Geometry of Interaction

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1 Geometry of Interaction

The product and coproduct of finitely many copies of the Hilbert space $\mathbb{H} = \ell^2$ are both given by the direct sum. So, any morphism $f : \mathbb{H}^n \longrightarrow \mathbb{H}^m$ can be decomposed according to the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{i=1}^n \mathbb{H} \arrow[r, "f"] & \bigoplus_{j=1}^m \mathbb{H} \\ \mathbb{H} \arrow[u] \arrow[r, "f_{i,j}"'] & \mathbb{H} \arrow[u] \end{tikzcd} \tag{1}$$

Thus, the data of such a morphism f is equivalent to the data of a set of morphisms $\{f_{i,j} : \mathbb{H} \longrightarrow \mathbb{H}\}$ which, if $\operatorname{End}\mathbb{H}$ denotes the space of endomorphisms on $\mathbb{H}$, can be written as an element of $M_{n,m}(\operatorname{End}\mathbb{H})$.

Fix a set of bijections

$$\mathscr{B} := \{\alpha_i : \mathbb{N} \longrightarrow \mathbb{N}^i \mid i > 0\} \tag{2}$$

and let $\mathscr{I}$ denote the corresponding set of isometric isomorphisms

$$\mathscr{I} := \{\hat{\alpha}_i = \bigoplus_{j=1}^{i} p_{ij} : \mathbb{H} \longrightarrow \mathbb{H}^i \mid \alpha_i \in \mathscr{B}\} \tag{3}$$

Notation a bit confusing. Using the isomorphisms $\hat{\alpha}_i$ we can associate to each multiplicative proof-net [4] an isometric isomorphism. Recall [4] that we denote the set of proofs in MLL by Σ.

Definition 1.0.1. We let

$$[\![\cdot]\!] : \Sigma \longrightarrow \operatorname{End}\mathbb{H} \tag{4}$$

denote the function defined inductively by associating to each multiplicative, linear logic deduction rule [4] an element of $\operatorname{End}\mathbb{H}$:

- Axiom:

$$\frac{}{\vdash A, \sim A}\,(\mathrm{ax}) \qquad\qquad \mathbb{H} \xrightarrow{\hat{\alpha}_2} \mathbb{H}^2 \xrightarrow{M} \mathbb{H}^2 \xrightarrow{\hat{\alpha}^{-1}} \mathbb{H}$$

$$\text{where } M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Cut:

Example 1.0.2. Consider the proof-net π_1, for clarity sakes we explicitly put in the occurrence names

$$(A,1) \qquad (\sim A,2) \qquad (A,3) \qquad (\sim A,4) \tag{5}$$

which is equivalent under cut-elimination to the following which we denote π_2:

$$(A,1) \qquad (\sim A,4) \tag{6}$$

We have

$$[\![\pi_1]\!] = \begin{array}{c|cccc} & (A,3) & (\sim A,2) & (A,1) & (\sim A,4) \\ \hline (A,3) & 0 & & & 1 \\ (\sim A,2) & & 0 & 1 & \\ (A,1) & & 1 & 0 & \\ (\sim A,4) & 1 & & & 0 \end{array} \tag{7}$$

We then perform matrix multiplication:

$$[\![\pi_1]\!]\sigma[\![\pi_1]\!] = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & 1 & 0 & \\ 1 & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & 1 & 0 & \\ 1 & & & 0 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix} \cong [\![\pi_2]\!] \tag{8}$$

Example 1.0.3.

Warning: have not done cut-elimination for proof-structures yet. The following crucial Lemma will be used to prove both GoI 0 and multiplicative GoI 1.

Lemma 1.0.4. *Let π be a proof-structure admitting a cut*

$$A \qquad \sim A \tag{9}$$

and write $A := A_1 \boxtimes_1 \dots \boxtimes_{n-1} A_n$, $B := B_1 \diamondtimes_1 \dots \diamondtimes_{n-1} B_n$ with $\boxtimes_i, \diamondtimes_i \in \{\otimes, \parr\}$ for each i. Then, once all tensor-par redexes have been eliminated from π (and no other redexes) yielding a proof π', for each i the following cut will be present in π':

$$A_i \qquad B_i \tag{10}$$

Proof. By induction on n, the base case is trivial and the inductive step follows by inspection of the tensor-par redex rule []need reference. $\square$

We now use this to prove GoI 0:

Definition 1.0.5. Let π be a proof-structure

Theorem 1.0.6 (Geometry of Interaction 0)**.** *Let π be a proof-structure*

References

[1] *Linear Logic*, J.Y. Girard

[2] *Geometry of Interaction I*, J.Y. Girard.

[3] *Intuitionistic, linear sequent calculus* W. Troiani.

[4] *Proof-nets*, W Troiani