We now have a notion of *proof* and a notion of *truth*. The obvious question to ask is: which of the provable formulas are true, and which of the true formulas are provable?

Theorem 0.0.1. *Let ϕ be a formula other than $\bot$. If $\Gamma \vdash \phi$ then $\Gamma \models \phi$.*

Proof. Let π be a proof of γ with set of hypotheses Γ. We proceed by induction on the height of π. If π has height 0, then π consists of a single assumption and single conclusion γ. This implies $\gamma \in \Gamma$. Thus, if $\mathcal{I} \models \Gamma$ then in particular $\mathcal{I} \models \gamma$.

Now say that π has heigh $n > 0$ and the result holds for all proofs π' with height $k < n$. We proceed by cases on the deduction rule of π. Most of these cases are trivial. For instance, say the final rule is $\wedge I$ so that $\gamma = \phi \wedge \psi$ for some ϕ, ψ.

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \phi\end{matrix} \qquad \begin{matrix}\pi_1 \\ \vdots \\ \phi\end{matrix}}{\phi \wedge \psi} \wedge I$$

By definition, $\Gamma \models \phi \wedge \psi$ if for all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \Gamma$ we have $\mathcal{I} \models \phi \wedge \psi$, in other words, $\mathcal{I}_\nu(\phi \wedge \psi) = 1$ for all valuations ν. By Definition **??** this holds if and only if $\mathcal{I}_\nu(\phi) = \mathcal{I}_\nu(\psi) = 1$ which holds by the inductive hypothesis.

The cases when the final rule is $\wedge E1, \wedge E2, \vee I1, \vee I2$ are similarly simple.

Say the final rule is $\wedge E^{i,j}$ for some i, j

$$\dfrac{\begin{matrix}\pi' \\ \vdots \\ \phi \vee \psi\end{matrix} \qquad \begin{matrix}[\phi]^i \\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\psi]^j \\ \vdots \\ \gamma\end{matrix}}{\gamma} \wedge E^{i,j}$$

Say $\mathcal{I}$ satisfies $\mathcal{I} \models \Gamma$ and let ν be an arbitrary valuation. By the existence of π' we have $\Gamma \vdash \phi \vee \psi$ and so by the inductive hypothesis $\Gamma \models \phi \vee \psi$, which is to say $\mathcal{I}(\phi \vee \psi) = 1$. Thus either $\mathcal{I}_\nu(\phi) = 1$ or $\mathcal{I}_\nu(\psi) = 1$. We also have that $\Gamma \cup \{\phi\} \vdash \gamma$ and so $\Gamma \cup \{\phi\} \models \gamma$. Thus, if $\mathcal{I}_\nu(\phi) = 1$ then $\Gamma \cup \{\phi\} \models \gamma$ implies $\mathcal{I}_\nu(\gamma) = 1$. Otherwise, we must have $\mathcal{I}_\nu(\psi) = 1$ and then $\Gamma \cup \{\psi\} \models \gamma$ implies $\mathcal{I}_\nu(\gamma) = 1$. It follows that $\Gamma \models \gamma$.

Say the final rule of π is $\Rightarrow I^i$ so that $\gamma = \phi \Rightarrow \psi$ for some ϕ, ψ and let $\mathcal{I}$ is such that $\mathcal{I} \models \phi$.

$$\dfrac{\begin{array}{c}[\phi]^i\\ \vdots\\ \psi\end{array}}{\phi \Rightarrow \psi} \Rightarrow I^i$$

We need to show $\mathcal{I}_\nu(\phi \Rightarrow \psi) = 1$ for each evaluation ν which amounts to showing that either $\mathcal{I}_\nu(\phi) = 0$ or $\mathcal{I}_\nu(\psi) = 1$. If $\mathcal{I}_\nu(\phi) \neq 0$ then $\mathcal{I}_\nu(\phi) = 1$. Thus, since $\Gamma \cup \{\phi\} \vdash \psi$ and so $\Gamma \cup \{\phi\} \models \psi$ we have $\mathcal{I}_\nu(\psi) = 1$.

Say the final rule of π is $\Rightarrow E$.

$$\dfrac{\begin{array}{c}\pi'\\ \vdots\\ \phi \Rightarrow \gamma\end{array} \qquad \begin{array}{c}\pi''\\ \vdots\\ \phi\end{array}}{\gamma} \Rightarrow E$$

We have $\Gamma \vdash \phi \Rightarrow \gamma$ and so $\Gamma \models \phi \Rightarrow \gamma$ which means $\mathcal{I}_\nu(\phi) = 0$ or $\mathcal{I}_\nu(\gamma) = 1$. Since $\Gamma \vdash \phi$ and hence $\Gamma \models \phi$ we have $\mathcal{I}_\nu(\phi) = 1$, which implies $\mathcal{I}_\nu(\gamma) = 1$.

Say the final rule of π is $\neg I^i$.

$$\dfrac{\begin{array}{c}[\phi]^i\\ \vdots\\ \bot\end{array}}{\neg\phi} \neg I^i$$

We need to show that $\mathcal{I}_\nu(\neg\phi) = 1$ which amounts to showing $\mathcal{I}_\nu(\phi) = 0$. We use proof by contradiction. Say $\mathcal{I}_\nu(\phi) = 1$. Since $\Gamma \cup \{\phi\} \vdash \bot$ we have $\Gamma \cup \{\phi\} \models \bot$ which implies $\mathcal{I}_\nu(\bot) = 1$, contradicting Definition **??**.

Now say the last rule is $\forall I$.

$$\dfrac{\begin{array}{c}\pi'\\ \vdots\\ \phi[x := y]\end{array}}{(\forall x : C)\phi} \forall I$$

Say $\Gamma \not\models (\forall x : C)\phi$. Let ν be a valuation such that $\mathcal{I}_\nu((\forall x : C)\phi) = 0$. Then there exists some $d \in \mathcal{I}(C)$ such that $\mathcal{I}_{\nu(x \mapsto d)}(\phi) = 0$. This means $\mathcal{I}_{\nu(y \mapsto d)}(\phi) = 0$, which means $\mathcal{I} \not\models \phi[x := y]$.

Say the last rule is $\forall E$.

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ (\forall x : C)\phi\end{array}}{\phi[x := t]}\ (\forall E)$$

Say $\mathcal{I} \not\models \phi[x := t]$ so there exists a valuation ν such that $\mathcal{I}_\nu(\phi[x := t]) = 0$. Then $d := \mathcal{I}_\nu(t)$ is some value in $\mathcal{I}(C)$ and we see $\mathcal{I}_{\nu(x \mapsto d)}(\phi) = 0$ and so $\mathcal{I} \not\models (\forall x : C)\phi$.

Say the last rule is $\exists I$.

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \phi[x := t]\end{array}}{(\exists x : C)\phi}\ \exists I$$

If ν is a valuation such that $\mathcal{I}_\nu \models \phi[x := t]$ then $\mathcal{I}_{\nu(x \mapsto \mathcal{I}_\nu(t))}(\phi) = 1$. Thus $\mathcal{I} \models (\exists x : C)\phi$.

Say the last rule is $\exists E^i$.

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ (\exists x : C)\phi\end{array} \qquad \begin{array}{c}[\phi[x := y]]^i\\ \vdots\\ \gamma\end{array}}{\gamma}\ \exists E^i$$

□

Theorem 0.0.2. *Let $\mathbb{T}$ be a first order theory, that is, a set of formulas in some first order language. There exists an interpretation $\mathcal{I}$ so that $\mathcal{I} \models \mathbb{T}$ if and only if for every finite subset $\mathbb{T}' \subseteq \mathbb{T}$ there exists an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \models \mathbb{T}'$.*

Proof. For convenience, if a theory $\mathbb{S}$ admits an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathbb{S}$ we will say that $\mathbb{S}$ **admits a model**.

We prove the contrapositive. Assume that $\mathbb{T}$ does not admit a model.

By the Completeness Theorem we have that $\mathbb{T}$ is inconsistent. Let A denote a formula such that $\mathbb{T} \vdash A$ and $\mathbb{T} \vdash \neg A$. Let π, π' respectively be proofs of $A, \neg A$. Since π, π' are finite there exists finite subsets $\mathbb{T}', \mathbb{T}'' \subseteq \mathbb{T}$ so that $\mathbb{T}' \vdash A$ and $\mathbb{T}'' \vdash \neg A$. This implies that $\mathbb{T}' \cup \mathbb{T}'' \vdash A \wedge \neg A$. Thus the finite subset $\mathbb{T}' \cup \mathbb{T}''$ is inconsistent and thus does not admit a model.

The other direction of the Theorem is trivial. □