1. Introduction

We provide an application of the internal logic of a topos, which we now explain. Proofs of statements about the topos Sets of sets are sometimes easier to construct than proofs of statements in arbitrary topoi because of the structure of Sets. In particular, the fact that sets contain *elements* can be a very helpful observation. The internal logic of a topos offers this structure to arbitrary topoi as the interal logic admits to each type X a (countably) infinite set of *variables* $x, y, ...$, see Definition **??** for a precise definition. Section 1 shows a concrete application of this structure. The general idea is the following: given a surjective family of functions of sets $\{t_i : A_i \longrightarrow A\}_{i=0}^{\infty}$ and a family of functions of sets $\{g_i : A_i \longrightarrow U\}_{i=0}^{\infty}$ we can define a function $f : A \longrightarrow U$ given by "choosing a lift" $a_i \in A_i$ of $a \in A$ along some t_i and defining $f(a) := g_i(a_i)$, provided that the functions t_i indeed can be suitably glued together. There seems to be no easy way to describe the map $f : A \longrightarrow U$ without using the internal logic. This is the content of Section 1 which indeed is self contained and does not require the content of Section **??**.

2. Surjective families

As another application of the Internal logic of a topos, we show how one can construct lifts along surjective families using the familiar idea from the topos Sets of sets and functions.

Given a surjective family of functions of sets $\{t_i : A_i \to A\}_{i=0}^{\infty}$ and a family of functions $\{g_i : A_i \to U\}_{i=0}^{\infty}$, if

$$\forall a_i \in A_i, \forall a_j \in A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{1}$$

then there exists a well defined function $f : A \to U$ which is given by "choosing a lift" $a_i \in A_i$ of $a \in A$ along some t_i and then defining $f(a) := g_i(a_i)$. In choosing a lift $a_i \in A_i$ of $a \in A$ there are two pieces of information, one is that $t_i(a_i) = a$ and the other is $g_i(a_i)$, which can be captured by the following subset of $A \times U$:

$$\left\{\vec{z} \in A \times U \mid \exists i \in \mathbb{N}, \exists a_i \in A_i, \vec{z} = \big(t_i(a_i), g_i(a_i)\big)\right\} \subseteq A \times U$$

which we denote by X. Thus there is the following diagram

$$\begin{tikzcd}
& X \arrow[d, tail] & \\
& A \times U \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_R"] & \\
A & & U
\end{tikzcd}$$

It then follows from Equation 1 that there exists a bijection $\hat{f} : X \xrightarrow{\sim} A$. The function $\pi_R \hat{f}^{-1}$ is then equal to f. The following Lemma generalises this description to an arbitrary elementary topos $\mathcal{E}$.

Lemma 2.0.1. *Let* $\{t_i(a_i) : A\}_{i=0}^{\infty}$ *be a finite set of terms satisfying the following sequent:*

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, t_i(a_i) = a \tag{2}$$

Let $\{g_i : A_i \to U\}_{i=0}^{\infty}$ *be a set of morphisms in* $\mathcal{E}$*, and assume the following sequent holds for each* i, j*:*

$$\vdash \forall a_i : A_i, \forall a_j : A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{3}$$

Then there exists a (necessarily unique) morphism $f : A \to U$ *such that for each* i*, the following diagram commutes*

$$\begin{tikzcd} A_i \arrow[d, "t_i"'] \arrow[dr, "g_i"] & \\ A \arrow[r, "f"'] & U \end{tikzcd}$$

The significance of Lemma 1.0.1 is emphasised by the fact that in the topos Sets both the coproduct of a collection of functions and the coequaliser of a pair of functions are examples of morphisms satisfying the hypotheses of Lemma 1.0.1 (and thus, so are all colimits) as Example 1.0.2 and 1.0.3 demonstrate.

Example 2.0.2. Say $\{g_i : A_i \to U\}_{i=0}^{\infty}$ is a family of functions, and $\iota_i : A_i \rightarrowtail \coprod_{i=0}^{\infty} A_i$ is the i^{th} inclusion map. Then there exists a unique morphism $f : \coprod_{i=0}^{\infty} A_i \to U$ such that $f\iota_i = g_i$ for each i, and this function f is given by:

$$f : \coprod_{i=0}^{\infty} A_i \to U$$
$$a_i \mapsto g_i(a_i), \text{ where } \iota_i(a_i) = a_i$$

Example 2.0.3. Say $g_0, g_1 : A'' \to A'$ and $e : A' \twoheadrightarrow \mathrm{Coeq}(g_0, g_1)$ is the coequaliser of g_0 and g_1. Then given a morphism $g : A' \to U$ such that $gg_0 = gg_1$ there exists a unique function $f : \mathrm{Coeq}(g_0, g_1) \to U$ such that $fe = g$, and this function f is given by:

$$f : \mathrm{Coeq}(g_0, g_1) \to U$$
$$[a] \mapsto g(a), \text{ where } e(a) = [a]$$

Remark 2.0.4. One might notice that Example 1.0.2 allows for a (countably) infinite disjoint union (and indeed, an even more general situation can be considered), so why does this paper only concern itself with *finite* colimits? The reason is simply because the internal logic only allows for the construction of product terms which are *finite* in size. One could allow for a more general language, however doing so is non-standard and so we omit this level of generality within this paper.

Lemma 2.0.5. *Let* $\{t_i(a_i) : A\}_{i=0}^{\infty}$ *be a finite set of terms satisfying the following sequent:*

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, t_i(a_i) = a \tag{4}$$

Let $\{g_i : A_i \to U\}_{i=0}^{\infty}$ *be a set of morphisms in* $\mathcal{E}$*, and assume the following sequent holds for each* i, j*:*

$$\vdash \forall a_i : A_i, \forall a_j : A_j, t_i(a_i) = t_j(a_j) \Rightarrow g_i(a_i) = g_j(a_j) \tag{5}$$

Then there exists a (necessarily unique) morphism $f : A \to U$ *such that for each* i*, the following diagram commutes*

$$\begin{tikzcd} A_i \arrow[d, "t_i"'] \arrow[dr, "g_i"] & \\ A \arrow[r, "f"'] & U \end{tikzcd}$$

Proof. First, define the following subobject

$$[\![(z : A \times U). \bigvee_{i=0}^{n} (\exists a_i : A_i, z = \langle t_i(a_i), g_i a_i \rangle)]\!] \overset{c}{\rightarrowtail} A \times U$$

To ease notation let $\phi_i(z, a_i)$ be the formula $z = \langle t_i(a_i), g_i a_i \rangle$ and $\phi(z)$ the formula $\bigvee_{i=0}^{\infty} \exists a_i : A_i, \phi_i(z, a_i)$. Then consider the composite

$$[\![(z : A \times U).\phi(z)]\!] \overset{c}{\rightarrowtail} A \times U \overset{\pi_A}{\twoheadrightarrow} A$$

It now suffices to show that this is an isomorphism, as then the morphism f can be taken to be $\pi_U c(\pi_A c)^{-1}$. Since every elementary topos is *balanced*, that is, any morphism in a topos which is both epic and monic is an isomorphism (see [**?**, §IV.1 Prop 2]), it suffices to show this of $\pi_A c$. By Lemma [**?**, 3.4.2] it suffices to show

$$\phi(z_1) \wedge \phi(z_2) \vdash_{z_1, z_2 : A \times U} \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \Rightarrow z_1 = z_2 \tag{6}$$

and

$$\vdash_{a:A} \exists z : A \times U, \phi(z) \wedge \mathrm{fst}(z) = a \tag{7}$$

Recall that the following Sequents hold in any elementary topos:

1. $$(\vec{x} = \vec{s}) \wedge \psi \vdash_{\vec{y}} \psi[\vec{x} := \vec{s}]$$

   where $\vec{x}$ is a string of variables, $\vec{s}$ is a string of terms with the same length and type as $\vec{x}$, and no free variable in ψ becomes bound in $\psi[\vec{x} := \vec{s}]$ (this follows from the *substitution* and *equality* axioms given in [**?**, §4.1 Definition 1.3.1]).
2. $$\vdash_{w : W_1 \times W_2} \langle \mathrm{fst}(w), \mathrm{scd}(w) \rangle = w$$

   (See [**?**, §D4.1 Lemma 4.1.6])
3. $$z_1 = z_2 \vdash_{z_1 : Z, z_2 : Z} \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

4.
$$\psi[x := t] \vdash_{\vec{y}} \exists x : X, \psi$$

To show that Sequent 6 holds, it suffices to show that for each i, j:

$$\exists a_i : A_i, \exists a_j : A_j, \phi_i(z_1, a_i) \wedge \phi_j(z_2, a_j) \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \vdash_{z_1, z_2 : A \times U} z_1 = z_2 \tag{8}$$

by 2 above, it suffices to show:

$$\exists a_i : A_i, \exists a_j : A_j, \phi_i(z_1, a_i) \wedge \phi_j(z_2, a_j) \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2)$$
$$\vdash_{z_1, z_2 : A \times U} \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

ie,

$$z_1 = \langle t_i(a_i), g_i a_i \rangle \wedge z_2 = \langle t_j(a_j), g_j a_j \rangle \wedge \mathrm{fst}(z_1) = \mathrm{fst}(z_2) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

where $\Gamma = (a_i : A_i, a_j : A_j, z_1 : A \times U, z_2 : A \times U)$. By 1 above, it suffices to show

$$\mathrm{fst}(\langle t_i(a_i), g_i a_i \rangle) = \mathrm{fst}(\langle t_j(a_j), g_j a_j \rangle) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

that is,

$$t_i(a_i) = t_j(a_j) \vdash_\Gamma \mathrm{scd}(z_1) = \mathrm{scd}(z_2)$$

Since $=$ is an equivalence relation [**?**, §4.1 Definition 1.3.1b] and using 3 above, we have

$$z_1 = \langle t_i(a_i), g_i a_i \rangle \wedge z_2 = \langle t_j(a_j), g_j a_j \rangle \vdash_\Gamma \mathrm{scd}(z_1) = g_i a_i \wedge \mathrm{scd}(z_2) = g_j a_j$$

thus it suffices to show:

$$t_i(a_i) = t_j(a_j) \vdash_\Gamma g_i a_i = g_j a_j$$

which is exactly Sequent 5.

To show that Sequent 7 holds, by 4 above, it suffices to show

$$\vdash_{a:A} \bigvee_{i=0}^{\infty} \exists a_i : A_i, \langle t_i(a_i), g_i a_i \rangle = \langle t_i(a_i), g_i a_i \rangle \wedge t_i a_i = a$$

which follows from Sequent 4. □

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