Elimination and cut-elimination in multiplicative linear logic

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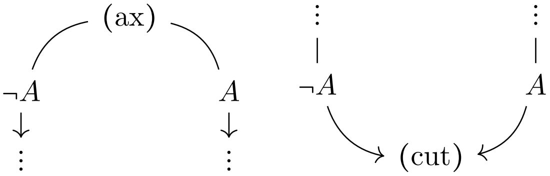
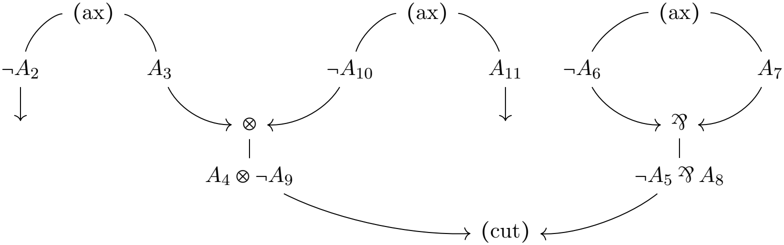
Geometry of Interaction

Figure: Identification of variables in an intuitionsitic sequent calculus proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : p \vdash p}\,(\mathrm{ax}) \qquad \dfrac{\dfrac{}{x : p \vdash p}\,(\mathrm{ax}) \qquad \dfrac{}{x : p \vdash p}\,(\mathrm{ax})}{y' : p \supset p, x : p \vdash p}\,(L \supset)}{y : p \supset p, y' : p \supset p, x : p \vdash p}\,(L \supset)}{y : p \supset p, x : p \vdash p}\,(\mathrm{ctr})}{y : p \supset p \vdash p \supset p}\,(R \supset)$$

Proof nets.

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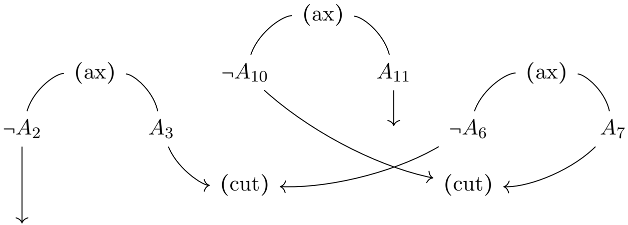
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The goals of this paper are to set up a basic dictionary between

- multiplicative proofs nets and ideals
- reduction sequences and monomial orders
- cut-elimination and elimination

The ideals I_π do not have a very interesting geometry: the associated affine variety is just an intersection of pairwise diagonals. In subsequent papers in this series we introduce, on top of the foundations laid here, more interesting algebra and geometry (see Section 8).

Formulas

Definition (Formulas)

- *Unoriented atoms* $X, Y, Z, \ldots$
- *An oriented atom* (or *atomic proposition*) is a pair $(X, +)$ or $(X, -)$ where X is an unoriented atom.

Pre-formulas:

- Any atomic proposition is a preformula.
- If A, B are pre-formulas then so are $A \otimes B$, $A \parr B$.
- If A is a pre-formula then so is $\neg A$.

Formulas: quotient of pre-formulas:

$$\neg(A \otimes B) \sim \neg B \parr \neg A \qquad \neg(A \parr B) \sim \neg B \otimes \neg A$$

$$\neg(X, +) \sim (X, -) \qquad \neg(X, -) \sim (X, +)$$

Polynomial ring of a proof structure

Definition (Sequence of (un)oriented atoms)

Let A be a formula with sequence of oriented atoms $\big((X_1, x_1), ..., (X_n, x_n)\big)$. The *sequence of unoriented atoms* of A is $(X_1, ..., X_n)$ and the *set of unoriented atoms* of A is the disjoint union $\{X_1\} \coprod \cdots \coprod \{X_n\}$.

Definition (Polynomial ring P_A of a formula A)

P_A is the free commutative k-algebra on the set of unoriented atoms of A:

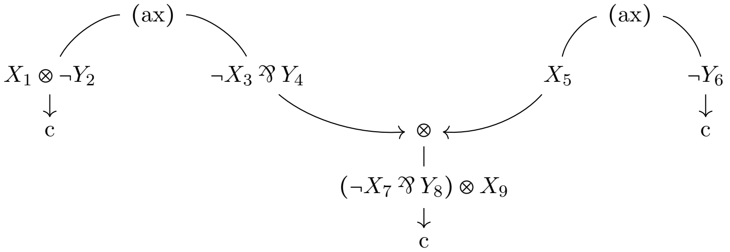
$$P_A = k[X_1, ..., X_n]$$

Let π be a proof structure with edge set E and denote by A_e the formula labelling edge $e \in E$. The *polynomial ring* of π, denoted P_π is the following, where U_e is the set of unoriented atoms of A_e.

$$P_\pi := \bigotimes_{e \in E} P_{A_e} \cong k[\coprod_{e \in E} U_e]$$

Polynomial ring example

Let π denote the following proof net.



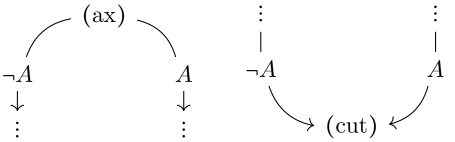
$$P_\pi =$$
$$k\Big[\{X\}\coprod\{Y\}\coprod\{X\}\coprod\{Y\}\coprod\{X\}\coprod\{Y\}\coprod\{X\}\coprod\{Y\}\coprod\{X\}\Big]$$
$$= k[X_1, Y_2, X_3, Y_4, X_5, Y_6, X_7, Y_8, X_9]$$

But what about the links?

Links

Definition (Link ideal I_l, link coordinate ring R_l)

Axiom/Cut link l:



$\big((X_1, x_1), \dots, (X_n, x_n)\big)$ is the sequence of oriented atoms of A.

$$\begin{gathered} I_l \subseteq P_A \otimes P_{\neg A} \\ I_l = (X_i - X_i')_{i=1}^n = (X_i \otimes 1 - 1 \otimes X_i)_{i=1}^n \end{gathered} \qquad R_l := P_A \otimes P_{\neg A} / I_l$$

Tensor/Par links

Tensor/Par link l:

$$\begin{tikzcd}[column sep=small]
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
A \arrow[dr, bend right] & & B \arrow[dl, bend left] & A \arrow[dr, bend right] & & B \arrow[dl, bend left] \\
& \otimes \arrow[d, dash] & & & \parr \arrow[d, dash] & \\
& A \otimes B \arrow[d] & & & A \parr B \arrow[d] & \\
& \vdots & & & \vdots &
\end{tikzcd}$$

Let $\boxtimes = \otimes$ if l is a tensor link, and $\boxtimes = \parr$ if l is a par link.

$$I_l \subseteq P_A \otimes P_B \otimes P_{A \boxtimes B}$$
$$I_l = (\{X_i - X_i'\}_{i=1}^{n} \cup \{Y_j - Y_j'\}_{j=1}^{m})$$
$$= (\{X_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes X_i\}_{i=1}^{n} \cup \{1 \otimes Y_j \otimes 1 - 1 \otimes 1 \otimes Y_j\}_{j=1}^{m})$$

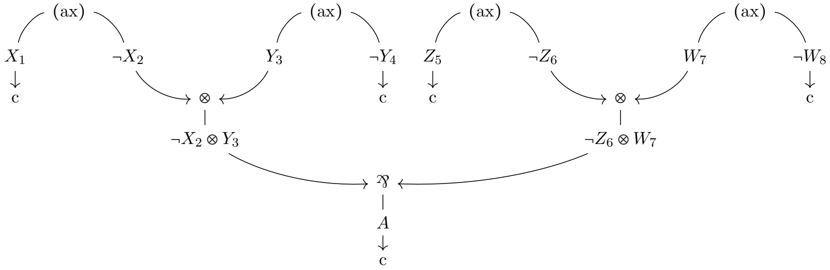
$$R_l = P_A \otimes P_B \otimes P_{A \boxtimes B} / I_l$$

Definition (Defining ideal I_π, coordinate ring R_π)

$I_\pi := \sum_l I_l \subseteq P_\pi$ where l ranges over all links of π. $R_\pi := P_\pi / I_\pi$.

Example of coordinate ring of a proof structure

$$A := (\neg X_2 \otimes Y_3) \parr (\neg Z_6 \otimes W_7)$$



$$\begin{aligned}
P_\pi &= k[X_1, X_2, X_2', X_2'', Y_3, Y_3', Y_3'', Y_4, Z_5, Z_6, Z_6', Z_6'', W_7, W_7', W_7'', W_8] \\
I_\pi &= (X_1 - X_2) + (Y_3 - Y_4) + (Z_5 - Z_6) + (W_7 - W_8) \\
&\quad + (X_2 - X_2', Y_3 - Y_3') + (Z_6 - Z_6', W_7 - W_7') \\
&\quad + (X_2' - X_2'', Y_3' - Y_3'', Z_6' - Z_6'', W_7' - W_7'') \\
R_\pi &= P_\pi / I_\pi \cong k[X, Y, Z, W]
\end{aligned}$$

Persistent paths

Let π be a proof net with single conclusion A, and let

$$(Z_1, z_1), ..., (Z_n, z_n) \tag{1}$$

be the sequence of oriented atoms of A. Then $n = 2m$ is even, there are an equal number of positive and negative atoms, and if $\mathbf{U} = U_1, \ldots, U_m$ denotes the subsequence of positive unoriented atoms and $\mathbf{V} = V_1, \ldots, V_m$ the subsequence of negative unoriented atoms then

(i) The inclusions $k[\mathbf{U}] \longrightarrow P_\pi$ and $k[\mathbf{V}] \longrightarrow P_\pi$ followed by the quotient $P_\pi \longrightarrow R_\pi$ are isomorphisms β_+, β_- as in the diagram

$$\begin{tikzcd}
k[\mathbf{U}] \arrow[d] \arrow[dr, "\beta_+"] & \\
P_\pi \arrow[r, two heads] & R_\pi \\
k[\mathbf{V}] \arrow[u] \arrow[ur, "\beta_-"'] &
\end{tikzcd} \tag{2}$$

(ii) The composite $\beta_-^{-1}\beta_+ : k[\mathbf{U}] \longrightarrow k[\mathbf{V}]$ is

$$\beta_-^{-1}\beta_+(U_i) = V_{\sigma(i)}, \quad 1 \le i \le m \tag{3}$$

for some permutation σ_π of $\{1, ..., m\}$.

(iii) Each equivalence class of the relation $\approx$ is the underlying set of a sequence

$$\mathscr{P} = (Z_1, \dots, Z_r) \tag{4}$$

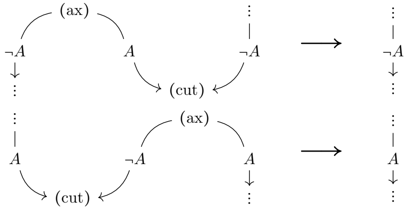
where for some $1 \le i \le m$ we have $Z_1 = V_{\sigma(i)}, Z_r = U_i$ and $Z_i \sim Z_{i+1}$ for $1 \le i < r$.

Definition

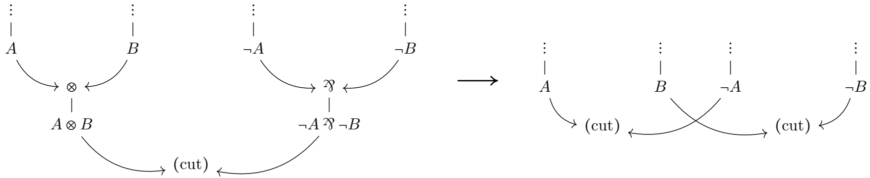
Let π be a proof net with single conclusion A. The sequences $\mathscr{P}$ of (4) whose underlying sets are the equivalence classes of $\approx$ are called *persistent paths*.

Cut reduction

a-redexes:



m-redex:



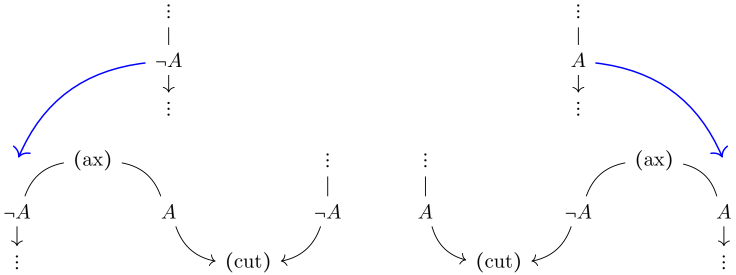
Modelling cut-reduction

Definition

Let $\gamma : \pi \longrightarrow \pi'$ be a reduction, there exists homomorphisms.

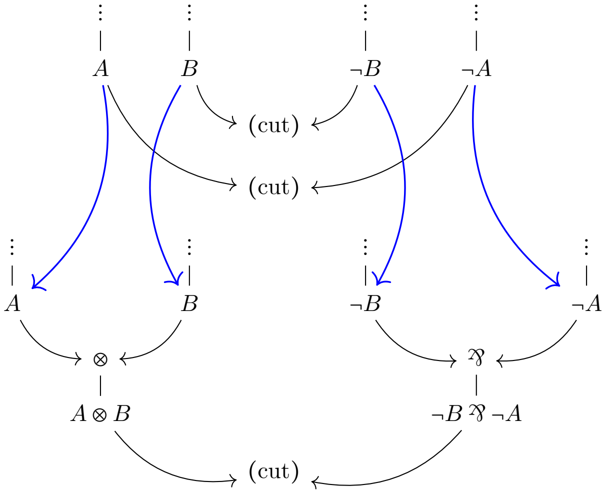
$$\begin{tikzcd} P_{\pi'} \arrow[r, bend left, "T_\gamma"] & P_\pi \arrow[l, bend left, "S_\gamma"] \end{tikzcd}$$

T_γ, γ reducing an a-redex:



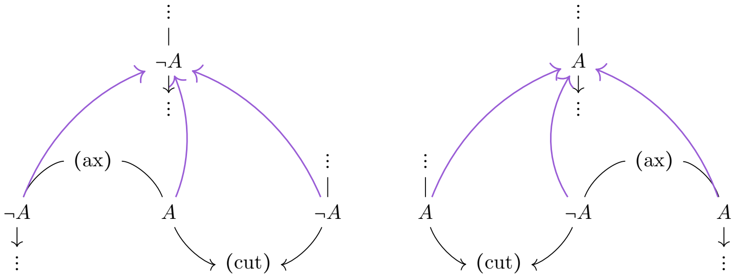
Modelling cut reduction

T_γ, γ reducing an m-redex:



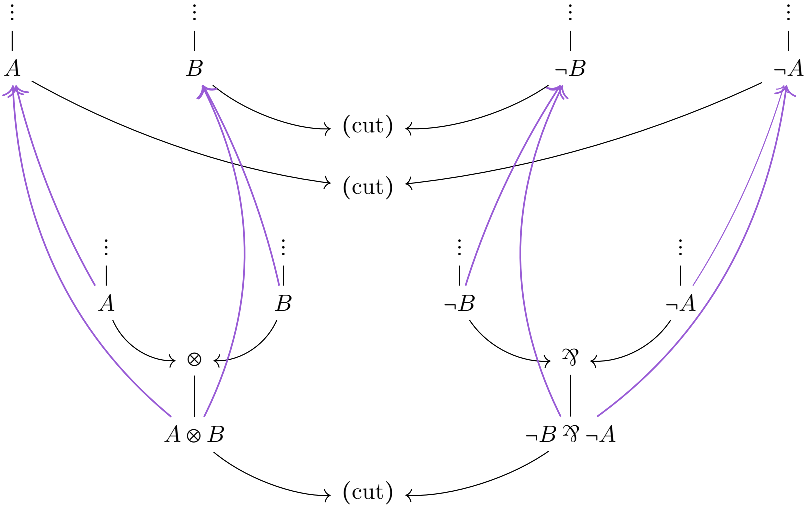
Modelling cut reduction

S_γ, γ reducing an a-redex.



Modelling cut reduction

S_γ, γ reducing an m-redex.



Cut elimination on the level of the coordinate rings

Proposition

Let γ be any reduction, we have $T_\gamma(I_{\pi'}) \subseteq I_\pi, S_\gamma(I_\pi) \subseteq I_{\pi'}$ and the induced morphisms of k-algebras $\overline{T}_\gamma, \overline{S}_\gamma$ making the following diagram commute, are mutually inverse isomorphisms. In the following, $p : P_\pi \twoheadrightarrow R_\pi$ and $p' : P_{\pi'} \twoheadrightarrow R_{P_{\pi'}}$ are projection maps.

$$
\begin{tikzcd}
I_\pi \arrow[r] & P_\pi \arrow[r, two heads, "p"] \arrow[d, bend right, "S_\gamma"'] & R_\pi \arrow[d, bend right, "\overline{S}_\gamma"'] \\
I_{\pi'} \arrow[r, tail] & P_{\pi'} \arrow[r, two heads, "p'"] \arrow[u, bend right, "T_\gamma"'] & R_{\pi'} \arrow[u, bend right, "\overline{T}_\gamma"']
\end{tikzcd}
$$

Reduction sequence

Definition

A *reduction sequence* $\Gamma : \pi \longrightarrow \pi'$ between proof structures π, π' is a nonempty sequence of reductions

$$\pi = \pi_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{n-1}} \pi_n = \pi' \,. \tag{5}$$

This induces a sequence of k-algebra morphisms

$$P_{\pi'} \xrightarrow{T_{\gamma_{n-1}}} \cdots \xrightarrow{T_{\gamma_1}} P_\pi \tag{6}$$

the composite of which we denote by $T_\Gamma : P_{\pi'} \longrightarrow P_\pi$.

Division algorithm for polynomials in multiple variables

Choose an order $x_1 < \cdots < x_n$, this induces lexicographic order on the monic monomials of $k[x_1, ..., x_n]$ with respect to the degrees. Consider $\mathbb{C}[x > y]$.

$$y < xy < x^2 < x^2y^{10} < x^3 < \cdots$$

Now, divide according to leading terms!

$$\begin{array}{cr}
q_0: & xy^2 \\
q_1: & \\
x^2y & \overline{)x^3y^3 + xy^2 - y} \\
x + y &
\end{array}$$

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q_0 : & \qquad xy^2 \\
q_1 : & \\
x^2y & \overline{x^3y^3 + xy^2 - y} \\
x + y & \big)x^3y^3 + xy^2 - y \\
 & \;x^3y^3 \\
\hline
 & \qquad\quad xy^2 - y
\end{array}$$

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$$y < xy < x^2 < x^2y^{10} < x^3 < \cdots$$

Now, divide according to leading terms!

$$\begin{array}{rl}
q_0 : & \quad xy^2 \\
q_1 : & \quad y^2 \\
\begin{matrix} x^2y \\ x+y \end{matrix} & \overline{)x^3y^3 + xy^2 - y} \\
& \underline{\; x^3y^3 \qquad\qquad\quad} \\
& \qquad\quad xy^2 - y
\end{array}$$

Division algorithm for polynomials in multiple variables

Choose an order $x_1 < \cdots < x_n$, this induces lexicographic order on the monic monomials of $k[x_1, ..., x_n]$ with respect to the degrees. Consider $\mathbb{C}[x > y]$.

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Now, divide according to leading terms!

$$\begin{array}{cl}
q_0 : & \qquad\qquad xy^2 \\
q_1 : & \qquad\qquad y^2 \\
x^2y & \overline{)x^3y^3 + xy^2 - y} \\
x + y & \\
 & \;\; x^3y^3 \\
\hline
 & \qquad\qquad xy^2 - y \\
 & \qquad\qquad xy^2 + y^3 \\
\cline{2-2}
 & \qquad\qquad\quad - y - y^3
\end{array}$$

Leading terms

Given polynomials $f_1, \ldots, f_n$ we have the following inclusion, where $\langle g_1, \ldots, g_m \rangle$ denotes the ideal generated by the polynomials $g_1, \ldots, g_m$.

$$\langle \mathrm{LT}\, f_1, \cdots, \mathrm{LT}\, f_n \rangle \subseteq \langle \mathrm{LT} \langle f_1, \ldots, f_n \rangle \rangle$$

This reverse inclusion does *not* hold in general. Indeed, consider the polynomial ring $k[x, y]$ with $y < x$. Let f_1, f_2 respectively denote the polynomials $x^3 - 2xy$ and $x^2y - 2y^2 + x$. We have:

$$\{\mathrm{LT}\, f_1, \mathrm{LT}\, f_2\} = \{x^3, x^2y\}$$

however, the following polynomial is in the ideal generated by $\{f_1, f_2\}$.

$$y(x^3 - 2xy) - x(x^2y - 2y^2 + x) = -x^2$$

Hence, x^2 is in the leading ideal. However, x^2 is not in the ideal generated by the polynomials x^3, x^2y.

Gröbner bases

Definition

A set of polynomials $\{f_1, ..., f_n\}$ satisfying the following:

$$\langle \mathrm{LT}\, f_1, \cdots \mathrm{LT}\, f_n \rangle = \langle \mathrm{LT} \langle f_1, ..., f_n \rangle \rangle$$

is a *Gröbner basis* for the ideal $\langle f_1, ..., f_n \rangle$ generated by $f_1, ..., f_n$.

Definition

The S-*polynomial* of polynomials $g, h \in k[x_1, ..., x_n]$ is defined to be the following, where $\beta = (\beta_1, ..., \beta_n)$ where $\beta_i = \max\big((\deg g)_i, (\deg h)_i\big)$..

$$S(g, h) := \frac{x^\beta}{\mathrm{LT}\, g} g - \frac{x^\beta}{\mathrm{LT}\, h} h$$

This is indeed a polynomial, and is designed to obtain cancellation of leading terms.

Euclidean division with early stopping

Algorithm 1 Euclidean Division with Early Stopping

```
Require: (f_1, ..., f_s), f
  p ← f
  q_1, ..., q_s ← 0, ... 0
  r ← 0
  while p ≠ 0 do
      DivOcc ← False
      i ← 1
      while i ≤ s and DivOcc = false do
          if LT f_i | LT p then
              q_i ← q_i + LT p / LT f_i
              p ← p − (LT p / LT f_i) f_i
              DivOcc ← True
          else
              i ← i + 1
          end if
      end while
      if DivOcc = false then
          r ← p
          p ← 0
      end if
  end while
  return (q_1, ..., q_s, r)
```

Buchberger with early stopping

Algorithm 2 Buchberger with Early Stopping

Require: $F = (f_1, \ldots, f_s)$, returns a Gröbner basis for $\langle f_1, \ldots, f_s \rangle$.

$B \leftarrow \{(i,j) \mid 1 \le i < j \le s\}$
$G \leftarrow F$
$t \leftarrow s$
while $B \neq \varnothing$ **do**
  Let $(i,j) \in B$ be first in the lexicographic order.
  if $\mathrm{LCM}(\mathrm{LM}(f_i), \mathrm{LM}(f_j)) \neq \mathrm{LM}(f_i)\,\mathrm{LM}(f_j)$ **and** $\mathrm{Criterion}(f_i, f_j, B)$ **is false then**
    $S \leftarrow \mathrm{div}_{es}(S(f_i, f_j), G)$
    if $S \neq 0$ **then**
      $t \leftarrow t + 1$
      $f_t \leftarrow \frac{1}{\mathrm{LC}(S)} S$
      $G \leftarrow G \cup \{f_t\}$
      $B \leftarrow B \cup \{(i,t) \mid 1 \le i \le t-1\}$
    end if
  end if
  $B \leftarrow B \setminus \{(i,j)\}$
end while
return G

where $\mathrm{Criterion}(f_i, f_j, B)$ is true provided that there is some $k \notin \{i,j\}$ for which the pairs $[i,k]$ and $[j,k]$ are *not* in B and $\mathrm{LM}(f_k)$ divides $\mathrm{LCM}(\mathrm{LM}(f_i), \mathrm{LM}(f_j))$.

Elimination Theory

Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ and $\mathbf{Y} = \{Y_1, \ldots, Y_m\}$ be variables. We suppose given an ideal $I \subseteq k[\mathbf{X}, \mathbf{Y}]$ and we are interested to know the equations between the $\mathbf{X}$-variables that are implied by the equations in I. We call this set of equations the elimination ideal:

Definition

The *elimination ideal* of I is the ideal $I \cap k[\mathbf{X}]$ in $k[\mathbf{X}]$.

Theorem (The Elimination Theorem)

Let $I \subseteq k[\mathbf{X}, \mathbf{Y}]$ be an ideal and G a Gröbner basis of I with respect to a lexicographic order where $X_i < Y_j$ for all $1 \leq i \leq n, 1 \leq j \leq m$. Then $G \cap k[\mathbf{X}]$ is a Gröbner basis for $I \cap k[\mathbf{X}]$.

A graphical presentation

Fix a polynomial ring $k[X_1, \ldots, X_n]$, let $<$ be a total order on the set $\{X_1, \ldots, X_n\}$ and take the lexicographic monomial order determined by $<$. Let σ be the permutation uniquely defined by

$$X_{\sigma^{-1}1} < X_{\sigma^{-1}2} < \cdots < X_{\sigma^{-1}n} .$$

The position of X_i in this sequence is $\sigma(i)$. We view σ as assigning a *height* to variables:

Definition

The *realisation* of $<$ is the oriented graph $\mathscr{R}_<$ with vertices

$$\big\{(i, \sigma i) \mid 1 \le i \le n\big\} \subseteq \mathbb{R}^2 \tag{7}$$

with an edge between $(i, \sigma i), (i+1, \sigma(i+1))$ for $i < n$, and $(i, \sigma i)$ decorated with X_i. The orientation of the edge is from $(i, \sigma i)$ to $(j, \sigma j)$ if $X_i < X_j$.

Definition

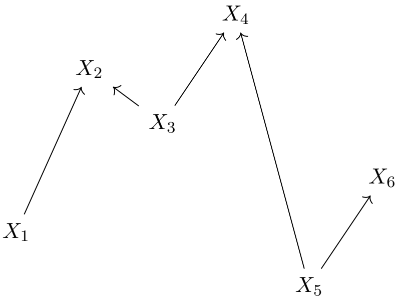
A $<$-*graph* is an oriented graph on the vertex set (7) with the property that if there is an edge from a vertex $(i, \sigma i)$ decorated with X_i to a vertex decorated with X_j then $X_i < X_j$.

Definition

A $<$-graph is *linear* if every vertex has valence at most two.

Example

Let $X_1, \ldots, X_6$ be ordered by $X_5 < X_1 < X_6 < X_3 < X_2 < X_4$.
Then $\mathscr{R}_<$ is



(8)

We simply write *graph* instead of $<$-graph.

Definition

A *roof* in a graph $\mathscr{S}$ is an ordered pair (e, e') of edges $e : X_i \longrightarrow X_l, e' : X_k \longrightarrow X_l$ with the same endpoint X_l and $X_i < X_k$. We call X_l the *tip* of the roof.

Definition

Given a graph $\mathscr{S}$ we define

$$G_{\mathscr{S}} = \left\{ X_j - X_i \mid e : X_i \longrightarrow X_j \text{ is an edge in } \mathscr{S} \right\}.$$

Standard monomial order

Let π be a proof net with single conclusion A. Let

$$\mathscr{P}_1, \ldots, \mathscr{P}_m \tag{9}$$

be the persistent paths of π ordered so that if U_i is the last unoriented atom in $\mathscr{P}_i$ then $U_1, \ldots, U_m$ is the order that these atoms appear in A. Let us name the variables X_i so that (9) is $X_1, \ldots, X_n$ and $P_\pi = k[X_1, \ldots, X_n]$.

Definition

We write $U <_0 V$ if U is before V in (9). The monomial order $<_0$ on P_π is the lexicographic order determined by $<_0$ on the variables.

Definition

Let $\mathscr{S}_0$ be the oriented graph with vertex set U_π where two variables $U, V \in U_\pi$ are connected by an edge $e : U \longrightarrow V$ if $U \sim V$ and $U <_0 V$.

Monomial order of a reduction sequence

$\Gamma : \pi \longrightarrow \pi'$ is a reduction sequence between proof nets with single conclusion A. Let $\mathscr{Q}_i$ be the subsequence of $\mathscr{P}_i$ consisting just of those unoriented atoms in π' (those in the image of T_Γ). Let $\mathscr{P}_i \setminus \mathscr{Q}_i$ denote the complement of the subsequence $\mathscr{Q}_i$. Then

$$\mathscr{Q}_1, \ldots, \mathscr{Q}_m, \mathscr{P}_1 \setminus \mathscr{Q}_1, \ldots, \mathscr{P}_m \setminus \mathscr{Q}_m \tag{10}$$

is the set of unoriented atoms of π arranged in an order that depends on the reduction Γ. Note that $\mathscr{P}_i \setminus \mathscr{Q}_i$ are the variables in $\mathscr{P}_i$ eliminated during the reduction sequence.

Definition

We write $U <_\Gamma V$ if U is before V in (10), reading from left to right. The monomial order $<_\Gamma$ on P_π is the lexicographic order determined by $<_\Gamma$ on the variables.

Definition

We denote by $G_\pi^{(0)}$ the ordered set of polynomials $G_{\mathscr{S}_0}$

$$G_\pi^{(0)} = \left\{ V - U \mid e : U \longrightarrow V \text{ is an edge in } \mathscr{S}_0 \right\}. \tag{11}$$

Definition

Let $\mathscr{S}_\Gamma$ be the oriented graph with vertex set U_π, where two variables $U, V \in U_\pi$ are connected by an edge $e : U \longrightarrow V$ if $U \sim V$ and $U <_\Gamma V$.

Definition

Given a sequence $F = (f_1, \ldots, f_s)$ of polynomials and a monomial order $<$ on $k[X_1, \ldots, X_n]$ we denote by $\mathbb{B}_{es}(F, <)$ the output of the Buchberger Algorithm with early stopping.

Theorem

There is an equality of sets

$$G_{\pi'}^{(0)} = \mathbb{B}_{es}(G_\pi^{(\Gamma)}, <_\Gamma) \cap P_{\pi'}. \tag{12}$$

Falling Roofs

Definition

A roof (e, e') precedes a roof (d, d') if $e < d$ or $e = d$ and $e' < d'$.

Algorithm 3 Falling Roofs

Require: Linear graph $\mathscr{S}$

$\mathscr{N} \leftarrow \mathscr{S}$

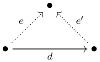
Mark all edges in $\mathscr{N}$ as live

while $\mathscr{N}$ contains a live roof **do**

  $(e, e') \leftarrow$ the first live roof in $\mathscr{N}$

  Mark e, e' as dead

  If it does not exist, add to $\mathscr{N}$ a live edge d as shown below:

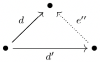


  while d is part of a live roof in $\mathscr{N}$ **do**

    if (d, e'') is a live roof in $\mathscr{N}$ **then**

      Mark e'' as dead

      If it does not exist, add to $\mathscr{N}$ a live edge d' as shown belo



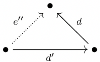
      Remove d from $\mathscr{N}$

      $d \leftarrow d'$

    else if (e'', d) is a live roof in $\mathscr{N}$ **then**

      Mark e'' as dead

      If it does not exist, add to $\mathscr{N}$ a live edge d' as shown below:



      Remove d from $\mathscr{N}$

      $d \leftarrow d'$

    end if

  end while

end while

return $\mathscr{N}$

Example

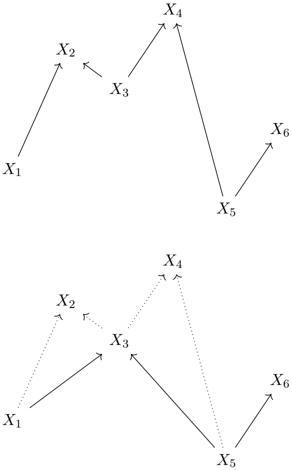


Figure: The falling roofs algorithm applied to the graph of Example 1, reading from left to right and top to bottom.

What about Buchberger *without* early stopping?

Let π denote the following proof net.

$$
\begin{tikzcd}
 & (\mathrm{ax}) \arrow[dl, no head, bend right] \arrow[dr, no head, bend left] & & & & (\mathrm{ax}) \arrow[dl, no head, bend right] \arrow[dr, no head, bend left] & & & & (\mathrm{ax}) \arrow[dl, no head, bend right] \arrow[dr, no head, bend left] & \\
X_1 \arrow[d] & & Y_1 \arrow[dr, bend right] & & Y_2 \arrow[dl, bend left] & & Y_3 \arrow[dr, bend right] & & Y_4 \arrow[dl, bend left] & & X_2 \arrow[d] \\
\mathrm{c} & & & (\mathrm{cut}) & & & & (\mathrm{cut}) & & & \mathrm{c}
\end{tikzcd}
$$

π reduces to π':

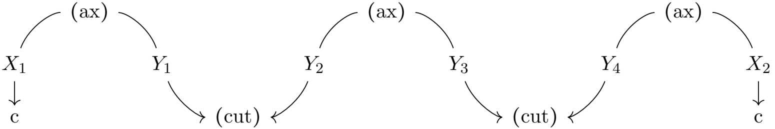
$$
\begin{tikzcd}
 & (\mathrm{ax}) \arrow[dl, no head, bend right] \arrow[dr, no head, bend left] & \\
X_1 \arrow[d] & & X_2 \arrow[d] \\
\mathrm{c} & & \mathrm{c}
\end{tikzcd}
$$

We now consider the sets of generators of the defining ideals of π and π'.

$$G_\pi := \{X_1 - Y_1, Y_1 - Y_2, Y_2 - Y_3, Y_3 - Y_4, Y_4 - X_2\}, \quad G_{\pi'} := \{X_1 - X_2\}$$

$$Y_1 > Y_2 > Y_3 > Y_4 > X_1 > X_2$$

There *is* something to do



$$G_\pi = \{f_1 = X_1 - Y_1,\ f_2 = Y_1 - Y_2,\ f_3 = Y_2 - Y_3,\ f_4 = Y_3 - Y_4,\ f_5 = Y_4 - X_2\}$$

$$Y_1 > Y_2 > Y_3 > Y_4 > X_1 > X_2$$

The leading terms of $f_1, \ldots, f_5$ respectively are $-Y_1, Y_1, Y_2, Y_3, Y_4$ and the leading term of $f_1 + \cdots + f_5$ is X_1. Hence:

$$X_1 \in \mathrm{LT}\langle G_\pi \rangle, \qquad X_1 \notin \langle \mathrm{LT}\, G_\pi \rangle$$

Thus, G_π is *not* Gröbner basis.

We now calculate the 10 S-polynomials which arise from G_π.

$$S(f_1,f_2) = Y_2 - X_1 \qquad S(f_1,f_3) = Y_1Y_3 - Y_2X_1 \qquad S(f_1,f_4) = Y_1Y_4 - X_1X_3$$

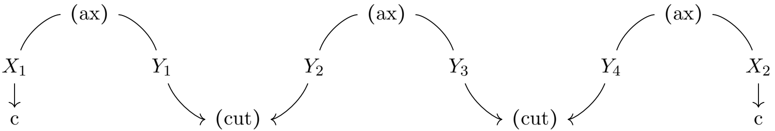
$$S(f_1,f_5) = Y_1X_2 - X_1Y_4 \qquad S(f_2,f_3) = Y_1Y_3 - Y_2^2 \qquad S(f_2,f_4) = Y_1Y_4 - Y_2Y_3$$

$$S(f_2,f_5) = Y_1X_2 - Y_2Y_4 \qquad S(f_3,f_4) = Y_2Y_4 - Y_2^2 \qquad S(f_3,f_5) = Y_2X_2 - Y_3Y_4$$

$$S(f_4,f_5) = Y_3X_2 - Y_4^2$$

For each $i > j$, $i,j \in \{1,...,5\}$ we now divide $S(f_i,f_j)$ by G. In fact, this always gives a remainder zero except for the particular case when $(i,j) = (1,2)$, which we show on the next slide.

Division



$$G_\pi = \{ f_1 = X_1 - Y_1,\ f_2 = Y_1 - Y_2,\ f_3 = Y_2 - Y_3,\ f_4 = Y_3 - Y_4,\ f_5 = Y_4 - X_2 \}$$

$$G_\pi \quad \begin{array}{l}
(0,0,1,1,1) \\
\overline{)\,Y_2 - X_1} \\
\ \ Y_2 - Y_3 \\
\hline
\qquad\quad Y_3 - Y_4 \\
\qquad\quad Y_3 - X_1 \\
\hline
\qquad\qquad\quad Y_4 - X_1 \\
\qquad\qquad\quad Y_4 - X_2 \\
\hline
\qquad\qquad\qquad\quad X_2 - X_1
\end{array}$$

$$\big(G_\pi \cup \{X_2 - X_1\}\big) \cap k[X_1, X_2] = G_{\pi'}$$

Summary

$$\pi \longmapsto (P_\pi, G_\pi, <_0) \tag{13}$$

There are many other monomial orders on P_π. With respect to some monomial orders G_π will be "stable" in the sense that it is a Gröbner basis, while it will be "unstable" (not a Gröbner basis) with respect to others and running the Buchberger algorithm makes nontrivial changes.

$$\begin{tikzcd}
\pi \arrow[r, "(78)"] \arrow[ddd, "\gamma"'] & (P_\pi, G_\pi, <_0) \arrow[r, "\text{destabilise}"] & (P_\pi, G_\pi, <_\Gamma) \arrow[d, "\text{Buchberger}"] \\
& & (P_\pi, G_{\mathcal{N}}, <_\Gamma) \arrow[d, "\text{intersection}"] \\
& & (P_{\pi'}, G_{\mathcal{N}} \cap P_{\pi'}, <_0) \arrow[d, equal, "\text{Theorem 7.2}"] \\
\pi' \arrow[rr, "(78)"'] & & (P_{\pi'}, G_{\pi'}, <_0)
\end{tikzcd}$$

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