Quantum Error Correcting Codes and Matrix Factorisations

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1 Quantum Error Correcting Codes

Definition 1.0.1. A **qubit** is a copy of the $\mathbb{C}$-Hilbert space $\mathbb{C}^2$.

The **state** of a qubit $\mathbb{C}^2$ is a vector $|\psi\rangle \in \mathbb{C}^2$ of norm 1.

A pair $(\mathbb{C}^2, |\psi\rangle)$ consisting of a qubit $\mathbb{C}^2$ and a state $|\psi\rangle \in \mathbb{C}^2$ is a **prepared qubit** and we say $\mathbb{C}^2$ has been **prepared** to $|\psi\rangle$.

Definition 1.0.2. Let $\mathcal{H}_1, \mathcal{H}_2$ be two state spaces. The **composite state space** is $\mathcal{H}_1 \otimes \mathcal{H}_2$. A **state** of a composite system is a vector $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ which can be written as a linear combination of pure tensors

$$\alpha_1 |\psi_1\rangle + \ldots + \alpha_n |\psi_n\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

where the coefficients satisfy $|\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1$.

We define a measurement as a family of possible outcomes with assocaited probabilities. We allow for the possibility that measurements effect the state, and so measurements are operators upon the state space.

Definition 1.0.3. A **measurement** on a state space $\mathcal{H}$ is a finite family of linear operators $\{M_m : \mathcal{H} \longrightarrow \mathcal{H}\}_{m \in \mathcal{M}}$ satisfying the **completeness condition**.

$$\sum_{m \in \mathcal{M}} M_m^\dagger M_m = I \tag{1}$$

An element $m \in \mathcal{M}$ is an **outcome**.

The **resulting state** after measurement $\{M_m\}_{m \in \mathcal{M}}$ and outcome m is:

$$\frac{M_m |\psi\rangle}{\sqrt{p(m)}} \tag{2}$$

Remark 1.0.4. Associated to every measurement and state vector $|\psi\rangle$ there is a value

$$p(m) := \langle\psi| M_m^\dagger M_m |\psi\rangle = \| M_m |\psi\rangle \|^2 \tag{3}$$

It follows from (1) that $p(m) \leq 1$ for all $m, |\psi\rangle$. We understand $p(m)$ as the probability of outcome m on the measurement $\{M_m\}_{m\in\mathcal{M}}$.

Definition 1.0.5. A linear transformation P is a **projector** if $P^2 = P$.

Definition 1.0.6. A **quantum error correcting code (QECC)** is a pair $\mathcal{Q} = (\mathcal{H}, S)$ consisting of a state space $\mathcal{H}$ along with a set of operators S on $\mathcal{H}$. The elements of S are the **stabilisers**. The **codespace** $\mathcal{H}^S$ of $\mathcal{Q}$ is the maximal subspace of $\mathcal{H}$ invariant under all the operators in S.

See Appendix A for an example of a simple QECC.

2 Matrix factorisations

2.1 Koszul Complex

Recall that a $\mathbb{Z}_2$-graded ring R comes equipt with a choice of isomorphism $R \cong R_0 \oplus R_1$ where R_0, R_1 are subgroups of R. The elements of R_1 are **odd**.

Definition 2.1.1. Let E be a $\mathbb{Z}_2$-graded ring and consider a set of odd elements $\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^* \in E$. These **satisfy the canonical anticommutation relations** if the following hold for all $i, j = 1, \ldots, n$.

- $\theta_i\theta_j + \theta_j\theta_i = 0$
- $\theta_i^*\theta_j^* + \theta_j^*\theta_i^* = 0$
- $\theta_i\theta_j^* + \theta_j\theta_i^* = \delta_{ij}$

When $A \cong A_0 \oplus A_1, B \cong B_1 \oplus B_2$ are $\mathbb{Z}_2$-graded modules (over a graded ring R, say), a homomorphism $\varphi : A \longrightarrow B$ can be written as a matrix

$$\begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix} \tag{4}$$

where $\varphi_{00}(A_0) \subseteq B_0, \varphi_{01}(A_0) \subseteq B_1, \varphi_{10}(A_1) \subseteq B_0, \varphi_{11}(A_1) \subseteq B_1$. Writing this matrix as a sum yields respectively an even and odd component of φ

$$\begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix} = \begin{pmatrix} \varphi_{00} & 0 \\ 0 & \varphi_{11} \end{pmatrix} + \begin{pmatrix} 0 & \varphi_{01} \\ \varphi_{10} & 0 \end{pmatrix} \tag{5}$$

In this way, $\mathrm{Hom}(A, B)$ is also a $\mathbb{Z}_2$-graded module over R.

When E is a $\mathbb{Z}_2$-graded ring of the form $E = \mathrm{End}(A)$ for some $\mathbb{Z}_2$-graded ring A, then E admitting a set of odd elements satisfying the anticommutation relations is sufficient for A to admit a Clifford algebra represenation [1, Lemma 5.6.2].

Definition 2.1.2. The Clifford Algebra C_n is generated by elements $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n$ subject to:

$$[\mu_i, \mu_j] = -2\delta_{ij} \quad [\nu_i, \nu_j] = 2\delta_{ij} \quad [\mu_i, \nu_j] = 0$$

where $[\xi, \zeta] = \xi\zeta + \zeta\xi$ for $\xi, \zeta \in \{\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n\}$.

There is a C_m action on S_m and hence on $S_m \otimes_{\mathbb{C}} (Y \otimes X)$. This is induced by two canonical endomorphisms which exist on S_m. The **wedge** and **contraction** maps.

$$\begin{aligned} \theta_i : \bigwedge^{d-1}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) &\longrightarrow \bigwedge^{d}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) \\ \theta_{j_1} \wedge \ldots \wedge \theta_{j_{d-1}} &\longmapsto \theta_i \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_{d-1}} \end{aligned}$$

and

$$\begin{aligned} \theta_i^* : \bigwedge^{d}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) &\longrightarrow \bigwedge^{d-1}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) \\ \theta_{j_1} \wedge \ldots \wedge \theta_{j_d} &\longmapsto \sum_{k=1}^{d} (-1)^{k+1} \delta_{j_k = i} \theta_{j_1} \wedge \ldots \wedge \hat{\theta}_{j_k} \wedge \ldots \wedge \theta_{j_d} \end{aligned}$$

Given a free R-module $A = R\theta_1 \oplus \ldots \oplus R\theta_r$ of rank r, the multiplication and contraction operators satisfy the canonical anticommutation relations. Thus, A admits a Clifford algebra representaiton.

2.2 Matrix factorisations

Let k denote a ring.

Recall that if A, B are $\mathbb{Z}$-graded k-modules then $\mathrm{Hom}(A, B)$ is also $\mathbb{Z}$-graded [3]. We have a similar definition for $\mathbb{Z}_2$-graded modules.

Definition 2.2.1. Let A, B be $\mathbb{Z}_2$-graded k-modules. A homomorphism $f : A \longrightarrow B$ is **even** if $f(A_0) \subseteq B_0$ and $f(A_1) \subseteq A_1$. The homomorphism f is **odd** if $f(A_0) \subseteq B_1$ and $f(A_1) \subseteq B_0$.

Remark 2.2.2. If $f : A \longrightarrow B$ is any module homomorphism then f can be written as a matrix
$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} \tag{6}$$
The morphism f is thus even if $f_{01} = f_{10} = 0$ and is odd if $f_{00} = f_{11} = 0$. This also shows that the morphism f can be written as the sum of an even and an odd component.

Definition 2.2.3. Let $f \in k$ be a non-zero divisor. A **linear factorisation of** $f \in k$ **over** k is a pair (X, ∂_X) consisting of a $\mathbb{Z}_2$-graded k-module $X = X_0 \oplus X_1$ and an odd homomorphism $\partial_X : X \longrightarrow X$ satisfying
$$\partial_X^2 = f \cdot \mathrm{id}_X \tag{7}$$
If X is free then (X, ∂_X) is a **matrix factorisation**.

The theory of Matrix factorisations is motivated by the search for square roots to operators. As a toy example, multiplication by $x^2 - y^2$ in $\mathbb{C}[x, y]$ does not admit a square root, but it does if we allow matrix solutions.
$$\begin{pmatrix} 0 & x-y \\ x+y & 0 \end{pmatrix}^2 = (x^2 - y^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$
A more serious example is given by the square root of the Laplacian operator, see the Introduction of [4].

Our interest in matrix factorsations comes from the fact that appropriate homotopy categories of matrix factorsations form the homcategories of the bicategory of Landau-Ginzburg models, which we anticipate to find within a model of multiplicative linear logic (proofs as hypersurface singularities).

Definition 2.2.4. A **morphism of linear factorisations**
$$\alpha : \left(X = X_0 \oplus X_1, d_X = \begin{pmatrix} 0 & p_X \\ q_X & 0 \end{pmatrix} \right) \longrightarrow \left(Y = Y_0 \oplus Y_1, d_Y = \begin{pmatrix} 0 & p_Y \\ q_Y & 0 \end{pmatrix} \right)$$
of $f \in R$ is a pair of morphisms $\alpha_0 : X_0 \longrightarrow Y_0, \alpha_1 : X_1 \longrightarrow Y_1$ rendering the following diagram commutative.
$$\begin{tikzcd} X_0 \arrow[r, "p_X"] \arrow[d, "\alpha_0"] & X_1 \arrow[r, "q_X"] \arrow[d, "\alpha_1"] & X_0 \arrow[d, "\alpha_0"] \\ Y_0 \arrow[r, "p_Y"] & Y_1 \arrow[r, "q_Y"] & Y_0 \end{tikzcd} \tag{9}$$

Given a matrix factorisation $(X = X_0 \oplus X_1, d_X) = \begin{pmatrix} 0 & p_X \\ q_X & 0 \end{pmatrix}$ there is a sequence

$$\dots \xrightarrow{p_X} X_1 \xrightarrow{q_X} X_0 \xrightarrow{p_X} X_1 \xrightarrow{q_X} \dots \tag{10}$$

however, we note that in general $d_X^2 = f \cdot I \neq 0$ and so strictly speaking this is *not* a chain complex.

Definition 2.2.5. We use the notation of Definition 2.2.4. Let $\beta = (\beta_0, \beta_1)$ be another morphism of linear factorisations $(X, d_X) \longrightarrow (Y, d_Y)$. The morphisms α, β are **homotopic** if there exists a pair of morphisms $h_0 : X_0 \longrightarrow Y_1, h_1 : X_1 \longrightarrow Y_0$ such that the following holds

$$\alpha_0 - \beta_0 = q_Y h_0 + h_1 p_X, \qquad \alpha_1 - \beta_1 = h_0 q_X + p_Y h_1 \tag{11}$$

$$\begin{tikzcd}
\dots \arrow[r, "q_X"] & X_0 \arrow[r, "p_X"] \arrow[dl, "h_0"'] \arrow[d, shift right, "\alpha_0"'] \arrow[d, shift left, "\beta_0"] & X_1 \arrow[r, "q_X"] \arrow[dl, "h_1"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & X_0 \arrow[r, "p_X"] \arrow[dl, "h_0"'] \arrow[d, shift right, "\alpha_0"'] \arrow[d, shift left, "\beta_0"] & X_1 \arrow[r, "q_X"] \arrow[dl, "h_1"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & \dots \arrow[dl, "h_0"'] \\
\dots \arrow[r, "q_Y"'] & Y_0 \arrow[r, "p_Y"'] & Y_1 \arrow[r, "q_Y"'] & Y_0 \arrow[r, "p_Y"'] & Y_1 \arrow[r, "q_Y"'] & \dots
\end{tikzcd}$$

The relation of homotopy defines an equivalence relation on the set of morphisms of linear factorisations.

Definition 2.2.6. A linear transformation whose underlying $\mathbb{Z}_2$-graded k-module is free and of finite rank is a **matrix factorisation**. There is a category $\mathrm{hmf}(k[\underline{x}], f)$ where the objects are matrix factorisations of f and the morphisms are homotopy equivalence classes of morphisms of matrix factorisations.

Definition 2.2.7. If (X, d_X) is a matrix factorisation then so is $(X[1], -d_X)$. If we denote this by $\Psi(X, d_X)$ then $\Psi : \mathrm{hmf}(k[\underline{x}], f) \longrightarrow \mathrm{hmf}(k[\underline{x}], f)$ is extends to an endofunctor which induces a supercategorical structure on $\mathrm{hmf}(k[\underline{x}], f)$ if we take $\xi : \Psi^2 \longrightarrow 1_{\mathrm{hmf}(k[\underline{x}], f)}$ to be the identity.

Definition 2.2.8. Let (X, ∂_X) be a linear factorisation of $f \in k$ over k and (Y, ∂_Y) a linear factorisation of $g \in k$ also over k. Then the **tensor product** of (X, ∂_X) and (Y, ∂_Y) consists of the following data:

$$X \otimes_k Y, \qquad \partial_{X \otimes_k Y} = d_X \otimes 1 + 1 \otimes d_Y \tag{12}$$

where $X \otimes_k Y$ is the *graded* tensor product, which satisfies the following for all $x_1, x_2 \in X, y_1, y_2 \in Y$.

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{\deg(x_2)\deg(y_1)}(x_1 x_2 \otimes y_1 y_2) \tag{13}$$

Lemma 2.2.9. *The tensor product $(X \otimes_k Y, \partial_{X \otimes_k Y})$ is a linear factorisation of $f + g$.*

Proof. See [1][Page 35]. $\square$

In the special case where there exists $f \in k[\underline{x}], g \in k[\underline{y}], h \in k[\underline{z}]$ and (X, ∂_X) is a linear factorisation of $f - g \in k[\underline{x}, \underline{y}]$ and (Y, ∂_Y) is a linear factorisation of $g - h \in k[\underline{y}, \underline{z}]$ then we also have the *cut* of (X, ∂_X) and (Y, ∂_Y).

Definition 2.2.10. For each $y_1, \dots, y_n \in \underline{y}$ let $\partial_{y_i} g$ denote the formal partial derivative of g with respect to y_i. Denote by J_g the following $k[\underline{y}]$-module.

$$J_g := k[\underline{y}]/(\partial_{y_1} g, \dots, \partial_{y_n} g) \tag{14}$$

The **cut** of $(X, \partial_X), (Y, \partial_Y)$ is the data of

$$X|Y := (X \otimes_{k[\underline{y}]} J_g \otimes_{k[\underline{y}]} Y), \qquad \partial_{X|Y} = d_X \otimes 1 \otimes 1 + 1 \otimes 1 \otimes d_Y \tag{15}$$

Lemma 2.2.11. *The cut $X|Y$ is a matrix factorisation of $f - h$.*

Proof. Check this. $\square$

We will use the following Lemma to indirectly talk about matrix factorisations using $\mathbb{Z}_2$-graded modules over Clifford algebras.

Definition 2.2.12. Let $k[\underline{x}], k[\underline{y}]$ denote polynomial rings over variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$ respectively. Let $U(\underline{x}) = \sum_{i=1}^{n} x_i^2$.

We let C_U denote the $\mathbb{Z}_2$-graded k-algebra with multiplicative generators $\mu_1, \dots, \mu_n, \nu_1, \dots, \mu_n$ satisfying the relations

$$[\mu_i, \mu_j] = -2\delta_{ij} \qquad [\mu_i, \nu_j] = 0 \qquad [\nu_i, \nu_j] = 2\delta_{ij} \tag{16}$$

where $\delta_{ij} = 1$ if and only if $i = j$ and $\delta_{ij} = 0$ otherwise is the Kronecker delta.

Lemma 2.2.13. *Let $\tilde{X}$ be a $\mathbb{Z}_2$-graded C_U-module which is free and finitely generated over k. Then $X := \tilde{X} \otimes_k k[\underline{x}, \underline{y}]$ coupled with the map*

$$\partial_X = \sum_{i=1}^{n} \mu_i x_i + \sum_{j=1}^{n} \nu_j y_j \tag{17}$$

is a matrix factorisation of $U(\underline{y}) - U(\underline{x}) \in k[\underline{x}, \underline{y}]$.

Proof. See [1][Lemma 5.6.1]. □

Remark 2.2.14. The map (17) is odd because we consider $k[\underline{x}, \underline{y}]$ to admit the $\mathbb{Z}_2$-grading

$$k[\underline{x}, \underline{y}] \oplus 0 \tag{18}$$

That is, $k[\underline{x}, \underline{y}]$ has $k[\underline{x}, \underline{y}]$ entirely in degree 0, and the zero module 0 in degree 1. For example, if $\underline{x}, \underline{y}$ are both singleton sets $\underline{x} = \{x\}, \underline{y} = \{y\}$ then

$$\begin{aligned} \deg(\partial_X(x \otimes p)) &= \deg(\mu x \otimes xp + \nu x \otimes y_j p) \\ &= \deg(\mu x) \ (= \deg(\nu x)) \\ &= \deg(x) + 1 \end{aligned}$$

Example 2.2.15. Let $\underline{x}$ be a set of variables $\{x_1, \ldots, x_n\}$ and $\sigma \in S_n$ a permutation on this set. Let $\tilde{X}$ denote the $\mathbb{Z}_2$-graded k-algebra

$$\tilde{X} := \bigwedge(k\theta_1 \oplus \ldots \oplus k\theta_n) \tag{19}$$

which is a C_U-algebra (Definition 2.2.12) with C_U-action induced by the following

$$\begin{aligned} \mu_i &= \theta_{\sigma^{-1}i} + \theta^*_{\sigma^{-1}i} \\ \nu_i &= \theta_i - \theta^*_i \end{aligned}$$

Thus, by Lemma 2.2.13 we obtain a matrix factorisation $(X = \tilde{X} \otimes k[\underline{x}, \underline{y}], \partial_X)$.

Now say we had another similar matrix factorisation; let $\underline{y} = \{y_1, \ldots, y_m\}$ be another set of variables and let τ be a permutation on this set. Let $\tilde{Y}$ denote the $\mathbb{Z}_2$-graded k-algebra

$$\tilde{Y} := \bigwedge(k\psi_1 \oplus \ldots \oplus k\psi_m) \tag{20}$$

This is a C_U-module with C_U-action induced by the following

$$\overline{\nu_i} = \psi_{\tau^{-1}i} + \psi^*_{\tau^{-1}i}$$
$$\omega_i = \psi_i - \psi^*_i$$

This induces a matrix factorisation $(Y = \tilde{Y} \otimes k[\underline{y}, \underline{z}], \partial_Y)$ of $U(\underline{y}) - U(\underline{z})$ where

$$\partial_Y := \sum_{i=1}^{n} \overline{\nu}_i y_i + \sum_{i=1}^{n} \omega_i z_i \tag{21}$$

and $\underline{z} = \{z_1, \ldots, z_n\}$ is another set of variables.

We will first consider the cut $X|Y$. The sequence of partial derivatives $(\partial_{y_1} U(\underline{y}), \ldots, \partial_{y_n} U(\underline{y})) = (2y_1, \ldots, 2y_n)$ and so

$$J_{U(\underline{y})} = k[\underline{y}]/(y_1, \ldots, y_n) = k \tag{22}$$

as a $k[\underline{y}]$-module with trivial $k[\underline{y}]$-action. We thus have

$$X|Y = X \otimes_{k[\underline{x},\underline{y}]} k \otimes_{k[\underline{y},\underline{z}]} Y, \qquad \partial_{X|Y} = d_X \otimes 1 \otimes 1 + 1 \otimes 1 \otimes d_Y \tag{23}$$

Say R is a commutative ring. We will consider an element $f \in R$ of the following particular form: say we have $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$ such that $f = \sum_{i=1}^{n} a_i b_n$. The guiding example of such an element is when R is a polynomial ring and f is a quadratic form.

Lemma 2.2.16. *Suppose M is a $\mathbb{Z}_2$-graded R-module with odd R-linear maps $\theta_i, \theta_i^* : M \longrightarrow M, i = 1, \ldots, n$ satisfying the canonical anticommutation relations. Then, setting $\delta_+ = \sum_{i=1}^{n} a_i \theta_i$ and $\delta_- = \sum_{i=1}^{n} b_i \theta_i^*$ we have that $(M, \delta_- + \delta_+)$ is a linear factorisation of f.*

Proof. See [1, Lemma 4.2.3]. □

Therefore, $(\bigwedge R^n, \delta_- + \delta_+)$ is a matrix factorisation of f. This is the **Koszul matrix factorisation of** f.

A Quantum error correcting codes examples

Definition A.0.1 (Bit flip correction)**.** Input: a received message $|\psi\rangle$,

1. perform the following projective measurements:

$$\langle\psi| Z_1Z_2 |\psi\rangle \text{ with resulting state } |\psi'\rangle\,, \tag{24}$$

followed by

$$\langle\psi'| Z_2Z_3 |\psi'\rangle \tag{25}$$

let (r_1, r_2) be the pair of results from these measurements.

2. It will be shown that $r_1, r_2 \in \{1, -1\}$, and the resulting state of the second measurement is $|\psi\rangle$.

3. Now retrieve $|\varphi\rangle$ based on the values of r_1, r_2:

   - if $(r_1, r_2) = (1, 1)$, return $|\psi\rangle$,
   - if $(r_1, r_2) = (-1, 1)$, return $X_1 |\psi\rangle$,
   - if $(r_1, r_2) = (1, -1)$, return $X_3 |\psi\rangle$,
   - if $(r_1, r_2) = (-1, -1)$, return $X_2 |\psi\rangle$

We now prove correctness of Algorithm A.0.1:

Proof. It will be helpful to first notice:

$$\begin{aligned}
Z_1Z_2 |000\rangle &= |000\rangle & Z_1Z_2 |001\rangle &= |001\rangle \\
Z_1Z_2 |010\rangle &= -|010\rangle & Z_1Z_2 |011\rangle &= -|011\rangle \\
Z_1Z_2 |100\rangle &= -|100\rangle & Z_1Z_2 |101\rangle &= -|101\rangle \\
Z_1Z_2 |110\rangle &= |110\rangle & Z_1Z_2 |111\rangle &= |111\rangle
\end{aligned}$$

Let $|\psi\rangle := a|010\rangle + b|101\rangle$ be a state, ie, an element of $\mathbb{H}^{\otimes 3}$. We perform the measurement Z_1Z_2 followed by Z_2Z_3:

$$\begin{aligned}
\langle\psi| Z_1Z_2 |\psi\rangle &= (a\langle 010| + b\langle 101|)Z_1Z_2(a|010\rangle + b|101\rangle) \\
&= (a\langle 010| + b\langle 101|)(-a|010\rangle - b|101\rangle) \\
&= -a^2 - b^2 = -1
\end{aligned}$$

and

$$\begin{aligned}
\langle\psi| Z_2Z_3 |\psi\rangle &= (a\langle 010| + b\langle 101|)Z_1Z_2(a|010\rangle + b|101\rangle) \\
&= (a\langle 010| + b\langle 101|)(-a|010\rangle - b|101\rangle) \\
&= -a^2 - b^2 = -1
\end{aligned}$$

We can infer from the fact that both of these came out as -1 that it was the second bit which was flipped, and so we can correct this. However, what is the impact of this measurement on the state? Again we calculate:

$$\begin{aligned} Z_1 Z_2(a\,|010\rangle + b\,|101\rangle) &= Z_1(-a\,|010\rangle + b\,|101\rangle) \\ &= -a\,|010\rangle - b\,|101\rangle \end{aligned}$$

and

$$\begin{aligned} Z_2 Z_3(-a\,|010\rangle - b\,|101\rangle) &= Z_2(-a\,|010\rangle + b\,|101\rangle) \\ &= a\,|010\rangle + b\,|101\rangle \end{aligned}$$

and so the measurements (in the end) did not impact our state. □

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