l"] & P \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \operatorname{im} f \arrow[r] & \operatorname{im} f \oplus \operatorname{im} h \arrow[r] & \operatorname{im} h \arrow[r] & 0 \end{tikzcd} \tag{26}$$

it then follows from the five Lemma that l is an isomorphism. The bottom row of (26) is clearly isomorphic to

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0 \tag{27}$$

Lastly, assume that f admits a right inverse $h : N \longrightarrow M$. Since $hf = \mathrm{id}_M$ we have that h is surjective. Thus $N/\ker h \cong M$ and similarly, $N/\ker g \cong P$. Now, there is a map $l : N \longrightarrow N/\ker h \oplus N/\ker g$ given by the sum of the respective projection maps which fits into a commutative diagram similar to (26). The result follows similarly to before. □

2.2 The tensor product

The tensor product admits the following universal property:

Lemma 2.2.1. *Let M, N, P be modules and denote the set of bilinear transformations $M \times N \longrightarrow P$ by* $\mathrm{Bil}(M \times N, P)$. *There is the following natural isomorphism*

$$\mathrm{Bil}(M \oplus N, P) \cong \mathrm{Hom}(M \otimes N, P) \tag{28}$$

Proof. Easy. □

The tensor product is distributive, that is:

Lemma 2.2.2. *Let M, N, P be modules, then*

$$M \otimes (N \oplus P) \cong (M \otimes N) \oplus (M \otimes P) \tag{29}$$

Proof. We define an explicit map and an inverse. By Lemma 2.2.1 it suffices to define the following bilinear map:

$$\begin{aligned} \varphi : M \oplus (N \oplus P) &\longrightarrow (M \otimes N) \oplus (M \otimes P) \\ (m, (n, p)) &\longmapsto (m \otimes n, m \otimes p) \end{aligned}$$

Let $\overline{\varphi}$ map induced by applying Lemma 2.2.1, we define an explicit inverse to $\overline{\varphi}$. Again, using Lemma 2.2.1 and the universal property of the direct sum, it suffices to define the following two maps

$$\begin{aligned} \psi_1 : M \oplus N &\longrightarrow M \otimes (N \oplus P) \\ (m, n) &\longmapsto m \otimes (n, 0) \end{aligned} \qquad \begin{aligned} \psi_2 : M \oplus P &\longrightarrow M \otimes (N \oplus P) \\ (m, p) &\longmapsto m \otimes (0, p) \end{aligned}$$

Let $\overline{\psi} : (M \otimes N) \oplus (M \otimes P) \longrightarrow M \otimes (N \oplus P)$ denote the induced map. We see:

$$\begin{aligned} \overline{\psi}\overline{\varphi}(m \otimes (n, p)) &= \overline{\psi}(m \otimes n, m \otimes p) \\ &= m \otimes (n, 0) + m \otimes (0, p) \\ &= m \otimes (n, p) \end{aligned}$$

and

$$\begin{aligned} \overline{\varphi}\overline{\psi}(m \otimes n, m' \otimes p) &= \overline{\varphi}(m \otimes (n, 0) + m' \otimes (0, p)) \\ &= (m \otimes n + m' \otimes 0, m \otimes 0 + m' \otimes p) \\ &= (m \otimes n, m' \otimes p) \end{aligned}$$

□

In fact, the proof of Lemma 2.2.2 generalises:

Lemma 2.2.3. *The tensor product commutes with arbitrary direct sum, more precisely, if $M_{i \in I}$ is is a collection of modules and N is also a module, then*

$$N \otimes \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i) \tag{30}$$

Proof. Following the proof of Lemma 2.2.2 we define

$$\begin{aligned} \varphi : N \oplus \bigoplus_{i \in I} M_i &\longrightarrow \bigoplus_{i \in I} (N \otimes M_i) \\ (n, (m_i)_{i \in I}) &\longmapsto (n \otimes m_i)_{i \in I} \end{aligned}$$

which is well defined as since $(m_i)_{i\in I}$ satisfies $m_i = 0$ for all but finitely many i, the same can be said of $(n \otimes m_i)_{i\in I}$. We also define an I-indexed family of maps

$$\begin{aligned} \psi_i : N \oplus M_i &\longrightarrow N \oplus \bigoplus_{i\in I} M_i \\ (n, m) &\longmapsto (n, \iota_i m) \end{aligned}$$

where

$$\iota_i : M_i \longrightarrow \bigoplus_{i\in I} M_i \tag{31}$$

is the canonical inclusion map. It is then easy to see that the induced maps $\overline{\varphi}$ and $\overline{\psi}$ are mutual inverse to each other. □

2.3 Flat modules

Definition 2.3.1. A module M is flat if given any short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0 \tag{32}$$

the induced sequence:

$$0 \longrightarrow N_1 \otimes M \longrightarrow N_2 \otimes M \longrightarrow N_3 \otimes M \longrightarrow 0 \tag{33}$$

is also short exact.

Below, Lemma 2.3.3 states that in the setting of Definition 2.3.1 the sequence

$$N_1 \otimes M \longrightarrow N_2 \otimes M \longrightarrow N_3 \otimes M \longrightarrow 0 \tag{34}$$

is always short exact.

Definition 2.3.2. Let $\underline{\mathrm{Mod}_A}$ denote the category of (left) A-modules.

A functor $F : \underline{\mathrm{Mod}_A} \longrightarrow \underline{\mathrm{Mod}_A}$ is **right exact** if given a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{35}$$

the induced sequence

$$F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3) \longrightarrow 0 \tag{36}$$

is exact.

Clearly, Definition 2.3.2 need not be bound to the particular category chosen, but we worth in this restricted setting for now.

Lemma 2.3.3. *For any module M, the functor $\_\_ \otimes M$ is right exact.*

Proof. Let

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0 \tag{37}$$

be an arbitrary short exact sequence and consider

$$N_1 \otimes M \xrightarrow{f\otimes \mathrm{id}} N_2 \otimes M \xrightarrow{g\otimes \mathrm{id}} N_3 \otimes M \longrightarrow 0 \tag{38}$$

It is clear that g surjective implies $g \otimes \mathrm{id}$ is surjective. It is also clear that $gf = 0 \Rightarrow (g \otimes \mathrm{id})(f \otimes \mathrm{id}) = 0$. Thus, it remains to show:

$$\mathrm{im}(f \otimes \mathrm{id}) \supseteq \ker(g \otimes \mathrm{id}) \tag{39}$$

We do this by showing there exists an isomorphism

$$(N_2 \otimes M)/(\mathrm{im}(f \otimes \mathrm{id})) \cong N_3 \otimes M \tag{40}$$

The map $g \otimes \mathrm{id}$ induces a homomorphism $\overline{g \otimes \mathrm{id}} : (N_2 \otimes M)/\operatorname{im}(f \otimes \mathrm{id}) \longrightarrow N_3 \otimes M$, we construct a right inverse.

Let $h : N_3 \otimes M \longrightarrow (N_2 \otimes M)/(\operatorname{im} f \otimes \mathrm{id})$ be such that $h(n \otimes m) = [n' \otimes m]_{\mathrm{im}(f \otimes \mathrm{id})}$. where n' is an arbitrary element of N_2 such that $g(n') = n$. This is well defined, as if $n'' \in N_2$ is also such that $g(n'') = n$, then $n' - n'' \in \ker g = \operatorname{im} f$ which means $[n' \otimes m]_{\mathrm{im}(f \otimes \mathrm{id})} = [n'' \otimes m]_{\mathrm{im}(f \otimes \mathrm{id})}$. Notice that h is clearly a right inverse to $\overline{g \otimes \mathrm{id}}$. $\square$

Thus, we have the following definition of a flat module:

Corollary 2.3.4. *A module M is flat if and only if it satisfies the following condition:*

for any injective morphism $f : N \longrightarrow N'$ the induced morphism $f \otimes \mathrm{id} : N \otimes M \longrightarrow N' \otimes M$ is injective.

Example 2.3.5. A *non-example* of a flat module, ie, a module which is not flat, is given by $\mathbb{Z}/n\mathbb{Z}$, for any n. Indeed, consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{41}$$

which induces the following sequence

$$\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{42}$$

which is isomorphic to

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{43}$$

and the map $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n\mathbb{Z}$ is clearly not injective.

Example 2.3.6. Free modules are flat. Indeed, if $f : N \longrightarrow N'$ is injective, then since the tensor product and direct sum commute (Lemma 2.2.3) we have the following commuting diagram where the vertical arrows are isomorphisms:

$$\begin{tikzcd}
N \otimes A^I \arrow[r] \arrow[d] & N' \otimes A^I \arrow[d] \\
(N \otimes A)^I \arrow[r] \arrow[d] & (N' \otimes A)^I \arrow[d] \\
N^I \arrow[r] & N'^I
\end{tikzcd} \tag{44}$$

Example 2.3.7. If M is flat and $M \cong N \oplus P$ then both N and P are flat. Indeed, assume $f : O \longrightarrow O'$ is injective and denote the inclusion $P \rightarrowtail M$ by i. Consider the following commuting diagram

$$\begin{tikzcd}
O \otimes P \arrow[r, "f \otimes \mathrm{id}_P"] \arrow[d, "\mathrm{id}_O \otimes i"'] & O' \otimes P \arrow[d, "\mathrm{id}_{O'} \otimes i"] \\
O \otimes M \arrow[r, "f \otimes \mathrm{id}_M"] & O' \otimes M
\end{tikzcd} \tag{45}$$

We have by assumption that $f \otimes \mathrm{id}_M$ is injective, we finish the proof by showing $\mathrm{id}_{O'} \otimes i$ is injective.

We have the following commutative diagram:

$$\begin{tikzcd}
O' \otimes P \arrow[d, "\mathrm{id}_{O'} \otimes i"'] \arrow[drr, tail] & & \\
O' \otimes M \arrow[r, "\sim"] & O' \otimes (N \oplus P) \arrow[r, "\sim"] & (O' \otimes N) \oplus (O' \otimes P)
\end{tikzcd} \tag{46}$$

2.4 Chain complexes

Throughout we work in an abelian category $\mathscr{A}$, first we relax the notion of an exact sequence:

Definition 2.4.1. A **chain complex** is a sequence

$$\dots \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n+1}} \dots \tag{47}$$

such that for all n, $\partial_{n-1}\partial_n = 0$. We denote a chain complex by $(M_\bullet, \partial_\bullet)$ or simply (M, ∂). The element in the i^{th} position in the complex consists of the **degree** i elements. We sometimes say that the module M_i is in **position** i.

Remark 2.4.2. We remark that this notion of degree is consistent with that of a *graded module* (see [2]).

Definition 2.4.3. A **morphism** of chain complexes $f_\bullet : (M_\bullet, \partial_\bullet) \longrightarrow (N_\bullet, \partial'_\bullet)$ is a set of morphisms $\{f_n : M_n \longrightarrow N_n$ such that for all n the following diagram commutes

$$\begin{tikzcd} M_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & M_{n-1} \arrow[d, "f_n"'] \\ N_n \arrow[r, "\partial'_n"] & N_{n-1} \end{tikzcd} \tag{48}$$

Definition 2.4.4. Given a chain complex $(M_\bullet, \partial_\bullet)$, the n**-th homology object** $H_n(M_\bullet, \partial_\bullet)$ is given as follows: since $\partial_n\partial_{n+1} = 0$ we have that $\partial n + 1$ factors through $\ker \partial_n$, ie, there is a morphism $j : M_{m+1} \longrightarrow \ker \partial_n$ such that the following diagram commutes:

$$\begin{tikzcd} & & \ker \partial_m \arrow[r] \arrow[d, tail] & \operatorname{coker} j & & \\ \dots \arrow[r] & M_{m+1} \arrow[ur, "j"] \arrow[r, "\partial_{m+1}"] & M_m \arrow[r, "\partial_m"] & M_{m-1} \arrow[r, "\partial_{m-1}"] & \dots \end{tikzcd} \tag{49}$$

we then define $H_n(M_\bullet, \partial_\bullet)$ to be the object $\operatorname{coker} j$.

Definition 2.4.5. A **chain homotopy** between two morphisms $f_\bullet, g_\bullet$ of chain complexes

$$\dots \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n+1}} \dots \tag{50}$$

and

$$\dots \xrightarrow{\partial'_{n+1}} N_n \xrightarrow{\partial'_n} N_{n-1} \xrightarrow{\partial'_{n+1}} \dots \tag{51}$$

is a set of maps $\{d_n : M_{n-1} \longrightarrow M_n\}_{n\in\mathbb{Z}}$ such that for all n,

$$\partial'_n d_n + d_{n+1}\partial_{n+1} = f - g \tag{52}$$

Remark 2.4.6. Any two chain homotopy morphisms induce the same map on homology.

We now specialise to the case where $\mathscr{A}$ is the category of left R-modules, with R a ring. We introduce two operations on chain complexes in $\mathscr{A}$, the *tensor product* and the *mapping cone*.

Definition 2.4.7. Given an arbitrary chain complex (M, ∂) we denote by $M(n)$ the chain complex identical to M but with the positions shifted by n. More precisely,

$$M(n)_m = M_{n+m} \tag{53}$$

If ∂ is the differential of M, then the differential for $M(n)$, denoted $\partial(n)$ is given by $\partial(n) = (-1)^n\partial$.

Let R be a ring considered as a module over itself. Denote the following chain complex:

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \dots \tag{54}$$

where R occurs in position zero.

Let R be a ring and $y \in R$ an element of R. We construct the following diagram:

$$\begin{tikzcd}
R(-1) & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\mathrm{id}_R"] & 0 \\
K(y) & \dots \arrow[r] & R \arrow[r, "y"] \arrow[d, "\mathrm{id}_R"] & R \arrow[r] \arrow[d] & 0 \\
R & \dots \arrow[r] & R \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{55}$$

where $K(y)$ denotes the Koszul complex corresponding to y (see [2]).

Now say we had an arbitrary complex $\mathscr{G}$:

$$\mathscr{G} \qquad \dots \longrightarrow G_i \xrightarrow{\psi_i} G_{i+1} \xrightarrow{\psi_{i+1}} G_{i+2} \xrightarrow{\psi_{i+2}} \dots \tag{56}$$

We tensor (55) with $\mathscr{G}$ to obtain:

$$\begin{tikzcd}
\mathscr{G}(-1) & \dots \arrow[r] & G_{i-1} \arrow[r] \arrow[d] & G_i \arrow[r] \arrow[d] & G_{i+1} \arrow[r] \arrow[d] & \dots \\
K(y) \otimes \mathscr{G} & \dots \arrow[r] & G_{i-1} \oplus G_i \arrow[r] \arrow[d] & G_i \oplus G_{i+1} \arrow[r] \arrow[d] & G_{i+1} \oplus G_{i+2} \arrow[r] \arrow[d] & \dots \\
\mathscr{G} & \dots \arrow[r] & G_i \arrow[r] & G_{i+1} \arrow[r] & G_{i+2} \arrow[r] & \dots
\end{tikzcd} \tag{57}$$

where the vertical maps are:

$$\begin{aligned} G_{i-1} &\longrightarrow G_{i-1} \oplus G_i \\ g &\longmapsto (g, 0) \end{aligned} \qquad\qquad \begin{aligned} G_{i-1} \oplus G_i &\longrightarrow G_i \\ (g_{i-1}, g_i) &\longmapsto g_i \end{aligned}$$

From Diagram (57) we then construct a long exact sequence on homology:

$$\dots \longrightarrow H^i(\mathscr{G}) \xrightarrow{\delta} H^i(\mathscr{G}) \longrightarrow H^{i+1}(K(y) \otimes \mathscr{G}) \longrightarrow H^{i+1}(\mathscr{G}) \xrightarrow{\delta} H^{i+1}(\mathscr{G}) \longrightarrow \dots \tag{58}$$

we now show that the connecting morphism δ_i is multiplication by $(-1)^i y$.

Let $g \in G_i$ by such that $\partial_{\mathscr{G}}(g) = 0$ (where ∂ is the differential of $\mathscr{G}$). The map $G_{i+1} \oplus G_i \longrightarrow G_i$ is surjective, and so we may pick a lift, an obvious choice is $(0, g)$. Now we calculate $\partial_{K(y) \otimes \mathscr{G}}(0, g)$. We have:

$$\begin{tikzcd}
G_{i-1} \oplus G_i \arrow[rr] \arrow[dd] & & G_i \oplus G_{i+1} \\
(0, g) \arrow[d, mapsto] & ((-1)^i g y, 0) & \\
(0, g \otimes 1) \mapsto \big((-1)^i g \otimes y, \partial_{\mathscr{G}}(g)\big) = \big((-1)^i g \otimes y, 0\big) \arrow[ur, mapsto] & & \\
(G_{i-1} \otimes R) \oplus (G_i \otimes R) \arrow[rr] & & (G_i \otimes R) \oplus (G_{i+1} \otimes R) \arrow[uuu]
\end{tikzcd} \tag{59}$$

where we have used the fact that $\partial_{\mathscr{G}}(g) = 0$.

Thus, the appropriate lift along $G_{i+1} \longrightarrow G_{i+1} \oplus G_{i+2}$ to take is $(-1)^i g y$, as claimed.

2.5 Injective modules

Definition 2.5.1. An A-module I is **injective** if every short exact sequence

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{60}$$

splits.

Example 2.5.2. All finite dimensional vector spaces are injective. To see this, let k be a field and V a k-vector space, denote by $v_1, ..., v_n$ a basis for V. Consider a short exact sequence

$$0 \longrightarrow V \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{61}$$

then the images of $v_1, , , .v_n$ under $V \longrightarrow N$ can be extended to a basis of M. These extra vectors span a subspace $M' \subseteq M$ such that $M \cong V \oplus M'$.

Lemma 2.5.3. *Let I be an A-module, then the following are equivalent:*

1. *I is injective.*
2. *if I is a submodule of some other R-module M, then there exists another submodule $M' \subseteq M$ such that $M = I + M'$ and $I \cap M' = \{0\}$ (in other words, M is the internal direct sum of I and M'),*
3. *given an injective homomorphism $f : M \longrightarrow N$ and $g : M \longrightarrow I$ an arbitrary morphims, then there exists a morphism $h : N \longrightarrow I$ such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, tail, "f"] \arrow[d, "g"] & N \arrow[dl, dashed] \\ I & \end{tikzcd} \tag{62}$$

Proof. $(1 \Rightarrow 2)$: Construct the following short exact sequence

$$0 \longrightarrow I \longrightarrow M \longrightarrow \operatorname{coker} M \longrightarrow 0 \tag{63}$$

which necessarily splits. We thus have $M = I + \operatorname{coker} M$.

That $2 \Rightarrow 1$ is obvious.

$(1 \Rightarrow 3)$: Consider the pushout $I \sqcup_M N$ and appeal to the fact that the resulting short exact sequence splits. Conversely, let a short exact sequence be given:

$$0 \longrightarrow I \xrightarrow{f} M \longrightarrow N \longrightarrow 0 \tag{64}$$

and consider the diagram

$$\begin{tikzcd} I \arrow[r, tail, "f"] \arrow[d, "\mathrm{id}"] & M \\ I & \end{tikzcd} \tag{65}$$

from which we obtain a left inverse to f. □

Remark 2.5.4. Notice that condition 3 of Lemma 2.5.3 is equivalent the following:

if $f : M \rightarrowtail N$ is injective then $(\_ \circ f) : \hom(N, I) \longrightarrow \hom(M, I)$ is surjective.

Thus, a module I is injective if and only if $\hom(\_, I)$ is exact.

For the next definition, recall that a subobject of a an object A in some category is a monomorphism $A' \rightarrowtail A$.

Definition 2.5.5. If the abelian category $\mathscr{A}$ is such that every object is isomorphic to a subobject of an injective object, then $\mathscr{A}$ **has enough injectives**.

If $\mathscr{A}$ has enough injectives then every object admits an injective resolution. To see this, let $A \in \mathscr{A}$ and let I_0 be such that $i_0 : A \rightarrowtail I_0$ is a subobject. By considering the morphism $I_0 \longrightarrow \operatorname{coker} i_0$ composed with $i_0' : \operatorname{coker} i \rightarrowtail I_1$, where I_1 is again injective, we obtain after repeating this process an injective resolution for A:

$$0 \longrightarrow A \xrightarrow{i_0} I_0 \xrightarrow{i_1' i_1} I_1 \xrightarrow{i_2' i_2} \dots \tag{66}$$

Remark 2.5.6. Note that it is true but non-trivial that the category of R-modules (R a commutative ring) has enough injectives. See for instance [1]

2.6 Projective modules

Definition 2.6.1. An A-module P is **projective** if every short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{67}$$

splits.

Example 2.6.2. Free modules are projective. Indeed, consider an arbitrary short exact sequence

$$0 \longrightarrow M \longrightarrow N \xrightarrow{f} A^S \longrightarrow 0 \tag{68}$$

then we define a right inverse of f by mapping the unit of the s^{th} copy of A to any lift along f of it. This induces a well defined homomorphism as A^S is free.

Example 2.6.3. If P is projective and $P \cong N \oplus O$ then both N and O are also projective.

Proof. Say we have the following short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow N \longrightarrow 0 \tag{69}$$

then the following sequence is also short exact

$$0 \longrightarrow M_1 \oplus O \longrightarrow M_2 \oplus O \longrightarrow N \oplus O \longrightarrow 0 \tag{70}$$

which is split by hypothesis. There thus exists a right inverse to $M_2 \oplus O \longrightarrow N \oplus O$ from which one can derive a right inverse to $M_2 \longrightarrow N$. $\square$

Lemma 2.6.4. *Let P be an A-module, then the following are equivalent:*

1. *P is projective,*
2. *there exists an A-module M such that $M \oplus P$ is free,*
3. *given a surjective homomorphism $f : N \twoheadrightarrow M$ and an arbitrary homomorphism $g : P \longrightarrow M$, there exists a homomorphism $h : P \longrightarrow N$ such that the following diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ N \arrow[r, two heads, "f"] & M \end{tikzcd} \tag{71}$$

Proof. Say P is projective and let S be a set of generators for P. and let $\varphi : A^S \longrightarrow P$ be such that $e_s \longmapsto s$ where e_s is the unit of the s^{th} copy of A. Then there is the following short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow A^S \xrightarrow{\varphi} P \longrightarrow 0 \tag{72}$$

which is split as P is projective. Thus $A^S \cong \ker\varphi \oplus P$. Thus (1) implies (2).

To see that (2) implies (1) we observe that free modules are projective (Example 2.6.2) and that summands of projective modules are projective (2.6.3).

Next we prove (1) implies (3). This is done by considering the fibred product $M \times_N P$. For the converse, say

$$0 \longrightarrow M \longrightarrow N \xrightarrow{g} P \longrightarrow 0 \tag{73}$$

is exact. Then $N \longrightarrow P$ is surjective, so we consider the following diagram:

$$\begin{tikzcd} & P \arrow[d, "\mathrm{id}_P"] \\ N \arrow[r, two heads, "g"] & P \end{tikzcd} \tag{74}$$

from which we obtain a left inverse to g. $\square$

Definition 2.6.5. A **free resolution** of a module M is an exact sequence

$$\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M \longrightarrow 0 \tag{75}$$

where each M_i for $i \geq 0$ is free. We denote this $\partial : M_\bullet \longrightarrow M$.

One defines similarly a **projective resolution** (which we also denote by $\partial : M_\bullet \longrightarrow M$).

Given two resolutions (free or projective) $\partial : M_\bullet \longrightarrow M, \partial' : N_\bullet \longrightarrow N$, there is an obvious notion of a **morphism** of (free or projective, the definition is identical) resolutions which we denote $f : \partial \longrightarrow \partial'$.

We write down some easy to prove facts:

Fact 2.6.6. *Every module admits a free resolution.*

and thus:

Fact 2.6.7. *Every module admits a projective resolution.*

Fact 2.6.8. *If $f : M \longrightarrow N$ is a homomorphism and say we have projective resolutions $\partial : M_\bullet \longrightarrow M$ and $\partial' : N_\bullet \longrightarrow N$, then there exists a morphism of resolutions $\partial \longrightarrow \partial'$.*

Let $f : M \twoheadrightarrow N$ be surjective and consider an arbitrary $g : P \longrightarrow N$. Since P is projective there exists a morphism $l : P \longrightarrow M$ rendering the following diagram commutative

$$\begin{tikzcd} & P \arrow[dl, "l"'] \arrow[d, "g"] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd} \tag{76}$$

In other words, the functor $\hom(P, \_)$ is exact. We now arrive at the following more general definition of a module being projective:

Definition 2.6.9. Let $\mathscr{A}$ be an abelian category. An object $P \in \mathscr{A}$ is **projective** if the functor $\hom(P, \_)$ is exact.

Definition 2.6.10. An abelian category $\mathscr{A}$ **has enough projectives** if for every object $A \in \mathscr{A}$ there exists a projective objects $P \in \mathscr{A}$ and an epimorphism $P \twoheadrightarrow A$.

If $\mathscr{A}$ has enough projectives then every object admits a projective resolution. To see this, let $A \in \mathscr{A}$ and let P_0 be projective and $p_0 : P_0 \twoheadrightarrow A$ an epimorphism. By considering the morphism $p_0' : \ker p_0 \rightarrowtail P_0$ pre composed with $P_1 \overset{p_1}{\twoheadrightarrow} \ker p_0$, where P_1 again is projective, we obtain after repeating this process a projective resolution for A:

$$\dots \xrightarrow{p_1' p_2} P_1 \xrightarrow{p_0' p_1} P_0 \xrightarrow{p_0} A \longrightarrow 0 \tag{77}$$

Definition 2.6.11. A **chain homotopy** between two resolutions

$$\dots \longrightarrow M_2 \xrightarrow{i_2} M_1 \xrightarrow{i_1} M_0 \xrightarrow{i_0} 0 \tag{78}$$

and

$$\dots \longrightarrow N_2 \xrightarrow{j_2} N_1 \xrightarrow{j_1} N_0 \xrightarrow{j_0} 0 \tag{79}$$

say is a set of maps $\{d_n : M_d$

Proposition 2.6.12. *Let $f : M \longrightarrow N$ be a morphism in an abelian category with enough projectives. Let $P_\bullet, P_\bullet'$ be projective resolutions respectively for M, N. Then there is a chain map $f_\bullet : (P_\bullet \longrightarrow M) \longrightarrow (P_\bullet' \longrightarrow M)$. Moreover, the following sequences*

$$\dots P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \tag{80}$$

and

$$\dots P_2' \longrightarrow P_1' \longrightarrow P_0' \longrightarrow 0 \tag{81}$$

may not be exact sequences (so may not be resolutions), however they are chain maps. Given two such chain maps $f_\bullet, g_\bullet$, there is a chain homotopy between (80) *and* (81).

Proof. We construct $f_\bullet$ inductively. Consider the diagram of solid arrows

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "i_0"] \arrow[d, dashed, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
P_0' \arrow[r, two heads, "i_0'"'] & N \arrow[r] & 0
\end{tikzcd} \tag{82}$$

Then since P_0 is projective, there exists a morphism $f_0 : P_0 \longrightarrow P_0'$ rending the diagram commutative. Now consider the following diagram

$$\begin{tikzcd}
P_{j+1} \arrow[rr, "i_{j+1}'"] \arrow[d, dashed] \arrow[dr, dotted] & & P_j \arrow[r, "i_j"] \arrow[d, "f_j"'] & P_{j-1} \\
P_{j+1}' \arrow[rr, "i_{j+1}'"] \arrow[dr, two heads] & & P_j' \arrow[r, "i_j'"] & P_{j-1}' \\
& \ker i_j \arrow[ur, tail] & &
\end{tikzcd} \tag{83}$$

the dotted arrow exists by commutativity and that $i_j' i_{j+1}' = 0$, and the dashed line exists by projectivity of P_{j+1}. $\square$

Remark 2.6.13. It is in Proposition 2.6.12 that we see why projective modules are a natural consideration. They are the minimal requirement to obtain this Proposition. Free modules also satisfy this, but we can ask for less.

For the second claim, consider the following Diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "i'_2"] \arrow[d, shift right, "g_2"'] \arrow[d, shift left, "f_2"] & P_1 \arrow[r, "i_1"] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "f_1"] & P_0 \arrow[r] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & 0 \\
\cdots \arrow[r] & P'_2 \arrow[r, "i'_2"] & P'_1 \arrow[r, "i'_1"] & P'_0 \arrow[r] & 0
\end{tikzcd}
\tag{84}
$$

We have that $i'_0(f_0 - g_0) = f - f = 0$ and so $f_0 - g_0$ factors through $\ker i'_0$. There is also a surjective map $P'_1 \twoheadrightarrow \ker i'_0$ and so there exists a map $d_1 : P_0 \longrightarrow P'_1$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & P_0 \arrow[r] \arrow[dl, "d_1"'] \arrow[d, "f_0-g_0"] & 0 \\
P'_1 \arrow[r, "i'_1"] & P'_0 \arrow[r] & 0
\end{tikzcd}
\tag{85}
$$

By setting $d_0 : 0 \longrightarrow P'_0$ and $j : P_0 \longrightarrow 0$ when then have that $f_0 - g_0 = d_0 j + i'_1 d_1$.

We then notice that

$$
i'_1(f_1 - g_1) = i'_1 f_1 - i'_1 g_1 = f_0 i_1 - g_0 i_1 = (d_0 i_0 + i'_1 d_1) i_1 = i'_1 d_1 i_1 \tag{86}
$$

and so $f_1 - g_1 - d_1 i_1$ factors through $\ker i'_1$. There is also a surjective map $P'_2 \twoheadrightarrow \ker i'_1$ so there we obtain the necessary map d_1. Continuing in this way we construct d_n.

Notice that we did not need that P'_n were projective.

Proposition 2.6.14. *In an abelian category with enough projectives, any two projective resolutions of the same object give rise to naturally isomorphic homology.*

Proof. First we construct the following diagram:

$$
\begin{tikzcd}
 & & \ker i_0 \arrow[rr, two heads, "l_0"] \arrow[dr, tail, "\iota_0"] \arrow[dd, "f_0^c"'] & & \operatorname{coker} j_0 \arrow[dd, "f_0^{\mathrm{ck}}"'] & \\
\cdots \arrow[r] & P_1 \arrow[ur, "j_0"] \arrow[rr, "i_1"] & & P_0 \arrow[rr] & & 0 \\
 & & \ker i'_0 \arrow[rr, two heads] \arrow[dr, tail, "\iota'_0"] \arrow[dd, "g_0^c"'] & & \operatorname{coker} j'_0 \arrow[dd, "g_0^{\mathrm{ck}}"'] & \\
\cdots \arrow[r] & P'_1 \arrow[ur, "j_0"] \arrow[rr, "i'_1"] & & P'_0 \arrow[rr] & & 0 \\
 & & \ker i_0 \arrow[rr, two heads] \arrow[dr, tail, "\iota_0"] & & \operatorname{coker} j_0 & \\
\cdots \arrow[r] & P_1 \arrow[ur, "j_0"] \arrow[rr, "i_1"] & & P_0 \arrow[rr] & & 0
\end{tikzcd}
\tag{87}
$$

We know from Remark 2.6.13 that there exists a chain homotopy $\{d_n : P_{n-1} \longrightarrow P_n\}$ (where $P_{-1} = 0$) between $g_\bullet f_\bullet$ and $\mathrm{id}_{P_\bullet}$. We calculate:

$$
\begin{aligned}
\iota_0 g_0^c f_0^c &= g_0 f_0 \iota_0 \\
&= (d_0 0 + i_1 d_1 + \mathrm{id}_{P_0}) \iota_0 \\
&= 0 + i_1 d_1 \iota_0 + \iota_0 \\
&= \iota_0
\end{aligned}
$$

so since ι_0 is monic, we have $g_0^c f_0^c = \mathrm{id}_{\ker P_0}$. It then follows that $g_0^{\mathrm{ck}} f_0^{\mathrm{ck}} l_0 = l_0$ which since l_0 is epic implies $g_0^{\mathrm{ck}} f_0^{\mathrm{ck}} = \mathrm{id}_{\operatorname{coker} j_0}$.

Next, we consider the following diagram:

$$
\begin{tikzcd}
 & & \ker i_2 \arrow[rr, two heads, "l_2"] \arrow[dr, tail, "\iota_1"] \arrow[dd, "f_1^c"'] & & \operatorname{coker} j_1 \arrow[dd, "f_1^{\mathrm{ck}}"'] & \\
\cdots \arrow[r] & P_1 \arrow[ur, "j_1"] \arrow[rr, "i_2"] & & P_1 \arrow[rr] & & P_0 \\
 & & \ker i_2' \arrow[rr, two heads] \arrow[dr, tail, "\iota_1'"] \arrow[dd, "g_1^c"'] & & \operatorname{coker} j_1' \arrow[dd, "g_1^{\mathrm{ck}}"'] & \\
\cdots \arrow[r] & P_1' \arrow[ur, "j_1"] \arrow[rr, "i_2'"] & & P_1' \arrow[rr] & & P_0 \\
 & & \ker i_2 \arrow[rr, two heads] \arrow[dr, tail, "\iota_1"] & & \operatorname{coker} j_1 & \\
\cdots \arrow[r] & P_1 \arrow[ur, "j_1"] \arrow[rr, "i_2"] & & P_1 \arrow[rr] & & P_0
\end{tikzcd}
\tag{88}
$$

We calculate, in the following, $\gamma : P_1 \longrightarrow \ker i_1$ is the map induced by the fact that $P_1 \longrightarrow P_1' \longrightarrow P_1 \longrightarrow P_0$ is 0:

$$
\begin{aligned}
\iota_1 g_1^c f_1^c &= g_1 f_1 \iota_1 \\
&= (d_1 i_1 + i_2 d_2 + \mathrm{id}_{P_1})\iota_1 \\
&= d_1 i_1 \iota_1 + i_2 d_2 \iota_1 + \iota_1 \\
&= i_2 d_2 \iota_1 + \iota_1 \\
&= \iota_1 \gamma \iota_1 + \iota_1
\end{aligned}
$$

and so $g_1^c f_1^c = \gamma \iota_1 + \mathrm{id}_{\ker i_1}$.

We now claim $g_1^{\mathrm{ck}} f_1^{\mathrm{ck}} = \mathrm{id}_{\operatorname{coker} j_1}$, since l_1 is epic, it suffices to show

$$
g_1^{\mathrm{ck}} f_1^{\mathrm{ck}} l_1 = l_1 \tag{89}
$$

We calculate:

$$
\begin{aligned}
g_1^{\mathrm{ck}} f_1^{\mathrm{ck}} l_1 &= l_1 g_1^c f_1^c \\
&= l_1(\gamma \iota_1 + \mathrm{id}_{\ker i_1}) \\
&= l_1 \gamma \iota_1 + l_1 \\
&= l_1
\end{aligned}
$$

Following in this way, we have $g_\bullet^{\mathrm{ck}} f_\bullet^{\mathrm{ck}} = \mathrm{id}_\bullet$. That $f_\bullet^{\mathrm{ck}} g_\bullet^{\mathrm{ck}} = \mathrm{id}_\bullet$ follows via a symmetric argument. □

2.7 Derived functors

References

[1] `https://sites.math.washington.edu/~mitchell/Algh/chung.pdf`

[2] *W. Troiani*, Notes on commutative algebra.