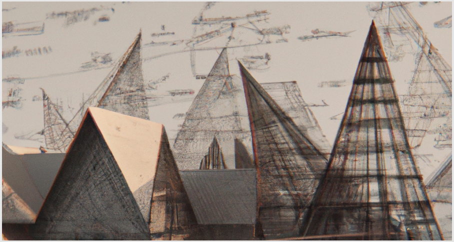

Computation in logic as the splitting of idempotents in algebraic geometry; two models of multiplicative linear logic.

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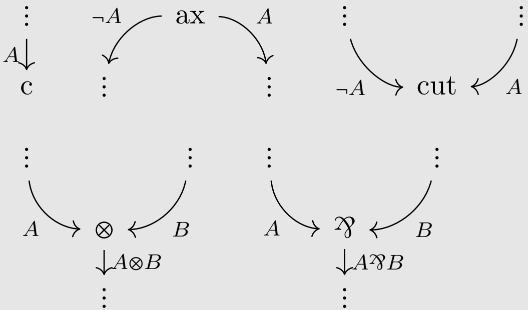
University of Melbourne, University of Sorbonne Paris Nord

2023

Logicomp

What is a logical system? What is a system of computation? Multiplicative Linear logic proof nets is a graphical system of logic, with vertices labelled by deduction rules and edges labelled by formulas.



Example of a proof

Proofs are static objects. As mathematical objects they are interesting because they encode patterns of equality.

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\mathrm{c} & & \otimes \arrow[dr, bend right, "A \otimes \neg A"'] & \mathrm{c} & \parr \arrow[dl, bend left, "A \parr \neg A"] \\
 & & & \mathrm{cut} &
\end{tikzcd}
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In logic, we study correctness or proofs, in various different systems.

Cut-elimination

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\begin{tikzcd}
\vdots \arrow[dr, bend right, "A"'] & & \mathrm{ax} \arrow[dl, "\neg A"] \arrow[dr, bend left, "A"] & & \vdots \arrow[d, "A"] \\
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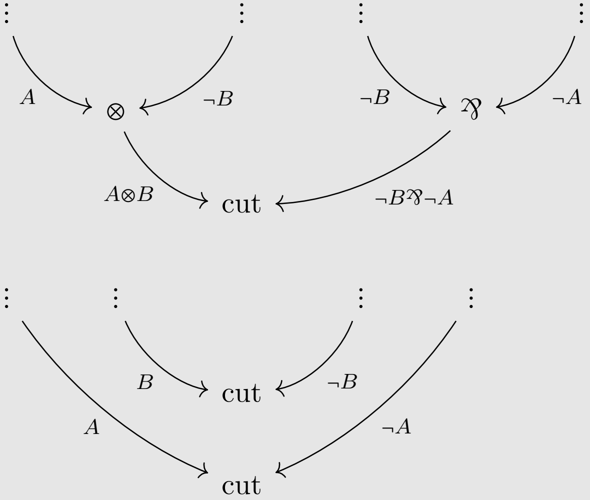
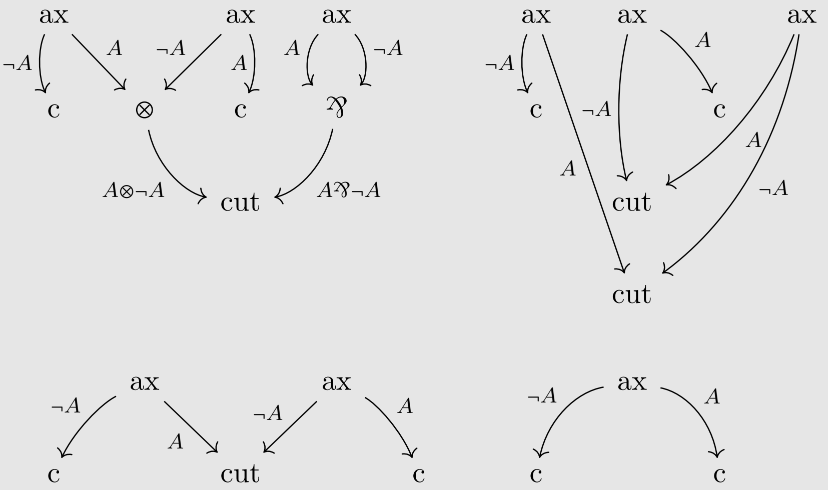
Cut-elimination



The dynamics of proofs



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& ? \arrow[dr] & & ? \arrow[dl, "?(X\otimes\neg X)"'] & \\
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& \parr \arrow[dr] & & & \\
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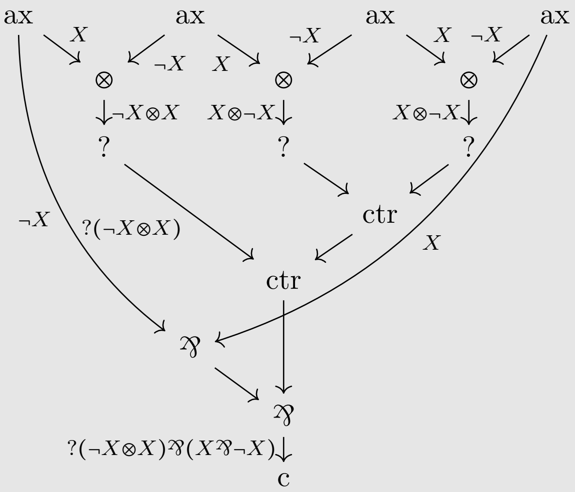
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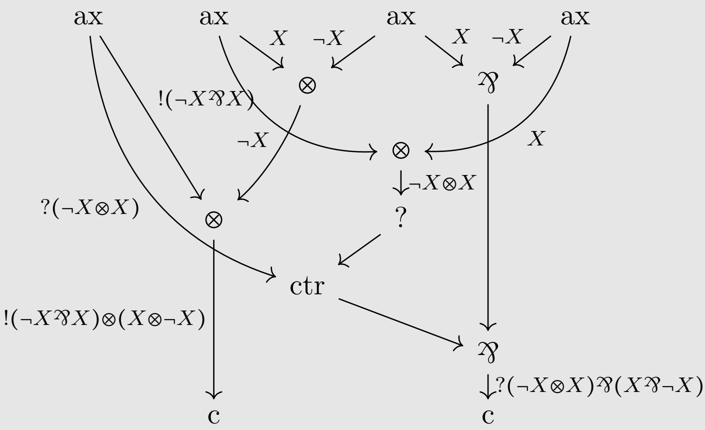

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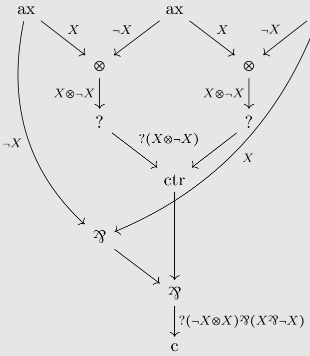
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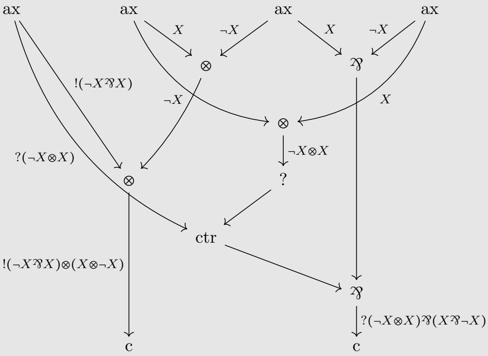
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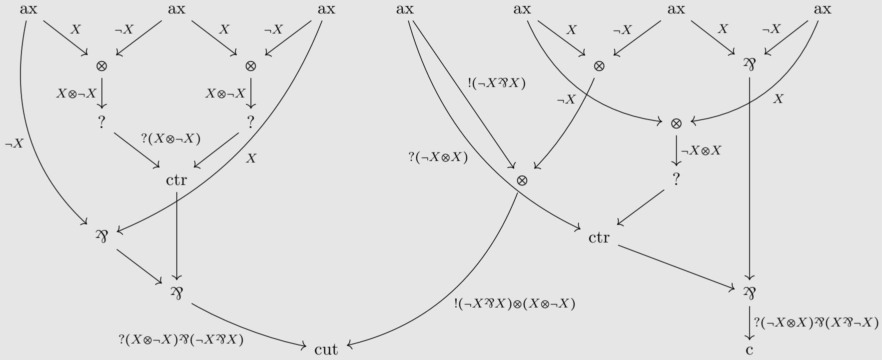

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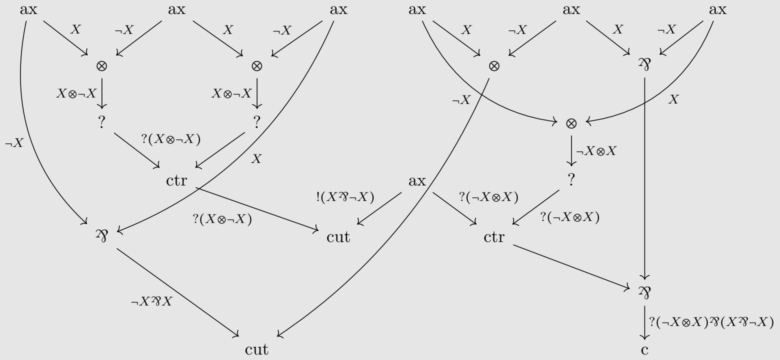

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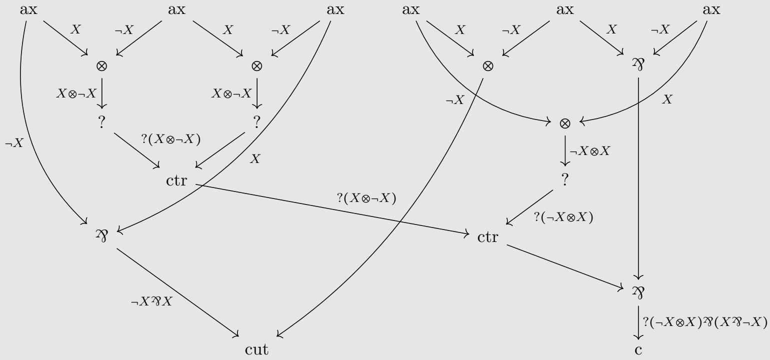
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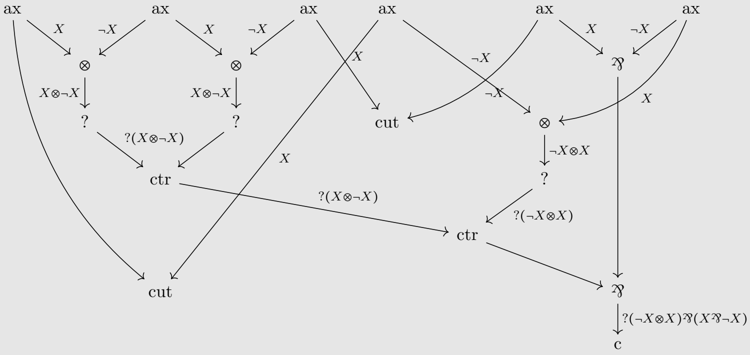
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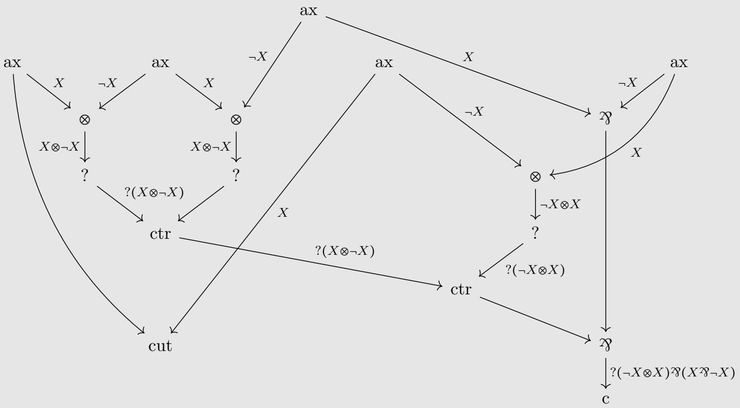

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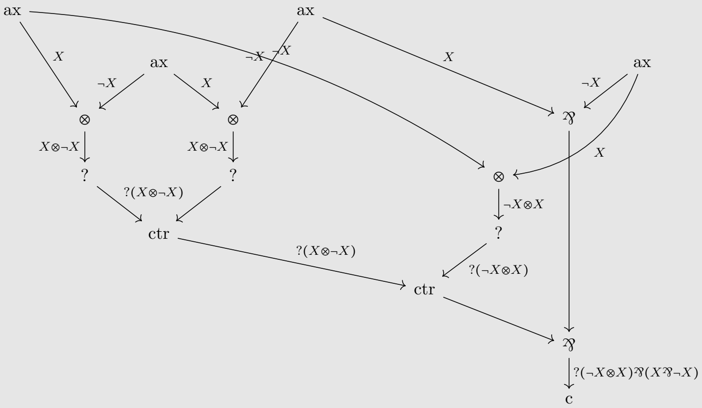
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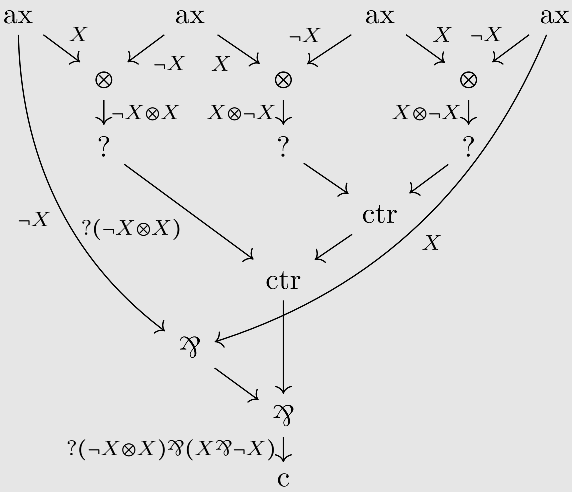
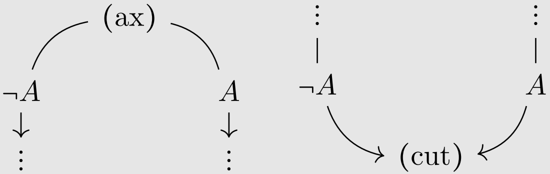
$\mathtt{Succ}(2) \longrightarrow_{\mathrm{cut}} 3$



Geometry of Interaction, patterns of equality

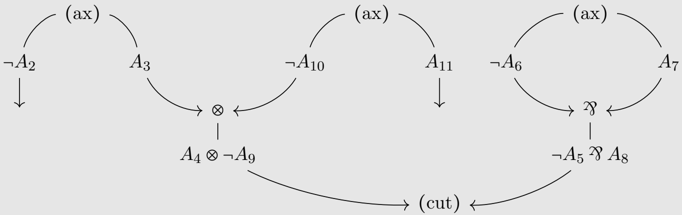
Proof nets.

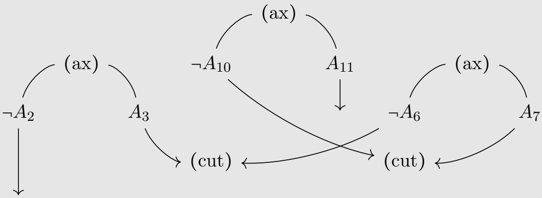


Say A has unoriented atoms $X_1, \ldots, X_n$, then an axiom links identifies each copy of X_i in A with the corresponding copy in $\neg A$.

Dynamics

We understand proofs as static objects quite well, but what about as *dynamic* objects?





Matrix factorisations

For a polynomial $U(\underline{x}) \in \mathbb{C}[\underline{x}]$ the equation

$$V(\underline{x})^2 = U(\underline{x})$$

may have no solution in polynomials, but it may acquire solutions when we enlarge our sphere of consideration to include *matrices*.

Example

The polynomial $U(x_1, x_2) = x_1^2 + x_2^2 \in \mathbb{C}[x_1, x_2]$ has no square root, but nonetheless

$$\begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix}^2 = (x_1^2 + x_2^2) \cdot I = U(x_1, x_2) \cdot I$$

where I is the 2×2 identity matrix.

Matrix factorisations, formally defined

Definition

A **matrix factorisation** of a polynomial $U(\underline{x}) \in \mathbb{C}[\underline{x}]$ is a pair (X, d_X) consisting of a $\mathbb{Z}_2$-graded, free, finitely generated k-module X and a **differential** d_X which is an odd linear transformation satisfying

$$d_X^2 = U(\underline{x}) \cdot I$$

Since X is $\mathbb{Z}_2$-graded we can write $X = X_0 \oplus X_1$. Since d_X is odd we have an object resembling a chain complex.

$$\ldots \xrightarrow{p_X} X_1 \xrightarrow{q_X} X_0 \xrightarrow{p_X} X_1 \xrightarrow{q_X} \ldots$$

Theorem

The category $\mathrm{hmf}(\underline{x}, U(\underline{x}))$ *is the zero category if and only if* $U(\underline{x})$ *has no singularities.*

A taste of the bicategory of Landau-Ginzburg models (over $\mathbb{C}$)

The objects are polynomials with isolated critical points.

$$(\underline{x}, U(\underline{x})) \qquad (\underline{y}, V(\underline{y})), \qquad (\underline{z}, W(\underline{z}))$$

The category of morphisms $(\underline{x}, U(\underline{x})) \longrightarrow (\underline{y}, V(\underline{y}))$ is

$$\mathrm{hmf}((\underline{x}, \underline{y}), U(\underline{x}) - V(\underline{y}))^{\omega}$$

$$\begin{tikzcd}
\dots \arrow[r, "q_X"] & X_0 \arrow[r, "p_X"] \arrow[dl, "h_0"'] \arrow[d, shift right, "\alpha_0"'] \arrow[d, shift left, "\beta_0"] & X_1 \arrow[r, "q_X"] \arrow[dl, "h_1"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & X_0 \arrow[r, "p_X"] \arrow[dl, "h_0"'] \arrow[d, shift right, "\alpha_0"'] \arrow[d, shift left, "\beta_0"] & X_1 \arrow[r, "q_X"] \arrow[dl, "h_1"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & \dots \arrow[dl, "h_0"'] \\
\dots \arrow[r, "q_Y"'] & Y_0 \arrow[r, "p_Y"'] & Y_1 \arrow[r, "q_Y"'] & Y_0 \arrow[r, "p_Y"'] & Y_1 \arrow[r, "q_Y"'] & \dots
\end{tikzcd}$$

Idempotent completion

Lemma

If $\mathcal{C}$ is a preadditive category then the following are equivalent

- *$\mathcal{C}$ is idempotent complete,*
- *all idempotents have a kernel,*
- *all idempotents have a cokernel*

Definition

The **idempotent completion** of $\mathcal{C}$ is an idempotent complete category $\mathcal{C}^\omega$ with a full and faithful functor $\mathcal{C} \longrightarrow \mathcal{C}^\omega$ which is universal amongst functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ where $\mathcal{D}$ is idempotent complete

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, "F"'] & \mathcal{C}^\omega \arrow[d, dashed, "F^\omega"] \\ & \mathcal{D} \end{tikzcd}$$

Split idempotent

Say $e : C \longrightarrow C$ is an idempotent. If $\mathcal{C}$ is idempotent complete, then e admits a kernel, meaning

$$\begin{tikzcd} & C \arrow[dl] \\ \ker e \arrow[r, tail] & C \arrow[u, "\mathrm{id} - e"'] \end{tikzcd}$$

So $\mathcal{C}$ being idempotent complete means all idempotents **split**, that is, there exists morphism $l : R \longrightarrow C, r : C \longrightarrow R$ such that $rl = \mathrm{id}_R$ and $lr = e$.
For today, *this* is the property we want all idempotents to have.

"Infinitary" compositions

Matrix factorisations can be composed using the tensor product *but only up to homotopy*:

$$(\mathbb{C}[\underline{x}], U(\underline{x})) \xrightarrow{(X, d_X)} (\mathbb{C}[\underline{y}], V(\underline{y})) \xrightarrow{(Y, d_Y)} (\mathbb{C}[\underline{z}], W(\underline{z}))$$

Define

$$Y \circ X = (Y \otimes_{\mathbb{C}[\underline{y}]} X, d_Y \otimes 1 + 1 \otimes d_X)$$

The resulting matrix factorisation $Y \circ X$ (of $W(\underline{z}) - U(\underline{x})$) is a free module of *possibly infinite rank* over $\mathbb{C}[\underline{x}, \underline{z}]$.

Example

Take $X = \mathbb{C}[\underline{x}, \underline{y}]^m, Y = \mathbb{C}[\underline{y}, \underline{z}]^{m'}$. Then

$$X \otimes_{\mathbb{C}[\underline{y}]} Y \cong \mathbb{C}[\underline{x}, \underline{y}, \underline{z}]^{mm'}$$

which is free, but not finitely generated over $\mathbb{C}[\underline{x}, \underline{z}]$.

Semantics of composition

A *computable* process for recovering a matrix factorisation proper which is homotopy equivalent to the composite is the contents of Murfet's paper [13].

Definition

Let Y be a matrix factorisation of the difference of two polynomials $U(\underline{x}) - V(\underline{y})$ and X of $V(\underline{y}) - W(\underline{z})$. Define

$$J_{V(\underline{y})} = \mathbb{C}[y_1, \ldots, y_n]/(\partial_{y_1} V(\underline{y}), \ldots, \partial_{y_n} V(\underline{y}))$$

The **cut** of X, Y is

$$Y \mid X = Y \otimes_{\mathbb{C}[\underline{y}]} J_{V(\underline{y})} \otimes_{\mathbb{C}[\underline{y}]} X$$

The cut is always finite rank, this is because in general, $J_{V(\underline{y})}$ is a finitely generated $\mathbb{C}$-module.

Extracting the composite from the cut

Let S_n denote $\bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n)$ where $\theta_1, \ldots, \theta_n$ are formal variables, and n is the length of the sequence $\underline{y}$.

Lemma

There exists a homotopy equivalence of matrix factorisations over $\mathbb{C}[\underline{x}, \underline{z}]$

$$Y \otimes_{\mathbb{C}[\underline{y}]} J_{V(\underline{y})} \otimes_{\mathbb{C}[\underline{y}]} X = Y \mid X \rightleftarrows S_n \otimes_{\mathbb{C}} (Y \otimes X)$$

For today, we will black box the definition of this homotopy, however it is worth noting that *explicit equations* exist which define it. See [13].

Clifford Algebras

Recall that the Clifford Algebra C_n is generated by elements $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n$ subject to:

$$[\mu_i, \mu_j] = -2\delta_{ij} \quad [\nu_i, \nu_j] = 2\delta_{ij} \quad [\mu_i, \nu_j] = 0$$

where $[\xi, \zeta] = \xi\zeta + \zeta\xi$ for $\xi, \zeta \in \{\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n\}$.
There is a C_n action on S_n and hence on $S_n \otimes_{\mathbb{C}} (Y \otimes X)$.

Clifford action

This is induced by two canonical endomorphisms which exist on $S_n = \wedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n)$. The **wedge** and **contraction** maps.

$$\begin{aligned}\theta_i : \bigwedge^{d-1}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) &\longrightarrow \bigwedge^{d}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n)\\ \theta_{j_1} \wedge \ldots \wedge \theta_{j_{d-1}} &\longmapsto \theta_i \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_{d-1}}\end{aligned}$$

and

$$\begin{aligned}\theta_i^* : \bigwedge^{d}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) &\longrightarrow \bigwedge^{d-1}(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n)\\ \theta_{j_1} \wedge \ldots \wedge \theta_{j_d} &\longmapsto \sum_{k=1}^{d}(-1)^{k+1}\delta_{j_k=i}\theta_{j_1} \wedge \ldots \wedge \hat{\theta}_{j_k} \wedge \ldots \wedge \theta_{j_d}\end{aligned}$$

Set $\mu_i = \theta_i - \theta_i^*, \nu_i = \theta_i + \theta_i^*$. Passing this through the homotopy of the previous slide, we obtain a Clifford algebra representation (up to homotopy) on the cut $Y \mid X$.

Recovering the composite

Consider the idempotent $e = \theta_1^* \dots \theta_n^* \theta_n \dots \theta_1 : S_n \longrightarrow S_n$. This is the projection onto k sitting inside S_n. Thus we obtain a pair of morphisms

$$S_n \otimes_{\mathbb{C}} (Y \otimes X) \underset{\iota}{\overset{e}{\rightleftarrows}} Y \otimes X$$

satisfying the properties that $e\iota = \mathrm{id}_{Y \otimes X}$ and $\iota e = e$. Carrying this through the homotopy

$$Y \mid X \simeq S_n \otimes_{\mathbb{C}} (Y \otimes X) \tag{1}$$

we see that extracting $Y \otimes X$ from $Y \mid X$ amounts to computing the image of the endomorphism corresponding to e.
Since (1) is an isomorphism of C_n-representations, it follows that e acting on the left splits to the same thing as e acting on the right.

Summary of the process

Finding a pair of maps ι and the space $Y \otimes X$ is a process referred to as *splitting the idempotent* e. Since $Y \mid X$ is a genuine matrix factorisation (that is, it is finitely generated), and since the splitting of e can be performed as a step-by-step process using explicit maps, we take $Y \circ X$ in the following diagram to be the finite model of $Y \otimes X$.

$$\begin{tikzcd}
Y \mid X \arrow[r, shift right] \arrow[d, shift left] & S_n \otimes_{\mathbb{C}} (Y \otimes X) \arrow[l, shift right] \arrow[d, shift left, "e"] \\
Y \circ X \arrow[u, shift left] & Y \otimes X \arrow[u, shift left, "\iota"]
\end{tikzcd}$$

Is this cut-elimination? This is how we motivate the search for a model of multiplicative linear logic in the setting of matrix factorisations.

Formulas

Definition (Formulas)

- *Unoriented atoms* $X, Y, Z, \ldots$
- *An oriented atom* (or *atomic proposition*) is a pair $(X, +)$ or $(X, -)$ where X is an unoriented atom.

Pre-formulas:

- Any atomic proposition is a preformula.
- If A, B are pre-formulas then so are $A \otimes B$, $A \parr B$.
- If A is a pre-formula then so is $\neg A$.

Formulas: quotient of pre-formulas:

$$\neg(A \otimes B) \sim \neg B \parr \neg A \qquad \neg(A \parr B) \sim \neg B \otimes \neg A$$

$$\neg(X, +) \sim (X, -) \qquad \neg(X, -) \sim (X, +)$$

The model, formulas

If $(\underline{x}, U(\underline{x})), (\underline{y}, V(\underline{y}))$ are pairs consisting of a sequence of variables and a polynomial over those variables (with base ring $\mathbb{C}$) then define

$$(\underline{x}, U(\underline{x})) \square (\underline{y}, V(\underline{y})) := ((\underline{x}, \underline{y}), U(\underline{x}) + V(\underline{y}))$$

Definition

Say A has oriented atoms $(X_1, x_1), \dots, (X_n, x_n)$. Then

$$\begin{aligned} [\![A]\!] &:= \Big((X_1, \dots, X_n), \sum_{i=1}^{n} x_i X_i^2\Big) \\ [\![\neg A]\!] &:= \Big((X_n, \dots, X_1), -\sum_{i=1}^{n} x_i X_i^2\Big) \\ [\![A \otimes B]\!] &:= [\![A]\!] \square [\![B]\!] \\ [\![A \parr B]\!] &:= [\![A]\!] \square [\![B]\!] \end{aligned}$$

Inducing matrix factorisations from sequences

Consider polynomials $\sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2, \in \mathbb{C}[x_1, \ldots, x_n, y_1 \ldots, y_n]$.

Lemma

As operators on $\bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) \otimes_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ *we have the following equality:*

$$\Big(\sum_{i=1}^n (x_i + y_i)\theta_i + \sum_{i=1}^n (x_i - y_i)\theta_i^*\Big)^2 = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2$$

We call this the **Koszul matrix factorisation** corresponding to the sequence

$$(x_1 - y_1, \ldots, x_n - y_n)$$

This sequence in turn should be thought of as a choice of *pairing* of the variables $x_1, \ldots, x_n$ with the variables $y_1, \ldots, y_n$.

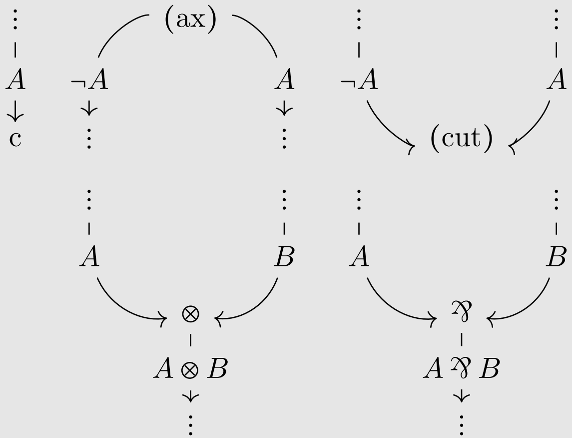
A fork in the road

There are now at least three different approaches we can take:

- Focus on the sequences which give rise to the matrix factorisations (this is done in "Elimination and cut-elimination in multiplicative linear logic", [14]).
- Focus on the Koszul complexes and use the fact that we have chosen specific polynomials (recall that the polynomial associated to each formula is the sum of squares of its unoriented atoms). This lead to "proofs as Quantum Error Correction Codes", to appear.
- Focus on the matrix factorisations themselves. Still a work in progress.

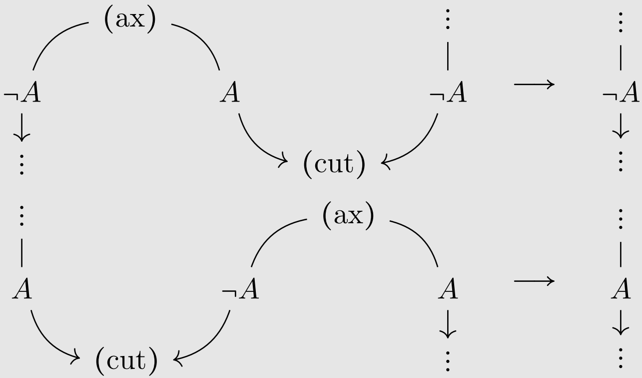
Proof nets

For our models we used proof nets.

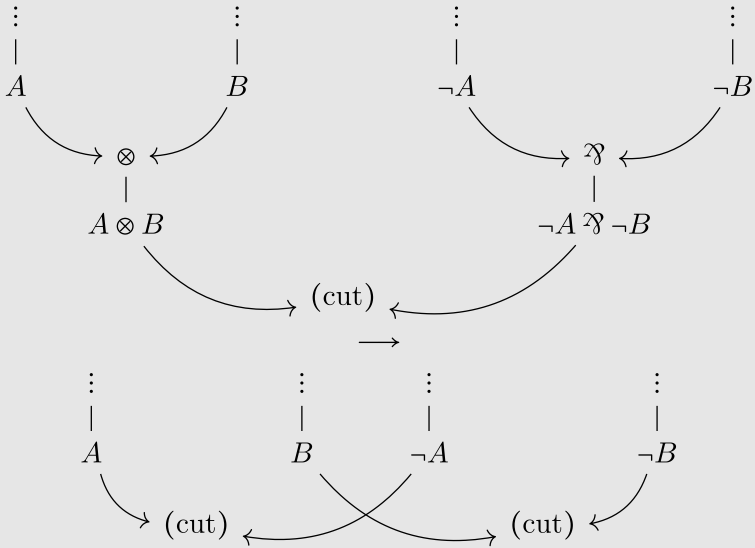


The dynamics

a-redexes:



m-redex:



Elimination, cut-elimination, and falling roofs

Definition (Polynomial ring P_A of a formula A)

P_A is the free commutative $\mathbb{C}$-algebra on the set of unoriented atoms of A:

$$P_A = \mathbb{C}[X_1, ..., X_n]$$

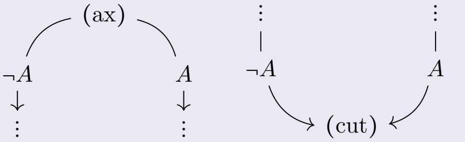
Let π be a proof structure with edge set E and denote by A_e the formula labelling edge $e \in E$. The *polynomial ring* of π, denoted P_π is the following, where U_e is the set of unoriented atoms of A_e.

$$P_\pi := \bigotimes_{e \in E} P_{A_e} \cong \mathbb{C}[\coprod_{e \in E} U_e]$$

Links

Definition (Link ideal I_l, link coordinate ring R_l)

Axiom/Cut link l:



$\big((X_1, x_1), ..., (X_n, x_n)\big)$ is the sequence of oriented atoms of A.

$$I_l \subseteq P_A \otimes P_{\neg A}$$
$$I_l = (X_i - X_i')_{i=1}^n = (X_i \otimes 1 - 1 \otimes X_i)_{i=1}^n$$

$$R_l := P_A \otimes P_{\neg A} / I_l$$

Tensor/Par links

Tensor/Par link l:

$$\begin{tikzcd}[column sep=small]
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
A \arrow[dr, bend right] & & B \arrow[dl, bend left] \\
& \otimes \arrow[d, dash] & \\
& A \otimes B \arrow[d] & \\
& \vdots &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
A \arrow[dr, bend right] & & B \arrow[dl, bend left] \\
& \parr \arrow[d, dash] & \\
& A \parr B \arrow[d] & \\
& \vdots &
\end{tikzcd}$$

Let $\boxtimes = \otimes$ if l is a tensor link, and $\boxtimes = \parr$ if l is a par link.

$$I_l \subseteq P_A \otimes P_B \otimes P_{A \boxtimes B}$$
$$I_l = (\{X_i - X_i'\}_{i=1}^n \cup \{Y_j - Y_j'\}_{j=1}^m)$$
$$= (\{X_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes X_i\}_{i=1}^n \cup \{1 \otimes Y_j \otimes 1 - 1 \otimes 1 \otimes Y_j\}_{j=1}^m)$$

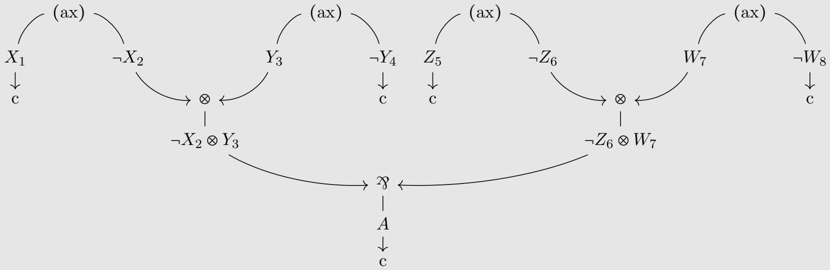
$$R_l = P_A \otimes P_B \otimes P_{A \boxtimes B} / I_l$$

Definition (Defining ideal I_π, coordinate ring R_π)

$I_\pi := \sum_l I_l \subseteq P_\pi$ where l ranges over all links of π. $R_\pi := P_\pi / I_\pi$.

Example of coordinate ring of a proof structure

$$A := (\neg X_2 \otimes Y_3) \parr (\neg Z_6 \otimes W_7)$$



$$\begin{aligned}
P_\pi &= \mathbb{C}[X_1, X_2, X_2', X_2'', Y_3, Y_3', Y_3'', Y_4, Z_5, Z_6, \\
&\qquad Z_6', Z_6'', W_7, W_7', W_7'', W_8] \\
I_\pi &= (X_1 - X_2) + (Y_3 - Y_4) + (Z_5 - Z_6) + (W_7 - W_8) \\
&\quad + (X_2 - X_2', Y_3 - Y_3') + (Z_6 - Z_6', W_7 - W_7') \\
&\quad + (X_2' - X_2'', Y_3' - Y_3'', Z_6' - Z_6'', W_7' - W_7'') \\
R_\pi &= P_\pi / I_\pi \cong \mathbb{C}[X, Y, Z, W]
\end{aligned}$$

Results

Definition

Given a sequence $F = (f_1, \ldots, f_s)$ of polynomials and a monomial order $<$ on $\mathbb{C}[X_1, \ldots, X_n]$ we denote by $\mathbb{B}_{es}(F, <)$ the output of the Buchberger Algorithm with early stopping.

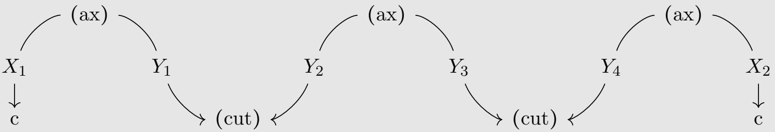
Theorem

There is an equality of sets

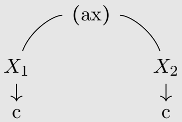
$$G_{\pi'}^{(0)} = \mathbb{B}_{es}(G_{\pi}^{(\Gamma)}, <_{\Gamma}) \cap P_{\pi'} \, .$$

Example

Let π denote the following proof net.



π reduces to π':

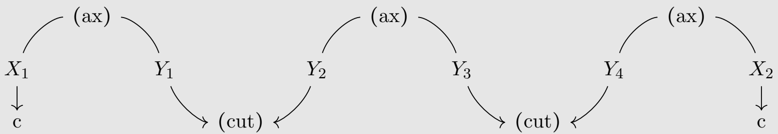


We now consider the sets of generators of the defining ideals of π and π'.

$$G_\pi := \{X_1 - Y_1, Y_1 - Y_2, Y_2 - Y_3, Y_3 - Y_4, Y_4 - X_2\}, \quad G_{\pi'} := \{X_1 - X_2\}$$

$$Y_1 > Y_2 > Y_3 > Y_4 > X_1 > X_2$$

Division



$$G_\pi = \{f_1 = X_1 - Y_1,\ f_2 = Y_1 - Y_2,\ f_3 = Y_2 - Y_3,\ f_4 = Y_3 - Y_4,\ f_5 = Y_4 - X_2\}$$

$$\begin{array}{rl} & \quad (0,0,1,1,1) \\ G_\pi & \overline{\big) Y_2 - X_1} \\ & \quad\ Y_2 - Y_3 \\ & \quad \overline{\qquad\qquad\qquad} \\ & \qquad\quad Y_3 - Y_4 \\ & \qquad\quad Y_3 - X_1 \\ & \qquad\quad \overline{\qquad\qquad\qquad} \\ & \qquad\qquad\quad Y_4 - X_1 \\ & \qquad\qquad\quad Y_4 - X_2 \\ & \qquad\qquad\quad \overline{\qquad\qquad\qquad} \\ & \qquad\qquad\qquad\quad X_2 - X_1 \end{array}$$

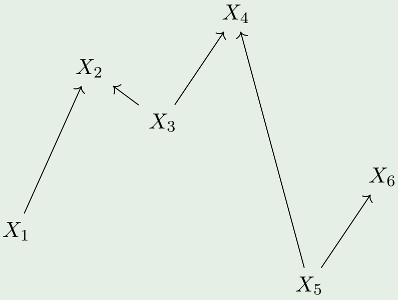
$$\big(G_\pi \cup \{X_2 - X_1\}\big) \cap k[X_1, X_2] = G_{\pi'}$$

Graphical presentation

Vertical axis (higher = greater): order <, horizontal axis: enumeration of variables (the suggested order here is meaningless).

Example

Let $X_1, \ldots, X_6$ be ordered by $X_5 < X_1 < X_6 < X_3 < X_2 < X_4$. Then $\mathscr{R}_<$ is



Falling roofs

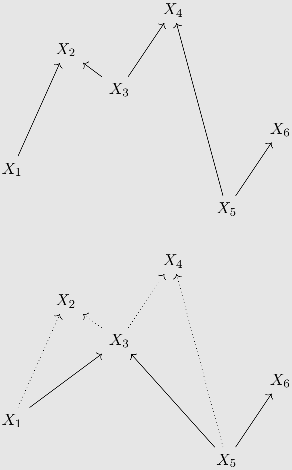


Figure: The falling roofs algorithm applied to the graph of Example 6, reading from left to right and top to bottom.

Example

As a simple example, consider $\mathbb{C}[Y > Z > X]$ with associated sequence $(Y - X, Y - Z)$.

$$\begin{tikzcd}
 & Y & \\
 & & Z \arrow[ul] \\
X \arrow[uur] & &
\end{tikzcd}$$

Falling Roofs terminates at the following

$$\begin{tikzcd}
 & Y & \\
 & & Z \arrow[ul] \\
X \arrow[uur, dashed] \arrow[urr] & &
\end{tikzcd}$$

from which we can extract the sequence $(Z - X, Y - Z)$.

Associated sequences

To the original sequence $(Y - X, Y - Z)$ there is an associated sequence $(Y + X, -Y - Z)$ so that

$$
\begin{aligned}
&(Y - X)(Y + X) + (Y - Z)(-Y - Z) \\
&= Z^2 - X^2 \\
&= (Z^2 - Y^2) + (Y^2 - X^2)
\end{aligned}
$$

From this pair of sequences and the sequence $(Z - X, Y - Z)$ obtained from falling roofs we can construct a fourth sequence $(Y + X, X - Z)$ which has the property

$$
\begin{aligned}
&(Z - X)(Y + X) + (Y - Z)(X - Z) \\
&= ZY + ZX - XY - X^2 + YX - ZY - ZX + Z^2 \\
&= Z^2 - X^2
\end{aligned}
$$

Isomorphisms of matrix factorisations

So Falling Roofs calculates a sequence of pairs of sequences

$$\big((\underline{f_1}, \underline{g_1}), (\underline{f_2}, \underline{g_2})\big)$$

with the property $\underline{f_1} \cdot \underline{g_1} = \underline{f_2} \cdot \underline{g_2} = Z^2 - X^2$.
We read each of these pairs of sequences $(\underline{f_i}, \underline{g_i})$ as the composition of two matrix factorisations:

$$\begin{aligned}\{g_i, f_i\} := \big(&(\textstyle\bigwedge(\mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2) \otimes_{\mathbb{C}} \mathbb{C}[X, Y, Z], \\ &\underline{g_i}^1\theta_1^* + \underline{g_i}^2\theta_2^* + \underline{f_i}^1\theta_1 + \underline{f_i}^2\theta_2)\big)\end{aligned}$$

The calculations above (which is the work of Falling Roofs) induces an isomorphism of matrix factorisations

$$\{\underline{g_1}, \underline{f_1}\} \cong \{\underline{g_2}, \underline{f_2}\}$$

Passing to the cut...

If we look at the cut rather than the composition, something interesting happens...

$$\overline{\{(\underline{g_i}, \underline{f_i})\}} := \{(\underline{g_i}, \underline{f_i})\} \otimes_{\mathbb{C}[Y]} \mathbb{C}$$

We have a family of maps

$$\begin{array}{c}
\text{Cut of } X \xrightarrow{\Delta} Y \xrightarrow{\Delta} Z = \{Z+Y, Z-Y\} \mid \{Y+X, Y-X\} \\
\Big\| \\
\overline{\{\underline{g_1}, \underline{f_1}\}} \\
\Big\downarrow \cong \\
\overline{\{\underline{g_2}, \underline{f_2}\}} \\
\Big\downarrow \simeq \\
\{Z+X, Z-X\}
\end{array}$$

QECC

The final isomorphism $\{Z+X, Z-X\} \longrightarrow \overline{\{\underline{g_2}, \underline{f_2}\}}$ maps $1 \longmapsto 1+\theta_1\theta_2$ which, by reading indices, can be thought of as the entangled qubit $|00\rangle + |11\rangle$.
Thus, it is *inevitable* that the organisation steps of Falling Roofs correspond to *something* in the Quantum Error Correcting Codes literature. When this is taken to its logical end, we find that cut-elimination corresponds to the quantum error correction process.

Dynamics

Theorem (The Reduction Theorem)

For each reduction $\gamma : \pi \longrightarrow \pi'$ there exists a subset $C_\pi \subseteq S_\pi$ and an isomorphism:

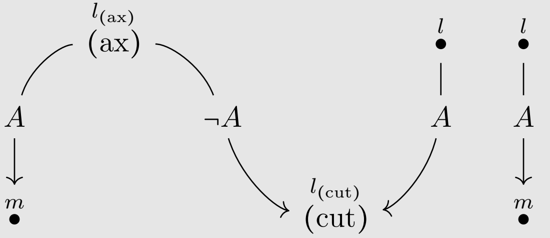
$$\hat{\gamma} : \mathcal{H}_{\pi'} \longrightarrow \mathcal{H}_\pi^{C_\pi}$$

such that for every $g \in S_\pi \setminus C_\pi$ there is a unique $g' \in S_{\pi'}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] \arrow[d, "g'"] & \mathcal{H}_\pi^{C_\pi} \arrow[d, "g"'] \\ \mathcal{H}_{\pi'} \arrow[r, "\hat{\gamma}"] & \mathcal{H}_\pi^{C_\pi} \end{tikzcd}$$

and this map $g \longmapsto g'$ is a bijection $S_\pi \setminus C_\pi \longrightarrow S_{\pi'}$.

We label the relevant links of π, π' according to the following diagram.

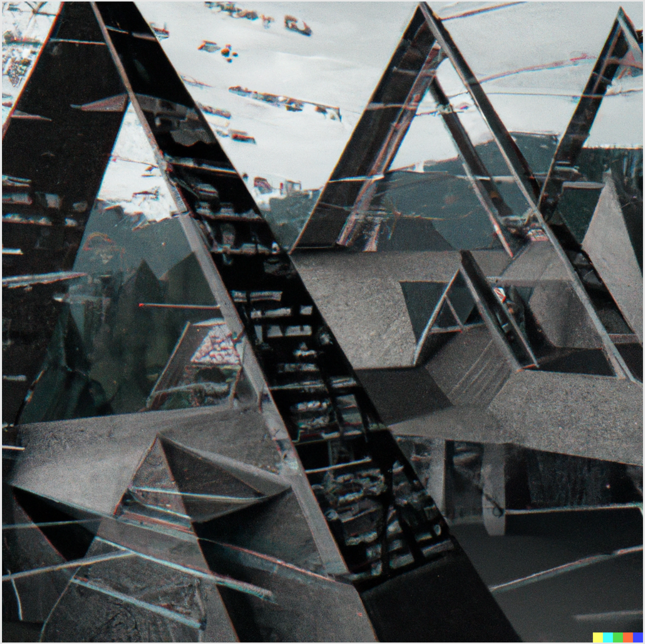


For each oriented atom (U, y) of A we define a $\mathbb{Z}_2$-degree zero map for $y = +$ by:

$$
\begin{aligned}
\gamma_U : \bigwedge \mathbb{C}\psi_U^{l} &\longrightarrow \bigwedge \mathbb{C}\psi_U^{l} \otimes \bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{cut})}} \otimes \bigwedge \mathbb{C}\psi_U^{l_{(\mathrm{ax})}} \\
|j\rangle &\longmapsto \frac{1}{\sqrt{2}}\left(|+++\rangle + (-1)^j |---\rangle\right)
\end{aligned}
$$

What is left to do?

- Today's talk has souly been about *multiplicative linear logic*, so what about *exponential* linear logic?
- The quantum error correction story can be totally recast in the guise of hamiltonians and renormalisation (another deep idea from physics).
- Categorifying our models.
- "Categorical elimination theory", coming from falling roofs.



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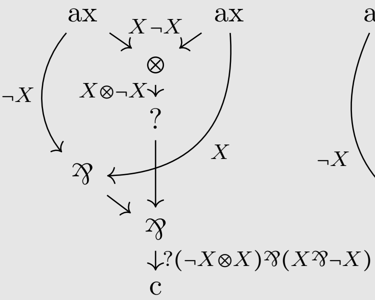

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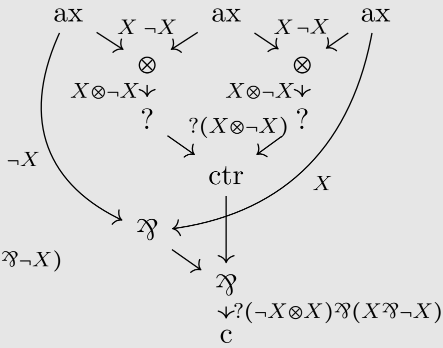

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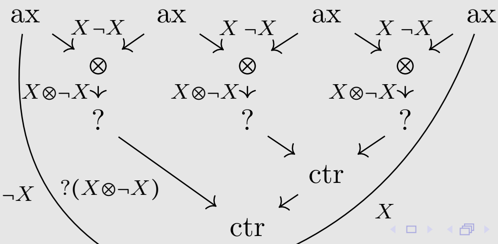

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