1 Bicategory Theory

2 Bicategory Theory

Often one motivates the notion of a *bicategory* by observing the definition of a category and then "suping up" the hom-sets. From this angle though there are multiple reasonable generalisations, ought associativity hold strictly or up to natural isomorphism?

So instead we seek our motivation from a different source, recall:

Definition 2.0.1. A **monoidal category** consists of

- a category $\mathscr{C}$,
- a special object $\mathbb{1} \in \mathscr{C}$,
- a functor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$

along with three natural isomorphisms $\lambda : \mathbb{1} \times \mathrm{id}_{\mathscr{C}} \longrightarrow \mathrm{id}_{\mathscr{C}}, \rho : \mathrm{id}_{\mathscr{C}} \times \mathbb{1} \longrightarrow \mathscr{C}, \alpha : (\mathrm{id}_{\mathscr{C}} \times \mathrm{id}_{\mathscr{C}}) \times \mathrm{id}_{\mathscr{C}} \longrightarrow \mathrm{id}_{\mathscr{C}} \times (\mathrm{id}_{\mathscr{C}} \times \mathrm{id}_{\mathscr{C}})$, with all of this data satisfying:

- the **pentagon diagram**, ie, commutativity for all $A, B, C, D \in \mathscr{C}$ of the following:

$$\begin{tikzcd} & A \otimes (B \otimes (C \otimes D)) \arrow[r] \arrow[dl] & A \otimes ((B \otimes C) \otimes D) \arrow[dr] & \\ (A \otimes B) \otimes (C \otimes D) \arrow[drr] & & & (A \otimes (B \otimes C)) \otimes D \arrow[dl] \\ & & ((A \otimes B) \otimes C) \otimes D & \end{tikzcd} \tag{1}$$

- the **identity diagrams**, ie, commutativity for all $A, B, C \in \mathscr{C}$ of the following:

$$\begin{tikzcd} (A \otimes \mathbb{1}) \otimes B \arrow[r] \arrow[dr] & A \otimes (\mathbb{1} \otimes B) \arrow[d] \\ & A \otimes B \end{tikzcd} \tag{2}$$

We wish to define a *bicategory* so that a monoidal category is a bicategory with a single object. Thus, we see that we take associativity and identity up to natural isomorphism, hence:

Definition 2.0.2. A **bicategory** $\mathscr{C}$ consists of

- a collection $\mathrm{Obj}\,\mathscr{C}$ of **objects**,
- for every pair X, Y of objects, a category

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \tag{3}$$

  whose objects are 1**-morphisms** with **domain** X and **codomain** Y, and whose morphisms are 2**-morphisms**,
- for every triple X, Y, Z of objects, a functor called **horizontal composition**

$$\begin{aligned} \circ_{X,Y,Z} : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Z) \\ (f, g) &\longmapsto f \circ g \\ (\alpha : f \Rightarrow g, \beta : h \Rightarrow j) &\longmapsto \beta * \alpha \end{aligned}$$

- for each tuple of objects X, Y, Z, W a natural isomorphism $\alpha_{X,Y,Z,W}$ from the functor defined by the composite:

$$\mathrm{Hom}_{\mathscr{C}}(Z,W) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \times \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Z,W) \times \mathrm{Hom}_{\mathscr{C}}(X,Z) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,W) \tag{4}$$

  and the functor defined by the composite

$$\mathrm{Hom}_{\mathscr{C}}(Z,W) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \times \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y,W) \times \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,W) \tag{5}$$

- for every object $X \in \mathscr{C}$ a functor $\mathbb{1}_X : \mathbf{1} \longrightarrow \mathrm{Hom}(X,X)$, where $\mathbf{1}$ is the category with a single object and single morphism,

- for every pair of objects X, Y a pair of natural isomorphisms, ρ which maps from the functor defined by the composite:

$$\mathbf{1} \times \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(Y,Y) \times \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X,Y) \tag{6}$$

  to the functor defined by the composite

$$\mathbf{1} \times \mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(X,Y) \tag{7}$$

  and λ, which is defined similarly,

satisfying:

- for all objects X, Y, Z, W the following diagram commutes

$$\begin{tikzcd}
& X \circ (Y \circ (Z \circ W)) \arrow[r] \arrow[dl] & X \circ ((Y \circ Z) \circ W) \arrow[dr] & \\
(X \circ Y) \circ (Z \circ W) \arrow[drr] & & & (X \circ (Y \circ Z)) \circ W \arrow[dl] \\
& & ((X \circ Y) \circ Z) \circ W &
\end{tikzcd} \tag{8}$$

- for all objects X, Y the following diagram commutes:

$$\begin{tikzcd}
(X \circ \mathbb{1}) \circ Y \arrow[r] \arrow[dr] & X \circ (\mathbb{1} \circ Y) \arrow[d] \\
& X \circ Y
\end{tikzcd} \tag{9}$$

Remark 2.0.3. The point is the following:

1. hom sets have become categories,
2. composition has become a functor
3. identity morphisms are not an element of a set now, they are an object of a category,
4. composition now has the freedom to hold only up to isomorphism,
5. associativity only holds up to isomorphism,
6. that units act like units only holds up to isomorphism,
7. these are 3-ary and 2-ary isomorphisms so we need compatibility diagrams.

The example we are chiefly concerned with is the *bicategory of Landau-Ginzburg models*, we now head towards this definition.

Definition 2.0.4. A **potential** is

3 The semantics

4 Relevant calculations

4.1 Calculating $\bigcap_{i=1}^{m} \mathrm{Ker}(\bar{\nu}_i + \nu_i)$

"A typical element of $\wedge k\underline{\psi} \otimes \wedge k\underline{\theta}$ annihilated by $\bar{\nu}_i + \nu_i$ is an *entangled bit*":

$$\begin{aligned}
(\bar{\nu}_i + \nu_i)(1 + \psi_i\theta_{\sigma^{-1}i}) &= \bar{\nu}_i + \nu_i + \bar{\nu}_i\psi\theta_{\sigma^{-1}i} + \nu_i\psi_i\theta_{\sigma^{-1}i} \\
&= \theta^*_{\sigma^{-1}i} + \theta_{\sigma^{-1}i} + \psi_i - \psi_i^* + \theta^*_{\sigma^{-1}i}\psi\theta_{\sigma^{-1}i} + \theta_{\sigma^{-1}i}\psi_i\theta_{\sigma^{-1}i} + \psi_i\psi_i\theta_{\sigma^{-1}i} - \psi_i^*\psi_i\theta_{\sigma^{-1}i} \\
&= \theta^*_{\sigma^{-1}i} + \theta_{\sigma^{-1}i} + \psi_i - \psi_i^* + \psi_i + 0 - \theta_{\sigma^{-1}i} - 0 \\
&= \theta^*_{\sigma^{-1}i} - \psi_i^*
\end{aligned}$$