# In the category of simplicial sets

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## 1 In the topos of simplicial sets

As a hands on example of the methodology presented in Section [1, §5] we consider the particular topos sSet of simplicial sets (Definition 1.0.4 below). Recall that associated to every simplicial set $S$ is its *Geometric Realisation* $|S|$ [20]. Although this notion will not be required for this Section, awareness of it will help with guiding intuition.

**Definition 1.0.1.** The **simplex category** $\Delta$ is the category whose objects are sets of the form $\{0, 1, ..., n\}$ for some $n$, these will be denoted $[n]$. The morphisms of this category are order preserving functions. For any positive integer $k$, let $\Delta_{\leq k}$ be the full subcategory of $\Delta$ with objects $\{[0], ..., [k]\}$.

There is a canonical way of factorising morphisms in the simplex category:

**Definition 1.0.2.** Define

$$\epsilon_n^i : [n-1] \to [n]$$
$$j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

and

$$\eta_n^i : [n+1] \to [n]$$
$$j \mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

**Theorem 1.0.3.** *Any morphism $[n] \to [m]$ in $\Delta$ can be written uniquely as*

$$\epsilon_m^{i_1}\epsilon_{m-1}^{i_2}...\epsilon_{m-k+1}^{i_l}\eta_{m-k}^{j_1}\eta_{m-k+1}^{j_2}...\eta_{m-1}^{j_{k-1}}\eta_m^{j_k}$$

*with $m \geq i_1 \geq i_2 \geq ... \geq i_l \geq 0$, and $0 \leq j_1 \leq j_2 \leq ... \leq j_k \leq n$.*

*Proof.* See [15, §VIII.5.1]. □

**Definition 1.0.4.** A **simplicial set** is a functor $\Delta^{\text{op}} \to \underline{\text{Set}}$, where Set is the category of sets. The collection of these, along with the collection of natural transformations between them, forms a category sSet, the category of simplicial sets.

**Example 1.0.5.** Consider the simplicial set $S$ given by the colimit of the following diagram, the geometric realisation of which is the interval.

$$\begin{tikzcd} {[0]} \arrow[d, "\epsilon_0^1"] \\ {[1]} \\ {[0]} \arrow[u, "\epsilon_1^1"'] \end{tikzcd}$$

We construct the diagram (**??**) in this setting. There are 5 morphisms in this diagram, including the identity morphisms. These induce two morphisms:

$$\Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[0]} \coprod \Omega^{[1]} \underset{s_1}{\overset{s_0}{\rightrightarrows}} \Omega^{[0]} \coprod \Omega^{[1]} \coprod \Omega^{[0]} \tag{1}$$

We now give the description of these morphisms $s_1, s_2$ as terms $t_1, t_2$. Let $z_1, z_2, z_3, z_4, z_5 : \Omega^{[0]}$, and let $z_6 : \Omega^{[1]}$. Here, the idea is that $z_1$ corresponds to $[0]^{\epsilon_0^1}$, $z_2$ corresponds to $[1]^{\mathrm{id}}$, $z_3$ to $[0]^{\mathrm{id}}$, $z_4$ to $[0]^{\epsilon_1^1}$, and $z_5$ to $[1]^{\mathrm{id}}$.

$$t_0 = \langle\langle z_1 \cup z_2, z_3 \cup z_4\rangle, z_5\rangle \tag{2}$$

$$t_1 = \langle\langle z_2, z_3\rangle, \epsilon_0^1(z_1) \cup \epsilon_1^1(z_5) \cup z_6\rangle \tag{3}$$

The interesting term here is $t_1$. Reading $t_1$ from left to right, we first read $\langle z_2, z_3\rangle$, indicating that the two copies of $[0]$ are not glued together, and the next component $\epsilon_0^1(z_1) \cup \epsilon_1^1(z_5) \cup z_6$ which describes how the 0 dimensional components are glued to the 1 dimensional component. This fits our intuition of how the geometric realisation of the simplicial set $S$ is constructed.

A more complicated example is given by the following.

**Example 1.0.6.** Let $S$ be the simplicial set given by the colimit of the following diagram, the geometric realisation of which is a triangle. We have artificially added labellings to the copies of objects in this diagram for clarity.

$$\begin{tikzcd}
 & & {[0]_1} \arrow[dl, "\epsilon_1^0"'] \arrow[dr, "\epsilon_1^1"] & & \\
 & {[1]_4} \arrow[r, "\epsilon_2^1"'] & {[2]_7} & {[1]_5} \arrow[l, "\epsilon_2^2"] & \\
{[0]_2} \arrow[ur, "\epsilon_1^0"] \arrow[rr, "\epsilon_1^0"'] & & {[1]_6} \arrow[u, "\epsilon_2^0"] & & {[0]_3} \arrow[ul, "\epsilon_1^0"'] \arrow[ll, "\epsilon_1^1"]
\end{tikzcd}$$

We define the following variables.

$$\begin{array}{ccc}
z_1^1, z_2^1 : \Omega^{[0]_1} & z_1^2, z_2^2 : \Omega^{[0]_2} & z_1^3, z_2^3 : \Omega^{[0]_3} \\
x_4 : \Omega^{[1]_4} & x_5 : \Omega^{[1]_5} & x_6 : \Omega^{[1]_6} \\
y_6 : \Omega^{[2]_7} & &
\end{array}$$

The term of interest is the following, we ignore the bracketting.

$$t_1 = \langle z_1^1 \cup z_2^1, z_1^2 \cup z_2^2, z_1^3 \cup z_2^3, \epsilon_1^0(z_2^1) \cup \epsilon_1^0(z_1^2), \epsilon_1^1(z_2^1) \cup \epsilon_1^0(z_1^3), \epsilon_1^0(z_2^2) \cup \epsilon_1^1(z_1^3), \epsilon_2^0(x_6) \cup \epsilon_2^1(x_4) \cup \epsilon_2^2(x_5), y_7\rangle \tag{4}$$

Which, as in Example 1.0.5, agrees with the glueing instructions corresponding to the geometric realisation of $S$.

# References


[1] W. Troiani, *Finite Colimits in the Internal Language of Topos*

[2] J. May, *A Crash Course in Algebraic Topology*, University of Chicago Press, Chicago, 1999.