1 Bicategory Theory

Often one motivates the notion of a *bicategory* by observing the definition of a category and then "suping up" the hom-sets. From this angle though there are multiple reasonable generalisations, ought associativity hold strictly or up to natural isomorphism?

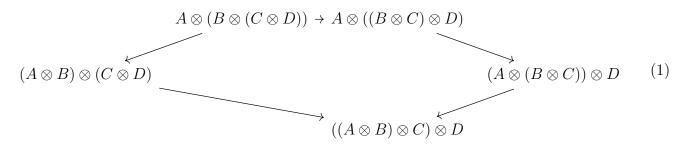
So instead we seek our motivation from a different source, recall:

Definition 1.0.1. A monoidal category consists of

- a category \mathscr{C} ,
- a special object $1 \in \mathscr{C}$,
- a functor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$

along with three natural isomorphisms $\lambda: \mathbb{1} \times id_{\mathscr{C}} \longrightarrow id_{\mathscr{C}}, \rho: id_{\mathscr{C}} \times \mathbb{1} \longrightarrow \mathscr{C}, \alpha: (id_{\mathscr{C}} \times id_{\mathscr{C}}) \times id_{\mathscr{C}} \longrightarrow id_{\mathscr{C}} \times (id_{\mathscr{C}} \times id_{\mathscr{C}})$, with all of this data satisfying:

• the **pentagon diagram**, ie, commutativity for all $A, B, C, D \in \mathcal{C}$ of the following:



• the identity diagrams, ie, commutativity for all $A, B, C \in \mathcal{C}$ of the following:

$$(A \otimes 1) \otimes B \longrightarrow A \otimes (1 \otimes B)$$

$$\downarrow \qquad \qquad \downarrow$$

$$A \otimes B$$

$$(2)$$

We wish to define a *bicategory* so that a monoidal category is a bicategory with a single object. Thus, we see that we take associativity and identity up to natural isomorphism, hence:

Definition 1.0.2. A bicategory \mathscr{C} consists of

- a collection Obj & of **objects**,
- for every pair X, Y of objects, a category

$$\operatorname{Hom}_{\mathscr{C}}(X,Y) \tag{3}$$

whose objects are 1-morphisms with domain X and codomain Y, and whose morphisms are 2-morphisms,

• for every triple X, Y, Z of objects, a functor called **composition**

$$\circ_{X,Y,Z}: \operatorname{Hom}_{\mathscr{C}}(Y,Z) \times \operatorname{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,Z)$$
$$(f,g) \longmapsto f \circ g$$
$$(\alpha: f \Rightarrow g, \beta: h \Rightarrow j) \longmapsto \beta * \alpha$$

• for each tuple of objects X, Y, Z, W a natural isomorphism $\alpha_{X,Y,Z,W}$ from the functor defined by the composite:

 $\operatorname{Hom}_{\mathscr{C}}(Z,W) \times \operatorname{Hom}_{\mathscr{C}}(Y,Z) \times \operatorname{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(Z,W) \times \operatorname{Hom}_{\mathscr{C}}(X,Z) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,W)$ (4) and the functor defined by the composite

$$\operatorname{Hom}_{\mathscr{C}}(Z,W) \times \operatorname{Hom}_{\mathscr{C}}(Y,Z) \times \operatorname{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(Y,W) \times \operatorname{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,W) \quad (5)$$

- for every object $X \in \mathscr{C}$ a functor $\mathbb{1}_X : \mathbf{1} \longrightarrow \operatorname{Hom}(X, X)$, where $\mathbf{1}$ is the category with a single object and single morphism,
- for every pair of objects X, Y a pair of natural isomorphisms, ρ which maps from the functor defined by the composite:

$$\mathbf{1} \times \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(Y, Y) \times \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(X, Y) \tag{6}$$

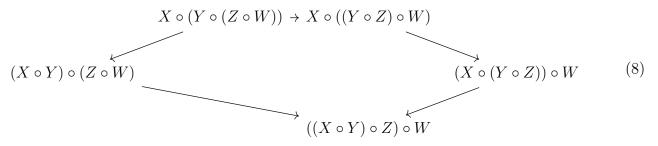
to the functor defined by the composite

$$\mathbf{1} \times \operatorname{Hom}(X, Y) \xrightarrow{\sim} \operatorname{Hom}(X, Y) \tag{7}$$

and λ , which is defined similarly,

satisfying:

• for all objects X, Y, Z, W the following diagram commutes



• for all objects X, Y the following diagram commutes:

$$(X \circ 1) \circ Y \longrightarrow X \circ (1 \circ Y)$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \circ Y$$

$$(9)$$

Remark 1.0.3. The point is the following:

- 1. hom sets have become categories,
- 2. composition has become a functor
- 3. identity morphisms are not an element of a set now, they are an object of a category,
- 4. composition now has the freedom to hold only up to isomorphism,
- 5. associativity only holds up to isomorphism,
- 6. that units act like units only holds up to isomorphism,
- 7. these are 3-ary and 2-ary isomorphisms so we need compatibility diagrams.

2 Landau-Ginzburg models

2.1 Preliminaries