

# Hartshorne Exercise Solutions

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## 1 Chapter I

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### 1.1 §1

1.1:

a) The affine coordinate ring is defined by the formula  $A(Y) = k[x, y]/I(Y)$ . In this instance,  $I(Y) = (y - x^2)$  as  $(y - x^2)$  is a radical ideal. Let  $\varphi : k[x, y] \rightarrow k[x]$  be the morphism defined by  $x \mapsto x$  and  $y \mapsto x^2$ . This is surjective and  $\ker(\varphi) = (y - x^2)$ , so that  $A(Y) \cong k[x]$ .

b) We have  $A(Z) = k[x, y]/(1 - xy)$ . This is in fact isomorphic to  $k[x]_x$ . To see this, define a morphism  $\varphi : k[x, y] \rightarrow k[x]_x$  by  $x \mapsto x$  and  $y \mapsto x^{-1}$ . Then  $\varphi$  is a surjection and its kernel is exactly  $(1 - xy)$ .

c) First note that if  $p(x, y)$  is a homogeneous polynomial of degree  $n$  in  $k[x, y]$ , where  $k$  is an algebraically closed field, then  $p$  splits into a product of linear factors. To see this write  $p = y^n g(\frac{x}{y})$ . Then  $g(\frac{x}{y})$  will split so we can write  $p = y^n (\frac{x}{y} - a_1) \dots (\frac{x}{y} - a_n) = (x - a_1 y) \dots (x - a_n y)$ .

Now, suppose that  $f(x, y)$  is an irreducible quadratic over an algebraically closed field  $k$ . Let  $p(x, y)$  be the degree 2 homogeneous part of  $f$ . By the above we can write  $p = (ax - by)(cx - dy)$ . Potentially swapping variables we can assume without loss of generality that  $a \neq 0$ . If these factors are linearly dependent, we can do a change of variables to replace  $x$  with  $ax - by$  (note that replacing  $x$  with a linear polynomial in  $x$  and  $y$  induces an automorphism of  $k[x, y]$ ). Then  $f(x, y) = x^2 + ax + by + c$ . We can then do a change of variables and replace  $ax + by + c$  with  $-y$ , giving  $f(x, y) = x^2 - y$ . Solving  $f = 0$  then gives  $y = x^2$ .

If both factors are linearly independent, we can assume that  $a, d \neq 0$ . Thus by a change of variables (replacing  $ax - by$  with  $x$  and  $cx - dy$  with  $y$ , which induces an automorphism of  $k[x, y]$  as these factors are

linearly independent) we can write  $f(x, y) = xy + ax + by + c$ . We then have  $f(x, y) = (x + b)(y + a) + c - ab$ . Another change of variables then allows us to write  $f(x, y) = xy - 1$ . Solving for  $f = 0$  then gives  $xy = 1$ .

**1.2:** For the first part, simply note that  $Y = Z(y - x^2, z - x^3)$ . Similarly to 1.1c, we can see that  $k[x, y, z]/(y - x^2, z - x^3) \cong k[x]$ . Since  $k[x]$  has no nilpotent elements,  $(y - x^2, z - x^3)$  is a radical ideal and is thus equal to  $I(Y)$ . Hence  $A(Y) \cong k[x]$ , as required.

**1.3:**  $Y = Z(y) \cup Z(x) \cup Z(x^2 - y)$  and the corresponding ideals are  $(y)$ ,  $(x)$ , and  $(x^2 - y)$ .

**1.4:** A basis for the closed sets of  $\mathbb{A}^1 \times \mathbb{A}^1$  is given by  $\{X \times Y \mid X \subseteq \mathbb{A}^1 \text{ closed}, Y \subseteq \mathbb{A}^1 \text{ closed}\}$  which means every closed set is finite. However, the set  $Z(y - x) \subseteq \mathbb{A}^2$  is closed and infinite ( $k$  is algebraically closed and thus infinite), thus these topologies are not equal.

**1.5:** If  $B$  is finitely generated then  $B \cong k[x_1, \dots, x_n]/\mathfrak{a}$  for some ideal  $\mathfrak{a}$ . Moreover, if  $B$  has no nilpotent elements then  $\mathfrak{a}$  is radical. Which means  $Z(\mathfrak{a})$  is such that

$$A(Z(\mathfrak{a})) = k[x_1, \dots, x_n]/IZ(\mathfrak{a}) = k[x_1, \dots, x_n]/\sqrt{\mathfrak{a}} = k[x_1, \dots, x_n]/\mathfrak{a} \cong B$$

The converse is obvious.

**1.6:** See [3].

**1.7:**

a) Routine, if one was only interested in the case of algebraic sets then use the bijection between algebraic sets and radical ideals coupled with the corresponding statements for Noetherian rings.

b) If  $X$  is not quasi-compact then one can construct from an infinite cover with no finite subcover a strictly ascending chain of open subsets, taking complements of which induces a strictly decreasing chain of closed sets.

c) Follows easily by considering the contrapositive.

d) Let  $X$  be Noetherian and Hausdorff. The space  $X$  decomposes into finitely many irreducible components  $X = X_1 \cup \dots \cup X_n$ . Each  $X_i$  is Noetherian, Hausdorff, and irreducible. By irreducibility, any two non-empty open sets of  $X_i$  have non-empty intersection, which contradicts the Hausdorff condition unless  $X_i$  consists of a single element. Thus  $X$  is finite. Lastly, any finite, Hausdorff space is discrete.

**1.8:**

Decompose  $Y \cap H$  into finitely many irreducibles  $Y \cap H = Y_1 \cup \dots \cup Y_n$  with no  $Y_i$  containing any other. Each  $Y_i$  is an irreducible subset of  $Y$  and so corresponds to a prime  $\mathfrak{p}_i$  of  $A(Y)$ . Since  $Y_i$  is also a subset of  $H$  it follows that  $\mathfrak{p}_i$  contains  $(IH)A(Y) = (IZ(f))A(Y) = (f)A(Y)$ . In fact, since there is no irreducible subset strictly between  $Y_i$  and  $Y$  it follows that  $\mathfrak{p}_i$  is minimal over  $(f)A(Y)$ , that is,  $\dim A(Y)/\mathfrak{p}_i = \dim A(Y) - 1 = r - 1$ . Since prime ideals of  $A(Y)/\mathfrak{p}_i$  correspond to irreducible subsets of  $Y_i$  we thus have  $\dim Y_i = r - 1$ .

**1.9:**

Decompose  $Z(\mathfrak{a})$  into finitely many irreducible components  $Z(\mathfrak{a}) = Y_1 \cup \dots \cup Y_n$  with no  $Y_i$  containing any other. Each  $Y_i$  corresponds to a prime ideal  $\mathfrak{p}_i$  which is minimal over  $\mathfrak{a}$ . By Krull's Principal Ideal Theorem,  $\text{ht. } \mathfrak{p} \leq r$ . We also know

$$\text{ht. } \mathfrak{p}_i + \dim A_n/\mathfrak{p}_i = \dim A_n$$

thus  $\dim Y_i = \dim A_n/\mathfrak{p}_i \geq n - r$ .

**1.10:**

a) Solved in [3].

b) Solved in [3].

c) Consider the Sierpinski space  $\Sigma := \{0, 1\}$  with topology  $\{\emptyset, \{0\}, \{0, 1\}\}$ . We have that  $\overline{\{0\}} = \Sigma$  so  $\{0\}$  is dense. Furthermore,  $\dim\{0\} = 0$ . However,  $\dim\Sigma = 1$  as demonstrated by the following sequence  $\{0\} \subseteq \Sigma$ .

d) This is obvious as if  $Y \neq X$  then any chain of irreducible, closed subsets of  $Y$  remain so as subsets of  $X$ . Since  $X$  itself is irreducible,  $Y \neq X \implies \dim Y < \dim X$ .

e) Consider  $\mathbb{N}$  with the topology whose closed sets are all initial segments.

1.12  $x^2 + y^2 + 1$ . We have that  $Z_{\mathbb{A}_{\mathbb{R}}^2}(x^2 + y^2 + 1) = \emptyset$  which by definition is not irreducible.

## 1.2 §2

Throughout,  $S = k[x_0, \dots, x_n]$

2.1:

For clarity, if  $\mathfrak{a} \subset S$  is an ideal we will write  $Z_{\mathbb{P}^n}(\mathfrak{a})$  for the zero set in  $\mathbb{P}^n$  and  $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$  for the zero set in  $\mathbb{A}^{n+1}$ .

Let  $\mathfrak{a} \subseteq S$  be homogeneous and say  $f \in S$  is a homogeneous polynomial such that  $\deg f > 0$  and for all  $P \in Z_{\mathbb{P}^n}(\mathfrak{a})$  we have that  $f(P) = 0$ . It follows that for all non-zero  $P \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$  we have that  $f(P) = 0$ . Moreover, since  $\deg f > 0$  and  $f$  is homogeneous it follows that  $f(0, \dots, 0) = 0$ . Thus for all  $P \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$  we have that  $f(P) = 0$  and so by the regular nullstellensatz we have that  $f^r \in \mathfrak{a}$  for some  $r > 0$ .

2.2:

Say  $Z_{\mathbb{P}^n}(\mathfrak{a}) = \emptyset$ . Then  $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$  is either empty or the singleton set  $\{(0, \dots, 0)\}$ . In the case that it is empty, it follows from the nullstellensatz that  $\mathfrak{a} = S$ , and in the case that it is the singleton set containing  $(0, \dots, 0)$  we have that  $\sqrt{\mathfrak{a}} = S_+$  again by the nullstellensatz, thus  $(i) \Rightarrow (ii)$ . Now say  $\sqrt{\mathfrak{a}} = S_+$  and let  $d$  be the least integer such that there exists a polynomial of degree  $d$  in  $\mathfrak{a}$ , we claim that  $S_d \subseteq \mathfrak{a}$ . For each  $i$  there exists  $d_i > 0$  such that  $x_i^{d_i} \in \mathfrak{a}$ , as  $\sqrt{\mathfrak{a}} = S_+$ . Let  $d = \max_i d_i$ . Then  $x_i^d \in \mathfrak{a}$  for all  $i$ , as these generate  $S_d$  we have that  $S_d \subseteq \mathfrak{a}$ . If  $\sqrt{\mathfrak{a}} = S$  then  $\mathfrak{a} = S$ . Thus  $(ii) \Rightarrow (iii)$ . Lastly, if  $\mathfrak{a} \supset S_d$  for some  $d$  then  $Z_{\mathbb{A}^{n+1}}(\mathfrak{a}) \subseteq Z_{\mathbb{A}^{n+1}}(S_d) = \{(0, \dots, 0)\}$  and so  $Z_{\mathbb{P}^n}(\mathfrak{a}) = \emptyset$ .

2.3:

a), b), c) are trivial.

d) First notice that if  $Z(\mathfrak{a}) = \emptyset$  then  $IZ(\mathfrak{a}) = S$ , but from the previous part it might be that  $\sqrt{\mathfrak{a}} = S_+$ , so we cannot assert that  $IZ(\mathfrak{a}) = \sqrt{\mathfrak{a}}$ . Assuming  $Z(\mathfrak{a}) \neq \emptyset$  then we have that  $I_{\mathbb{A}^{n+1}}Z_{\mathbb{A}^{n+1}}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$ . Notice that all elements of  $\sqrt{\mathfrak{a}}$  are homogeneous, and so  $I_{\mathbb{P}^n}Z_{\mathbb{P}^n}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$ .

e) Let  $W \supseteq Y$  be closed, we show  $ZI(Y) \subseteq W$ . Write  $W = Z(\mathfrak{a})$ . By a) it suffices to show  $I(Y) \supseteq \mathfrak{a}$ . This holds as  $W \supseteq Y$  implies  $I(Y) \supseteq I(W) = IZ(\mathfrak{a})$ , which by d) is equal to  $\sqrt{\mathfrak{a}}$ . The result then follows as  $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$ .

2.4:

a) The previous exercise implies that there is a one-to-one order reversing bijection between proper radical ideals of  $S$  not equal to  $S_+$  and non-empty closed subsets of  $\mathbb{P}^n$ . We then notice that  $I(\emptyset) = S$  and  $Z(S) = \emptyset$ , so this bijection extends to that as stated in the question.

b) Immediate from the fact that the bijection is order reversing.

c)  $I(\mathbb{P}^n) = (0)$  which is prime.

2.5: a): Every descending chain of algebraic sets corresponds to an ascending chain of ideals of  $k[x_0, \dots, x_n]$

which is Noetherian.

**b)** Follows from Proposition [1, §I Prop1.5]

**2.6:**

We will use the following lemma:

**Lemma 1.2.1.** *If a ring map  $f : A \longrightarrow B$  is injective and extends to a map  $F : A[\{x_i\}_{i \in I}] \longrightarrow B$  such that the ideal generated by  $\{x_i\}_{i \in I}$  has empty intersection with  $\ker F$ , then  $F$  is injective.*

*Proof.* Clearly a non-zero element of  $A[\{x_i\}_{i \in I}]$  maps to a non-zero element of  $B$ . □

There is a map

$$\begin{aligned} S &\longrightarrow S_{(x_i)} \\ f &\mapsto f(x_0/x_i, \dots, x_n/x_i) \end{aligned}$$

and thus a composite

$$\psi_i : A \xrightarrow{\beta_i} S \longrightarrow S_{(x_i)}$$

given by  $f \mapsto x_i^{\deg f} f(x_0/x_i, \dots, x_n/x_i) \mapsto f(x_0/x_i, \dots, x_n/x_i)$  (with  $x_i/x_i$  omitted). This map is clearly an isomorphism as it is just a relabelling of indeterminants. In fact, we have:

**Lemma 1.2.2.** *Let  $Y \subseteq \mathbb{P}^n$  be a projective variety,  $f \in I(Y_i)$ , and  $P \in Y \cap U_i$ . Then*

$$f(\varphi_i(P)) = 0 \iff (\beta_i f)(P) = 0$$

*Moreover, if  $P \notin U_i$  then  $P_i = 0$  and so  $(\beta_i f)(P) = 0$ . Thus  $f \in I(Y_i) \Rightarrow \beta_i(f) \in I(Y)$ .*

Thus  $\psi_i(I(Y_i)) = I(Y)S_{(x_i)}$ , and so

$$\varphi_i^* : A(Y_i) \longrightarrow S_{(x_i)}/(I(Y)S_{(x_i)}) \cong S(Y)_{(x_i)}$$

is an isomorphism.

This extends naturally to a surjective map  $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$ , the image of  $x_i$  under which is a unit, we thus have a map

$$\delta_i : (A(Y_i)[x_i])_{x_i} \longrightarrow S(Y)_{x_i}$$

our next claim is that this is an isomorphism. This maps onto a set of generators and is thus surjective. For injectivity, as  $A(Y_i)[x_i]$  is an integral domain, it suffices to show  $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$  is injective, which follows from Lemma 1.2.1.

We now show  $\dim S(Y)_{x_i} = \dim S(Y) - 1$ . By (1.8A) this equality is equivalent to  $\text{tr. deg}_k S(Y)_{x_i} = \text{tr. deg}_k S(Y) - 1$ . We have

$$\text{Frac } S(Y) \cong \text{Frac } S(Y)_{x_i} \cong \text{Frac } (A(Y_i)[x_i])_{x_i} \cong \text{Frac } (A(Y_i)[x_i]) = (A(Y_i))(x_i) \cong (S(Y)_{x_i})_0(x_i)$$

thus

$$\text{tr. deg}_k S(Y) = \text{tr. deg}_k (S(Y)_{x_i})_0(x_i) = \text{tr. deg}_k (S(Y)_{x_i})_0 + 1$$

We also have that

$$\dim (S(Y)_{x_i})_0 = \dim A(Y_i) = \dim(Y \cap U_i)$$

Thus  $\dim S(Y) = \dim(Y \cap U_i) + 1$  for all  $i$ , notice this value is independent of  $i$  and so by exercise 1.10b) we have  $\dim S(Y) = \dim Y + 1$ .

**2.7:**

**a)** Cover  $\mathbb{P}^n$  by open affines  $\{U_i\}_{i=0}^n$ , by exercise 1.10 we have that  $\dim \mathbb{P}^n = \sup_i \dim U_i$ . For each  $U_i$  we have  $\dim U_i = \dim \mathbb{A}^n = n$ .

**b)** We make use of the following fact from topology:

**Fact 1.** Let  $X, Y$  be topological spaces,  $Z \subseteq X$  a subset, and  $U \subseteq X, V \subseteq Y$  open subsets. If  $\varphi : U \rightarrow V$  is a homeomorphism then  $\varphi(U \cap \text{cl}_X(Z)) = \text{cl}_V(\varphi(U \cap Z))$ .

$Y$  is an open subset of an affine space and so is irreducible. This in turn implies that  $\bar{Y}$  is irreducible and thus affine. The previous exercise then applies, so we have  $\dim \bar{Y} = \dim(\bar{Y})_i$ , where we recall that  $(\bar{Y})_i = \varphi_i(\text{cl}_{\mathbb{P}^n}(Y) \cap U_i)$ . We have  $\varphi_i(\text{cl}_{\mathbb{P}^n}(Y) \cap U_i) = \text{cl}_{\varphi_i(U_i)} \varphi_i(Y \cap U_i)$  by Fact 1 and this in turn is just  $\text{cl}_{\mathbb{A}^n} \varphi_i(Y \cap U_i)$ . In other notation, we have  $\overline{(Y_i)} = (\bar{Y})_i$ . It follows from Proposition [1, §1 1.10] that  $\dim Y_i = \dim \overline{(Y_i)}$ . It remains to show that  $\dim Y_i = \dim Y$ . By exercise 1.10 it suffices to show for all  $i \neq j$  such that neither  $Y \cap U_i$  nor  $Y \cap U_j$  are empty that  $\dim Y_i = \dim Y_j$ . We have:

$$\dim \overline{(Y_i)} = \dim(\bar{Y})_i = \dim \bar{Y}$$

finishing the proof.

## 2.9:

**a)** First we claim  $I(\bar{Y}) \subseteq \beta I(Y)$ . Let  $f = f(x_0, \dots, x_n) \in I(\bar{Y})$  be homogeneous and consider  $f(1, x_1, \dots, x_n)$ . This is such that  $\beta f(1, x_1, \dots, x_n) = f$  and so lies in the image of  $\beta$ . Moreover, if  $P = (P_1, \dots, P_n) \in Y$  then the element  $\bar{P}$  of  $\bar{Y}$  given by the set of homogeneous coordings  $(1, P_1, \dots, P_n)$  is such that  $f(\bar{P}) = 0$ , or equivalently,  $f(1, P_1, \dots, P_n) = 0$ . That is,  $f \in \beta I(Y)$ . Since the homogeneous elements generate  $I(\bar{Y})$  and  $\beta I(Y)$  is an ideal, this establishes the claim.

Conversely, let  $f \in \beta I(Y)$  and let  $g \in I(Y)$  be such that  $x_0^{\deg g} g(x_1/x_0, \dots, x_n/x_0) = f$ . For clarity, we distinguish  $Y$  from  $\varphi_0(Y)$ . Since  $g \in I(Y)$  we have for any  $P = (P_1, \dots, P_n) \in Y$  that  $g(P) = 0$ , in other words,  $1^{\deg g} g(P_1/1, \dots, P_n/1) = 0$ , and thus  $Z(f) \supseteq \varphi_0^{-1}Y$ . Since  $Z(f)$  is closed this implies  $Z(f) \supseteq \bar{Y}$ , that is,  $f \in I(\bar{Y})$ .

**b)** From the previous part, we have that  $I(\bar{Y})$  is equal to the ideal generated by  $\beta I(Y)$ , thus  $(\beta f_1, \dots, \beta f_n) \subseteq I(\bar{Y})$ . So the statement of the question is true if and only if the ideal generated by  $\beta(f_1, \dots, f_r)$  is not contained in  $(\beta f_1, \dots, \beta f_r)$ . Specialising now to the question at hand, we have  $I(Y) = IZ(y - x^2, z - x^3)$  which is radical, and so is equal to  $(y - x^2, z - x^3)$ . We need an element of the ideal generated by  $\beta(y - x^2, z - x^3)$  which is not in  $(wy - x^2, w^2z - x^3)$ . Consider  $x(y - x^2) - (z - x^3) = xy - z \in (y - x^2, z - x^3)$  so that  $xy - wz$  is in the ideal generated by  $\beta(y - x^2, z - x^3)$ . This element is not in  $(wy - x^2, w^2z - x^3)$ . **It remains to find generators for  $I\bar{Y}$  but I think  $I\bar{Y} = (wy - x^2, xz - y^2, xy - zw)$  works.**

## 2.10:

**a)** Let  $S = k[x_0, \dots, x_n]$ . First notice by Exercise 2.2 we have for any ideal  $\mathfrak{a} \subseteq S$  with  $IZ_{\mathbb{P}^n}(\mathfrak{a}) \neq \emptyset$  that  $IZ_{\mathbb{P}^n}(\mathfrak{a}) \cap k = \{0\}$ . We therefore assume  $I(Y) \cap k = \{0\}$ . If  $I(Y) = \{0\}$  then  $Y = \mathbb{P}^n$  and so  $C(Y) = \mathbb{A}^{n+1}$  which is algebraic. If  $I(Y) \supseteq \{0\}$  then any non-zero  $f \in I(Y)$  has strictly positive degree and so admits  $(0, \dots, 0) \in \mathbb{A}^{n+1}$  as a zero. Thus if  $Y = Z_{\mathbb{P}^n}(T)$  then  $C(Y) = Z_{\mathbb{A}^{n+1}}(T)$ . Moreover,  $IC(Y) = I(Y)$ .

**b)**  $Y$  is irreducible iff  $I(Y)$  is prime iff  $IC(Y)$  is prime iff  $C(Y)$  is irreducible.

**c)** In the case where  $Y$  is a projective variety we have

$$\dim C(Y) = \dim S(C(Y)) = \dim S(Y) = \dim Y + 1$$

For the general case, we use exercise 2.7.

## 2.11:

**a)** Say  $I(Y)$  can be generated by linear polynomials. Since  $S$  is noetherian we can assume there are finitely many such generators,  $f_1, \dots, f_m$ . We have

$$Y = ZI(Y) = Z(f_1, \dots, f_m) = Z(f_1) \cap \dots \cap Z(f_m)$$

where each  $Z(f_i)$  is a hyperplane.

Conversely, notice that since  $\mathbb{P}^n$  is noetherian, we can assume  $Y$  can be written as the finite intersection of hyperplanes  $Z(f_1) \cap \dots \cap Z(f_m)$ , the result follows from the same calculation as above.

b) We begin by establishing the following lemma:

**Lemma 1.2.3.** *Let  $f_1, \dots, f_m$  be a set of linear polynomials in  $S$ . Then  $\dim S/(f_1, \dots, f_m) = n + 1 - m$ .*

*Proof.* Since  $S/(f_1, \dots, f_m) \cong (S/(f_1, \dots, f_{m-1})) / (\overline{f_m})$  it suffices to prove the case when there is a single  $f_i$ , say  $f$ . Write  $f = \alpha_0 x_0 + \dots + \alpha_n x_n$  and by reordering the variables if necessary assume  $\alpha_0 \neq 0$ . Consider the map  $k[x_0, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$  which maps  $x_i \mapsto x_i$  for  $i \geq 1$  and  $x_0 \mapsto \alpha_0^{-1}(-\alpha_1 x_1 - \dots - \alpha_n x_n)$ . This induces an isomorphism  $k[x_0, \dots, x_n]/(f) \cong k[x_1, \dots, x_n]$  and the result follows.  $\square$

Now proceeding with the question at hand. Let  $Y$  have dimension  $r$  and write  $Y = Z(f_1) \cap \dots \cap Z(f_m) = Z(f_1, \dots, f_m)$  where each  $Z(f_i)$  is a hyperplane, and moreover assume  $m$  is minimal amongst such decompositions. We have:

$$r + 1 = \dim Y + 1 = \dim S(Y) = n + 1 - m$$

and thus  $m = n - r$ .

c): The solution to this question essentially comes down to the following observation:

**Lemma 1.2.4.** *A linear variety  $Y$  in  $\mathbb{P}^n$  is a  $k$ -vector subspace of  $\mathbb{A}^{n+1}$  and the dimension of  $Y$  as a variety is one less than its dimension as a vector space.*

*Proof.* That  $Y$  is a  $k$ -vector subspace is obvious, we prove the second claim by induction on  $n - \dim Y$ . If  $\dim Y = n$  then  $Y$  is the whole space and so as a subspace of  $\mathbb{A}^{n+1}$  has dimension  $n + 1$ . For the inductive step, assume  $\dim Y = k$  and that  $\{y_1, \dots, y_{k+1}\}$  is a basis for  $Y$  as a subspace of  $\mathbb{A}^{n+1}$ . For a linear polynomial  $f$  such that  $Z(f) \cap Y \neq Y$  we have  $Y \cap Z(f) = \text{Span}\{y_1, \dots, y_{k+1}\} \cap Z(f)$ . Write  $f = \alpha_0 x_0 + \dots + \alpha_n x_n$ , then  $Y \cap Z(f)$  is the span of the vectors  $y_1, \dots, y_{k+1}$  subject to the condition  $y_i^0 = \alpha_0^{-1}(-\alpha_1 y_i^1 - \dots - \alpha_n y_i^n)$ , and so has dimension 1 less than that of  $Y$ . What we have shown is that as  $Y$  decreases by 1 in dimension as a variety, so to does it decrease by 1 in dimension as a subspace.  $\square$

The question at hand is now reduced to elementary linear algebra.

## 2.12:

a) We show that  $\mathfrak{a} = \sum_{d \geq 0} (S_d \cap \mathfrak{a})$ . The  $\supseteq$  direction is obvious. For the reverse, let  $f \in \mathfrak{a}$  and write  $f = \sum_{j \geq 0} f_j$  where all but finitely many  $f_j = 0$  and  $\deg f_j = j$  for all  $j$ . It suffices to show that  $\theta(f_j) = 0$  for all  $j$ , but this follows from  $\theta(f) = 0$  as  $i \neq j \Rightarrow \deg \theta(f_i) \neq \deg \theta(f_j)$ . That  $\mathfrak{a}$  is prime follows from the fact that  $\theta$  is a ring homomorphism with codomain an integral domain.

b) Here we follow the convention that  $M_i = x_i^d$  for  $i = 0, \dots, n$ . That  $\text{im } \rho_d \subseteq Z(\alpha)$  is obvious. For the converse we come up with a description for  $\mathfrak{a}$ : for every sequence  $(j_0, \dots, j_n)$  of integers such that  $j_k < d$  and  $\sum_{k=0}^n j_k = d$  we have that  $y_0^{j_0} \dots y_n^{j_n}$  maps under  $\theta$  to a degree  $d^2$  homogeneous element of  $k[x_0, \dots, x_n]$ . Thus there exists some  $m_{(j_0, \dots, j_n)} > n$  such that  $y_0^{j_0} \dots y_n^{j_n} - y_{m_{(j_0, \dots, j_n)}}^d$  maps to zero under  $\theta$ . Thus if  $P \in \mathbb{P}^N$  is such that  $P \in Z(\ker \theta)$  we have that  $P$  is a root of a polynomials of the form

$$y_0^{j_0} \dots y_n^{j_n} - y_{m_{(j_0, \dots, j_n)}}^d \quad (1)$$

First consider the case where  $d$  is even. The equations (1) show that for  $l > n$  the element  $a_l$  is determined by  $a_0, \dots, a_n$ . Thus  $P = \rho_d([\sqrt[d]{a_0} : \dots : \sqrt[d]{a_n}])$ . Now consider the case when  $d$  is odd. Again we obtain a family of equations which show that for  $l > n$  the element  $a_l$  is determined up to sign by  $a_0, \dots, a_n$ . Now, by considering  $a_0 a_1^{d-2} a_i = a_{m_{1,d-2,0,\dots,1,\dots,0}}^d$  we see that  $a_0$  and  $a_i$  have the same sign. A similar argument shows  $a_0$  and  $a_1$  have the same sign. Thus by multiplying  $(a_0, \dots, a_N)$  by  $-1$  if necessary we again see  $P = \rho_d([\sqrt[d]{a_0} : \dots : \sqrt[d]{a_n}])$ .

c) In the case that  $d$  is odd the preimage of a point  $P \in \text{im } \rho_d$  can be recovered by the first  $n$  elements of  $P$  and so  $\rho_d$  is injective. In the case when  $d$  is even we can recover the preimage up to sign and then the

argument given above shows the first  $n$  elements all have the same sign, thus  $\rho_d$  is injective.

If  $P \in \mathbb{P}^n$  and  $f \in k[x_0, \dots, x_N]$  a polynomial such that  $f(\rho_d(P)) = 0$  then the polynomial  $f(M_0, \dots, M_N)$  vanishes at  $P$  and conversely. So if we write  $\text{mon } f$  for  $f(M_0, \dots, M_N)$  and  $\text{mon } I(Y)$  for the ideal generated by  $\{\text{mon } f \mid f \in I\}$  then it follows that for an algebraic set  $Z(\mathfrak{b})$  we have  $\rho_d^{-1}(Z(\mathfrak{b})) = Z(\text{mon } \mathfrak{b})$ , thus  $\rho_d$  is continuous.

Next we show this map is closed. Let  $Z(\mathfrak{b})$  be an algebraic subset of  $\mathbb{P}^n$ . Then  $\rho_d(Z(\mathfrak{b})) = Z(\theta^{-1}(\mathfrak{b})) \cap Z(\ker \theta)$ . This is true because for all  $g \in \theta^{-1}(\mathfrak{b})$  and all  $P \in Z(\mathfrak{b})$  we have  $\theta(g)(P) = 0$  if and only if  $g(\rho_d(P)) = 0$ .

**d)** This amounts to calculating the kernel in this specific case which should be what is written in red in 2.9b.

**2.13:** Since  $Z$  is of dimension 1 which is 1 less than  $2 = \dim \mathbb{P}^2$  we have that  $Z = Z(f)$  for some irreducible  $f \in S^2$ . Let  $M_0, \dots, M_5$  be the degree 2 homogeneous monomials of  $S^2$  and write  $f = \sum_{j=0}^5 \alpha_j M_j$ . Then let  $g = \sum_{j=0}^5 \alpha_j y_j$ , we claim  $Z(g) \cap Y = \rho_2(Z(f))$ . By the solution to the previous question this amounts to showing  $Z(g) \cap \text{im } \rho_2 = Z(\theta^{-1}(f)) \cap Z(\ker \theta)$ . For  $P \in \mathbb{P}^2$  and  $h \in \theta^{-1}(f)$  we have

$$\begin{aligned} h(\rho_2(P)) = 0 &\iff \theta(h)(P) = 0 \\ &\iff f(P) = 0 \\ &\iff g(\rho_2(P)) = 0 \end{aligned}$$

from which the result follows.

**2.14:**

Let  $\theta : k[\{z_{ij}\}_{0 \leq i \leq r, 0 \leq j \leq s}] \longrightarrow k[x_0, \dots, x_r, y_0, \dots, y_s]$  be the ring homomorphism given by  $z_{ij} \mapsto x_i y_j$ . Say  $P \in \mathbb{P}^{r+s+rs}$  is such that  $P \in Z(\ker \theta)$ . Then in particular,  $P$  is a root of every polynomial of the form  $z_{ij} z_{kl} - z_{il} z_{kj}$ , where  $0 \leq i, k \leq r$  and  $0 \leq j, l \leq s$ . Let  $\{P_{ij}\}$  be a set of homogeneous coordinates for  $P$  and now fix a pair of integers  $(a, b)$  such that  $P_{ab} \neq 0$ . For all  $0 \leq k \leq r$  and all  $0 \leq j \leq s$  we have  $P_{aj}/P_{ab} = P_{kj}/P_{kb}$  which implies:

$$\frac{P_{aj}}{P_{ab}} P_{kb} = P_{kj}$$

Thus we can recover all  $P_{kj}$  from the set  $\{P_{a0}, \dots, P_{as}, P_{0b}, \dots, P_{rb}\}$ . We write  $P$  as

$$P = \left[ \frac{P_{aj}}{P_{ab}} P_{kb} \right]_{0 \leq k \leq r, 0 \leq j \leq s} = \psi \left( [P_{0b} : \dots : P_{rb}], \left[ \frac{P_{a0}}{P_{ab}} : \dots : \frac{P_{as}}{P_{ab}} \right] \right)$$

which shows  $Z(\ker \theta) \subseteq \text{im } \psi$ . The other direction is trivial.

We observe that the above also implies that  $\psi$  is injective: let  $(P, Q), (P', Q') \in \mathbb{P}^r \times \mathbb{P}^s$  whose image under  $\psi$  are equal, for clarity we write

$$\psi(P, Q) = [P_0 Q_0 : \dots : P_0 Q_s : \dots : P_r Q_0 : \dots : P_r Q_s] = [P'_0 Q'_0 : \dots : P'_0 Q'_s : \dots : P'_r Q'_0 : \dots : P'_r Q'_s] = \psi(P', Q') \quad (2)$$

and let  $\lambda \neq 0$  be such that

$$(P_0 Q_0 : \dots : P_0 Q_s : \dots : P_r Q_0 : \dots : P_r Q_s) = \lambda (P'_0 Q'_0 : \dots : P'_0 Q'_s : \dots : P'_r Q'_0 : \dots : P'_r Q'_s) \quad (3)$$

From the above, there exists pairs of integers  $(a, b), (a', b')$  such that

$$\frac{P_a Q_j}{P_a Q_b} P_k Q_b = P_k Q_j \quad \text{and} \quad \frac{P'_{a'} Q'_j}{P'_{a'} Q'_b} P'_k Q'_b = P'_k Q'_j \quad (4)$$

Thus for all  $0 \leq k \leq r, 0 \leq j \leq s$ :

$$P_k Q_j = \frac{P_a Q_j}{P_a Q_b} P_k Q_b \quad \text{by (4)}$$

$$= \frac{\lambda P'_{a'} Q'_j}{P_a Q_b} \lambda P'_k Q'_{b'} \quad \text{by (3)}$$

$$= \lambda^2 \frac{P'_{a'} Q'_{b'}}{P_a Q_b} \left( \frac{P'_{a'} Q'_j}{P'_{a'} Q'_{b'}} P'_k Q'_{b'} \right)$$

$$= \lambda^2 \frac{P'_{a'} Q'_{b'}}{P_a Q_b} P'_k Q'_j \quad \text{by (4)}$$

proving  $(P, Q) = (P', Q')$ .

### 2.15:

a) Since  $\text{im } \psi = Z(\ker \theta)$  ( $\theta$  as in the previous question) it suffices to show  $\ker \theta = (z_{00}z_{11} - z_{01}z_{10})$ . Let  $f \in \ker \theta$ . We write  $f = (z_{00}z_{11} - z_{01}z_{10})^m f_1 + f_2$  for the largest possible integer  $m$ . Let  $\alpha^{d_1 d_2 d_3 d_4}$  be the coefficient in front of  $f_2$  in front of  $z_{00}^{d_1} z_{01}^{d_2} z_{10}^{d_3} z_{11}^{d_4}$  and let  $\beta^{d_1 d_2 d_3 d_4}$  be the coefficient of  $\theta(f_2)$  in front of  $(x_0 y_0)^{d_1} (x_0 y_1)^{d_2} (x_1 y_0)^{d_3} (x_1 y_1)^{d_4}$ . We have  $\theta(f_2) = 0$  and so by linear independence  $\beta^{d_1 d_2 d_3 d_4} = 0$  for all sequences  $d_1 d_2 d_3 d_4$ . We have  $\beta^{1111} = \alpha^{1001} + \alpha^{0110} = 0$  and so  $\alpha^{1001} = -\alpha^{0110}$  so either both are zero or neither are. If neither are then  $f_2 = (z_{00}z_{11} - z_{01}z_{10})f_3 + f_4$  contradicting maximality of  $n$ . Thus both are zero. The final claim is for all sequences  $d_1 d_2 d_3 d_4$  other than 1111 we have  $\alpha^{d_1 d_2 d_3 d_4} = \beta^{d_1 d_2 d_3 d_4}$  which can be proved by induction on such sequences in lexicographic order. Thus  $f_2 = 0$  and  $f \in (z_{00}z_{11} - z_{01}z_{10})$ .

### 2.16:

a) We have

$$Q_1 \cap Q_2 = Z(x^2 - yw) \cap Z(xy - zw) = Z(x^2 - yw, xy - zw)$$

Multiplying  $xy - zw = 0$  by  $y$  we have  $xy^2 - zyw = 0$ . Substituting  $x^2 - yw = 0$  into  $xy^2 - zyw$  we get

$$xy^2 - zx^2 \implies x(y^2 - zx)$$

which means either  $x = 0$  or  $y^2 - zx = 0$ , we will show that  $x = 0$  corresponds to the line, and  $y^2 - zx$  corresponds to the twisted cubic curve.

Say  $x = 0$ . Then since  $x^2 - yw = 0$  we have that either  $y = 0$  or  $w = 0$ . If  $y = 0$  then since  $xy - zw = 0$  we have either  $z = 0$  or  $w = 0$  with the other variable arbitrary, this corresponds to a line. If  $y \neq 0$  then multiplying  $xy - zw = 0$  by  $x^2$  we have  $x^3 y - x^2 zw = 0$  which by substituting  $yw$  for  $x^2$  gives

$$x^3 y - z y w^2 = 0 \implies y(x^3 - z w^2) = 0$$

which since  $y \neq 0$  implies  $z w^2 = 0$  so either  $z = 0$  or  $w = 0$  with the other arbitrary. This also corresponds to a line.

Now say  $x \neq 0$  so  $y^2 - zx = 0$ . Then we have

$$Q_1 \cap Q_2 = Z(x^2 - yw, xy - zw, y^2 - zw)$$

which **assuming the postulated solution of Exercise 2.9b is correct**, gives the twisted cubic curve.

b)  $I(C) = (x^2 - yz)$ ,  $I(L) = (y)$ , and  $I(C \cap L) = (x, y)$ . Thus we need to show  $(x^2 - yz) + (y) \neq (x, y)$  which is clear as  $x \neq (x^2 - yz) + (y)$ .



### 1.3 §3

#### 3.1

a) We saw in exercise 1.1 that there are two possibilities up to isomorphism for the affine coordinate rings, and so there are two possibilities up to isomorphism of corresponding conics. Since  $\mathbb{A}^1$  and  $\mathbb{A}^1 \setminus \{(0, 0)\}$  are conics, we are done.

b) Any open subset of  $\mathbb{A}^1$  is equal to  $\mathbb{A}^1 \setminus V$  where  $V$  is a finite set of points. Let  $v \in V$ , then  $1/(x - v)$  is an invertible element in  $\mathcal{O}(\mathbb{A}^1 \setminus V)$  and is not in  $k$ , thus  $\mathcal{O}(\mathbb{A}^1 \setminus V) \not\cong \mathcal{O}(\mathbb{A}^1)$ .

c) Let  $f \in k[x_0, x_1, x_2]$  be homogeneous, irreducible and degree 2. Then  $f$  can be written as  $x^T M x$  where  $x^T = (x_0, x_1, x_2)$  and  $M$  is some symmetric matrix. Since  $M$  is symmetric and  $k$  is algebraically closed, there exists an orthogonal matrix  $Q$  such that  $Q^T M Q$  is diagonal. The matrix  $Q$  corresponds to a linear isomorphism  $\varphi_Q : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  and so is an isomorphism of varieties such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\varphi_Q} & \mathbb{P}^2 \\ \uparrow & & \uparrow \\ Z(x^T M x) & \xrightarrow{\varphi_Q|_{Z(x^T M x)}} & Z(x^T Q^T M Q x) \end{array}$$

Moreover,  $\varphi_Q(Z(x^T M x)) = Z(x^T Q^T M Q x)$  because  $P \in Z(x^T Q^T M Q x)$  if and only if  $QP \in Z(x^T M x)$  (both of these are the statement:  $P^T Q^T M Q P = 0$ ). Thus  $\varphi_Q|_{Z(x^T M x)}$  is an isomorphism of varieties.

The upshot is that we may assume  $f = \lambda_1 x_0^2 + \lambda_2 x_1^2 + \lambda_3 x_2^2$ . There is another linear transformation given by the diagonal matrix with  $ii$  entry equal to  $1/\lambda_i$  which shows that in fact we can assume  $f = x_0^2 + x_1^2 + x_2^2$ , that is, all conics are isomorphic to one in particular, thus are all isomorphic to each other. To finish the question, we can simply observe that  $\mathbb{P}^1$  is isomorphic to its image under the 2-uple embedding and thus is isomorphic to all conics.

e) Follows from Theorems 3.2 and 3.4.

#### 3.2

a) This is clearly bijective. To show bicontinuity it suffices to show that every proper, closed subset of  $Z(y^2 - x^3)$  is finite. Let  $T$  be such a closed set, then  $T = Z(y^2 - x^3) \cap T'$  for some closed set  $T'$  which can be written as a finite union of irreducible components,  $T' = T'_1 \cup \dots \cup T'_n$ . Since this union is finite it suffices to show  $Z(y^2 - x^3) \cap T'_i$  is finite for each  $i$ . Fix an  $i$ . This set can itself be written as the finite union of irreducible elements,  $Z(y^2 - x^3) \cap T'_i = Y_1 \cup \dots \cup Y_m$  say. We show  $\dim Y_i = 0$ . Since  $T$  is a proper subset,  $Y_i \subsetneq Z(y^2 - x^3)$  and so it is sufficient to show  $\dim Z(y^2 - x^3) \leq 1$ . By considering the map  $k[x, y] \rightarrow k[t]$  such that  $x \mapsto t^3$  and  $y \mapsto t^2$  we see that  $(y^2 - x^3)$  is prime, and thus  $Z(y^2 - x^3)$  is irreducible. This is a proper subset of  $\mathbb{A}^2$  which has dimension 2 and so  $\dim Z(y^2 - x^3) \leq 1$ .

Now, to see that this is not an isomorphism, we assume to the contrary that it is. The map  $\mathbb{A}^1 \xrightarrow{\varphi} Z(y^2 - x^3) \xrightarrow{\varphi^{-1}} \mathbb{A}^1$  is regular and so  $t = \varphi^{-1}\varphi(t) = \varphi^{-1}(t^2, t^3)$ , where  $\varphi^{-1}$  must be a polynomial. No such polynomial exists so this is a contradiction.

b) This is bijective and thus bicontinuous. That it is not an isomorphism follows from the fact that  $t \mapsto t^{1/p}$  is not a polynomial.

#### 3.3:

a) For every open set  $U \subseteq Y$  there is a map

$$\begin{aligned} \hat{\varphi} : \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(\varphi^{-1}(U)) \\ f &\mapsto f \circ \varphi \end{aligned}$$

and  $\mathcal{O}_X(\varphi^{-1}(U))$  maps to  $\text{Colim}_{U \ni p}(\varphi^{-1}(U))$  which by the universal property of this colimit maps to  $\mathcal{O}_{X,P}$ . Similarly,  $\mathcal{O}_Y(U)$  maps to  $\mathcal{O}_{Y,\varphi(P)}$  which by the universal property of this colimit maps into  $\text{Colim}_{U \ni p}(\varphi^{-1}(U))$  hence we get a map  $\mathcal{O}_{Y,\varphi(P)} \rightarrow \mathcal{O}_{X,P}$  given by  $[f] \mapsto [f \circ \varphi]$ . It remains to show this is a homomorphism of local rings, but this is clear as if  $[f] \in \mathcal{O}_{Y,\varphi(P)}$  is such that  $f(\varphi(P)) = 0$  then  $(f \circ \varphi)(P) = 0$ .

**b)** First we show that  $\varphi$  is a morphism. Let  $U \subseteq Y$  be open, and  $f : U \rightarrow \mathbb{A}^1$  regular. We need to show  $f \circ \varphi$  is regular at every point. Let  $P \in \varphi^{-1}(U)$  and consider  $[f] \in \mathcal{O}_{Y,\varphi(P)}$ . The image of  $[f]$  under  $\varphi_P^*$  is represented by  $f \circ \varphi$  suitably restricted, thus there is some open subset  $W \subseteq X$  containing  $P$  such that  $(f \circ \varphi)|_W$  is regular, that is to say,  $f \circ \varphi$  is regular at  $P$ .

Now we show  $\varphi^{-1}$  is a morphism. First notice that by uniqueness of inverses,  $\varphi^{-1}$  can be given explicitly by  $[f] \mapsto [f \circ \varphi^{-1}]$ . The argument is identical to above.

**c)** Let  $[f] \neq [g] \in \mathcal{O}_{Y,\varphi(P)}$  be represented by  $f : U_1 \rightarrow \mathbb{A}^1$  and  $g : U_2 \rightarrow \mathbb{A}^1$  respectively. Since  $[f] \neq [g]$  we have that  $f$  and  $g$  are not equal on  $U_1 \cap U_2$  so we can assume  $U_1 = U_2$ , let  $U$  denote this set. We see that since  $f, g$  are regular, the fact they're unequal on  $U$  implies they're unequal on  $U \cap \varphi(X)$ . This holds true for all  $U$  and so  $\varphi_P^*[f] \neq \varphi_P^*[g]$ , thus  $\varphi_P^*$  maps distinct elements to distinct elements and so is injective.

### 3.4:

We will make use of the map  $\theta : S^N \rightarrow S^n$  with kernel  $\mathfrak{a}$  given in the statement of Exercise 2.12. We have already shown in exercise 2.12 that  $\rho_d$  is a homeomorphism, so by the previous exercise it suffices to show  $\rho_d^* : \mathcal{O}_{\text{im } \rho_d, \rho_d(P)} \rightarrow \mathcal{O}_{\mathbb{P}^n, P}$  is an isomorphism for all  $P \in \mathbb{P}^n$ . Let  $P \in \mathbb{P}^n$  and write  $Q$  for  $\rho_d(P)$ . By Theorem [1, §I 3.3 3.5] we have that  $\mathcal{O}_{\text{im } \rho_d, Q} \cong (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)}$  and  $\mathcal{O}_{\mathbb{P}^n, P} \cong S^n_{(\mathfrak{m}_P)}$  where  $S^m = k[x_0, \dots, x_m]$ . So the problem is reduced to finding an isomorphism  $\eta : (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \rightarrow S^n_{(\mathfrak{m}_P)}$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{\text{im } \rho_d, Q} & \xrightarrow{\rho_d^*} & \mathcal{O}_{\mathbb{P}^n, P} \\ \downarrow & & \downarrow \\ (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} & \xrightarrow{\eta} & S^n_{(\mathfrak{m}_P)} \end{array} \quad (5)$$

There is an injective map  $\bar{\theta} : S^N/\mathfrak{a} \rightarrow S^n$  such that  $\bar{\theta}(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$ , so this induces a map  $(S^N/\mathfrak{a})_{\mathfrak{m}_Q} \rightarrow (S^n)_{\mathfrak{m}_P}$  which since  $S^N/\mathfrak{a}$  and  $S^n$  are integral domains is also injective. Lastly,  $\theta$  maps degree  $e$  elements to degree  $de$  elements, thus the elements of degree 0 map injectively to those of degree 0, we thus have a map  $(S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} \rightarrow (S^n)_{(\mathfrak{m}_P)}$  which we take to be  $\eta$ . Notice that the collection of rational functions in  $(S^n)_{(\mathfrak{m}_P)}$  are generated by the quotient of two degree  $d$  monomials of  $S^n$ , which lie in the image of  $\eta$ , thus this map is surjective and thus an isomorphism.

It remains to show commutativity of (5). For any  $m \geq 0$  denote  $k[x_1, \dots, x_m]$  by  $A^m$ , and let pick  $i$  such that  $P \in U_i$ , we have the following isomorphisms:  $\mathcal{O}_{\text{im } \rho_d, Q} \xrightarrow{\sim} A^N((\text{im } \rho_d)_i)_{\mathfrak{m}'_Q}$  and  $\mathcal{O}_{\mathbb{P}^n, P} \xrightarrow{\sim} (A^n)_{\mathfrak{m}'_P}$  where  $\mathfrak{m}'_Q$  is the maximal ideal corresponding to  $Q$  and similarly for  $\mathfrak{m}'_P$ . Now (5) can then be extended to the following commuting diagram:

$$\begin{array}{ccc} \mathcal{O}_{\text{im } \rho_d, Q} & \xrightarrow{\rho_d^*} & \mathcal{O}_{\mathbb{P}^n, P} \\ \downarrow & & \downarrow \\ A^N((\text{im } \rho_d)_i)_{\mathfrak{m}'_Q} & \dashrightarrow & (A^n)_{\mathfrak{m}'_P} \\ \downarrow & & \downarrow \\ (S^N/\mathfrak{a})_{(\mathfrak{m}_Q)} & \xrightarrow{\eta} & S^n_{(\mathfrak{m}_P)} \end{array} \quad (6)$$

where the dashed arrow is induced by  $\theta$  and the vertical arrows are isomorphism.

**Remark 1.3.1.** *Commutativity of the top square of (6) (arguably) should be justified:*

**Lemma 1.3.1.** *Let  $\varphi : X \longrightarrow Y$  be a morphism of varieties with  $X, Y$  affine, then for all  $P \in X$  the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_{Y, \varphi(P)} & \xrightarrow{\varphi_P^*} & \mathcal{O}_{X, P} \\ \uparrow & & \uparrow \\ A(Y)_{\mathfrak{m}_{\varphi(P)}} & \xrightarrow{\hat{\varphi}_{\mathfrak{m}_P}} & A(X)_{\mathfrak{m}_P} \end{array} \quad (7)$$

*Proof.* The morphism  $A(Y)_{\mathfrak{m}_{\varphi(P)}} \longrightarrow \mathcal{O}_{Y, \varphi(P)}$  is given by  $[f]/[g] \mapsto [\gamma_{f/g}]$  where  $\gamma_{f/g} : Y \longrightarrow \mathbb{A}^1$  is given by  $y \mapsto f(y)/g(y)$ . The map  $\hat{\varphi}_{\mathfrak{m}_P}$  maps  $[f]/[g]$  to  $[f \circ \varphi]/[g \circ \varphi]$ . Denote by  $\gamma_{f \circ \varphi / g \circ \varphi} : X \longrightarrow \mathbb{A}^1$  the map given by  $x \mapsto (f \circ \varphi)(x)/(g \circ \varphi)(x)$ . Then the image of  $[f \circ \varphi]/[g \circ \varphi]$  under the right, vertical map of (7) is  $[\gamma_{f \circ \varphi / g \circ \varphi}]$ . It remains to show  $[\gamma_{f/g} \circ \varphi] = [\gamma_{f \circ \varphi / g \circ \varphi}]$  which is clear.  $\square$

### 3.5:

Let  $f \in S^n$  be a homogeneous, irreducible polynomial such that  $H = Z(f)$ . Write  $f = \sum_{j=0}^N \alpha_j M_j$ . Then by the solution to Exercise 2.12c) we have that  $\rho_d(Z(f)) = Z(\theta^{-1}(f)) \cap \ker \theta$  so it remains to calculate  $\theta^{-1}(f)$ . This is just the ideal generated by  $\sum_{j=0}^N \alpha_j y_j$  which is linear.

There exists a rotation matrix  $R_\theta : \mathbb{P}^N \longrightarrow \mathbb{P}^N$  which maps the hyperplane to  $Z(x_i)$  for some  $x_i$ . Multiplication by this matrix gives a family of polynomials and so zero sets are sent to zero sets and regular functions are mapped to regular functions. Thus this is an isomorphism.

### 3.6:

First we show that  $\mathcal{O}(X) \cong k[x, y]$ , this isomorphism might seem strange at first because surely  $1/(x^2 + y^2)$  is a unit in  $\mathcal{O}(X)$  but not in  $k[x, y]$ , however,  $1/(x^2 + y^2)$  is not an element of  $\mathcal{O}(X)$  as we are working with an algebraically closed field  $k$ , and so in fact has infinitely many solutions, not just  $(0, 0)$ .

First notice that if  $Y$  is an affine variety and  $Y'$  is an open subset then  $K(Y) \cong K(Y')$ . Thus  $K(X) \cong K(\mathbb{A}^2) \cong k(x, y)$ , also,  $\mathcal{O}(X)$  embeds into  $k(x, y)$ . Now, let  $f/g \in \mathcal{O}(X)$  be arbitrary.  $g$  can only be 0 when  $f$  is which is finitely many times and so  $g$  is a constant **this statement follows from Bezout's Theorem**. Thus  $\mathcal{O}(X) \cong k[x, y]$ .

To finish the question, we notice that the identity map  $k[x, y] \longrightarrow k[x, y]$  corresponds under the equivalence  $\text{Hom}(\mathbb{A}^2, X) \cong \text{Hom}(k[x, y], k[x, y])$  to the inclusion function  $X \hookrightarrow \mathbb{A}^2$  which is clearly not an isomorphism.

### 3.7:

b) (which implies a) we make use of the following lemma:

**Lemma 1.3.2.** *If  $Y$  is an irreducible subset of a topological space  $X$  and  $Y' \subseteq Y$  is also an irreducible subset of  $X$  then  $Y'$  is irreducible as a subset of  $Y$ .*

*Proof.* Let  $Y' = U \cup V$  where  $U = U' \cap Y'$ ,  $V = V' \cap Y'$  with  $U', V' \subseteq X$  closed. Then

$$Y = Y' \cup Y = ((U' \cap Y') \cup (V' \cap Y')) \cup Y = (U' \cap Y) \cup (V' \cup Y) = U' \cup V'$$

which implies that  $U' = Y$ , say. Thus  $Y' = Y' \cap U' = U$  which shows that  $Y'$  is irreducible.  $\square$

Now onto the question at hand. Say  $H \cap Y = \emptyset$ . Then  $Y \subseteq \mathbb{P}^n \setminus H$ . By Lemma 1.3.2 we have that  $Y$  is an irreducible, closed subset of  $\mathbb{P}^n \setminus H$  which by Exercise 3.5 is affine. Thus  $Y$  is both affine and projective so by 3.1e it is thus a point. This means  $\dim Y = 0$ .

### 3.8:

We prove something more general, that if  $Y \subseteq \mathbb{P}^n$  is an open set then the regular functions on  $Y$  are

constants. First notice that in this setting,  $K(Y) \cong K(\mathbb{P}^n)$ . We also have that  $\mathcal{O}(Y)$  embeds into  $K(Y)$ , so since  $K(\mathbb{P}^n) \cong S^n_{((0))}$ , a regular function  $f : Y \rightarrow \mathbb{A}^1$  can be thought of as a fraction  $f_1/f_2$  where  $f_1, f_2 \in S^n$  and  $\deg f_1 = \deg f_2$ . Using that  $k$  is infinite and again **using Bezout's Theorem** we have that  $f_2$  is a constant which implies  $\deg f_1 = 0$  and so is also a constant.

**3.9:**

$S(X) \cong S^1$  and  $S(Y) \cong S^2/(x_0x_1 - x_2^2)$ , the former is a UFD and the latter is not, as  $x_2^2 = x_0x_1$ .

**3.10:**

Let  $U \subseteq Y'$  be open and  $f : U \rightarrow \mathbb{A}^1$  regular. Write  $f = f_1/f_2$  where  $f_2$  is nowhere zero on  $U$ , and  $U = U' \cap Y'$  where  $U' \subseteq Y$  is open. Then  $U' \cap Z(f_2)^c$  is an open subset which extends  $f$ , and so  $f \circ \varphi : U' \cap Z(f_2)^c \rightarrow \mathbb{A}^1$  is regular as  $\varphi$  is a morphism and thus so is its restriction to  $X'$ .

**Observation:** The fact that  $X', Y'$  are locally closed is not integral to the restriction of  $\varphi$  respecting regular functions, this assumption is here so that  $X', Y'$  are varieties in their own right.

**3.11:**

For each closed subvariety  $X' \subseteq X$  containing  $P$  define the set  $\mathfrak{p}_{X'} := \{[(U, f)] \in \mathcal{O}_P \mid f|_{X'} = 0\}$ , we claim the map given by  $X' \rightarrow \mathfrak{p}_{X'}$  is a bijection.

We use the following Lemma:

**Lemma 1.3.3.** *Let  $X$  be an affine variety and  $U \subseteq X$  a quasi-affine variety. Write  $U = Z(\mathfrak{a})^c$ . There is a bijection:*

$$\begin{aligned} \psi : \{\text{Irreducible, closed subsets } V \subseteq U\} &\longrightarrow \{\text{Irreducible closed subsets } V \subseteq X \text{ such that } V \not\subseteq Z(\mathfrak{a})\} \\ V &\mapsto \text{Cl}_X(V) \end{aligned}$$

*Proof.* First we show this map is well defined. Irreducibility is transitive (Lemma [2, §Irreducible sets]) so since  $V$  is an irreducible subset of  $U$  it is also of  $X$ , moreover the closure of an irreducible space is irreducible, thus  $\bar{V}$  is irreducible. It is clearly also closed and not contained in  $Z(\mathfrak{a})$  otherwise it must have been the empty set which is not irreducible.

There is an inverse  $\varphi$  to this function which maps  $V$  to  $V \cap U$ . This is also clearly well defined, where we note that  $V \cap U \neq \emptyset$  as  $V \not\subseteq Z(\mathfrak{a})$ .

Now we show this is in fact a bijection.  $\varphi\psi(V) = \text{Cl}_X(V) \cap U$ . Since  $V \subseteq U$  is closed, write  $V = V' \cap U$  where  $V' \subseteq X$  is closed. We claim  $\text{Cl}_X(V' \cap U) \cap U = V$ . We have  $V \subseteq U$  and  $V = V' \cap U$  so  $V \subseteq \text{Cl}_X(V' \cap U) \cap U$ . We show the reverse inclusion.  $V'$  is a closed set containing  $V' \cap U$  and so  $\text{Cl}_X(V' \cap U) \subseteq V'$ , thus  $\text{Cl}_X(V' \cap U) \cap U \subseteq V' \cap U = V$ . Thus  $\varphi\psi(V) = V$ .

Conversely, we need to show  $\text{Cl}_X(W \cap U) = W$ , but this is true as  $U$  is open and thus dense.  $\square$

In particular, Lemma 1.3.3 implies that for any  $P \in U$ , there is a bijection between the irreducible, closed neighbourhoods of  $P \in U$  and the irreducible, closed neighbourhoods of  $P \in X$ .

Now back to the question at hand. Assume  $X$  is affine. There is a bijection between the prime ideals of  $A(X)$  containing  $\mathfrak{m}_P$  and the irreducible, closed neighbourhoods of  $P$  in  $X$ , so the affine and quasi-affine cases are solved.

In the projective case, for any  $U_i$  such that  $P \in U_i$  we have:

$$\begin{aligned} \psi' : \{\text{Irreducible, closed nbhds } V \subseteq U_i \text{ of } P\} &\rightarrow \{\text{Irreducible, closed nbhds } V \subseteq X \text{ of } P\} \\ V &\mapsto \text{Cl}_X(V) \end{aligned}$$

which is a bijection (proof left to reader). Since  $U_i$  is affine this reduces to the previous case.

### 3.12:

There are three cases to consider. First assume  $X$  is a quasi-affine variety. Then  $\dim X = \dim \bar{X}$  by Prop 1.10 and  $\dim \bar{X} = \dim \mathcal{O}_{\bar{X},P}$  by 3.2c and stalks can be calculated locally so  $\dim \mathcal{O}_{\bar{X},P} = \dim \mathcal{O}_{X,P}$ .

Say  $X$  is a projective variety. Then cover  $X$  by affine  $U_i$  and note that from Exercise 2.6 we have  $\dim X = \dim X_i$ . We thus have by 3.2c that  $\dim X_i = \dim \mathcal{O}_{X_i, \varphi_i(P)}$  and again stalks can be calculated locally so  $\dim \mathcal{O}_{X_i, \varphi_i(P)} = \dim \mathcal{O}_{X,P}$ .

Lastly, say  $X$  is a quasi-projective variety. Then by Exercise 2.7b we have  $\dim X = \dim \bar{X}$  and so we have reduced to the previous case.

**3.13:** Define  $\mathfrak{m}_Y := \{[(U, f)] \in \mathcal{O}_{Y,X} \mid f|_Y = 0\}$ . We claim this is the unique maximal ideal of  $\mathcal{O}_{Y,X}$ . Let  $[(U, f)] \in \mathcal{O}_{Y,X}$  which is not an element of  $\mathfrak{m}_Y$ , then there exists some  $y \in Y$  such that  $f(y) \neq 0$ , let  $V_y \ni y$  be an open neighbourhood of  $y$  such that  $f = f_1/f_2$  in  $V_y$ . Then  $V_y \cap Y \cap Z(f_2)^c$  is an open set containing  $y$  and so in particular is non-empty. Thus  $[(V_y \cap Y \cap Z(f_2)^c, f_2/f_1)]$  is inverse to  $[(U, f)]$ .

There is a ring homomorphism  $\mathcal{O}_{Y,X} \rightarrow K(Y)$  such that  $[(U, f)] \mapsto [(U \cap Y), f|_{U \cap Y}]$ . Say we have a representative  $(U, f)$  of an element  $[(U, f)] \in K(Y)$ . There exists an open subset  $U' \subseteq U$  on which  $f = f_1/f_2$  with  $f_2$  nowhere zero on  $U'$ .  $U' = U'' \cap Y$  for some open subset  $U'' \subseteq Y$  and so  $f$  extends to a regular function  $\hat{f}$  on the open subset  $U'' \cap Z(f_2)^c$  of  $X$ . The element  $[(U'' \cap Z(f_2)^c, \hat{f})]$  maps to  $[(U, f)]$  and so this map is surjective. The kernel is  $\mathfrak{m}_Y$  and so we have  $\mathcal{O}_{Y,X}/\mathfrak{m}_Y \cong K(Y)$ .

For the dimension claim, we cover  $X$  with open affines and appeal to Exercise 2.6 and Proposition 1.10 to reduce to the case where  $X$  is affine. We use Proposition 1.10 again to replace  $Y$  with  $\bar{Y}$  which is to say we can assume  $Y$  is also affine.

First notice that there is a projection map  $A(X) \rightarrow A(Y)$  with kernel  $\mathfrak{m}_Y$  and so  $A(X)/\mathfrak{m}_Y \cong A(Y)$ , so in particular  $\dim A(X)/\mathfrak{m}_Y = \dim Y$ . Next we have  $\text{ht. } \mathfrak{m}_Y + \dim A(X)/\mathfrak{m}_Y = \dim A(X)$ , and so  $\text{ht. } \mathfrak{m}_Y = \dim X - \dim Y$ . It remains to show  $\text{ht. } \mathfrak{m}_Y = \dim \mathcal{O}_{Y,X}$  but this follows from  $\mathcal{O}_{Y,X}/\mathfrak{m}_Y \cong K(Y)$  just established.

### 3.15:

**a)** Let  $X \times Y = Z_1 \cup Z_2$  with  $Z_i$  closed. Write  $Z_i = Z(\mathfrak{a}_i)$  where the  $\mathfrak{a}_i$  are ideals in  $k[x_1, \dots, x_n]$  and  $k[x_1, \dots, x_m]$  respectively.

Consider  $X_i := \{x \in X \mid \{x\} \times Y \subseteq Z_i\}$ . First we show  $X_1 \cup X_2 = X$ . Let  $\alpha \in X$  and consider the sets  $Y_i^\alpha = \{y \in Y \mid (\alpha, y) \in Z_i\}$ . These are closed as  $Y_i^\alpha = Z(\text{ev}_\alpha \mathfrak{a}_i)$  where  $\text{ev}_\alpha \mathfrak{a}_i := \{f(\alpha, y) \mid f \in \mathfrak{a}_i\}$ . Since  $Y$  is irreducible we have  $Y_1^\alpha = Y$  say, and so  $\alpha \in X_1 \subseteq X_1 \cup X_2$ .

Now we show that  $X_i$  are closed. This is easy as  $X_i = Z(\cup_{\beta \in Y} \text{ev}_\beta \mathfrak{a}_i)$ . Thus  $X_1 = X$  say (as  $X$  is irreducible) and so  $X \times Y = Z_1$ .

**b)** We show that  $A(X \times Y)$  along with the obvious projection maps satisfy the universal property of the coproduct in the category of commutative  $k$ -algebras.

Assume given maps  $\varphi_1 : A(X) \rightarrow B$  and  $\varphi_2 : A(Y) \rightarrow B$  where  $B$  is some  $k$ -algebra. Let  $\psi : A(X \times Y) \rightarrow B$  be the map satisfying  $[x_i] \mapsto \varphi_1([x_i])$  for  $i \leq n$  and  $[x_i] \mapsto \varphi_2([x_i])$  if  $i > n$ . This is well defined as if  $f \in I(X \times Y)$  then for each monomial  $[x_1^{j_1} \dots x_{n+m}^{j_{n+m}}]$  we have

$$f([x_1^{j_1} \dots x_{n+m}^{j_{n+m}}]) = f([x_1]^{j_1} \dots f([x_{n+m}]^{j_{n+m}}) = \varphi_1[x_1]^{j_1} \dots \varphi_2[x_{n+m}]^{j_{n+m}} = 0$$

Uniqueness of this map follows from linearity and commutativity with the projection maps. Thus  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .

**c)** Follows from Proposition 3.5 and the previous part.

**d)** We need:

**Lemma 1.3.4.** *Let  $A \rightarrow B$  be integral where  $A, B$  are  $k$ -algebras. Then  $\text{Frac } A \rightarrow \text{Frac } B$  is algebraic.*

*Proof.* Let  $a/b \in \text{Frac } A$  and  $f = x^n + \sum_{j=0}^{n-1} \alpha_j x^j \in k[x]$  such that  $f(a) = 0$ . Then

$$0 = (1/b^n)(a^n/1) + (1/b^n) \sum_{j=0}^{n-1} \alpha_j (a^j/1) = (a/b)^n + \sum_{j=0}^{n-1} \alpha_j / b^{n-j} (a/b)^j$$

□

This problem reduces to proving  $\dim(A \otimes_k B) = \dim A + \dim B$  for finitely generated  $k$ -integral domains  $A, B$ . Notice that we have known  $A \otimes_k B$  is an integral domain by part b). Using Noether Normalisation there exists sets of algebraically independent elements  $\gamma_1, \dots, \gamma_r \in A$  and  $\delta_1, \dots, \delta_s \in B$  with  $\dim A = r$  and  $\dim B = s$  such that  $A$  is a finitely generated  $k[\gamma_1, \dots, \gamma_r]$ -module and  $B$  is a finitely generated  $k[\delta_1, \dots, \delta_s]$ -module. We next claim the map determined by

$$\begin{aligned} k[x_1, \dots, x_r, y_1, \dots, y_s] &\longrightarrow A \otimes_k B \\ x_i &\mapsto \gamma_i \otimes_k 1 \\ y_i &\mapsto 1 \otimes_k \delta_i \end{aligned}$$

is injective. **Say we have this.** Thus we have an  $(r+s)$ -variable polynomial subalgebra of  $A \otimes_k B$ . It remains to show that  $\text{tr.deg}_k(A \otimes_k B) = r + s$ . Since  $A \otimes_k B$  is an integral domain (see the comment at the start of this proof), we reduce to showing  $k[\{\gamma_i \otimes_k 1\}, \{1 \otimes_k \delta_i\}] \longrightarrow A \otimes_k B$  is an integral extension, in fact we show it is a finite morphism. We know that all products of all powers of elements in  $\{\gamma_i \otimes_k 1\} \cup \{1 \otimes_k \delta_i\}$  form a generating set for  $A \otimes_k B$ , it remains to show that a finite subset will do. The modules  $A$  and  $B$  over  $k[\gamma_1, \dots, \gamma_r]$  and  $k[\delta_1, \dots, \delta_s]$  are finite, thus for all pairs  $(\gamma_i, \delta_j)$  there exists a least integer  $n_{ij}$  such that  $\gamma_i^{n_{ij}}$  and  $\delta_j^{n_{ij}}$  can both be written as a linear combination of products of powers of the  $\gamma_i$  and  $\delta_i$  respectively with powers less than  $n_{ij}$ . Thus finitely many elements generate all elements of the form  $(\gamma_i \otimes_k \delta_j)^n$ . Thus finitely many elements generate all products of such elements. Thus finitely many elements generate all of  $A \otimes_k B$ .

### 3.16:

**a), b)** Both a) and b) follow from the following observation: let  $X = Z(\mathfrak{a})$ ,  $Y = Z(\mathfrak{b})$ ,  $(P_1, P_2) \in X \times Y$ ,  $(f_1, f_2) \in \mathfrak{a} \times \mathfrak{b}$ . Then write  $f_1(x_0, \dots, x_n) f_2(y_0, \dots, y_m)$  as  $\sum_{i=0}^n \sum_{j=0}^m \alpha_{ij} x_i y_j$ . Define  $g(\{z_{ij}\}) = \sum_{i=0}^n \sum_{j=0}^m \alpha_{ij} z_{ij}$ . We have  $f_1(P_1) f_2(P_2) = 0$  if and only if  $g(\psi(P_1, P_2)) = 0$ .

### 3.17:

**a)** By Exercise 3.3b) it suffices to consider an isomorphic variety. By Exercise 3.1c we know that every conic in  $\mathbb{P}^2$  is isomorphic to  $\mathbb{P}^1$  so it suffices to show this is normal. Indeed  $\mathcal{O}_{\mathbb{P}^1, P} \cong k[x_0, x_1]_{(\mathfrak{m}_P)}$  which is normal if  $k[x_0, x_1]$  is. Indeed  $k[x_0, x_1]$  is normal as it is a UFD.

**b) Attempt at a direct approach:** First notice that  $(x_0 x_1 - x_2 x_3)$  is prime and so  $S(Q_1)_{(\mathfrak{m}_P)} \cong k[x_0, x_1, x_2, x_3] / (x_0 x_1 - x_2 x_3)_{(\mathfrak{m}_P)}$ . Let  $f \in S(Q_1)_{(\mathfrak{m}_P)}[X]$  by a monic polynomial and  $g \in S(Q_1)_{(0)}$  be such that  $f(g) = 0$ . We write  $g = g_1/g_2$  with  $g_2 \neq 0$  so that:

$$f(g) = (g_1/g_2)^n + \sum_{j=0}^{n-1} \alpha_j (g_1/g_2)^j = 0$$

We clear denominators to obtain

$$-g_1^n = \sum_{j=0}^{n-1} \alpha_j g_2^{n-j} g_1^j = g_2 \sum_{j=0}^{n-1} \alpha_j g_2^{n-j-1} g_1^j$$

and so  $g_2(P) = 0 \Rightarrow g_1(P) = 0$ . It thus remains to show  $g_1(P) \neq 0$  and to show this we claim  $g_1(P) = 0 \Rightarrow g_1 = 0$ , that is  $g_1 \in (x_0 x_1 - x_2 x_3)$  (by sloppy notation). **Incomplete.**

c) We claim this variety is not normal at the point  $P = (0, 0)$ . We need to come up with a monic polynomial  $f \in A(y^2 - x^3)_{\mathfrak{m}_P}[X]$  and  $a \in \text{Frac } A(y^2 - x^3)$  such that  $f(a) = 0$ , with  $a \notin A(y^2 - x^3)_{\mathfrak{m}_P}[X]$ . Take  $f = X^2 - x^2$  and  $a = y/x$ , we have

$$f(a) = a^2 - x^2 = y^2/x^2 - x^2 = y^2/x^2 - x^3/x = (y^2 - x^3)/x = 0$$

**3.21:**

- a) This reduces to showing that for polynomials  $f_1, f_2 \in k[x]$  we have  $f_1(-x)/f_2(-x)$  is a quotient of polynomials.
- b) This reduces to showing that for polynomials  $f_1, f_2 \in k[x]$  we have  $f_1(x^{-1})/f_2(x^{-1})$  is a quotient of polynomials which is true as this equals  $x^n f_3(x)/f_4(x)$  for polynomials  $f_3, f_4 \in k[x]$ .
- c) Given  $\varphi_1, \varphi_2 \in \text{Hom}(X, G)$  we define  $\varphi_1 \cdot \varphi_2 : X \rightarrow G$  to have action on  $x \in X$  given by  $\varphi_1(x) \cdot \varphi_2(x)$ .
- d) We know  $\text{Hom}(X, \mathbb{A}^1) \cong \text{Hom}(k[x], \mathcal{O}(X)) \cong \mathcal{O}(X)$  so it remains to show this is a group homomorphism which is an easy check.
- e) Similar to d).

## 1.4 §4

**4.1** Let  $h$  be the function described by the question. Let  $P \in U \cup V$  and assume without loss of generality that  $P \in U$ . Since  $f$  is regular on  $U$  there exists an open neighbourhood  $V \subseteq U$  of  $P$  for which  $f|_V = f_1/f_2$ , with  $f_2$  nowhere zero on  $V$ . This same neighbourhood  $V \subseteq U \subseteq U \cup V$  can be taken to show that  $h$  is regular at  $P$ .

**4.2** First we show the same claim for morphisms. Let  $X, Y$  be varieties,  $U_1, U_2 \subseteq X$  be open subsets of  $X$  and let  $\varphi_i : U_i \rightarrow Y$ ,  $i = 1, 2$ , be morphisms of varieties which agree on  $U_1 \cap U_2$ . Let  $h$  denote the function which is equal to  $\varphi_i$  on  $U_i$ . Say  $V \subseteq Y$  is an open subset and  $\gamma : V \rightarrow k$  a regular function. We obtain regular functions  $\gamma \circ \varphi_i : U_i \rightarrow k$  which glue to a regular function  $U_1 \cup U_2 \rightarrow k$  by the previous question. Thus  $h$  is a morphism.

The question at hand reduces to this previous considering by picking representatives of the two rational maps.

**4.3:**

- a) This function is defined on  $U_0$  and the corresponding regular function is given by the same rule.
- b) This extends to

$$\mathbb{P}^2 \setminus \{[0 : 0 : 1]\} \rightarrow \mathbb{P}^1, [P_0, P_1, P_2] \mapsto [P_0, P_1]$$

This cannot be extended further lest  $[0 : 0 : 1] \mapsto [0 : 0] \notin \mathbb{P}^1$ .

**4.4:**

- a) Recall that any conic is isomorphic to  $\mathbb{P}^1$  and so in particular is birationally equivalent to it.
- b) Define the map

$$\begin{aligned} Z(y^2 - x^3) \setminus \{(0, 0)\} &\rightarrow \mathbb{P}^1 \\ (x, y) &\mapsto [x : y] \end{aligned}$$

This clearly pulls back regular maps. Define also isomorphism with inverse:

$$\begin{aligned} \mathbb{P}^1 \setminus Z(x) &\rightarrow Z(y^2 - x^3) \\ [x : y] &\mapsto ((y/x)^2, (y/x)^3) \end{aligned}$$

These maps induce birational maps.

c) This map can be given explicitly. If  $[P_0 : P_1 : P_2] \in Y$  then its image is

## 1.5 §5

**5.9:** Using Exercise 2.5b we write  $Z(f) = Z(f_1) \cup \dots \cup Z(f_r) = Z(f_1 \dots f_r)$ , assume that  $r > 1$ . Now, using exercise 3.7 we have that  $Z(f_1) \cap Z(f_2) \neq \emptyset$ , so let  $P \in Z(f_1) \cap Z(f_2)$ . We have:

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x}(f_2 \dots f_r) + \dots + (f_1 \dots f_{r-1}) \frac{\partial f_r}{\partial x} \quad (8)$$

Evaluating (8) at  $P$  yields the value 0. Likewise,  $\frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ , contradicting the hypothesis. Thus  $r = 1$ .

# 2 Chapter 2

## 2.1 §1

The question labelling is taken from [1, II §1]

**1.1:**

We denote the constant presheaf associated to  $A$  by  $C_A$  and the constant sheaf  $\mathcal{A}$ . We construct a third sheaf  $\mathcal{F}$  and show  $C_A^+ \cong \mathcal{F} \cong \mathcal{A}$ .

For an open set  $U$  with connected components  $\{U_i\}_{i \in I}$  define  $\mathcal{F}(U) = \coprod_{i \in I} A$ . Let  $V \supseteq U$  is an open superset of  $U$  with connected components  $\{V_j \in J\}_{j \in J}$ . There is a collection of maps  $\varphi_{ij} : \mathcal{F}(V_j) = A \rightarrow A = \mathcal{F}(U_i)$  which is the identity if  $U_i \subseteq V_j$  and the zero map otherwise. Composing these with the inclusions  $\mathcal{F}(U_i) \hookrightarrow \mathcal{F}(U)$  induces a morphism  $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$  which we take as the restriction map corresponding to  $U \subseteq V$ . This is clearly a sheaf.

To see that  $\mathcal{F} \cong \mathcal{A}$ , notice that a function  $s : U \rightarrow A$  in  $\mathcal{A}(U)$  is clearly equivalent to giving an element of  $A$  for each connected component of  $U$ .

To see that  $C_A^+ \cong \mathcal{F}$  let  $U$  be a connected open subset and  $s$  an element of  $C_A^+(U)$ . There exists a cover of opens  $\{U_i\}_{i \in I}$  and elements  $a_i \in A$  such that if  $u \in U_i$  then  $s(u) = (a_i)_u$ . For all  $U_i \cap U_j \neq \emptyset$  we have  $a_i = a_j$  and  $U$  is connected, so the data of  $s$  amounts to a single element  $a \in A$ .

**1.2a:**

By essential uniqueness of colimits it suffices to show that  $\text{im } \varphi_p$  is a colimit  $\text{Colim}_{U \ni p} \text{im } \varphi^+(U)$ . Let  $s \in \text{im } \varphi^+(U)$  and take  $V \ni p$  and  $t \in \text{im } \varphi(U)$  to be such that for all  $v \in V$  we have  $s(v) = t_v$ . Then the equivalence class  $[(V, t)]$  gives an element of  $\text{im } \varphi_p$  and so we have a collection of maps  $\text{im } \varphi^+(U) \rightarrow \text{im } \varphi_p$ . Thus  $\text{im } \varphi_p$  is a cocone. Now say that  $K$  were any abelian group and there was a collection of morphisms  $\psi_U : (\text{im } \varphi^+)U \rightarrow K$  coherent with the restriction morphisms. Coherency here ensures that the image of any lift  $t \in \text{im } \varphi(V)$  of any  $[(V, t)] \in \text{im } \varphi_p$  under  $\text{im } \varphi(U) \rightarrow \text{im } \varphi^+(U) \rightarrow K$  is mapped to the same element. That is, there is a well defined morphism  $\text{im } \varphi_p \rightarrow K$ , which indeed is unique.

**1.2b**

This follows easily from the definition of monomorphism/epimorphism combined with the fact that for any pair of morphisms  $\gamma, \gamma' : \mathcal{H} \rightarrow \mathcal{J}$  subject to  $\gamma_p = \gamma'_p$  for all  $p$  then  $\gamma = \gamma'$ .

**1.2c**

Essentially an application of the previous two parts. The forward direction is by 1.2a: taking stalks at  $p$  at all parts of the diagram yields a sequence

$$\dots \longrightarrow \mathcal{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathcal{F}_p^i \xrightarrow{\varphi_p^i} \mathcal{F}_p^{i+1} \longrightarrow \dots$$



Since  $\ker \varphi^i = \text{im } \varphi^{i-1}$  it follows that  $\ker \varphi_p^i \cong (\ker \varphi^i)_p = (\text{im } \varphi^{i-1})_p \cong \text{im } \varphi_p^{i-1}$ .

The converse is by 1.2b: since  $(\ker \varphi^i)_p \cong (\text{im } \varphi^{i-1})_p$  for all  $p$ , we have that  $\ker \varphi^i = \text{im } \varphi^{i-1}$ .

## 2.2 §2

### 2.1

Let  $l : A \rightarrow A_f$  be the localisation map, and  $\hat{l} : \text{Spec } A_f \rightarrow \text{Spec } A$  the induced map on spectrum. This map is continuous and open, and thus is a homeomorphism onto its image, which is  $D(f)$ , from now on,  $\hat{l}$  will refer to this homeomorphism.

Since basic opens form a topology and  $\mathcal{O}_X \upharpoonright_{D(f)}$  and  $\mathcal{O}_{\text{Spec } A_f}$  are both sheaves, it suffices to specify  $\hat{l}^\#$  it suffices to define  $\hat{l}^\# D(gf)$  for each basic open  $D(gf)$  of  $D(f)$ . To do this, we first observe that

$$\mathcal{O}_X \upharpoonright_{D(f)} D(g) = \mathcal{O}_X(D(fg)) \cong A_{fg}$$

and

$$\mathcal{O}_{\text{Spec } A_f} \hat{l}_*(D(g)) = \mathcal{O}_{\text{Spec } A_f}(\hat{l}^{-1}(D(g))) = \mathcal{O}(D(g/1)) \cong (A_f)_{g/1}$$

so it suffices to give a local ring isomorphism  $A_{fg} \rightarrow (A_f)_{g/1}$ . We define such a map  $\frac{a}{f^n g^m} \mapsto \frac{a}{f^n} / \frac{g^m}{1}$ .

### 2.4

Let  $\varphi \in \text{Hom}_{\text{Ring}}(A, \Gamma(X, \mathcal{O}_X))$ , we define a corresponding morphism of schemes  $\beta(\varphi) = (\psi, \psi^\#)$ . Fix an open affine cover  $\{U_i = \text{Spec } A_i\}$  of  $X$  and for each pair  $(i, j)$  let  $\{U_k^{ij} = \text{Spec } A_k^{ij}\}$  be open affines covering  $U_i \cap U_j$ . By Proposition [1, 2.3] the ring homomorphisms

$$\varphi_i : A \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\text{Res}_{U_i}^X} A_i$$

give rise to a family of morphisms  $(\gamma_i, \gamma_i^\#)$  of schemes  $\text{Spec } A_i \rightarrow \text{Spec } A$ .

Since  $\text{Res}_{U_k^{ij}}^{U_i} \varphi_i = \text{Res}_{U_k^{ij}}^{U_j} \varphi_j$  and the  $U_k^{ij}$  cover  $U_i \cap U_j$  we have that  $\gamma_i \upharpoonright_{U_i \cap U_j} = \gamma_j \upharpoonright_{U_i \cap U_j}$ , thus we have a well defined continuous function  $\psi : X \rightarrow \text{Spec } A$ .

Now we define  $\psi^\# : \mathcal{O}_{\text{Spec } A} \rightarrow \psi_* \mathcal{O}_X$  for which by the sheaf condition on  $\mathcal{O}_X$  it suffices to give a family  $\psi_i^\# : \mathcal{O}_{\text{Spec } A} \rightarrow \psi_* \mathcal{O}_X \rightarrow \text{Res}_{U_i}^X \psi_* \mathcal{O}_X$  such that  $\text{Res}_{U_i \cap U_j}^{U_i} \psi_i^\# = \text{Res}_{U_i \cap U_j}^{U_j} \psi_j^\#$ . However this is exactly given by the  $\gamma_i^\#$ .

### 2.7

Let  $(f, f^\#) : \text{Spec } K \rightarrow X$  be a morphism of schemes. Write  $x := f((0))$ . We have a ring homomorphism  $f_x^\# : \mathcal{O}_{X, x} \longrightarrow K_{(0)} \cong K$ . This is a local ring homomorphism and so  $(f_x^\#((0)))^{-1} = \ker(f_x^\#) = \mathfrak{m}_x$  and so we have a homomorphism  $k(x) \longrightarrow K$  which being a ring homomorphism with domain a field, is injective.

Conversely, a point  $x \in X$  is equivalent to a continuous function  $f : \text{Spec } K \rightarrow X$ . Given an open subset  $U \subseteq X$  which does not contain  $x$  the function  $f_U^\# : \mathcal{O}_X(U) \rightarrow f_* \mathcal{O}_{\text{Spec } K} U = \mathcal{O}_{\text{Spec } K}(\emptyset) = 0$  is the unique such. If  $x \in U$  then we have the function  $f_U^\# : \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X, x} \rightarrow k(x) \rightarrow K \cong \mathcal{O}_{\text{Spec } K}(\text{Spec } K) = \mathcal{O}_{\text{Spec } K}(f_*(U))$ .

### 2.16

a)

Let  $\varphi : U \longrightarrow \text{Spec } B$  be an isomorphism. For all  $x$  we have an isomorphism  $\mathcal{O}_{X, x} \cong B_{\varphi(x)}$ . Thus  $f_x \notin \mathfrak{m}_x \Leftrightarrow \bar{f} \notin \varphi(x)$  and so  $U \cap X_f \cong D(\bar{f})$ .

b)

Let  $\{U_i = \text{Spec } A_i\}_{i=1}^n$  be a finite open affine cover of  $X$ . From part (a) we know  $X_f \cap U_i = D(f_i)$ , where  $f_i$  is the image of  $f$  under  $A \longrightarrow A_i$ , thus  $a \upharpoonright_{D(f_i)} = 0$  for all  $i$ , that is,  $\frac{a|_{U_i}}{1} = 0$  in  $(A_i)_{f_i}$ . Thus there exists  $n_i > 0$

such that  $f_i^{n_i}a \upharpoonright_{U_i} = 0$ . Since there are finitely many  $U_i$  we can set  $n = \max_i n_i$  so that for each  $i$  we have  $f_i^n a \upharpoonright_{U_i} = 0$ . We then have by the sheaf condition that  $f^n a = 0$ .

c)

We need to define an element  $a \in \Gamma(X, \mathcal{O}_X)$ , we do this by defining an element of  $A_i$  for each  $i$  which agree on the overlaps. Consider  $b \upharpoonright_{X_f \cap U_i}$  for each  $i$ . We know that  $X_f \cap U_i = D(f \upharpoonright_{U_i})$  so we can write  $b \upharpoonright_{X_f \cap U_i} = \frac{a_i}{f \upharpoonright_{U_i}^{n_i}} \in (A_i)_{f \upharpoonright_{U_i}}$ . Since there are finitely many  $U_i$  we can write  $n = \sum_i n_i$  and let  $b_i = f \upharpoonright_{U_i}^{n-n_i} a_i \in A_i$ . Let  $W_{ij} = X_f \cap U_i \cap U_j$  and notice that

$$(b_i - b_j) \upharpoonright_{W_{ij}} = (f^{n-n_i} f^{n_i} b - f^{n-n_j} f^{n_j} b) \upharpoonright_{W_{ij}} = 0$$

So by part (b) there is  $d_{ij} > 0$  such that  $f^{d_{ij}}(b_i - b_j) \upharpoonright_{U_i \cap U_j} = 0$  as an element of  $\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$ . Letting  $d = \max_{i,j} \{d_{ij}\}$  we have  $f^d(b_i - b_j) \upharpoonright_{U_i \cap U_j} = 0$ , so by the sheaf condition there is an element  $a \in \Gamma(X, \mathcal{O}_X)$  such that  $a \upharpoonright_{U_i} = f^d b_i$  and  $a \upharpoonright_{X_f} = b$ .

## 2.17

a)

The collection of continuous functions  $(f \upharpoonright_{U_i})^{-1} : U_i \rightarrow f^{-1}(U_i) \hookrightarrow X$  agree on overlaps as they are the inverse of restrictions of a common function. Thus we obtain a continuous function  $Y \rightarrow X$  which is locally an inverse and thus an inverse to  $f$ .

Let  $g_i$  denote the inverse of  $f^\# \upharpoonright_{U_i} : \mathcal{O}_Y \upharpoonright_{U_i} \rightarrow f_* \mathcal{O}_X \upharpoonright_{U_i}$ . We need to show that  $(g_i)_{U_i \cap U_j} = (g_j)_{U_i \cap U_j}$ . Both of these maps are equal to  $(f^\# \upharpoonright_{U_i \cap U_j})^{-1}$  so we are done.

Notice that a corollary of the proof of this exercise is the following:

**Lemma 2.2.1.** *Let  $\{U_i\}$  be an open cover of  $Y$  and  $f_i : X \upharpoonright_{f^{-1}(U_i)} \rightarrow Y \upharpoonright_{U_i}$  a collection of scheme morphisms such that  $(f_i) \upharpoonright_{U_i \cap U_j} = (f_j) \upharpoonright_{U_i \cap U_j}$ . Then there exists a morphism  $f : X \rightarrow Y$  such that  $f \upharpoonright_{U_i} = f_i$ . Moreover,  $f$  is an isomorphism if and only if all the  $f_i$  are.*

b)

For any sheaf  $X$  there is the unit map  $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ . This morphism is an isomorphism if  $X$  is affine, thus we have a collection of isomorphisms  $X_{f_i} \rightarrow \text{Spec } \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$ . Since  $f_1, \dots, f_r$  generate 1 we have that  $\text{Spec } X_{f_i}$  cover  $\text{Spec } X$ . The result then follows from part (a).

## 2.18b)

We let  $\hat{\varphi} : \text{Spec } B \rightarrow \text{Spec } A$  denote the continuous map induced by  $\varphi : A \rightarrow B$ . Assume that  $\varphi$  is injective. As the collection  $\{D(f)\}_{f \in A}$  form a base for the topology on  $\text{Spec } A$ , it suffices to show that for all  $f \in A$ , the morphism  $\hat{\varphi}_{D(f)}^\# : \mathcal{O}_{\text{Spec } A} D(f) \rightarrow \hat{\varphi}_* \mathcal{O}_{\text{Spec } B} D(f) = \mathcal{O}_{\text{Spec } B} D(\varphi(f))$  is injective. Let  $f \in A$ . It's easy to show that since  $\varphi : A \rightarrow B$  is injective, so is  $\varphi_f : A_f \rightarrow B_{\varphi(f)}$ . Thus it remains to show commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{\text{Spec } A} D(f) & \xrightarrow{\hat{\varphi}_{D(f)}^\#} & \mathcal{O}_{\text{Spec } B} D(\varphi(f)) \\ \cong \uparrow & & \uparrow \cong \\ A_f & \xrightarrow{\varphi_f} & B_{\varphi(f)} \end{array}$$

Which can be established by a direct calculation.

## 2.3 §3

### Exercise 1. Hartshorne 3.1

*Proof.* We use the following fact from commutative algebra:

**Lemma 2.3.1.** *Let  $A, B$  be rings and  $f \in A$  an element of  $A$ . Then  $B$  is a finitely generated  $A$ -algebra if and only if it is a finitely generated  $A_f$ -algebra. (Note: we mean finitely generated as algebras, the corresponding statement for modules is false)*

Throughout, a *cover* of an open set  $U$  means a collection of open subsets  $\{U_i \subseteq U\}_{i \in I}$  of  $U$  such that  $\bigcup_{i \in I} U_i = U$ . For an open affine subset  $U = \text{Spec } A$  of  $Y$  let  $P(U)$  be the proposition “there exists a cover  $\{\text{Spec } B_i\}_{i \in I}$  of  $f^{-1}(U)$  such that each  $B_i$  is a finitely generated  $A$ -algebra”. Let  $\{U_i = \text{Spec } A_i\}_{i \in I}$  be an open affine cover of  $Y$  such that  $P(U_i)$  holds for each  $i$ , and let  $U = \text{Spec } A$  be an open affine subset of  $Y$ . First we show that  $U$  can be covered by open affines  $\{U_i\}_{i \in I}$  satisfying  $P(U_i)$  for each  $i$ .

Fix  $i \in I$ , let  $\{\text{Spec } B_{ij}\}_{j \in J}$  be a cover of  $f^{-1}(U_i)$  such that each  $B_{ij}$  is a finitely generated  $A_i$ -algebra, and let  $a_i \in A_i$  be such that  $D(a_i) \subseteq U_i$ . Let  $\varphi_{ij} : A_i \rightarrow B_{ij}$  be the ring homomorphism corresponding to the scheme morphism  $\text{Spec } B_{ij} \rightarrow \text{Spec } A_i$ .  $B_{ij}$  is a finitely generated  $A_i$ -algebra, so by Lemma 2.3.1,  $B_{ij, \varphi_{ij}(a_i)}$  is a finitely generated  $A_i$ -algebra. The collection  $\{\text{Spec } B_{ij, \varphi_{ij}(a_i)}\}$  cover  $f^{-1}(D(a_i))$  and so proposition  $P(D(a_i))$  holds.

We now have the following statement to prove: let  $U = \text{Spec } A \subseteq Y$  be an open affine subset of  $Y$  which can be covered by open affines  $U_i = \text{Spec } A_i$  such that  $P(U_i)$  holds for all  $i$ , then  $P(U)$  holds. But this follows easily from Lemma 2.3.1.  $\square$

**Exercise 2** (Hartshorne 3.14). *Let  $X$  be a scheme of finite type over a field  $k$ . Then the closed points of  $X$  are dense.*

*Proof.* We cover  $X$  by finitely many open affines  $\{U_i = \text{Spec } A_i\}_{i=1}^n$  where each  $A_i$  is a finitely generated  $k$ -algebra. Notice that by Theorem ?? each  $A_i$  is jacobson. Fix an  $i$  and let  $f \in A_i$  be such that  $D(f) \subseteq U_i$ . Assume that  $x$  is closed in  $D(f)$ , that is,  $x$  is a maximal ideal of  $(A_i)_f$ . We show first that  $x$  is closed in  $X$ . The inclusion  $D(f) \subseteq \text{Spec } A_i$  induces a ring homomorphism  $A_i \rightarrow (A_i)_f$  which in fact is a  $k$ -algebra homomorphism as  $X$  is over  $k$ . Combining this with the fact that  $(A_i)_f$  is a finitely generated  $k$ -algebra gives that  $(A_i)_f$  is a finitely generated  $A_i$ -algebra and so the preimage of  $x$  in  $A_i$  is maximal, by Theorem ??. This holds for any  $i$ , and so  $x$  is closed in all  $U_i \ni x$ , and thus is closed in  $X$  (this step here doesn't seem to require that there were finitely many such  $U_i$ ). It thus suffices to show that every  $D(f)$  contains a maximal ideal. If  $f$  is contained in every maximal ideal then it is nilpotent (Lemma ??) and thus  $D(f)$  is empty.  $\square$

## References

- [1] Hartshorne
- [2] Notes on Algebraic Geometry *Troiani*.
- [3] Varieties *Troiani* **Fix these references**