Execution formula for *all* of MELL

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Definition 0.0.1. There is an infinite set of **unoriented atoms** $X, Y, Z, \ldots$ and an **oriented atom** (or **atomic proposition**) is a pair $(X, +)$ or $(X, -)$ where X is an unoriented atom. Let $\mathcal{A}$ denote the set of oriented atoms.

For $x \in \{+, -\}$ we write $\overline{x}$ for the negation, so $\overline{+} = -, \overline{-} = +$.

Definition 0.0.2. The set of **pre-formulas** is defined as follows:

- Any atomic proposition is a preformula.
- If A, B are pre-formulas then so are $A \otimes B, A \parr B$.
- If A is a pre-formula then so are $\neg A, !A, ?A$.

The set of **multiplicative exponential linear logic formulas (MELL formulas)** is the quotient of the set of pre-formulas by the equivalence relation generated, for arbitrary formulas A, B and unoriented atom X, by

$$\neg(A \otimes B) = \neg B \parr \neg A, \quad \neg(A \parr B) = \neg B \otimes A, \quad \neg(X, x) = (X, \overline{x})$$
$$\neg !A = ?\neg A, \quad \neg ?A = !\neg A$$

Recall that in the multiplicative case, the set of words $\mathcal{A}^*$ over $\mathcal{A}$ forms a monoid under the operation of concatination. This monoidal structure extends to maps reflecting the connectives $\otimes, \parr, \neg$, for instance if $c : \mathcal{A}^* \times \mathcal{A}^* \longrightarrow \mathcal{A}^*$ denotes concatination, and $\otimes : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ is the map sending a pair of formulas A, B to the formula $A \otimes B$ then the following diagram commutes for some unique map $a : \mathcal{F} \longrightarrow \mathcal{A}^*$.

$$\begin{tikzcd}
\mathcal{F} \times \mathcal{F} \arrow[r, "a \times a"] \arrow[d, "\otimes"'] & \mathcal{A}^* \times \mathcal{A}^* \arrow[d, "c"] \\
\mathcal{F} \arrow[r, "a"] & \mathcal{A}^*
\end{tikzcd} \tag{1}$$

See [24, Definition 3.5, 3.6] for the full list of commutative diagrams. This map a induces the **sequence of oriented atoms** of a formula A.

$$a(A) = (X_1, x_1), \ldots, (X_n, x_n) \tag{2}$$

The **set of unoriented atoms** of A is then the disjoint union

$$U_A = \{X_1\} \coprod \ldots \coprod \{X_n\} \tag{3}$$

We use this notation in the following Definition.

Definition 0.0.3. Let A be a multiplicative exponential linear logic formula. The **set of unoriented atoms** U_A of A is defined by induction on the structure of A as follows.

- If $A = A_1 \otimes A_2$ or $A_1 \parr A_2$ then $U_A := U_{A_1} \coprod U_{A_2}$.
- If $A = \neg A'$ then $U_A := U_{A'}$.
- If $A =?A'$ or $A =!A'$ then $U_A := \coprod_{j=0}^{\infty} U_{A'}$.

The set of unoriented atoms only depends on the formula, and not its placement inside some proof. Next we take into account the *depth* of a formula inside a proof net.

Definition 0.0.4. Let A be an occurrence of a formula inside a proof net and say A has depth d. Then we define the set

$$\mathbb{N}^d \times U_A \tag{4}$$

Definition 0.0.5. Let π be a proof net and let E be its set of edges. Let k denote a ring. The **polynomial ring of** π P_π is

$$P_\pi = \bigotimes_{e \in E} k[\operatorname{Dep} U_{A_e}] \tag{5}$$

where A_e is the formula labelling edge e.

Recall that in [1] we defined MELL proof nets and we had the following clause for promotion links: Each promotion link must come equipt with a subset (V, E) of the links and edges of the proof structure such that the following conditions hold:

- The following process must result in a proof structure: for every edge $e \in E$ such that the target $t(e)$ is *not* an element of V, we introduce a conclusion vertex and set $t(e)$ to be this conclusion vertex.
- All edges $e \in E$ such that $t(e) \notin V$, the label of e is $?A$ for some A.
- The premise to the promotion link is an element of E.
- Each vertex labelled c has exactly one premise and no conclusion. Such a premise of a vertex labelled c is called a **conclusion** of the proof structure.

Definition 0.0.6. If $e \in E$ is an edge in a subset (V, E) of π corresponding to some promotion link, and the target of e does not lie in V, then the source of e is on the **boarder of a box**.

Remark 0.0.7. We notice that all vertices labelled ! are on the boarder of a box.

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