Hartshorne Exercise Solutions

Will Troiani

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1 Chapter I

Contents

1.1 §1

1.1:
a) The affine coordinate ring is defined by the formula $A(Y) = k[x, y]/I(Y)$. In this instance, $I(Y) = (y - x^2)$ as $(y - x^2)$ is a radical ideal. Let $\varphi : k[x, y] \to k[x]$ be the morphism defined by $x \mapsto x$ and $y \mapsto x^2$. This is surjective and $\ker(\varphi) = (y - x^2)$, so that $A(Y) \cong k[x]$.

b) We have $A(Z) = k[x, y]/(1 - xy)$. This is in fact isomorphic to $k[x]_x$. To see this, define a morphism $\varphi : k[x, y] \to k[x]_x$ by $x \mapsto x$ and $y \mapsto x^{-1}$. Then φ is a surjection and its kernel is exactly $(1 - xy)$.

c) First note that if $p(x, y)$ is a homogeneous polynomial of degree n in $k[x, y]$, where k is an algebraically closed field, then p splits into a product of linear factors. To see this write $p = y^n g(\frac{x}{y})$. Then $g(\frac{x}{y})$ will split so we can write $p = y^n(\frac{x}{y} - a_1)...(\frac{x}{y} - a_n) = (x - a_1 y)...(x - a_n y)$.

Now, suppose that $f(x, y)$ is an irreducible quadratic over an algebraically closed field k. Let $p(x, y)$ be the degree 2 homogeneous part of f. By the above we can write $p = (ax - by)(cx - dy)$. Potentially swapping variables we can assume without loss of generality that $a \neq 0$. If these factors are linearly dependent, we can do a change of variables to replace x with $ax - by$ (note that replacing x with a linear polynomial in x and y induces an automorphism of $k[x, y]$). Then $f(x, y) = x^2 + ax + by + c$. We can then do a change of variables and replace $ax + by + c$ with $-y$, giving $f(x, y) = x^2 - y$. Solving $f = 0$ then gives $y = x^2$.

If both factors are linearly independent, we can assume that $a, d \neq 0$. Thus by a change of variables (replacing $ax - by$ with x and $cx - dy$ with y, which induces an automorphism of $k[x, y]$ as these factors are linearly independent) we can write $f(x,y) = xy + ax + by + c$. We then have $f(x,y) = (x+b)(y+a) + c - ab$. Another change of variables then allows us to write $f(x,y) = xy - 1$. Solving for $f = 0$ then gives $xy = 1$.

1.2: For the first part, simply note that $Y = Z(y - x^2, z - x^3)$. Similarly to 1.1c, we can see that $k[x,y,z]/(y - x^2, z - x^3) \cong k[x]$. Since $k[x]$ has no nilpotent elements, $(y - x^2, z - x^3)$ is a radical ideal and is thus equal to $I(Y)$. Hence $A(Y) \cong k[x]$, as required.

1.3: $Y = Z(y) \cup Z(x) \cup Z(x^2 - y)$ and the corresponding ideals are $(y), (x)$, and $(x^2 - y)$.

1.4: A basis for the closed sets of $\mathbb{A}^1 \times \mathbb{A}^1$ is given by $\{X \times Y \mid X \subseteq \mathbb{A}^1 \text{ closed}, Y \subseteq \mathbb{A}^1 \text{ closed}\}$ which means every closed set is finite. However, the set $Z(y - x) \subseteq \mathbb{A}^2$ is closed and infinite (k is algebraically closed and thus infinite), thus these topologies are not equal.

1.5: If B is finitely generated then $B \cong k[x_1, ..., x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Moreover, if B has no nilpotent elements then $\mathfrak{a}$ is radical. Which means $Z(\mathfrak{a})$ is such that

$$A(Z(\mathfrak{a})) = k[x_1, ..., x_n]/IZ(\mathfrak{a}) = k[x_1, ..., x_n]/\sqrt{\mathfrak{a}} = k[x_1, ..., x_n]/\mathfrak{a} \cong B$$

The converse is obvious.

1.6: See [3].

1.7:
a) Routine, if one was only interested in the case of algebraic sets then use the bijection between algebraic sets and radical ideals coupled with the corresponding statements for Noetherian rings.

b) If X is not quasi-compact then one can construct from an infinite cover with no finite subcover a strictly ascending chain of open subsets, taking complements of which induces a strictly decreasing chain of closed sets.

c) Follows easily by considering the contrapositive.

d) Let X be Noetherian and Hausdorff. The space X decomposes into finitely many irreducible components $X = X_1 \cup \ldots \cup X_n$. Each X_i is Noetherian, Hausdorff, and irreducible. By irreducibility, any two non-empty open sets of X_i have non-empty intersection, which contradicts the Hausdorff condition unless X_i consists of a single element. Thus X is finite. Lastly, any finite, Hausdorff space is discrete.

1.8:
Decompose $Y \cap H$ into finitely many irreducibles $Y \cap H = Y_1 \cup ... \cup Y_n$ with no Y_i containing any other. Each Y_i is an irreducible subset of Y and so corresponds to a prime $\mathfrak{p}_i$ of $A(Y)$. Since Y_i is also a subset of H it follows that $\mathfrak{p}_i$ contains $(IH)A(Y) = \big(IZ(f)\big)A(Y) = (f)A(Y)$. In fact, since there is no irreducible subset strictly between Y_i and Y it follows that $\mathfrak{p}_i$ is minimal over $(f)A(Y)$, that is, $\dim A(Y)/\mathfrak{p}_i = \dim A(Y) - 1 = r - 1$. Since primes ideals of $A(Y)/\mathfrak{p}_i$ correspond to irreducible subsets of Y_i we thus have $\dim Y_i = r - 1$.

1.9:
Decompose $Z(\mathfrak{a})$ into finitely many irreducible components $Z(\mathfrak{a}) = Y_1 \cup \ldots \cup Y_n$ with no Y_i containing any other. Each Y_i corresponds to a prime ideal $\mathfrak{p}_i$ which is minimal over $\mathfrak{a}$. By Krull's Principal Ideal Theorem, $\operatorname{ht.} \mathfrak{p} \leq r$. We also know

$$\operatorname{ht.} \mathfrak{p}_i + \dim A_n/\mathfrak{p}_i = \dim A_n$$

thus $\dim Y_i = \dim A_n/\mathfrak{p}_i \geq n - r$.

1.10:

a) Solved in [3].

b) Solved in [3].

c) Consider the Sierpinski space $\Sigma := \{0, 1\}$ with topology $\{\varnothing, \{0\}, \{0, 1\}\}$. We have that $\overline{\{0\}} = \Sigma$ so $\{0\}$ is dense. Furthermore, $\dim\{0\} = 0$. However, $\dim \Sigma = 1$ as demonstrated by the following sequence $\{0\} \subsetneq \Sigma$.

d) This is obvious as if $Y \neq X$ then any chain of irreducible, closed subsets of Y remain so as subsets of X. Since X itself is irreducible, $Y \neq X \implies \dim Y < \dim X$.

e) Consider $\mathbb{N}$ with the the topology whose closed sets are all initial segments.

1.12 $x^2 + y^2 + 1$. We have that $Z_{\mathbb{A}^2_{\mathbb{R}}}(x^2 + y^2 + 1) = \varnothing$ which by definition is not irreducible.

1.2 §2

Throughout, $S = k[x_0, ..., x_n]$

2.1:

For clarity, if $\mathfrak{a} \subset S$ is an ideal we will write $Z_{\mathbb{P}^n}(\mathfrak{a})$ for the zero set in $\mathbb{P}^n$ and $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ for the zero set in $\mathbb{A}^{n+1}$.

Let $\mathfrak{a} \subseteq S$ be homogeneous and say $f \in S$ is a homogeneous polynomial such that $\deg f > 0$ and for all $P \in Z_{\mathbb{P}^n}(\mathfrak{a})$ we have that $f(P) = 0$. It follows that for all non-zero $P \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ we have that $f(P) = 0$. Moreover, since $\deg f > 0$ and f is homogeneous it follows that $f(0, ..., 0) = 0$. Thus for all $P \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ we have that $f(P) = 0$ and so by the regular nullstellensatz we have that $f^r \in \mathfrak{a}$ for some $r > 0$.

2.2:

Say $Z_{\mathbb{P}^n}(\mathfrak{a}) = \varnothing$. Then $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ is either empty or the singleton set $\{(0, ..., 0)\}$. In the case that it is empty, it follows from the nullstellensatz that $\mathfrak{a} = S$, and in the case that it is the singleton set containing $(0, ..., 0)$ we have that $\sqrt{\mathfrak{a}} = S_+$ again by the nullstellensatz, thus $(i) \Rightarrow (ii)$. Now say $\sqrt{\mathfrak{a}} = S_+$ and let d be the least integer such that there exists a polynomial of degree d in $\mathfrak{a}$, we claim that $S_d \subseteq \mathfrak{a}$. For each i there exists $d_i > 0$ such that $x_i^{d_i} \in \mathfrak{a}$, as $\sqrt{\mathfrak{a}} = S_+$. Let $d = \max_i d_i$. Then $x_i^d \in \mathfrak{a}$ for all i, as these generate S_d we have that $S_d \subseteq \mathfrak{a}$. If $\sqrt{\mathfrak{a}} = S$ then $\mathfrak{a} = S$. Thus $(ii) \Rightarrow (iii)$. Lastly, if $\mathfrak{a} \supset S_d$ for some d then $Z_{\mathbb{A}^{n+1}}(\mathfrak{a}) \subseteq Z_{\mathbb{A}^{n+1}}(S_d) = \{(0, ..., 0)\}$ and so $Z_{\mathbb{P}^n}(\mathfrak{a}) = \varnothing$.

2.3:

a),b),c) are trivial.

d) First notice that if $Z(\mathfrak{a}) = \varnothing$ then $IZ(\mathfrak{a}) = S$, but from the previous part it might be that $\sqrt{\mathfrak{a}} = S_+$, so we cannot assert that $IZ(\mathfrak{a}) = \sqrt{\mathfrak{a}}$. Assuming $Z(\mathfrak{a}) \neq \varnothing$ then we have that $I_{\mathbb{A}^{n+1}}Z_{\mathbb{A}^{n+1}}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$. Notice that all elements of $\sqrt{\mathfrak{a}}$ are homogeneous, and so $I_{\mathbb{P}^n}Z_{\mathbb{P}^n}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$.

e) Let $W \supseteq Y$ be closed, we show $ZI(Y) \subseteq W$. Write $W = Z(\mathfrak{a})$. By $a)$ it suffices to show $I(Y) \supseteq \mathfrak{a}$. This holds as $W \supseteq Y$ implies $I(Y) \supseteq I(W) = IZ(\mathfrak{a})$, which by $d)$ is equal to $\sqrt{\mathfrak{a}}$. The result then follows as $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$.

2.4:

a) The previous exercise implies that there is a one-to-one order reversing bijection between proper radical ideals of S not equal to S_+ and non-empty closed subsets of $\mathbb{P}^n$. We then notice that $I(\varnothing) = S$ and $Z(S) = \varnothing$, so this bijection extends to that as stated in the question.

b) Immediate from the fact that the bijection is order reversing.

c) $I(\mathbb{P}^n) = (0)$ which is prime.

2.5: **a)**: Every descending chain of algebraic sets corresponds to an ascending chain of ideals of $k[x_0, ..., x_n]$

which is Noetherian.

b) Follows from Proposition [1, §I Prop1.5]

2.6:
We will use the following lemma:

Lemma 1.2.1. *If a ring map $f : A \longrightarrow B$ is injective and extends to a map $F : A[\{x_i\}_{i\in I}] \longrightarrow B$ such that the ideal generated by $\{x_i\}_{i\in I}$ has empty intersection with $\ker F$, then F is injective.*

Proof. Clearly a non-zero element of $A[\{x_i\}_{i\in I}]$ maps to a non-zero element of B. □

There is a map

$$S \longrightarrow S_{(x_i)}$$
$$f \mapsto f(x_0/x_i, ..., x_n/x_i)$$

and thus a composite

$$\psi_i : A \xrightarrow{\beta_i} S \longrightarrow S_{(x_i)}$$

given by $f \mapsto x_i^{\deg f} f(x_0/x_i, ..., x_n/x_i) \mapsto f(x_0/x_i, ..., x_n/x_i)$ (with x_i/x_i omitted). This map is clearly an isomorphism as it is just a relabelling of indeterminants. In fact, we have:

Lemma 1.2.2. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety, $f \in I(Y_i)$, and $P \in Y \cap U_i$. Then*

$$f(\varphi_i(P)) = 0 \iff (\beta_i f)(P) = 0$$

Moreover, if $P \notin U_i$ then $P_i = 0$ and so $(\beta_i f)(P) = 0$. Thus $f \in I(Y_i) \Rightarrow \beta_i(f) \in I(Y)$.

Thus $\psi_i(I(Y_i)) = I(Y)S_{(x_i)}$, and so

$$\varphi_i^* : A(Y_i) \longrightarrow S_{(x_i)}/(I(Y)S_{(x_i)}) \cong S(Y)_{(x_i)}$$

is an isomorphism.

This extends naturally to a surjective map $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$, the image of x_i under which is a unit, we thus have a map

$$\delta_i : \big(A(Y_i)[x_i]\big)_{x_i} \longrightarrow S(Y)_{x_i}$$

our next claim is that this is an isomorphism. This maps onto a set of generators and is thus surjective. For injectivity, as $A(Y_i)[x_i]$ is an integral domain, it suffices to show $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$ is injective, which follows from Lemma 1.2.1.

We now show $\dim S(Y)_{x_i} = \dim S(Y) - 1$. By $(1.8A)$ this equality is equivalent to $\operatorname{tr.deg}_k S(Y)_{x_i} = \operatorname{tr.deg}_k S(Y) - 1$. We have

$$\operatorname{Frac} S(Y) \cong \operatorname{Frac} S(Y)_{x_i} \cong \operatorname{Frac}\big(A(Y_i)[x_i]\big)_{x_i} \cong \operatorname{Frac}\big(A(Y_i)[x_i]\big) = \big(A(Y_i)\big)(x_i) \cong \big(S(Y)_{x_i}\big)_0(x_i)$$

thus

$$\operatorname{tr.deg}_k S(Y) = \operatorname{tr.deg}_k \big(S(Y)_{x_i}\big)_0(x_i) = \operatorname{tr.deg}\big(S(Y)_{x_i}\big)_0 + 1$$

We also have that

$$\dim\big(S(Y)_{x_i}\big)_0 = \dim A(Y_i) = \dim(Y \cap U_i)$$

Thus $\dim S(Y) = \dim(Y \cap U_i) + 1$ for all i, notice this value is independent of i and so by exercise 1.10b) we have $\dim S(Y) = \dim Y + 1$.

2.7:
a) Cover $\mathbb{P}^n$ by open affines $\{U_i\}_{i=0}^n$, by exercise 1.10 we have that $\dim \mathbb{P}^n = \sup_i \dim U_i$. For each U_i we have $\dim U_i = \dim \mathbb{A}^n = n$.
b) We make use of the following fact from topology:

Fact 1. *Let X, Y be topological spaces, $Z \subseteq X$ a subset, and $U \subseteq X, V \subseteq Y$ open subsets. If $\varphi : U \to V$ is a homeomorphism then $\varphi(U \cap \mathrm{cl}_X(Z)) = \mathrm{cl}_V(\varphi(U \cap Z))$.*

Y is an open subset of an affine space and so is irreducible. This in turn implies that $\bar{Y}$ is irreducible and thus affine. The previous exercise then applies, so we have $\dim \bar{Y} = \dim(\bar{Y})_i$, where we recall that $(\bar{Y})_i = \varphi_i(\mathrm{cl}_{\mathbb{P}^n}(Y) \cap U_i)$. We have $\varphi_i(\mathrm{cl}_{\mathbb{P}^n}(Y) \cap U_i) = \mathrm{cl}_{\varphi_i(U_i)} \varphi_i(Y \cap U_i)$ by Fact 1 and this in turn is just $\mathrm{cl}_{\mathbb{A}^n} \varphi_i(Y \cap U_i)$. In other notation, we have $\overline{(Y_i)} = (\bar{Y})_i$. It follows from Proposition [1, §1 1.10] that $\dim Y_i = \dim \overline{(Y_i)}$. It remains to show that $\dim Y_i = \dim Y$. By exercise 1.10 it suffices to show for all $i \neq j$ such that neither $Y \cap U_i$ nor $Y \cap U_j$ are empty that $\dim Y_i = \dim Y_j$. We have:

$$\dim \overline{(Y_i)} = \dim(\bar{Y})_i = \dim \bar{Y}$$

finishing the proof.

2.9:

a) First we claim $I(\bar{Y}) \subseteq \beta I(Y)$. Let $f = f(x_0, ..., x_n) \in I(\bar{Y})$ be homogeneous and consider $f(1, x_1, ..., x_n)$. This is such that $\beta f(1, x_1, ..., x_n) = f$ and so lies in the image of β. Moreover, if $P = (P_1, ..., P_n) \in Y$ then the element $\bar{P}$ of $\bar{Y}$ given by the set of homogeneous coordings $(1, P_1, ..., P_n)$ is such that $f(P) = 0$, or equivalently, $f(1, P_1, ..., P_n) = 0$. That is, $f \in \beta I(Y)$. Since the homogeneous elements generate $I(\bar{Y})$ and $\beta I(Y)$ is an ideal, this establishes the claim.

Conversely, let $f \in \beta I(Y)$ and let $g \in I(Y)$ be such that $x_0^{\deg g} g(x_1/x_0, ..., x_n/x_0) = f$. For clarity, we distinguish Y from $\varphi_0(Y)$. Since $g \in I(Y)$ we have for any $P = (P_1, ..., P_n) \in Y$ that $g(P) = 0$, in other words, $1^{\deg g} g(P_1/1, ..., P_n/1) = 0$, and thus $Z(f) \supseteq \varphi_0^{-1} Y$. Since $Z(f)$ is closed this implies $Z(f) \supseteq \bar{Y}$, that is, $f \in I(\bar{Y})$.

b) From the previous part, we have that $I(\bar{Y})$ is equal to the ideal generated by $\beta I(Y)$, thus $(\beta f_1, ..., \beta f_n) \subseteq I(\bar{Y})$. So the statement of the question is true if and only if the ideal generated by $\beta(f_1, ..., f_r)$ is not contained in $(\beta f_1, ..., \beta f_r)$. Specialising now to the question at hand, we have $I(Y) = IZ(y - x^2, z - x^3)$ which is radical, and so is equal to $(y - x^2, z - x^3)$. We need an element of the ideal generated by $\beta(y - x^2, z - x^3)$ which is not in $(wy - x^2, w^2 z - x^3)$. Consider $x(y - x^2) - (z - x^3) = xy - z \in (y - x^2, z - x^3)$ so that $xy - wz$ is in the ideal generated by $\beta(y - x^2, z - x^3)$. This element is not in $(wy - x^2, w^2 z - x^3)$.

It remains to find generators for $I\bar{Y}$. This is difficult, we invoke the general theory of Gröbner bases. By [3, Lemma 3.2.17], it can be shown that

$$I\bar{Y} = (wy - x^2, xz - y^2, xy - zw) \tag{1}$$

2.10:

a) Let $S = k[x_0, ..., x_n]$. First notice by Exercise 2.2 we have for any ideal $\mathfrak{a} \subseteq S$ with $IZ_{\mathbb{P}^n}(\mathfrak{a}) \neq \varnothing$ that $IZ_{\mathbb{P}^n}(\mathfrak{a}) \cap k = \{0\}$. We therefore assume $I(Y) \cap k = \{0\}$. If $I(Y) = \{0\}$ then $Y = \mathbb{P}^n$ and so $C(Y) = \mathbb{A}^{n+1}$ which is algebraic. If $I(Y) \supsetneq \{0\}$ then any non-zero $f \in I(Y)$ has strictly positive degree and so admits $(0, ..., 0) \in \mathbb{A}^{n+1}$ as a zero. Thus if $Y = Z_{\mathbb{P}^n}(T)$ then $C(Y) = Z_{\mathbb{A}^{n+1}}(T)$. Moreoever, $IC(Y) = I(Y)$.

b) Y is irreducible iff $I(Y)$ is prime iff $IC(Y)$ is prime iff $C(Y)$ is irreducible.

c) In the case where Y is a projective variety we have

$$\dim C(Y) = \dim S(C(Y)) = \dim S(Y) = \dim Y + 1$$

For the general case, we use exercise 2.7.

2.11:

a) Say $I(Y)$ can be generated by linear polynomials. Since S is noetherian we can assume there are finitely many such generators, $f_1, ..., f_m$. We have

$$Y = ZI(Y) = Z(f_1, ..., f_m) = Z(f_1) \cap ... \cap Z(f_m)$$

where each $Z(f_i)$ is a hyperplane.

Conversely, notice that since $\mathbb{P}^n$ is noetherian, we can assume Y can be written as the finite intersection of hyperplanes $Z(f_1) \cap ... \cap Z(f_m)$, the result follows from the same calculation as above.

b) We begin by establishing the following lemma:

Lemma 1.2.3. *Let $f_1, ..., f_m$ be a set of linear polynomials in S. Then $\dim S/(f_1, ..., f_m) = n + 1 - m$.*

Proof. Since $S/(f_1, ..., f_m) \cong \big(S/(f_1, ..., f_{m-1})\big)/\overline{(f_m)}$ it suffices to prove the case when there is a single f_i, say f. Write $f = \alpha_0 x_0 + \ldots + \alpha_n x_n$ and by reordering the variables if necessary assume $\alpha_0 \neq 0$. Consider the map $k[x_0, ..., x_n] \to k[x_1, ..., x_n]$ which maps $x_i \mapsto x_i$ for $i \geq 1$ and $x_0 \mapsto \alpha_0^{-1}(-\alpha_1 x_2 - ... - \alpha_n x_n)$. This induces an isomorphism $k[x_0, ..., x_n]/(f) \cong k[x_1, ..., x_n]$ and the result follows. □

Now proceeding with the question at hand. Let Y have dimension r and write $Y = Z(f_1) \cap ... \cap Z(f_m) = Z(f_1, ..., f_m)$ where each $Z(f_i)$ is a hyperplane, and moreover assume m is minimal amongst such decompositions. We have:

$$r + 1 = \dim Y + 1 = \dim S(Y) = n + 1 - m$$

and thus $m = n - r$.

c): The solution to this question essentially comes down to the following observation:

Lemma 1.2.4. *A linear variety Y in $\mathbb{P}^n$ is a k-vector subspace of $\mathbb{A}^{n+1}$ and the dimension of Y as a variety is one less than its dimension as a vector space.*

Proof. That Y is a k-vector subspace is obvious, we prove the second claim by induction on $n - \dim Y$. If $\dim Y = n$ then Y is the whole space and so as a subspace of $\mathbb{A}^{n+1}$ has dimesion $n + 1$. For the inductive step, assume $\dim Y = k$ and that $\{y_1, ..., y_{k+1}\}$ is a basis for Y as a subspace of $\mathbb{A}^{n+1}$. For a linear polynomial f such that $Z(f) \cap Y \neq Z(f)$ we have $Y \cap Z(f) = \mathrm{Span}\{y_1, ..., y_{k+1}\} \cap Z(f)$. Write $f = \alpha_0 x_0 + ... + \alpha_n x_n$, then $Y \cap Z(f)$ is the span of the vectors $y_1, ..., y_{k+1}$ subject to the condition $y_i^0 = \alpha_0^{-1}(-\alpha_1 y_i^1 - ... - \alpha_n y_i^n)$, and so has dimension 1 less than that of Y. What we have shown is that as Y decreases by 1 in dimension as a variety, so to does it decrease by 1 in dimension as a subspace. □

The question at hand is now reduced to elementary linear algebra.

2.12:

a) We show that $\mathfrak{a} = \sum_{d \geq 0}(S_d \cap \mathfrak{a})$. The $\supseteq$ direction is obvious. For the reverse, let $f \in \mathfrak{a}$ and write $f = \sum_{j \geq 0} f_j$ where all but finitely many $f_j = 0$ and $\deg f_j = j$ for all j. It suffices to show that $\theta(f_j) = 0$ for all j, but this follows from $\theta(f) = 0$ as $i \neq j \Rightarrow \deg \theta(f_i) \neq \deg \theta(f_j)$. That $\mathfrak{a}$ is prime follows from the fact that θ is a ring homomorphism with codomain an integral domain.

b) Here we follow the convention that $M_i = x_i^d$ for $i = 0, ..., n$. That $\operatorname{im} \rho_d \subseteq Z(\alpha)$ is obvious. For the converse we come up with a description for $\mathfrak{a}$: for every sequence $(j_0, ..., j_n)$ of integers such that $j_k < d$ and $\sum_{k=0}^n j_k = d$ we have that $y_0^{j_0} ... y_n^{j_n}$ maps under θ to a degree d^2 homogeneous element of $k[x_0, ..., x_n]$. Thus there exists some $m_{(j_0,...,j_n)} > n$ such that $y_0^{j_0} ... y_n^{j_n} - y^d_{m_{(j_0,...,j_n)}}$ maps to zero under θ. Thus if $P \in \mathbb{P}^N$ is such that $P \in Z(\ker \theta)$ we have that P is a root of a polynomials of the form

$$y_0^{j_0} ... y_n^{j_n} - y^d_{m_{(j_0,...,j_n)}} \tag{2}$$

First consider the case where d is even. The equations (2) show that for $l > n$ the element a_l is determined by $a_0, ..., a_n$. Thus $P = \rho_d([\sqrt[d]{a_0} : ... : \sqrt[d]{a_n}])$. Now consider the case when d is odd. Again we obtain a family of equations which show that for $l > n$ the element a_l is determined *up to sign* by $a_0, ..., a_n$. Now, by considering $a_0 a_1^{d-2} a_i = a^d_{m_{1,d-2,0,...,1,...,0}}$ we see that a_0 and a_i have the same sign. A similar argument shows a_0 and a_1 have the same sign. Thus by multiplying $(a_0, ..., a_N)$ by -1 if necessary we again see $P = \rho_d([\sqrt[d]{a_0} : ... : \sqrt[d]{a_n}])$.

In the case that d is odd the preimage of a point $P \in \operatorname{im} \rho_d$ can be recovered by the first n elements of P and so ρ_d is injective. In the case when d is odd we can recover the preimage *up to sign* and then the argument given above shows the first n elements all have the same sign, thus ρ_d is injective.

c) If $P \in \mathbb{P}^n$ and $f \in k[x_0, ..., x_N]$ a polynomial such that $f(\rho_d(P)) = 0$ then the polynomial $f(M_0, ..., M_N)$ vanishes at P and conversely. So if we write $\operatorname{mon} f$ for $f(M_0, ..., M_N)$ and $\operatorname{mon} I(Y)$ for the ideal generated by $\{\operatorname{mon} f \mid f \in I\}$ then it follows that for an algebraic set $Z(\mathfrak{b})$ we have $\rho_d^{-1}(Z(\mathfrak{b})) = Z(\operatorname{mon} \mathfrak{b})$, thus ρ_d is continuous.

Next we show this map is closed. Let $Z(\mathfrak{b})$ be an algebraic subset of $\mathbb{P}^n$. Then $\rho_d(Z(\mathfrak{b})) = Z(\theta^{-1}(\mathfrak{b})) \cap Z(\ker \theta)$. This is true because for all $g \in \theta^{-1}(\mathfrak{b})$ and all $P \in Z(\mathfrak{b})$ we have $\theta(g)(P) = 0$ if and only if $g(\rho_d(P)) = 0$.

d) Define θ to be

$$\theta : k[z, y, x, w] \longrightarrow k[x_0, x_1, x_2, x_3]$$

which maps $z \longmapsto x_0^3, y \longmapsto x_0^2 x_1, x \longmapsto x_0 x_1^2, w \longmapsto x_1^3$. Then

$$(wy - x^2, xz - y^2, xy - zw) \subseteq \ker \theta \tag{3}$$

These generators give the only three degree 2 polynomials p such that $\theta(p) = 0$. Thus $\operatorname{LT} \ker \theta = (\operatorname{LT}(wy - x^2), \operatorname{LT}(xz - y^2), \operatorname{LT}(xy - zw))$. Since the listed generators form a Gröbner basis, it follows that this inequality is in fact equality.

2.13: Since Z is of dimension 1 which is 1 less than $2 = \dim \mathbb{P}^2$ we have that $Z = Z(f)$ for some irreducible $f \in S^2$. Let $M_0, ..., M_5$ be the degree 2 homogeneous monomials of S^2 and write $f = \sum_{j=0}^{5} \alpha_j M_j$. Then let $g = \sum_{j=0}^{5} \alpha_j y_j$, we claim $Z(g) \cap Y = \rho_2(Z(f))$. By the solution to the previous question this amounts to showing $Z(g) \cap \operatorname{im} \rho_2 = Z(\theta^{-1}(f)) \cap Z(\ker \theta)$. For $P \in \mathbb{P}^2$ and $h \in \theta^{-1}(f)$ we have

$$\begin{aligned} h(\rho_2(P)) = 0 &\iff \theta(h)(P) = 0 \\ &\iff f(P) = 0 \\ &\iff g(\rho_2(P)) = 0 \end{aligned}$$

from which the result follows.

2.14:
Let $\theta : k[\{z_{ij}\}_{0 \le i \le r, 0 \le j \le s}] \longrightarrow k[x_0, ..., x_r, y_0, ..., y_s]$ be the ring homomorphism given by $z_{ij} \mapsto x_i y_j$. Say $P \in \mathbb{P}^{r+s+rs}$ is such that $P \in Z(\ker \theta)$. Then in particular, P is a root of every polynomial of the form $z_{ij} z_{kl} - z_{il} z_{kj}$, where $0 \le i, k \le r$ and $0 \le j, l \le s$. Let $\{P_{ij}\}$ be a set of homogeneous coordinates for P and now fix a pair of integers (a, b) such that $P_{ab} \ne 0$. For all $0 \le k \le r$ and all $0 \le j \le s$ we have $P_{aj}/P_{ab} = P_{kj}/P_{kb}$ which implies:

$$\frac{P_{aj}}{P_{ab}} P_{kb} = P_{kj}$$

Thus we can recover all P_{kj} from the set $\{P_{a0}, ..., P_{as}, P_{0b}, ..., P_{rb}\}$. We write P as

$$P = \Big[\frac{P_{aj}}{P_{ab}} P_{kb}\Big]_{0 \le k \le r, 0 \le j \le s} = \psi\Big(\big[P_{0b} : ... : P_{rb}\big], \big[\frac{P_{a0}}{P_{ab}} : ... : \frac{P_{as}}{P_{ab}}\big]\Big)$$

which shows $Z(\ker \theta) \subseteq \operatorname{im} \psi$. The other direction is trivial.

We observe that the above also implies that ψ is injective: let $(P,Q),(P',Q') \in \mathbb{P}^r \times \mathbb{P}^s$ whose image under ψ are equal, for clarity we write

$$\begin{aligned}\psi(P,Q) &= [P_0Q_0 : \dots : P_0Q_s : \dots\dots : P_rQ_0 : \dots : P_rQ_s] \\ &= [P_0'Q_0' : \dots : P_0'Q_s' : \dots\dots : P_r'Q_0' : \dots : P_r'Q_s'] = \psi(P',Q')\end{aligned}$$

and let $\lambda \neq 0$ be such that

$$(P_0Q_0 : \dots : P_0Q_s : \dots\dots : P_rQ_0 : \dots : P_rQ_s) = \lambda(P_0'Q_0' : \dots : P_0'Q_s' : \dots\dots : P_r'Q_0' : \dots : P_r'Q_s') \tag{4}$$

From the above, there exists pairs of integers $(a,b),(a',b')$ such that

$$\frac{P_aQ_j}{P_aQ_b}P_kQ_b = P_kQ_j \qquad \text{and} \qquad \frac{P_{a'}'Q_j'}{P_{a'}'Q_b'}P_k'Q_b' = P_k'Q_j' \tag{5}$$

Thus for all $0 \leq k \leq r, 0 \leq j \leq s$:

$$\begin{aligned} P_kQ_j &= \frac{P_aQ_j}{P_aQ_b}P_kQ_b && \text{by (5)} \\ &= \frac{\lambda P_{a'}'Q_j'}{P_aQ_b}\lambda P_k'Q_{b'}' && \text{by (4)} \\ &= \lambda^2\frac{P_{a'}'Q_{b'}'}{P_aQ_b}\left(\frac{P_{a'}'Q_j'}{P_{a'}'Q_{b'}'}P_k'Q_{b'}'\right) \\ &= \lambda^2\frac{P_{a'}'Q_{b'}'}{P_aQ_b}P_k'Q_j' && \text{by (5)} \end{aligned}$$

proving $(P,Q) = (P',Q')$.

2.15:
a) Since $\operatorname{im}\psi = Z(\ker\theta)$ (θ as in the previous question) it suffices to show $\ker\theta = (z_{00}z_{11} - z_{01}z_{10})$. Let $f \in \ker\theta$. We write $f = (z_{00}z_{11} - z_{01}z_{10})^m f_1 + f_2$ for the largest possible integer m. Let $\alpha^{d_1d_2d_3d_4}$ be the coefficient in front of f_2 in front of $z_{00}^{d_1}z_{01}^{d_2}z_{10}^{d_3}z_{11}^{d_4}$ and let $\beta^{d_1d_2d_3d_4}$ be the coefficient of $\theta(f_2)$ in front of $(x_0y_0)^{d_1}(x_0y_1)^{d_2}(x_1y_0)^{d_3}(x_1y_1)^{d_4}$. We have $\theta(f_2) = 0$ and so by linear independence $\beta^{d_1d_2d_3d_4} = 0$ for all sequences $d_1d_2d_3d_4$. We have $\beta^{1111} = \alpha^{1001} + \alpha^{0110} = 0$ and so $\alpha^{1001} = -\alpha^{0110}$ so either both are zero or neither are. If neither are then $f_2 = (z_{00}z_{11} - z_{01}z_{10})f_3 + f_4$ contradicting maximality of n. Thus both are zero. The final claim is for all sequences $d_1d_2d_3d_4$ other than 1111 we have $\alpha^{d_1d_2d_3d_4} = \beta^{d_1d_2d_3d_4}$ which can be proved by induction on such sequences in lexicographic order. Thus $f_2 = 0$ and $f \in (z_{00}z_{11} - z_{01}z_{10})$.

2.16:
a) We have

$$Q_1 \cap Q_2 = Z(x^2 - yw) \cap Z(xy - zw) = Z(x^2 - yw, xy - zw)$$

Multiplying $xy - zw = 0$ by y we have $xy^2 - zyw = 0$. Substituting $x^2 - yw = 0$ into $xy^2 - zyw$ we get

$$xy^2 - zx^2 \Longrightarrow x(y^2 - zx)$$

which means either $x = 0$ or $y^2 - zx = 0$, we will show that $x = 0$ corresponds to the line, and $y^2 - zx$ corresponds to the twisted cubic curve.

Say $x = 0$. Then since $x^2 - yw = 0$ we have that either $y = 0$ or $w = 0$. If $y = 0$ then since $xy - zw = 0$ we have either $z = 0$ or $w = 0$ with the other variable arbitrary, this corresponds to a line. If $y \neq 0$ then multiplying $xy - zw = 0$ by x^2 we have $x^3y - x^2zw = 0$ which by substituting yw for x^2 gives

$$x^3y - zyw^2 = 0 \Longrightarrow y(x^3 - zw^2) = 0$$

which since $y \neq 0$ implies $zw^2 = 0$ so either $z = 0$ or $w = 0$ with the other arbitrary. This also corresponds to a line.

Now say $x \neq 0$ so $y^2 - zx = 0$. Then we have

$$Q_1 \cap Q_2 = Z(x^2 - yw, xy - zw, y^2 - zw)$$

which, gives the twisted cubic curve.

b) $I(C) = (x^2 - yz)$, $I(L) = (y)$, and $I(C \cap L) = (x, y)$. Thus we need to show $(x^2 - yz) + (y) \neq (x, y)$ which is clear as $x \neq (x^2 - yz) + (y)$.

1.3 §3

3.1

a) We saw in exercise 1.1 that there are two possibilities up to isomorphism for the affine coordinate rings, and so there are two possibilities up to isomorphism of corresponding conics. Since $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{(0,0)\}$ are conics, we are done.

b) Any open subset of $\mathbb{A}^1$ is equal to $\mathbb{A}^1 \setminus V$ where V is a finite set of points. Let $v \in V$, then $1/(x - v)$ is an invertible element in $\mathcal{O}(\mathbb{A}^1 \setminus V)$ and is not in k, thus $\mathcal{O}(\mathbb{A}^1 \setminus V) \not\cong \mathcal{O}(\mathbb{A}^1)$.

c) Let $f \in k[x_0, x_1, x_2]$ be homogeneous, irreducible and degree 2. Then f can be written as $x^T M x$ where $x^T = (x_0, x_1, x_2)$ and M is some symmetric matrix. Since M is symmetric and k is algebraically closed, there exists an orthogonal matrix Q such that $Q^T M Q$ is diagonal. The matrix Q corresponds to a linear isomorphism $\varphi_Q : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ and so is an isomorphism of varieties such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\varphi_Q} & \mathbb{P}^2 \\ \uparrow & & \uparrow \\ Z(x^T M x) & \xrightarrow{\varphi_Q \restriction_{Z(x^T M x)}} & Z(x^T Q^T M Q x) \end{array}$$

Moreover, $\varphi_Q(Z(x^T M x)) = Z(x^T Q^T M Q x)$ because $P \in z(x^T Q^T M Q x)$ if and only if $QP \in Z(x^T M x)$ (both of these are the statement: $P^T Q^T M Q P = 0$). Thus $\varphi_Q \restriction_{Z(x^T M x)}$ is an isomorphism of varieties.

The upshot is that we may assume $f = \lambda_1 x_0^2 + \lambda_2 x_1^2 + \lambda_3 x_2^2$. There is another linear transformation given by the diagonal matrix with ii entry equal to $1/\lambda_i$ which shows that in fact we can assume $f = x_0^2 + x_1^2 + x_2^2$, that is, all conics are isomorphic to one in particular, thus are all isomorphic to each other. To finish the question, we can simply observe that $\mathbb{P}^1$ is isomorphic to its image under the 2-uple embedding and thus is isomorphic to all conics.

e) Follows from Theorems 3.2 and 3.4.

3.2

a) This is clearly bijective. To show bicontinuity it suffices to show that every proper, closed subset of $Z(y^2 - x^3)$ is finite. Let T be such a closed set, then $T = Z(y^2 - x^3) \cap T'$ for some closed set T' which can be written as a finite union of irreducible components, $T' = T_1' \cup ... \cup T_n'$. Since this union is finite it suffices to show $Z(y^2 - x^3) \cap T_i'$ is finite for each i. Fix an i. This set can itself be written as the finite union of irreducible elements, $Z(y^2 - x^3) \cap T_i' = Y_1 \cup ... \cup Y_m$ say. We show $\dim Y_i = 0$. Since T is a proper subset, $Y_i \subsetneq Z(y^2 - x^3)$ and so it is sufficient to show $\dim Z(y^2 - x^3) \leq 1$. By considering the map $k[x, y] \to k[t]$ such that $x \mapsto t^3$ and $y \mapsto t^2$ we see that $(y^2 - x^3)$ is prime, and thus $Z(y^2 - x^3)$ is irreducible. This is a proper subset of $\mathbb{A}^2$ which has dimension 2 and so $\dim Z(y^2 - x^3) \leq 1$.

Now, to see that this is not an isomorphism, we assume to the contrary that it is. The map $\mathbb{A}^1 \xrightarrow{\varphi} Z(y^2 - x^3) \xrightarrow{\varphi^{-1}} \mathbb{A}^1$ is regular and so $t = \varphi^{-1}\varphi(t) = \varphi^{-1}(t^2, t^3)$, where φ^{-1} must be a polynomial. No such polynomial exists so this is a contradiction.

b) This is bijective and thus bicontinuous. That it is not an isomorphism follows from the fact that $t \mapsto t^{1/p}$ is not a polynomial.

3.3:
a) For every open set $U \subseteq Y$ there is a map

$$\begin{gathered}\hat{\varphi} : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U)) \\ f \mapsto f \circ \varphi\end{gathered}$$

and $\mathcal{O}_X(\varphi^{-1}(U))$ maps to $\mathrm{Colim}_{U \ni p}(\varphi^{-1}(U))$ which by the universal property of this colimit maps to $\mathcal{O}_{X,P}$. Similarly, $\mathcal{O}_Y(U)$ maps to $\mathcal{O}_{Y,\varphi(P)}$ which by the universal property of this colimit maps into $\mathrm{Colim}_{U \ni p}(\varphi^{-1}(U))$ hence we get a map $\mathcal{O}_{Y,\varphi(P)} \longrightarrow \mathcal{O}_{X,P}$ given by $[f] \mapsto [f \circ \varphi]$. It remains to show this is a homomorphism of local rings, but this is clear as if $[f] \in \mathcal{O}_{Y,\varphi(P)}$ is such that $f(\varphi(P)) = 0$ then $(f \circ \varphi)(P) = 0$.

b) First we show that φ is a morphism. Let $U \subseteq Y$ be open, and $f : U \longrightarrow \mathbb{A}^1$ regular. We need to show $f \circ \varphi$ is regular at every point. Let $P \in \varphi^{-1}(U)$ and consider $[f] \in \mathcal{O}_{Y,\varphi(P)}$. The image of $[f]$ under φ_P^* is represented by $f \circ \varphi$ suitably restricted, thus there is some open subset $W \subseteq X$ containing P such that $(f \circ \varphi) \restriction_W$ is regular, that is to say, $f \circ \varphi$ is regular at P.

Now we show φ^{-1} is a morphism. First notice that by uniqueness of inverses, φ^{-1} can be given explicity by $[f] \mapsto [f \circ \varphi^{-1}]$. The argument is identical to above.

c) Let $[f] \neq [g] \in \mathcal{O}_{Y,\varphi(P)}$ be represented by $f : U_1 \longrightarrow \mathbb{A}^1$ and $g : U_2 \longrightarrow \mathbb{A}^1$ respectively. Since $[f] \neq [g]$ we have that f and g are not equal on $U_1 \cap U_2$ so we can assume $U_1 = U_2$, let U denote this set. We see that since f, g are regular, the fact they're unequal on U implies they're unequal on $U \cap \varphi(X)$. This holds true for all U and so $\varphi_P^*[f] \neq \varphi_P^*[g]$, thus φ_P^* maps distinct elements to distinct elements and so in injective.

3.4:
We will make use of the map $\theta : S^N \longrightarrow S^n$ with kernel $\mathfrak{a}$ given in the statement of Exercise 2.12. We have already shown in exercise 2.12 that ρ_d is a homeomorphism, so by the previous exercise it suffices to show $\rho_d^* : \mathcal{O}_{\mathrm{im}\,\rho_d, \rho_d(P)} \longrightarrow \mathcal{O}_{\mathbb{P}^n,P}$ is an isomorphism for all $P \in \mathbb{P}^n$. Let $P \in \mathbb{P}^n$ and write Q for $\rho_d(P)$. By Theorem [1, §I 3.3 3.5] we have that $\mathcal{O}_{\mathrm{im}\,\rho_d,Q} \cong (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)}$ and $\mathcal{O}_{\mathbb{P}^n,P} \cong S^n_{(\mathfrak{m}_P)}$ where $S^m = k[x_0, ..., x_m]$. So the problem is reduced to finding an isomorphism $\eta : (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \longrightarrow S^n_{(\mathfrak{m}_P)}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_{\mathrm{im}\,\rho_d,Q} \arrow[r, "\rho_d^*"] \arrow[d] & \mathcal{O}_{\mathbb{P}^n,P} \arrow[d] \\ (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \arrow[r, "\eta"] & S^n_{(\mathfrak{m}_P)} \end{tikzcd} \tag{6}$$

There is an injective map $\bar{\theta} : S^N/\mathfrak{a} \longrightarrow S^n$ such that $\bar{\theta}(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$, so this induces a map $(S^N/\mathfrak{a})_{\mathfrak{m}_Q} \longrightarrow (S^n)_{\mathfrak{m}_P}$ which since $S^N/\mathfrak{a}$ and S^n are integral domains is also injective. Lastly, θ maps degree e elements to degree de elements, thus the elements of degree 0 map injectively to those of degree 0, we thus have a map $(S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \longrightarrow (S^n)_{(\mathfrak{m}_P)}$ which we take to be η. Notice that the collection of rational functions in $(S^n)_{(\mathfrak{m}_P)}$ are generated by the quotient of two degree d monomials of S^n, which lie in the image of η, thus this map is surjective and thus an isomorphism.

It remains to show commutativity of (6). For any $m \geq 0$ denote $k[x_1, ..., x_m]$ by A^m, and let pick i such that $P \in U_i$, we have the following isomorphisms: $\mathcal{O}_{\operatorname{im}\rho_d,Q} \xrightarrow{\sim} A^N((\operatorname{im}\rho_d)_i)_{\mathfrak{m}'_Q}$ and $\mathcal{O}_{\mathbb{P}^n,P} \xrightarrow{\sim} (A^n)_{\mathfrak{m}'_P}$ where $\mathfrak{m}'_Q$ is the maximal ideal corresponding to Q and similarly for $\mathfrak{m}'_P$. Now (6) can then be extended to the following commuting diagram:

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{im}\rho_d,Q} \arrow[r, "\rho_d^*"] \arrow[d] & \mathcal{O}_{\mathbb{P}^n,P} \arrow[d] \\
A^N((\operatorname{im}\rho_d)_i)_{\mathfrak{m}'_Q} \arrow[r, dashed] \arrow[d] & (A^n)_{\mathfrak{m}'_P} \arrow[d] \\
(S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \arrow[r, "\eta"] & S^n_{(\mathfrak{m}_P)}
\end{tikzcd} \tag{7}$$

where the dashed arrow is induced by θ and the vertical arrows are isomorphism.

Remark 1.3.1. *Commutativity of the top square of (7) (arguably) should be justified:*

Lemma 1.3.1. *Let $\varphi : X \longrightarrow Y$ be a morphism of varieties with X, Y affine, then for all $P \in X$ the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}_{Y,\varphi(P)} \arrow[r, "\varphi_P^*"] & \mathcal{O}_{X,P} \\
A(Y)_{\mathfrak{m}_{\varphi(P)}} \arrow[r, "\hat{\varphi}_{\mathfrak{m}_P}"] \arrow[u] & A(X)_{\mathfrak{m}_P} \arrow[u]
\end{tikzcd} \tag{8}$$

Proof. The morphism $A(Y)_{\mathfrak{m}_{\varphi(P)}} \longrightarrow \mathcal{O}_{Y,\varphi(P)}$ is given by $[f]/[g] \mapsto [\gamma_{f/g}]$ where $\gamma_{f/g} : Y \longrightarrow \mathbb{A}^1$ is given by $y \mapsto f(y)/g(y)$. The map $\hat{\varphi}_{\mathfrak{m}_P}$ maps $[f]/[g]$ to $[f \circ \varphi]/[g \circ \varphi]$. Denote by $\gamma_{f\varphi/g\varphi} : X \longrightarrow \mathbb{A}^1$ the map given by $x \mapsto (f \circ \varphi)(x)/(g \circ \varphi)(x)$. Then the image of $[f \circ \varphi]/[g \circ \varphi]$ under the right, vertical map of (8) is $[\gamma_{f\varphi/g\varphi}]$. It remains to show $[\gamma_{f/g} \circ \varphi] = [\gamma_{f\varphi/g\varphi}]$ which is clear. □

3.5:

Let $f \in S^n$ be a homogeneous, irreducible polynomial such that $H = Z(f)$. Write $f = \sum_{j=0}^N \alpha_j M_j$. Then by the solution to Exercise 2.12c) we have that $\rho_d(Z(f)) = Z(\theta^{-1}(f)) \cap \ker\theta$ so it remains to calculate $\theta^{-1}(f)$. This is just the ideal generated by $\sum_{j=0}^N \alpha_j y_j$ which is linear.

There exists a rotation matrix $R_\theta : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ which maps the hyperplane to $Z(x_i)$ for some x_i. Multiplication by this matrix gives a family of polynomials and so zero sets are sent to zero sets and regular functions are mapped to regular functions. Thus this is an isomorphism.

3.6:

First we show that $\mathcal{O}(X) \cong k[x, y]$, this isomorphism might seem strange at first because surely $1/(x^2 + y^2)$ is a unit in $\mathcal{O}(X)$ but not in $k[x, y]$, however, $1/(x^2 + y^2)$ is not an element of $\mathcal{O}(X)$ as we are working with an algebraically closed field k, and so in fact has infinitely many solutions, not just $(0, 0)$.

First notice that if Y is an affine variety and Y' is an open subset then $K(Y) \cong K(Y')$. Thus $K(X) \cong K(\mathbb{A}^2) \cong k(x, y)$, also, $\mathcal{O}(X)$ embedds into $k(x, y)$. Now, let $f/g \in \mathcal{O}(X)$ be arbitrary. g can only be 0 when f is which is finitely many times and so g is a constant this statement follows from Bezout's Theorem. Thus $\mathcal{O}(X) \cong k[x, y]$.

To finish the question, we notice that the identity map $k[x, y] \longrightarrow k[x, y]$ corresponds under the equivalnce $\operatorname{Hom}(\mathbb{A}^2, X) \cong \operatorname{Hom}(k[x, y], k[x, y])$ to the inclusion function $X \rightarrowtail \mathbb{A}^2$ which is clearly not an isomorphism.

3.7:

b) (which implies a) we make use of the following lemma:

Lemma 1.3.2. *If Y is an irreducible subset of a toplogical space X and $Y' \subseteq Y$ is also an irreducible subset of X then Y' is irreducible as a subset of Y.*

Proof. Let $Y' = U \cup V$ where $U = U' \cap Y'$, $V = V' \cap Y'$ with $U', V' \subseteq X$ closed. Then

$$Y = Y' \cup Y = \big((U' \cap Y') \cup (V' \cap Y')\big) \cup Y = (U' \cap Y) \cup (V' \cup Y) = U' \cup V'$$

which implies that $U' = Y$, say. Thus $Y' = Y' \cap U' = U$ which shows that Y' is irreducible. □

Now onto the quesiton at hand. Say $H \cap Y = \varnothing$. Then $Y \subseteq \mathbb{P}^n \setminus H$. By Lemma 1.3.2 we have that Y is an irreducible, closed subset of $\mathbb{P}^n \setminus H$ which by Exercise 3.5 is affine. Thus Y is both affine and projective so by 3.1e it is thus a point. This means $\dim Y = 0$.

3.8:
We prove something more general, that if $Y \subseteq \mathbb{P}^n$ is an open set then the regular functions on Y are constants. First notice that in this setting, $K(Y) \cong K(\mathbb{P}^n)$. We also have that $\mathcal{O}(Y)$ embeds into $K(Y)$, so since $K(\mathbb{P}^n) \cong S^n_{((0))}$, a regular function $f : Y \longrightarrow \mathbb{A}^1$ can be thought of as a fraction f_1/f_2 where $f_1, f_2 \in S^n$ and $\deg f_1 = \deg f_2$. Using that k is infinite and again using Bezout's Theorem we have that f_2 is a constant which implies $\deg f_1 = 0$ and so is also a constant.

3.9:
$S(X) \cong S^1$ and $S(Y) \cong S^2/(x_0x_1 - x_2^2)$, the former is a UFD and the latter is not, as $x_2^2 = x_0x_1$.

3.10:
Let $U \subseteq Y'$ be open and $f : U \longrightarrow \mathbb{A}^1$ regular. Write $f = f_1/f_2$ where f_2 is nowhere zero on U, and $U = U' \cap Y'$ where $U' \subseteq Y$ is open. Then $U' \cap Z(f_2)^c$ is an open subset which extends f, and so $f \circ \varphi : U' \cap Z(f_2)^c \longrightarrow \mathbb{A}^1$ is regular as φ is a morphism and thus so is its restriction to X'.

Observation: The fact that X', Y' are locally closed is not integral to the restriction of φ respecting regular functions, this assumption is here so that X', Y' are varieties in their own right.

3.11:
For each closed subvariety $X' \subseteq X$ containing P define the set $\mathfrak{p}_{X'} := \{[(U, f)] \in \mathcal{O}_P \mid f \restriction_{X'} = 0\}$, we claim the map given by $X' \longrightarrow \mathfrak{p}_{X'}$ is a bijection.

We use the following Lemma:

Lemma 1.3.3. *Let X be an affine variety and $U \subseteq X$ a quasi-affine variety. Write $U = Z(\mathfrak{a})^c$ There is a bijection:*

$$\psi : \{\textit{Irreducible, closed subsets } V \subseteq U\} \longrightarrow \{\textit{Irreducible closed subsets } V \subseteq X \textit{ such that } V \not\subseteq Z(\mathfrak{a})\}$$
$$V \mapsto \mathrm{Cl}_X(V)$$

Proof. First we show this map is well defined. Irreducibility is transitive (Lemma [2, §Irreducible sets]) so since V is an irreducible subset of U it is also of X, moreover the closure of an irreducible space is irreducible, thus $\bar{V}$ is irreducible. It is clearly also closed and not contained in $Z(\mathfrak{a})$ otherwise it must have been the empty set which is not irreducible.

There is an inverse φ to this function which maps V to $V \cap U$. This is also clearly well defined, where we note that $V \cap U \neq \varnothing$ as $V \not\subseteq Z(\mathfrak{a})$.

Now we show this is in fact a bijection. $\varphi\psi(V) = \mathrm{Cl}_X(V) \cap U$. Since $V \subseteq U$ is closed, write $V = V' \cap U$ where $V' \subseteq X$ is closed. We claim $\mathrm{Cl}_X(V' \cap U) \cap U = V$. We have $V \subseteq U$ and $V = V' \cap U$ so $V \subseteq \mathrm{Cl}_X(V' \cap U) \cap U$. We show the reverse inclusion. V' is a closed set containing $V' \cap U$ and so $\mathrm{Cl}_X(V' \cap U) \subseteq V'$, thus $\mathrm{Cl}_X(V' \cap U) \cap U \subseteq V' \cap U = V$. Thus $\varphi\psi(V) = V$.

Conversely, we need to show $\mathrm{Cl}_X(W \cap U) = W$, but this is true as U is open and thus dense. □

In particular, Lemma 1.3.3 implies that fpr any $P \in U$, there is a bijection between the irreducible, closed neighbourhoods of $P \in U$ and the irreducible, closed neighbourhoods of $P \in X$.

Now back to the question at hand. Assume X is affine. There is a bijection between the prime ideals of $A(X)$ containing $\mathfrak{m}_P$ and the irreducible, closed neighbourhoods of P in X, so the affine and quasi-affine cases are solved.

In the projective case, for any U_i such that $P \in U_i$ we have:

$$\psi' : \{\text{Irreducible, closed nbhds } V \subseteq U_i \text{ of } P\} \to \{\text{Irreducible, closed nbhds } V \subseteq X \text{ of } P\}$$
$$V \mapsto \mathrm{Cl}_X(V)$$

which is a bijection (proof left to reader). Since U_i is affine this reduces to the previous case.

3.12:

There are three cases to consider. First assume X is a quasi-affine variety. Then $\dim X = \dim \bar{X}$ by Prop 1.10 and $\dim \bar{X} = \dim \mathcal{O}_{\bar{X},P}$ by 3.2c and stalks can be calculated locally so $\dim \mathcal{O}_{\bar{X},P} = \dim \mathcal{O}_{X,P}$.

Say X is a projective variety. Then cover X by affine U_i and note that from Exercise 2.6 we have $\dim X = \dim X_i$. We thus have by 3.2c that $\dim X_i = \dim \mathcal{O}_{X_i,\varphi_i(P)}$ and again stalks can be calculated locally so $\dim \mathcal{O}_{X_i,\varphi_i(P)} = \dim \mathcal{O}_{X,P}$.

Lastly, say X is a quasi-projective variety. Then by Exercise 2.7b we have $\dim X = \dim \bar{X}$ and so we have reduced to the previous case.

3.13: Define $\mathfrak{m}_Y := \{[(U, f)] \in \mathcal{O}_{Y,X} \mid f \restriction_Y = 0\}$. We claim this is the unique maximal ideal of $\mathcal{O}_{Y,X}$. Let $[(U, f)] \in \mathcal{O}_{Y,X}$ which is not an element of $\mathfrak{m}_Y$, then there exists some $y \in Y$ such that $f(y) \neq 0$, let $V_y \ni y$ be an open neighbourhood of y such that $f = f_1/f_2$ in V_y. Then $V_y \cap Y \cap Z(f_2)^c$ is an open set containing y and so in particular is non-empty. Thus $[(V_y \cap Y \cap Z(f_2)^c, f_2/f_1)]$ is inverse to $[(U, f)]$.

There is a ring homomorphism $\mathcal{O}_{Y,X} \longrightarrow K(Y)$ such that $[(U, f)] \mapsto [(U \cap Y), f \restriction_{U\cap Y}]$. Say we have a representative (U, f) of an element $[(U, f)] \in K(Y)$. There exists an open subset $U' \subseteq U$ on which $f = f_1/f_2$ with f_2 nowhere zero on U'. $U' = U'' \cap Y$ for some open subset $U'' \subseteq Y$ and so f extends to a regular function $\hat{f}$ on the open subset $U'' \cap Z(f_2)^c$ of X. The element $[(U'' \cap Z(f_2)^c, \hat{f})]$ maps to $[(U, f)]$ and so this map is surjective. The kernel is $\mathfrak{m}_Y$ and so we have $\mathcal{O}_{Y,X}/\mathfrak{m}_Y \cong K(Y)$.

For the dimension claim, we cover X with open affines and appeal to Exercise 2.6 and Proposition 1.10 to reduce to the case where X is affine. We use Proposition 1.10 again to replace Y with $\bar{Y}$ which is to say we can assume Y is also affine.

First notice that there is a projection map $A(X) \longrightarrow A(Y)$ with kernel $\mathfrak{m}_Y$ and so $A(X)/\mathfrak{m}_Y \cong A(Y)$, so in particular $\dim A(X)/\mathfrak{m}_Y = \dim Y$. Next we have $\mathrm{ht.}\,\mathfrak{m}_Y + \dim A(X)/\mathfrak{m}_Y = \dim A(X)$, and so $\mathrm{ht.}\,\mathfrak{m}_Y = \dim X - \dim Y$. It remains to show $\mathrm{ht.}\,\mathfrak{m}_Y = \dim \mathcal{O}_{Y,X}$ but this follows from $\mathcal{O}_{Y,X}/\mathfrak{m}_Y \cong K(Y)$ just established.

3.15:

a) Let $X \times Y = Z_1 \cup Z_2$ with Z_i closed. Write $Z_i = Z(\mathfrak{a}_i)$ where the $\mathfrak{a}_i$ are ideals in $k[x_1, ..., x_n]$ and $k[x_1, ..., x_m]$ respectively.

Consider $X_i := \{x \in X \mid \{x\} \times Y \subseteq Z_i\}$. First we show $X_1 \cup X_2 = X$. Let $\alpha \in X$ and consider the sets $Y_i^\alpha = \{y \in Y \mid (\alpha, y) \in Z_i\}$. These are closed as $Y_i^\alpha = Z(\mathrm{ev}_\alpha\, \mathfrak{a}_i)$ where $\mathrm{ev}_\alpha\, \mathfrak{a}_i := \{f(\alpha, y) \mid f \in \mathfrak{a}_i\}$. Since Y is irreducible we have $Y_1^\alpha = Y$ say, and so $\alpha \in X_1 \subseteq X_1 \cup X_2$.

Now we show that X_i are closed. This is easy as $X_i = Z(\cup_{\beta \in Y} \mathrm{ev}_\beta\, \mathfrak{a}_i)$. Thus $X_1 = X$ say (as X is irreducible) and so $X \times Y = Z_1$.

b) We show that $A(X \times Y)$ along with the obvious projection maps satisfy the universal property of the coproduct in the category of commutative k-algebras.

Assume given maps $\varphi_1 : A(X) \longrightarrow B$ and $\varphi_2 : A(Y) \longrightarrow B$ where B is some k-algebra. Let $\psi : A(X \times Y) \longrightarrow B$ be the map satisfying $[x_i] \mapsto \varphi_1([x_i])$ for $i \leq n$ and $[x_i] \mapsto \varphi_2([x_i])$ if $i > n$. This is well defined as if $f \in I(X \times Y)$ then for each monomial $[x_1^{j_1}...x_{n_m}^{j_{n+m}}]$ we have

$$f([x_1^{j_1}...x_{n_m}^{j_{n+m}}]) = f([x_1])^{j_1}...f([x_{n_m}])^{j_{n+m}} = \varphi_1[x_1]^{j_1}...\varphi_2[x_{n_m}]^{j_{n+m}} = 0$$

Uniqueness of this map follows from linearity and commutativity with the projection maps. Thus $A(X \times Y) \cong A(X) \otimes_k A(Y)$.
c) Follows from Proposition 3.5 and the previous part.

d) We need:

Lemma 1.3.4. *Let $A \longrightarrow B$ be integral where A, B are k-algebras. Then* $\operatorname{Frac} A \longrightarrow \operatorname{Frac} B$ *is algebraic.*

Proof. Let $a/b \in \operatorname{Frac} A$ and $f = x^n + \sum_{j=0}^{n-1} \alpha_j x^j \in k[x]$ such that $f(a) = 0$. Then

$$0 = (1/b^n)(a^n/1) + (1/b^n)\sum_{j=0}^{n-1} \alpha_j (a^j/1) = (a/b)^n + \sum_{j=0}^{n-1} \alpha_j/b^{n-j}(a/b)^j$$

□

This problem reduces to proving $\dim(A \otimes_k B) = \dim A + \dim B$ for finiately generated k-integral domains A, B. (Notice that we know $A \otimes_k B$ is an integral domain by part b). Using Noether Normalisation there exists sets of algebraically independent elements $\gamma_1, ..., \gamma_r \in A$ and $\delta_1, ..., \delta_s \in B$ with $\dim A = r$ and $\dim B = s$ such that A is a finitely generated $k[\gamma_1, ..., \gamma_r]$-module and B is a finitely generated $k[\delta_1, ..., \delta_s]$-module. The map determined by

$$\begin{aligned} k[x_1, ..., x_r, y_1, ..., y_s] &\longrightarrow A \otimes_k B \\ x_i &\mapsto \gamma_i \otimes_k 1 \\ y_i &\mapsto 1 \otimes_k \delta_i \end{aligned}$$

is injective. Thus we have an $(r + s)$-variable polynomial subalgebra of $A \otimes_k B$. It remains to show that $\operatorname{tr.deg}_k(A \otimes_k B) = r + s$. Since $A \otimes_k B$ is an integral domain (see the comment at the start of this proof), we reduce to showing $k[\{\gamma_i \otimes_k 1\}, \{1 \otimes_k \delta_i\}] \longrightarrow A \otimes_k B$ is an integral extension, in fact we show it is a finite morphism. We know that all products of all powers of elements in $\{\gamma_i \otimes_k 1\} \cup \{1 \otimes_k \delta_i\}$ form a generating set for $A \otimes_k B$, it remains to show that a finite subset will do. The modules A and B over $k[\gamma_1, ..., \gamma_r]$ and $k[\delta_1, ..., \delta_s]$ are finite, thus for all pairs (γ_i, δ_j) there exists a least integer n_{ij} such that $\gamma_i^{n_{ij}}$ and $\delta_j^{n_{ij}}$ can both be written as a linear combination of products of powers of the γ_i and δ_i respectively with powers less than n_{ij}. Thus finitely many elements generate all elements of the form $(\gamma_i \otimes_k \delta_j)^n$. Thus finitely many elements generate all products of such elements. Thus finitely many elements generate all of $A \otimes_k B$.

3.16:
a), b) Both a) and b) follow from the following observation: let $X = Z(\mathfrak{a})$, $Y = Z(\mathfrak{b})$, $(P_1, P_2) \in X \times Y$, $(f_1, f_2) \in \mathfrak{a} \times \mathfrak{b}$. Then write $f_1(x_0, ..., x_n) f_2(y_0, ..., y_m)$ as $\sum_{i=0}^n \sum_{j=0}^m \alpha_{ij} x_i y_j$. Define $g(\{z_{ij}\}) = \sum_{i=0}^n \sum_{j=0}^m \alpha_{ij} z_{ij}$. We have $f_1(P_1) f_2(P_2) = 0$ if and only if $g(\psi(P_1, P_2)) = 0$.

3.17:
a) By Exercise 3.3b) it suffices to consider an isomorphic variety. By Exercise 3.1c we know that every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ so it suffices to show this is normal. Indeed $\mathcal{O}_{\mathbb{P}^1, P} \cong k[x_0, x_1]_{(\mathfrak{m}_P)}$ which is normal if $k[x_0, x_1]$ is. Indeed $k[x_0, x_1]$ is normal as it is a UFD.

b) Attempt at a direct approach: First notice that $(x_0 x_1 - x_2 x_3)$ is prime and so

$$S(Q_1)_{(\mathfrak{m}_p)} \cong k[x_0, x_1, x_2, x_3]/(x_0 x_1 - x_2 x_3)_{(\mathfrak{m}_P)} \tag{9}$$

Let $f \in S(Q_1)_{(\mathfrak{m}_p)}[X]$ by a monic polynomial and $g \in S(Q_1)_{(0)}$ be such that $f(g) = 0$. We write $g = g_1/g_2$ with $g_2 \neq 0$ so that:

$$f(g) = (g_1/g_2)^n + \sum_{j=0}^{n-1} \alpha_j (g_1/g_2)^j = 0$$

We clear denominators to obtain

$$-g_1^n = \sum_{j=0}^{n-1} \alpha_j g_2^{n-j} g_1^j = g_2 \sum_{j=0}^{n-1} \alpha_j g_2^{n-j-1} g_1^j$$

and so $g_2(P) = 0 \Rightarrow g_1(P) = 0$. It thus remains to show $g_1(P) \neq 0$ and to show this we claim $g_1(P) = 0 \Rightarrow g_1 = 0$, that is $g_1 \in (x_0x_1 - x_2x_3)$ (by sloppy notation). Incomplete.

c) We claim this variety is not normal at the point $P = (0, 0)$. We need to come up with a monic polynomial $f \in A(y^2 - x^3)_{\mathfrak{m}_P}[X]$ and $a \in \operatorname{Frac} A(y^2 - x^3)$ such that $f(a) = 0$, with $a \notin A(y^2 - x^3)_{\mathfrak{m}_P}[X]$. Take $f = X^2 - x^2$ and $a = y/x$, we have

$$f(a) = a^2 - x^2 = y^2/x^2 - x^2 = y^2/x^2 - x^3/x = (y^2 - x^3)/x = 0$$

3.21:
a) This reduces to showing that for polynomials $f_1, f_2 \in k[x]$ we have $f_1(-x)/f_2(-x)$ is a quotient of polynomials.
b) This reduces to showing that for polynomials $f_1, f_2 \in k[x]$ we have $f_1(x^{-1})/f_2(x^{-1})$ is a quotient of polynomials which is true as this equals $x^n f_3(x)/f_4(x)$ for polynomials $f_3, f_4 \in k[x]$.
c) Given $\varphi_1, \varphi_2 \in \operatorname{Hom}(X, G)$ we define $\varphi_1 \cdot \varphi_2 : X \to G$ to have action on $x \in X$ given by $\varphi_1(x) \cdot \varphi_2(x)$.
d) We know $\operatorname{Hom}(X, \mathbb{A}^1) \cong \operatorname{Hom}(k[x], \mathcal{O}(X)) \cong \mathcal{O}(X)$ so it remains to show this is a group homomorphism which is an easy check.
e) Similar to d).

1.4 §4

4.1 Let h be the function described by the question. Let $P \in U \cup V$ and assume without loss of generality that $P \in U$. Since f is regular on U there exists an open neighbourhood $V \subseteq U$ of P for which $f \restriction_V = f_1/f_2$, with f_2 nowhere zero on V. This same neighbourhood $V \subseteq U \subseteq U \cup V$ can be taken to show that h is regular at P.

4.2 First we show the same claim for morphisms. Let X, Y be varieties, $U_1, U_2 \subseteq X$ be open subsets of X and let $\varphi_i : U_i \longrightarrow Y$, $i = 1, 2$, be morphisms of varieties which agree on $U_1 \cap U_2$. Let h denote the function which is equal to φ_i on U_i. Say $V \subseteq Y$ is an open subset and $\gamma : V \longrightarrow k$ a regular function. We obtain regular functions $\gamma \circ \varphi_i : U_i \longrightarrow k$ which glue to a regular function $U_1 \cup U_2 \longrightarrow k$ by the previous question. Thus h is a morphism.

The question at hand reduces to this previous considering by picking representatives of the two rational maps.

4.3:
a) This function is defined on U_0 and the corresponding regular function is given by the same rule.

b) This extends to

$$\mathbb{P}^2 \setminus \{[0:0:1]\} \longrightarrow \mathbb{P}^1, [P_0, P_1, P_2] \longmapsto [P_0, P_1]$$

This cannot be extended further lest $[0:0:1] \mapsto [0:0] \notin \mathbb{P}^1$.

1.5 §5

5.9: Using Exercise 2.5b we write $Z(f) = Z(f_1) \cup \ldots \cup Z(f_r) = Z(f_1 ... f_r)$, assume that $r > 1$. Now, using exercise 3.7 we have that $Z(f_1) \cap Z(f_2) \neq \varnothing$, so let $P \in Z(f_1) \cap Z(f_2)$. We have:

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x}(f_2 \ldots f_r) + \ldots + (f_1 \ldots f_{r-1})\frac{\partial f_r}{\partial x} \tag{10}$$

Evaluating (10) at P yields the value 0. Likewise, $\frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$, contradicting the hypothesis. Thus $r = 1$.

2 Chapter 2

2.1 §1

The question labelling is taken from [1, II §1]

1.1:

We denote the constant presheaf associated to A by C_A and the constant sheaf $\mathscr{A}$. We construct a third sheaf $\mathscr{F}$ and show $C_A^+ \cong \mathscr{F} \cong \mathscr{A}$.

For an open set U with connected components $\{U_i\}_{i \in I}$ define $\mathscr{F}(U) = \coprod_{i \in I} A$. Let $V \supseteq U$ is an open superset of U with connected components $\{V_j \in J\}_{j \in J}$. There is a collection of maps $\varphi_{ij} : \mathscr{F}(V_j) = A \to A = \mathscr{F}(U_i)$ which is the identity if $U_i \subseteq V_j$ and the zero map otherwise. Composing these with the inclusions $\mathscr{F}(U_i) \rightarrowtail \mathscr{F}(U)$ induces a morphism $\mathscr{F}(V) \to \mathscr{F}(U)$ which we take as the restriction map corresponding to $U \subseteq V$. This is clearly a sheaf.

To see that $\mathscr{F} \cong \mathscr{A}$, notice that a function $s : U \to A$ in $\mathscr{A}(U)$ is clearly equivalent to giving an element of A for each connected component of U.

To see that $C_A^+ \cong \mathscr{F}$ let U be a connected open subset and s an element of $C_A^+(U)$. There exists a cover of opens $\{U_i\}_{i \in I}$ and elements $a_i \in A$ such that if $u \in U_i$ then $s(u) = (a_i)_u$. For all $U_i \cap U_j \neq \varnothing$ we have $a_i = a_j$ and U is connected, so the data of s amounts to a single element $a \in A$.

1.2a:

By essential uniquenes of colimits it suffices to show that $\operatorname{im} \varphi_p$ is a colimit $\operatorname{Colim}_{U \ni p} \operatorname{im} \varphi^+(U)$. Let $s \in \operatorname{im} \varphi^+(U)$ and take $V \ni p$ and $t \in \operatorname{im} \varphi(U)$ to be such that for all $v \in V$ we have $s(v) = t_v$. Then the equivalence class $[(V, t)]$ gives an element of $\operatorname{im} \varphi_p$ and so we have a collection of maps $\operatorname{im} \varphi^+(U) \to \operatorname{im} \varphi_p$. Thus $\operatorname{im} \varphi_p$ is a cocone. Now say that K were any abelian group and there was a collection of morphisms $\psi_U : (\operatorname{im} \varphi^+)U \to K$ coherent with the restriction morphisms. Coherency here ensures that the image of any lift $t \in \operatorname{im} \varphi(V)$ of any $[(V, t)] \in \operatorname{im} \varphi_p$ under $\operatorname{im} \varphi(U) \longrightarrow \operatorname{im} \varphi^+(U) \longrightarrow K$ is mapped to the same element. That is, there is a well defined morphism $\operatorname{im} \varphi_p \to K$, which indeed is unique.

1.2b

This follows easily from the definition of monomorphism/epimorphism combined with the fact that for any pair of morphisms $\gamma, \gamma' : \mathscr{H} \to \mathscr{J}$ subject to $\gamma_p = \gamma'_p$ for all p then $\gamma = \gamma'$.

1.2c

Essentially an application of the previous two parts. The forward direction is by 1.2a: taking stalks at p at all parts of the diagram yields a sequence

$$\ldots \longrightarrow \mathscr{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathscr{F}_p^i \xrightarrow{\varphi_p^i} \mathscr{F}_p^{i+1} \longrightarrow \ldots$$

Since $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$ it follows that $\ker \varphi_p^i \cong (\ker \varphi^i)_p = (\operatorname{im} \varphi^{i-1})_p \cong \operatorname{im} \varphi_p^{i-1}$.

The converse is by 1.2b: since $(\ker \varphi^i)_p \cong (\operatorname{im} \varphi^{i-1})_p$ for all p, we have that $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$.

2.2 §2

2.1

Let $l : A \to A_f$ be the localisation map, and $\hat{l} : \operatorname{Spec} A_f \to \operatorname{Spec} A$ the induced map on spectrum. This map is continuous and open, and thus is a homeomorphism onto its image, which is $D(f)$, from now on, $\hat{l}$ will refer to this homeomorphism.

Since basic opens form a topology and $\mathcal{O}_X \restriction_{D(f)}$ and $\mathcal{O}_{\operatorname{Spec} A_f}$ are both sheaves, it suffices to specify $\hat{l}^{\#}$ it suffices to define $\hat{l}^{\#}D(gf)$ for each basic open $D(gf)$ of $D(f)$. To do this, we first observe that

$$\mathcal{O}_X \restriction_{D(f)} D(g) = \mathcal{O}_X(D(fg)) \cong A_{fg}$$

and

$$\mathcal{O}_{\operatorname{Spec} A_f} \hat{l}_*(D(g)) = \mathcal{O}_{\operatorname{Spec} A_f}(\hat{l}^{-1}(D(g))) = \mathcal{O}(D(g/1)) \cong (A_f)_{g/1}$$

so it suffices to give a local ring isomorphism $A_{fg} \to (A_f)_{g/1}$. We define such a map $\frac{a}{f^n g^m} \mapsto \frac{a}{f^n}/\frac{g^m}{1}$.

2.4

Let $\varphi \in \operatorname{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X))$, we define a corresponding morphism of schemes $\beta(\varphi) = (\psi, \psi^{\#})$. Fix an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X and for each pair (i,j) let $\{U_k^{ij} = \operatorname{Spec} A_k^{ij}\}$ be open affines covering $U_i \cap U_j$. By Proposition [1, 2.3] the ring homomorphisms

$$\varphi_i : A \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\operatorname{Res}_{U_i}^X} A_i$$

give rise to a family of morphisms $(\gamma_i, \gamma_i^{\#})$ of schemes $\operatorname{Spec} A_i \to \operatorname{Spec} A$.

Since $\operatorname{Res}_{U_k^{ij}}^{U_i} \varphi_i = \operatorname{Res}_{U_k^{ij}}^{U_j} \varphi_j$ and the U_k^{ij} cover $U_i \cap U_j$ we have that $\gamma_i \restriction_{U_i \cap U_j} = \gamma_j \restriction_{U_i \cap U_j}$, thus we have a well defined continuous function $\psi : X \to \operatorname{Spec} A$.

Now we define $\psi^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to \psi_* \mathcal{O}_X$ for which by the sheaf condition on $\mathcal{O}_X$ it suffices to give a family $\psi_i^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to \psi_* \mathcal{O}_X \to \operatorname{Res}_{U_i}^X \psi_* \mathcal{O}_X$ such that $\operatorname{Res}_{U_i \cap U_j}^{U_i} \psi_i^{\#} = \operatorname{Res}_{U_i \cap U_j}^{U_j} \psi_i^{\#}$. However this is exactly given by the $\gamma_i^{\#}$.

2.7

Let $(f, f^{\#}) : \operatorname{Spec} K \to X$ be a morphism of schemes. Write $x := f((0))$. We have a ring homomorphism $f_x^{\#} : \mathcal{O}_{X,x} \longrightarrow K_{(0)} \cong K$. This is a local ring homomorphism and so $\left(f_x^{\#}((0))\right)^{-1} = \ker(f_x^{\#}) = \mathfrak{m}_x$ and so we have a homomorphism $k(x) \longrightarrow K$ which being a ring homomorphism with domain a field, is injective.

Conversely, a point $x \in X$ is equivalent to a continuous function $f : \operatorname{Spec} K \to X$. Given an open subset $U \subseteq X$ which does not contain x the function $f_U^{\#} : \mathcal{O}_X(U) \to f_* \mathcal{O}_{\operatorname{Spec} K} U = \mathcal{O}_{\operatorname{Spec} K}(\varnothing) = 0$ is the unique such. If $x \in U$ then we have the function $f_U^{\#} : \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \to K \cong \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) = \mathcal{O}_{\operatorname{Spec} K}(f_*(U))$.

2.16

a)

Let $\varphi : U \longrightarrow \operatorname{Spec} B$ be an isomorphism. For all x we have an isomorphism $\mathcal{O}_{X,x} \cong B_{\varphi(x)}$. Thus $f_x \notin \mathfrak{m}_x \Leftrightarrow \bar{f} \notin \varphi(x)$ and so $U \cap X_f \cong D(\bar{f})$.

b)

Let $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$ be a finite open affine cover of X. From part (a) we know $X_f \cap U_i = D(f_i)$, where f_i is the image of f under $A \longrightarrow A_i$, thus $a \restriction_{D(f_i)} = 0$ for all i, that is, $\frac{a \restriction_{U_i}}{1} = 0$ in $(A_i)_{f_i}$. Thus there exists $n_i > 0$ such that $f_i^{n_i} a \restriction_{U_i} = 0$. Since there are finitely many U_i we can set $n = \max_i n_i$ so that for each i we have $f_i^n a \restriction_{U_i} = 0$. We then have by the sheaf condition that $f^n a = 0$.

c)

We need to define an element $a \in \Gamma(X, \mathcal{O}_X)$, we do this by defining an element of A_i for each i which agree on the overlaps. Consider $b \restriction_{X_f \cap U_i}$ for each i. We know that $X_f \cap U_i = D(f \restriction_{U_i})$ so we can write $b \restriction_{X_f \cap U_i} = \frac{a_i}{f \restriction_{U_i}^{n_i}} \in (A_i)_{f \restriction_{U_i}}$. Since there are finitely many U_i we can write $n = \sum_i n_i$ and let $b_i = f \restriction_{U_i}^{n-n_i} a_i \in A_i$.

Let $W_{ij} = X_f \cap U_i \cap U_j$ and notice that

$$(b_i - b_j) \restriction_{W_{ij}} = (f^{n-n_i} f^{n_i} b - f^{n-n_j} f^{n_j} b) \restriction_{W_{ij}} = 0$$

So by part (b) there is $d_{ij} > 0$ such that $f^{d_{ij}}(b_i - b_j) \restriction_{U_i \cap U_j} = 0$ as an element of $\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$. Letting $d = \max_{i,j}\{d_{ij}\}$ we have $f^d(b_i - b_j) \restriction_{U_i \cap U_j} = 0$, so by the sheaf condition there is an element $a \in \Gamma(X, \mathcal{O}_X)$ such that $a \restriction_{U_i} = f^d b_i$ and $a \restriction_{X_f} = b$.

2.17

a)

The collection of continuous functions $(f \restriction_{U_i})^{-1} : U_i \longrightarrow f^{-1}(U_i) \rightarrowtail X$ agree on overlaps as they are the inverse of restrictions of a common function. Thus we obtain a continuous function $Y \to X$ which is locally an inverse and thus an inverse to f.

Let g_i denote the inverse of $f^\# \restriction_{U_i} : \mathcal{O}_Y \restriction_{U_i} \longrightarrow f_*\mathcal{O}_X \restriction_{U_i}$. We need to show that $(g_i)_{U_i \cap U_j} = (g_j)_{U_i \cap U_j}$. Both of these maps are equal to $(f^\# \restriction_{U_i \cap U_j})^{-1}$ so we are done.

Notice that a corollary of the proof of this exercise is the following:

Lemma 2.2.1. *Let $\{U_i\}$ be an open cover of Y and $f_i : X \restriction_{f^{-1}(U_i)} \to Y \restriction_{U_i}$ a collection of scheme morphisms such that $(f_i) \restriction_{U_i \cap U_j} = (f_j) \restriction_{U_i \cap U_j}$. Then there exists a morphism $f : X \to Y$ such that $f \restriction_{U_i} = f_i$. Moreover, f is an isomorphism if and only if all the f_i are.*

b)

For any sheaf X there is the unit map $X \longrightarrow \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$. This morphism is an isomorphism if X is affine, thus we have a collection of isomorphisms $X_{f_i} \longrightarrow \operatorname{Spec}\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$. Since $f_1, ..., f_r$ generate 1 we have that $\operatorname{Spec} X_{f_i}$ cover $\operatorname{Spec} X$. The result then follows from part (a).

2.18b)

We let $\hat{\varphi} : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ denote the continuous map induced by $\varphi : A \longrightarrow B$. Assume that φ is injective. As the collection $\{D(f)\}_{f \in A}$ form a base for the topology on $\operatorname{Spec} A$, it suffices to show that for all $f \in A$, the morphism $\hat{\varphi}^\#_{D(f)} : \mathcal{O}_{\operatorname{Spec} A} D(f) \longrightarrow \hat{\varphi}_* \mathcal{O}_{\operatorname{Spec} B} D(f) = \mathcal{O}_{\operatorname{Spec} B} D(\varphi(f))$ is injective. Let $f \in A$. It's easy to show that since $\varphi : A \longrightarrow B$ is injective, so is $\varphi_f : A_f \longrightarrow B_{\varphi(f)}$. Thus it remains to show commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A} D(f) \arrow[r, "\hat{\varphi}^\#_{D(f)}"] & \mathcal{O}_{\operatorname{Spec} B} D(\varphi(f)) \\
A_f \arrow[u, "\cong"] \arrow[r, "\varphi_f"'] & B_{\varphi(f)} \arrow[u, "\cong"']
\end{tikzcd}$$

Which can be established by a direct calculation.

2.3 §3

Exercise 1. *Hartshorne 3.1*

Proof. We use the following fact from commutative algebra:

Lemma 2.3.1. *Let A, B be rings and $f \in A$ an element of A. Then B is a finitely generated A-algebra if and only if it is a finitely generated A_f-algebra. (Note: we mean finitely generated as* algebras*, the corresponding statement for modules is false)*

Throughout, a *cover* of an open set U means a collection of open subsets $\{U_i \subseteq U\}_{i \in I}$ of U such that $\bigcup_{i \in I} U_i =$ U. For an open affine subset $U = \operatorname{Spec} A$ of Y let $P(U)$ be the proposition "there exists a cover $\{\operatorname{Spec} B_i\}_{i \in I}$ of $f^{-1}(U)$ such that each B_i is a finitely generated A-algebra". Let $\{U_i = \operatorname{Spec} A_i\}_{i \in I}$ be an open affine cover of Y such that $P(U_i)$ holds for each i, and let $U = \operatorname{Spec} A$ be an open affine subset of Y. First we show that U can be covered by open affines $\{U_i\}_{i\in I}$ satisfying $P(U_i)$ for each i.

Fix $i \in I$, let $\{\operatorname{Spec} B_{ij}\}_{j\in J}$ be a cover of $f^{-1}(U_i)$ such that each B_{ij} is a finitely generated A_i-algebra, and let $a_i \in A_i$ be such that $D(a_i) \subseteq U_i$. Let $\varphi_{ij} : A_i \to B_{ij}$ be the ring homomorphism corresponding to the scheme morphism $\operatorname{Spec} B_{ij} \to \operatorname{Spec} A_i$. B_{ij} is a finitely generated A_i-algebra, so by Lemma 2.3.1, $B_{ij,\varphi_{ij}(a_i)}$ is a finitely generated A_i-algebra. The collection $\{\operatorname{Spec} B_{ij,\varphi_{ij}(a_i)}\}$ cover $f^{-1}(D(a_i))$ and so proposition $P(D(a_i))$ holds.

We now have the following statement to prove: let $U = \operatorname{Spec} A \subseteq Y$ be an open affine subset of Y which can be covered by open affines $U_i = \operatorname{Spec} A_i$ such that $P(U_i)$ holds for all i, then $P(U)$ holds. But this follows easily from Lemma 2.3.1. □

Exercise 2 (Hartshorne 3.14)**.** *Let X be a scheme of finite type over a field k. Then the closed points of X are dense.*

Proof. We cover X by finitely many open affines $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$ where each A_i is a finitely generated k-algebra. Notice that by Theorem **??** each A_i is jacobson. Fix an i and let $f \in A_i$ be such that $D(f) \subseteq U_i$. Assume that x is closed in $D(f)$, that is, x is a maximal ideal of $(A_i)_f$. We show first that x is closed in X. The inclusion $D(f) \subseteq \operatorname{Spec} A_i$ induces a ring homomorphism $A_i \to (A_i)_f$ which in fact is a k-algebra homomorphism as X is over k. Combining this with the fact that $(A_i)_f$ is a finitely generated k-algebra gives that $(A_i)_f$ is a finitely generated A_i-algebra and so the preimage of x in A_i is maximal, by Theorem **??**. This holds for any i, and so x is closed in all $U_i \ni x$, and thus is closed in X (this step here doesn't seem to require that there were finitely many such U_i). It thus suffices to show that every $D(f)$ contains a maximal ideal. If f is contained in every maximal ideal then it is nilpotent (Lemma **??**) and thus $D(f)$ is empty. □

References

[1] Hartshorne

[2] Notes on Algebraic Geometry *Troiani*.

[3] Varieties *Troiani* Fix these references