Categorical Geometry of Interaction

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Throughout we work over the complex numbers $\mathbb{C}$.

Notation 0.0.1. Fix integers $n, m, l > 0$ and denote the following polynomials (with all coefficients equal to the complex number 1):

$$W := \sum_{i=1}^{n} x_i^2, \quad V := \sum_{i=1}^{m} y_i^2, \quad U := \sum_{i=1}^{l} z_i^2 \tag{1}$$

Recall that any nondegenerate, symmetric, bilinear form over a finite dimensional complex vector space is admits a matrix representation

$$\begin{pmatrix} I_q & 0 \\ 0 & -I_r \end{pmatrix} \tag{2}$$

for some positive integers q, r. See [1] for more details. Thus the setting we are working in is quite general.

Notation 0.0.2. For $X, Y \in \{W, U, V\}$ we denote by C_{XY} the Clifford algebra associated to the polynomial $X - Y$.

Example 0.0.3. Explicitly, to construct C_{UV} we consider the complex vector space $\mathbb{C}^{m+l}$ equipped with the following symmetric bilinear form B (written with respect to the standard basis $e_1, ..., e_{m+l}$ of $\mathbb{C}^{m+l}$):

$$\begin{pmatrix} I_l & 0 \\ 0 & -I_m \end{pmatrix} \tag{3}$$

Then $C_{UV} := C_B$.

This Clifford algebra is generated by the elements $e_1, ..., e_{m+l}$. It is helpful to ascribe new names to these elements, namely $\mu_1, ..., \mu_l$ for $e_1, ..., e_l$ and $\nu_1, ..., \nu_m$ for $e_{i+1}, ..., e_m$. These elements satisfy the following properties (note, all commutators are graded)

$$[\mu_i, \mu_j] = 2\delta_{ij}, \qquad [\mu_i, \nu_j] = 0, \qquad [\nu_i, \nu_j] = -2\delta_{ij} \tag{4}$$

So indeed, one may think of C_{UV} as the free $\mathbb{C}$-algebra generated by $\{\mu_1, ..., \mu_l, \nu_1, ..., \nu_m\}$ subject to (4).

Notation 0.0.4. We denote by $\bar{\nu}_1, ..., \bar{\nu}_m, \omega_1, ..., \omega_n$ the multiplicative generators of the Clifford algebra C_{VW} subject to the relations:

$$[\bar{\nu}_i, \bar{\nu}_j] = 2\delta_{ij}, \qquad [\bar{\nu}_i, \omega_j] = 0, \qquad [\omega_i, \omega_j] = -2\delta_{ij} \tag{5}$$

We now construct a matrix factorisation, for an introduction to matrix factorisations see [1].

Let $\tilde{X}$ be a $\mathbb{Z}_2$-graded C_{VW}-module which is also a k-module, where k is some commutative $\mathbb{Q}$-algebra. Denote by X the $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_m]$-algebra $k[x_1, ..., x_n, y_1, ..., y_m] \otimes \tilde{X}$ and let ∂ be given by

$$\partial = \sum_{i=1}^{n} x_i \omega_i + \sum_{j=1}^{m} y_j \bar{\nu}_j \tag{6}$$

Notice:

$$\begin{aligned}
\partial^2 &= \Big(\sum_{i=1}^{n} x_i\omega_i + \sum_{j=1}^{m} y_j\bar{\nu}_j\Big)^2 \\
&= \sum_{i,i'=1}^{n} x_i\omega_i x_{i'}\omega_{i'} + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i\omega_i y_j\bar{\nu}_j + \sum_{j=1}^{m}\sum_{i=1}^{n} y_j\bar{\nu}_j x_i\omega_i + \sum_{j,j'=1}^{m} y_j\bar{\nu}_j y_{j'}\bar{\nu}_{j'} \\
&= \sum_{i<i'} x_i x_{i'}(\omega_i\omega_{i'} + \omega_{i'}\omega_i) + \sum_{i=1}^{n} x_i^2\omega_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j(\omega_i\bar{\nu}_j + \bar{\nu}_j\omega_i) + \sum_{j<j'} y_j y_{j'}(\bar{\nu}_j\bar{\nu}_{j'} + \bar{\nu}_{j'}\bar{\nu}_j) \\
&= \sum_{i<i'} x_i x_{i'}[\omega_i, \omega_{i'}] + \sum_{i=1}^{n} x_i^2\omega_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j[\omega_i, \bar{\nu}_j] + \sum_{j<j'} y_j y_{j'}[\bar{\nu}_j, \bar{\nu}_{j'}] + \sum_{j=1}^{n} y_j^2\bar{\nu}_j^2 \\
&= 0 - \sum_{i=1}^{n} x_i^2 + 0 + 0 + \sum_{j=1}^{n} y_j^2 \\
&= (V - W)\,\mathrm{id}
\end{aligned}$$

so indeed we have a matrix factorisation.

References

[1] W. Troiani, *Commutative algebra*