λ-terms as polynomials

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Let t be a λ-term and let $\{x_1, \ldots, x_n\}$ be a valid context for t, that is

$$\mathrm{FV}(t) \subseteq \{x_1, \ldots, x_n\} \tag{1}$$

Let $m \geq 0$ be an integer. We define an integer m' and an interpretation for t as a polynomial map:

$$\begin{aligned} [\![x_1, \ldots, x_n \mid t]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m'} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_{m'}) &\longmapsto q(X_1, \ldots, X_{m'}) \end{aligned}$$

What we mean by $[\![t]\!]$ being a polynomial, is that

$$q(X_1, \ldots, X_m) \in (\mathbb{N}[x_1, \ldots, x_n])[X_1, \ldots, X_m] \tag{2}$$

Definition 0.0.1. Say $t = x_i$ **is a variable**. Then $m' = m$ and:

$$\begin{aligned} [\![x_1, \ldots, x_n \mid x_i]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m} = \mathbb{N}[x_1, \ldots, x_n]^{m} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto x_i \end{aligned}$$

Say $t = \lambda x_{n+1}.t$ **is an abstraction**: assume we have

$$\begin{aligned} [\![x_1, \ldots, x_n, x_{n+1} \mid t]\!] : \mathbb{N}[x_1, \ldots, x_n, x_{n+1}]^{m} &\longmapsto \mathbb{N}[x_1, \ldots, x_n, x_{n+1}] \\ (X_1, \ldots, X_m) &\longmapsto q(X_1, \ldots, X_m) \end{aligned}$$

We notice that

$$q(X_1, \ldots, X_m) \in (\mathbb{N}[x_1, \ldots, x_{n+1}])[X_1, \ldots, X_m] \tag{3}$$

and so there exists a polynomial $q' \in \mathbb{N}[x_1, \ldots, x_{n+1}, X_1, \ldots, X_m]$ such that

$$q'(x_1, \ldots, x_{n+1}, X_1, \ldots, X_m) = q(X_1, \ldots, X_m) \tag{4}$$

We introduce a new variable X_{m+1} and consider

$$q'(x_1, \ldots, x_n, X_{m+1}, X_1, \ldots, X_m) \in (\mathbb{N}[x_1, \ldots, x_n])[X_1, \ldots, X_{m+1}] \tag{5}$$

There exists a polynomial $q'' \in (\mathbb{N}[x_1, \ldots, x_n])[X_1, \ldots, X_{m+1}]$ such that

$$q''(X_1, \ldots, X_{m+1}) = q'(x_1, \ldots, x_n, X_1, \ldots, X_{m+1}) \tag{6}$$

We define

$$\begin{aligned} [\![x_1, \ldots, x_n \mid \lambda x_{n+1}.t]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m+1} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_{m+1}) &\longmapsto q''(X_1, \ldots, X_{m+1}) \end{aligned}$$

We notice that the construction just given is independent of the choice of variable x_{n+1}.

Say $t = uv$ is an application: say we have

$$\begin{aligned} [\![x_1, \ldots, x_n \mid u]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m_1} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_{m_1}) &\longrightarrow q_1(X_1, \ldots, X_{m_1}) \end{aligned}$$

and

$$\begin{aligned} [\![x_1, \ldots, x_n \mid v]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m_2} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_{m_2}) &\longmapsto q_2(X_1, \ldots, X_{m_2}) \end{aligned}$$

We define

$$\begin{aligned} [\![x_1, \ldots, x_n \mid uv]\!] : \mathbb{N}[x_1, \ldots, x_n]^{m_1+m_2-1} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_{m_1+m_2-1}) &\longmapsto q_1(X_{m_2+1}, \ldots, X_{m_2+m_1-1}, q_2(X_1, \ldots, X_{m_2})) \end{aligned}$$

Proposition 0.0.2. *This is a model of the untyped λ-calculus.*

Proof. We show that

$$[\![x_1, \ldots, x_n \mid (\lambda x_{n+1}.t)s]\!] = [\![x_1, \ldots, x_n \mid t[x_{n+1} := s]]\!] \tag{7}$$

We prove this by induction on the length of t.

Say $t = x_i$ is a variable. If $i \neq n+1$ then $t[x_{n+1} := s] = x_i$ and

$$\begin{aligned} [\![x_1, \ldots, x_n \mid x_i]\!] : \mathbb{N}[x_1, \ldots, x_n]^m = \mathbb{N} &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto x_i \end{aligned}$$

On the other hand,

$$\begin{aligned} [\![x_1, \ldots, x_n \mid \lambda x_{n+1}.x_i]\!] : \mathbb{N}[x_1, \ldots, x_n]^m &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto x_i \end{aligned}$$

and so

$$\begin{aligned} [\![x_1, \ldots, x_n \mid (\lambda x_{n+1}.x_i)s]\!] : \mathbb{N}[x_1, \ldots, x_n]^m &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto x_i \end{aligned}$$

If $i = n+1$ then $t[x_{n+1} := s] = s$ and

$$\begin{aligned} [\![x_1, \ldots, x_n \mid \lambda x_{n+1}.x_{n+1}]\!] : \mathbb{N}[x_1, \ldots, x_n]^m &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto X_m \end{aligned}$$

Thus

$$\begin{aligned} [\![x_1, \ldots, x_n \mid (\lambda x_{n+1}.x_{n+1})s]\!] : \mathbb{N}[x_1, \ldots, x_n]^m &\longrightarrow \mathbb{N}[x_1, \ldots, x_n] \\ (X_1, \ldots, X_m) &\longmapsto [\![x_1, \ldots, x_n \mid s]\!](X_1, \ldots, X_m) \end{aligned}$$

Say $t = \lambda x_{n+2}.u$ is an abstraction. Write

$$[\![x_1, \ldots, x_n \mid u]\!] = q(X_1, \ldots, X_m), \quad [\![x_1, \ldots, x_n \mid s]\!] = p(X_1, \ldots, X_{m'}) \tag{8}$$

Then

$$\begin{aligned}&[\![x_1,\dots,x_n \mid \lambda x_{n+2}.(\lambda x_{n+1}.u)s]\!]\\&\qquad= q(x_1,\dots,x_n,p(X_1,\dots,X_{m'}),X_{m+m'+1},X_{m'+1},\dots,X_{m'+m})\end{aligned}$$

Also,

$$[\![x_1,\dots,x_n \mid \lambda x_{n+1}x_{n+2}.u]\!] = q(x_1,\dots,x_n,X_{m+2},X_{m+1},X_1,\dots,X_m) \tag{9}$$

it follows that

$$\begin{aligned}&[\![x_1,\dots,x_n \mid (\lambda x_{n+1}x_{n+2}.u)s]\!]\\&\qquad= q(x_1,\dots,x_n,p(X_1,\dots,X_{m'}),X_{m'+m+1},X_{m'+1},\dots,X_{m'+m})\end{aligned}$$

By the inductive hypothesis, we have

$$[\![x_1,\dots,x_n,x_{n+2} \mid u[x_1 := s]]\!] = [\![x_1,\dots,x_n,x_{n+2} \mid (\lambda x_{n+1}.u)s]\!] \tag{10}$$

It follows that

$$[\![x_1,\dots,x_n \mid \lambda x_{n+2}(u[x_1 := s])]\!] = [\![x_1,\dots,x_n \mid \lambda x_{n+2}.(\lambda x_{n+1}.u)s]\!] \tag{11}$$

Combining this with the above, we have

$$\begin{aligned}[\![x_1,\dots,x_n \mid (\lambda x_{n+1}x_{n+2}.u)s]\!] &= [\![x_1,\dots,x_n \mid \lambda x_{n+2}.(u[x_1 := s])]\!]\\&= [\![x_1,\dots,x_n \mid (\lambda x_{n+2}.u)[x_1 := s]]\!]\end{aligned}$$

as required. **Say $t = t_1t_2$ is an application**. Write

$$[\![x_1,\dots,x_n \mid t_i]\!] = q_1(X_1,\dots,X_{m_i}),\text{ for } i = 1,2 \tag{12}$$

and again we write

$$[\![x_1,\dots,x_n \mid s]\!] = p(x_1,\dots,x_n,X_1,\dots,X_{m'}) \tag{13}$$

For $i = 1,2$ we have

$$\begin{aligned}&[\![x_1,\dots,x_n \mid (\lambda x_{n+1}.t_i)s]\!]\\&= q_i(x_1,\dots,x_n,p(x_1,\dots,x_n,X_1,\dots,X_{m'}),X_{m'+1},\dots,X_{m'+m_i})\end{aligned}$$

Thus,

$$\begin{aligned}&[\![x_1,\dots,x_n \mid \big[(\lambda x_{n+1}.t_1)s\big]\big[(\lambda x_{n+1}.t_2)s\big]]\!]\\&= q_1(x_1,\dots,x_n,p(x_1,\dots,x_n,X_1,\dots,X_{m'}),\\&\qquad X_{m'+m_2+1},\dots,X_{m'+m_2+m_1-1},q_2(x_1,\dots,x_n,p(x_1,\dots,x_n,X_1,\dots,X_{m'})\\&\qquad X_{m'+1},\dots,X_{m'+m_2}))\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&[\![x_1,\dots,x_n,x_{n+1} \mid t_1t_2]\!]\\&= q_1(x_1,\dots,x_n,x_{n+1},X_{m_2+1},\dots,X_{m_2+m_1-1},q_2(x_1,\dots,x_n,x_{n+1},X_1,\dots,X_{m_2}))\end{aligned}$$

Thus,

$$\begin{aligned}&[\![x_1,\dots,x_n \mid (\lambda x_{n+1}.t_1t_2)s]\!]\\&= q_1(x_1,\dots,x_n,p(x_1,\dots,x_n,X_1,\dots,X_{m'}),\\&\qquad X_{m'+m_2+1},\dots,X_{m'+m_2+m_1-1},q_2(x_1,\dots,x_n,p(x_1,\dots,x_n,X_1,\dots,X_{m'}),\\&\qquad X_{m'+1},\dots,X_{m'+m_2}))\end{aligned}$$

By the inductive hypothesis, we have for $i = 1, 2$:

$$[\![x_1, \ldots, x_n \mid (\lambda x_{n+1}.t_i)s]\!] = [\![x_1, \ldots, x_n \mid t_i[x_{n+1} := s]]\!] \tag{14}$$

It follows that

$$[\![x_1, \ldots, x_n \mid \big[(\lambda x_{n+1}.t_1)s\big]\big[(\lambda x_{n+1}.t_2)s\big] = [\![x_1, \ldots, x_n \mid t_1[x_{n+1} := s]t_2[x_{n+1} := s]]\!] \tag{15}$$

Combining this with above we have

$$\begin{aligned} [\![x_1, \ldots, x_n \mid (\lambda x_{n+1}.t_1 t_2)s]\!] &= [\![x_1, \ldots, x_n \mid t_1[x_{n+1} := s]t_2[x_{n+1} := s]]\!] \\ &= [\![x_1, \ldots, x_n \mid (t_1 t_2)[x_{n+1} := s]]\!] \end{aligned}$$

as required. $\square$