Geometry of Interaction One for Multiplicative Proof Structures

Will Troiani

February 2021

Contents

1 Geometry of Interaction One

We consider conclusion-conclusion paths in a proof net and associate to this collection a bounded linear operator upon a Hilbert space. First, we recall some general theory from functional analysis.

1.1 Internalisation of direction sum and tensor product

We focus on the specific Hilbert space $\mathbb{H} = \ell^2$ of sequences $\underline{z} = (z_0, z_1, ...)$ of complex numbers which are square summable, ie, $\sum_{n=0}^{\infty} |z_n|^2$ converges. This has an inner product given as follows.

$$\langle \underline{z}, \underline{w} \rangle = \sum_{n=0}^{\infty} z_n \overline{w}_n \tag{1}$$

In fact, the sum $\mathbb{H}^m$ of m copies of $\mathbb{H}$ also has an inner product structure, defined by

$$\Big\langle \big(\underline{z}^1, ..., \underline{z}^m\big), \big(\underline{w}^1, ..., \underline{w}^m\big) \Big\rangle_{\mathbb{H}^m} = \sum_{j=1}^{m} \langle \big(\underline{z}^j, \underline{w}^j\big) \rangle_{\mathbb{H}} \tag{2}$$

We fix the standard basis for ℓ^2 consisting of sequences $\underline{e}^i$ such that all entries are equal to 0 except for the i^{th} which is equal to 1. We note that this basis is countably infinite. A basis for $\ell^2 \oplus \ell^2$ is given by all $(\underline{e}^i, 0)$ and $(0, \underline{e}^i)$ which is also countable, thus, bijections $\alpha : \mathbb{N} \coprod \mathbb{N} \longrightarrow \mathbb{N}$ induce isomorphisms $\ell^2 \longrightarrow \ell^2 \oplus \ell^2$. More explicitly, if $\alpha : \mathbb{N} \coprod \mathbb{N} \longrightarrow \mathbb{N}$ is such a bijection then there exists injective functions $\alpha_1, \alpha_2 : \mathbb{N} \longrightarrow \mathbb{N}$ which make the following diagram commute.

$$\begin{tikzcd}
\mathbb{N} \arrow[d, hook] \arrow[dr, "\alpha_1"] & \\
\mathbb{N} \coprod \mathbb{N} \arrow[r, "\alpha"] & \mathbb{N} \\
\mathbb{N} \arrow[u, hook] \arrow[ur, "\alpha_2"'] &
\end{tikzcd} \tag{3}$$

The induced isomorphism $\hat{\alpha} : \ell^2 \longrightarrow \ell^2 \oplus \ell^2$ is then given by the following explicit formula, where $z = \sum_{i=0}^{\infty} z_i \underline{e}^i$:

$$\hat{\alpha}(z) = \sum_{i=0}^{\infty} \left(z_{\alpha_1(i)} \underline{e}^i, z_{\alpha_2(i)} \underline{e}^i \right) \tag{4}$$

The following calculation shows that $\hat{\alpha}$ is an isometry:

$$\begin{aligned}
\langle \hat{\alpha}(\underline{z}), \hat{\alpha}(\underline{w}) \rangle &= \Big\langle \sum_{i=0}^{\infty} \big(z_{\alpha_1(i)} \underline{e}^i, z_{\alpha_2(i)} \underline{e}^i \big), \sum_{i=0}^{\infty} \big(w_{\alpha_1(i)} \underline{e}^i, w_{\alpha_2(i)} \underline{e}^i \big) \Big\rangle \\
&= \Big\langle \sum_{i=0}^{\infty} z_{\alpha_1(i)} \underline{e}^i, \sum_{i=0}^{\infty} w_{\alpha_1(i)} \underline{e}^i \Big\rangle + \Big\langle \sum_{i=0}^{\infty} z_{\alpha_2(i)} \underline{e}^i, \sum_{i=0}^{\infty} w_{\alpha_2(i)} \underline{e}^i \Big\rangle \\
&= \sum_{i=0}^{\infty} z_{\alpha_1(i)} \overline{w}_{\alpha_i(i)} + \sum_{i=0}^{\infty} z_{\alpha_2(i)} \overline{w}_{\alpha_2(i)} \\
&= \sum_{i=0}^{\infty} z_i \overline{w}_i \\
&= \langle \underline{z}, \underline{w} \rangle
\end{aligned}$$

We claim that (4) can also be written as $\hat{\alpha}(z) = \big(p^*(z), q^*(z)\big)$ for operators $p, q : \ell^2 \longrightarrow \ell^2$ determined by continuity and the following conditions.

$$p(\underline{e}^i) = \underline{e}^{\alpha_1(i)}, \qquad q(\underline{e}^i) = \underline{e}^{\alpha_2(i)} \tag{5}$$

These maps are norm preserving and so are clearly bounded, thus we have well defined bounded, linear operators. It can be established by a direct calculation that these have adjoints respectively determined by continuity and the following conditions.

$$p^*(\underline{e}^i) = \underline{e}^{\alpha_1^{-1}(i)} \text{ if } \alpha_1^{-1}(i) \text{ exists, otherwise } p^*(\underline{e}^i) = 0 \tag{6}$$

$$q^*(\underline{e}^i) = \underline{e}^{\alpha_2^{-1}(i)} \text{ if } \alpha_2^{-1}(i) \text{ exists, otherwise } p^*(\underline{e}^i) = 0 \tag{7}$$

For example: let $w = \sum_{i=0}^{\infty} w_i \underline{e}^i$, then

$$\big\langle p(z), w \big\rangle = \sum_{i=0}^{\infty} z_i \overline{w}_{\alpha_1(i)} = \big\langle z, p^*(w) \big\rangle$$

we thus have the following formula.

$$\hat{\alpha} = p^* \oplus q^* \tag{8}$$

In a similar way, given any $n > 0$ along with a bijection $\alpha : \mathbb{N} \longrightarrow \coprod_{i=1}^{n} \mathbb{N}$, there is a corresponding induced isometric isomorphism $\hat{\alpha} : \mathbb{H} \longrightarrow \mathbb{H}^n$ which has an explicit formula, where $z = \sum_{i=0}^{\infty} z_i \underline{e}_i$:

$$\hat{\alpha}(z) = \sum_{i=0}^{\infty} \Big(z_{\alpha_1(i)} \underline{e}^i, ..., z_{\alpha_n(i)} \underline{e}^i \Big) \tag{9}$$

Example 1.1.1. A simple example is given by the following:

$$\begin{aligned} \alpha_1 : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n \end{aligned} \qquad\qquad \begin{aligned} \alpha_2 : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n+1 \end{aligned}$$

which induces $\alpha : \mathbb{N} \coprod \mathbb{N} \longrightarrow \mathbb{N}$, defined by $\alpha(n,1) = 2n$ and $\alpha(n,2) = 2n+1$. The functions $\alpha_1, \alpha_2, \alpha$ make the following a coproduct diagram:

$$\begin{tikzcd}
\mathbb{N} \arrow[d, tail] \arrow[dr, "\alpha_1"] & \\
\mathbb{N} \coprod \mathbb{N} \arrow[r, "\alpha"] & \mathbb{N} \\
\mathbb{N} \arrow[u, tail] \arrow[ur, "\alpha_2"'] &
\end{tikzcd} \tag{10}$$

and indeed α is a bijection. We thus have two functions:

$$\begin{aligned} p : \ell^2 &\longrightarrow \ell^2 \\ (z_0, z_1, ...) &\longmapsto (z_0, 0, z_1, 0, z_2, ...) \end{aligned} \qquad \begin{aligned} q : \ell^2 &\longrightarrow \ell^2 \\ (z_1, z_2, ...) &\longmapsto (0, z_0, 0, z_1, 0, ...) \end{aligned}$$

which have the following adjoints:

$$\begin{aligned} p^* : \ell^2 &\longmapsto \ell^2 \\ (z_0, z_1, ...) &\longmapsto (z_0, z_2, ...) \end{aligned} \qquad \begin{aligned} q^* : \ell^2 &\longrightarrow \ell^2 \\ (z_0, z_1, ...) &\longmapsto (z_1, z_3, ...) \end{aligned}$$

Aside 1.1.2. The following calculation shows that p^* is adjoint to p, the corresponding calculation for q is similar:

$$\begin{aligned} \big\langle p(z_0, z_1, ...), (w_0, w_1, ...)\big\rangle &= \big\langle (z_0, 0, z_1, 0, ...), (w_1, w_2, ...)\big\rangle \\ &= \big\langle (z_0, z_1, ...), (w_0, w_2, ...)\big\rangle \\ &= \big\langle (z_0, z_1, ...), p^*(w_0, w_1, ...)\big\rangle \end{aligned}$$

The function p^*, q^* induce $\hat{\alpha} = p^* \oplus q^* : \ell^2 \longrightarrow \ell^2 \oplus \ell^2$ defined by

$$\hat{\alpha}(z_0, z_1, ...) = \big((z_0, z_2, ...), (z_1, z_3, ...)\big) \tag{11}$$

We make a few observations:

Lemma 1.1.3. *The functions p, q, p^*, q^* satisfy the following:*

- $p^*p = \mathrm{id}_{\ell^2} = q^*q$,
- $pp^* + qq^* = \mathrm{id}_{\ell^2}$,
- $p^*q = 0 = q^*p$.

Corollary 1.1.4. *Let w be a word over the following alphabet $\{p, q, p^*, q^*, \mathrm{id}, \mathrm{id}^*\}$, and let $\hat{w}$ denote the induced function given by reading w as a composite of functions. Then $\hat{w} = 0$ if and only w admits either p^*q or q^*p as a subword.*

Proof. By lemma 1.1.3, it is clear that if p^*q or q^*p appears as a subword of w then $\hat{w} = 0$.

Conversely, say $\hat{w} = 0$. First, we can clearly assume that w is a word over the alphabet $\{p, q, p^*, q^*\}$. If w admits no occurrence of r^* where $r = p, q$ then $\hat{w} \neq 0$, this can be seen by considering $\hat{w}(\underline{e}^1)$. Hence, there is some occurrence of r^* for $r = p, q$ in w. By the first dotpoint of Lemma 1.1.3 we can assume that w does not admit p^*p nor q^*q as a subword.

We proceed by induction on the integer n given by the number of occurrences of letters r^* for $r = p, q$ in w.

Say there is only one such occurrence. Now, pick some occurrence of r^* for $r = p, q$. If w is a word of length 1 then clearly $\hat{w} \neq 0$, so we can pick r' to be a letter in w so that either $r'r$ or rr' appears as a subword of w. Say $r'r$ appears as a subword. Then if $r = p$ then $\hat{w}(\underline{e}^0) \neq 0$, if $r = q$ then $\hat{w}(\underline{e}^1) \neq 0$. Hence rr' appears as a subword of w and w admits no subwords of the from p^*p nor of the form q^*q, we necessarily have that w admits either p^*q or q^*p as a subword.

Now for the inductive case. We pick an arbitrary choice of r^* in w where $r = p, q$. Just as in the base case we see that there exists some letter r such that either $r'r$ or rr' appears as a subword in w. Say we are in the case of $r'r$. Then the function induced by this word cannot be the 0 function, so we can remove it and appeal to the inductive hypothesis. If we are in the case of rr', then either the function induced by rr' is 0 in which case we can appeal to the base case of this argument and deduce that rr' is either q^*p or p^*q. If the function induced by rr' is not the zero function, then we can remove it and appeal to the inductive hypothesis. $\square$

Corollary 1.1.5. *Let $w_1, ..., w_n$ be a set of words over the alphabet $\{p, q, p^*, q^*, \mathrm{id}, \mathrm{id}^*\}$ and consider their induced function $\hat{w}_1, ..., \hat{w}_n$ as in Corollary 1.1.4.*

Then $\sum_{i=1}^n \hat{w}_i \neq 0$ if and only if for some $i = 1, ...n$ the subword p^*q and the subword q^*p does not appear in w_i, then .

Proof. Clearly, if for all $i = 1, ..., n$ the word w_i admits either p^*q or q^*p as a subword, then $\sum_{i=1}^n \hat{w}_i = 0$, so it remains to show the converse.

Another way to state the conclusion of this Corollary is that $\sum_{i=1}^n \hat{w}_i = 0$ if and only if $\hat{w}_1 = \ldots = \hat{w}_n = 0$. We prove this claim.

Say there exists w_i such that $\hat{w}_i \neq 0$. Then by Corollary 1.1.4 the word w_i admits either p^*q or q^*p as a subword.

Conversely, say there exists some w_i such that $\hat{w}_i \neq 0$. Let $\underline{z}$ be such that $\hat{w}_i(\underline{z}) \neq 0$. The key observation is that if r_1, r_2 are any of p, q, p^*, q^* then if x_j denotes the j^{th} element of $r_1(\underline{z})$ and y_j the j^{th} element of $r_2(\underline{z})$ then $x_j \neq 0$ implies $x_j + y_j \neq 0$. The general statement follows from this. □

1.2 Proofs as operators

Definition 1.2.1. Let l be an axiom or cut link in a proof net π. Let $A, \neg A$ denote the conclusions of l in the case where l is a cut link, or the premises in the case where l is a cut link. In both cases, the order of the sequences of oriented axioms of A and $\neg A$ induces a bijection J_l from the set of unoriented axioms of A to the that of $\neg A$.

Say l is a tensor or par link, with premises A, B and conclusion C, say. The order of the sequences of oriented axioms of A, B, C induce two injective functions I_l^1, I_l^2 both with codomain the set of unoriented axioms of C. The domain of I_l^1 is the set of unoriented axioms of A and I_l^2 has domain the set of unoriented axioms of B.

It is sometimes convenient to leave the label of the link l implicit, in which case we simply write J for J_l and I^1, I^2 for I_l^1 or I_l^2 respectively.

Definition 1.2.2. Let π be a proof structure admitting a conclusion A. Choose also an unoriented atom X in A. A **persistent walk** of X is a walk in π satisfying the following conditions.

1. The formula A labels some edge e. The first edge e_1 of ν is e.

2. The unoriented axiom X uniquely determines an edge $e_2 \neq e_1$ adjacent to a common vertex with e_1, and so that the label of e_2 admits a copy of X amongst its set of unoriented axioms corresponding under one of J, I^1, I^2 to the copy of X associated to e_1.

3. If $i > 2$ then X_i determines at most one edge e_i such that $e_i \neq e_{i-1}, e_i \neq e_{i-2}$, the edges e_{i-1}, e_i are adjacent to a common vertex, and the label of e_i admits a copy of X amongst its set of unoriented axioms corresponding under one of J, I^1, I^2 to the copy of X associated to e_{i-1}. If no such edge exists then we define e_{i-1} to be the final edge in ν.

It is not obvious from Definition 1.2.2 but the set of persistent walks of X is finite, and so there exists a maximal length persistent walk which will be of particular interest. First we prove that indeed this set is finite which will follow as a Corollary to the following Lemma.

Lemma 1.2.3. *Let π be a proof structure and ν be the persistent walk of an unoriented atom X of a conclusion A. For $i = 1, ..., n$ say e_i is labelled by A_i. Implicit in Definition 1.2.2 is an association of a copy of X_i in the set of unoriented atoms of A_i to each edge e_i in the walk ν. If $i \neq j$ are such that $e_i = e_j$ then $X_i \neq X_j$.*

Proof. Let e_i denote the i^{th} element of the sequence ν. Let $p(i)$ denote the following property.

$$p(i) := \exists j > i \text{ such that } e_i = e_j \text{ and } X_i = X_j \tag{12}$$

First we claim that there exists i such that $p(e_i)$ does *not* hold. Say $p(e_1)$ holds. Then there exists j such that $i \neq j$ and $e_i = e_j$. This means that the path ν eventually returns to the edge e_1 which by Definition 1.2.2 is incident to a conclusion vertex. Hence so is e_j, and so ν is finite and e_j is the final element of the walk. It then follows that $p(e_j)$ does not hold as there is no $k > j$ such that e_k exists.

On the other hand, if it were the case that for all i the statement $p(e_i)$ does not hold, then there is nothing to prove, so assume this is not the case.

Hence we can let i be the maximal integer such that $p(e_i)$ holds and $p(e_{i+1})$ does not hold. Let j be such that $e_i = e_j$ and $X_i = X_j$. Let e denote the edge $e_i = e_j$. Then e is either a conclusion to a tensor or par link, or it is not. First consider the case when it is. Definition 1.2.1 makes it clear that $\operatorname{im} I_l^1 \cap \operatorname{im} I_l^2 = \varnothing$, and so the statement $X_i = X_j$ is contradictory. Hence, the edge e is not a conclusion to a tensor link nor to a par link.

Say e is premise to a cut link. Since e is not the conclusion to a tensor link nor a par link, e must be the conclusion to an axiom link. Denote the other conclusion to this axiom link by e'. We either have that $e' = e_{i-1}$ or $e' = e_{i+1}$. Say $e' = e_{i+1}$, then $X_1 = X_2$ implies that X_{i+1} is either equal to X_{j+1} or X_{j-1}, depending on the direction of the path at the point e_j, in either case though we contradict maximality of the integer i.

Now say $e' = e_{i-1}$, we can similarly contradict maximality of i by considering the other premise to the cut link which must be the edge e_{i+1}.

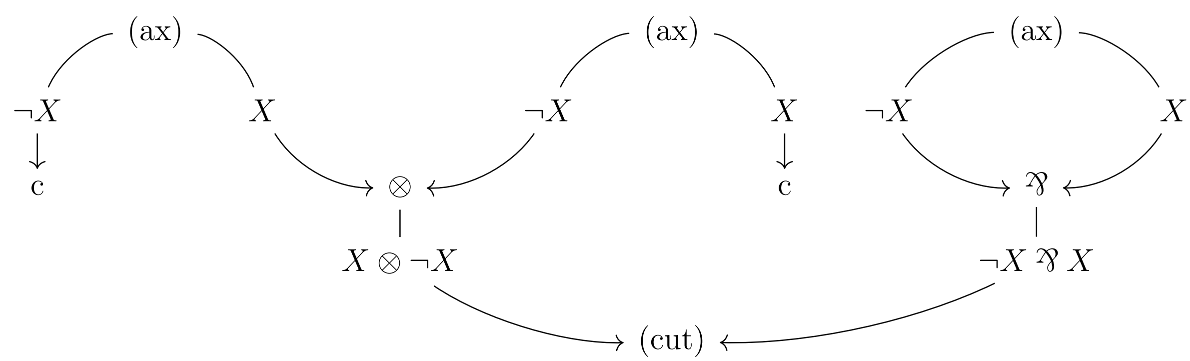
The final case is when e is conclusion to an axiom link. Then e_{i+1} is either the other conclusion e' to the axiom link, or a premise to some other link l', each case contradicts maximality of i. $\square$

Corollary 1.2.4. *There is exactly one maximal persistent walk of X.*

Proof. Given a persistent walk ν let $l(\nu)$ denote its length. Say ν, ν' are persistent walk of X and $l(\nu) < l(\nu')$, we first claim that ν is a subwalk of ν'. This follows from an easy induction on $l(\nu')$.

Hence it remains to show that persistent walks of X cannot be arbitrarily long. Although it is the case that persistent walks can admit cycles, they cannot cycle indefinitely, this is because the set of unoriented atoms of each formula in π is a finite set, and Lemma 1.2.3 shows that a distinct copy of X is associated to each edge in a persistent walk. $\square$

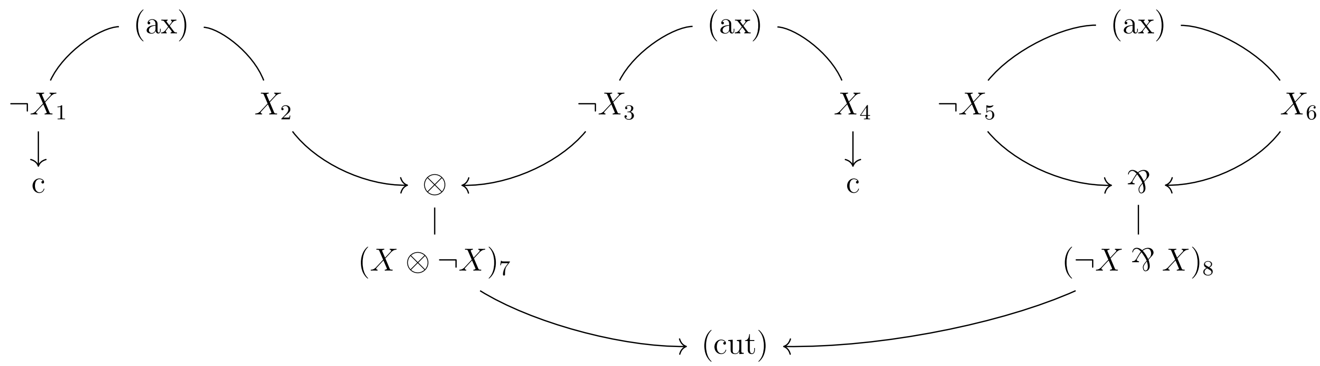
Example 1.2.5. Let π denote the following proof net.



(13)

Indeed, π is a proof *net*, but we will not use this structure here. For clarity, we artificially place labels on the formulas so that we can refer to particular edges, but for all $i = 1, \ldots, 8$ the notation Z_i, where

$Z = X, \neg X, X \otimes \neg X, \neg X \parr X$, denotes the formula Z.



(14)

Consider the conclusion $\neg X_1$. This is conclusion to an axiom link, and so the next element of the persistent path of $\neg X_1$ is the edge labelled X_2. The unique directed path from X_2 to either a conclusion or a cut link is the path $X_2, (X \otimes \neg X)_7$. The formula $(X \otimes \neg X)_7$ is premise to a cut link with other premise $(\neg X \parr X)_8$. Now, the oriented atoms of $(\neg X \parr X)_8$ is the sequence $(\neg X, X)$ and $\neg X$ is the first element, hence the following edge in the persistent walk is $\neg X_5$. We continue in this way to eventually obtain the following walk, which is the unique maximal length persistent walk of $\neg X_1$.

$$\neg X_1, X_2 (X \otimes \neg X)_7, (\neg X \parr X)_8, \neg X_5, X_6, (\neg X \parr X)_8, (X \otimes \neg X)_7, \neg X_3, X_4 \tag{15}$$

Definition 1.2.6. Let π be a proof structure, for each cut link l with premises $A, \neg A$ replace the vertex labelled (cut) in l by two vertices labelled c and make the edges $A, \neg A$ respectively be incident to one of these vertices labelled c. The result is a proof structure π', say this proof structure has conclusions $A_1, ..., A_n$. Let $m < n$ be an integer and assume that $A_1, ..., A_n$ are ordered so that $A_1, ..., A_m$ are conclusions of π' but *not* conclusions of π (in other words, $A_1, ..., A_m$ are the formulas appearing in cut links of π), assume also that for all odd $k < m$ there exists a cut link in π with premises A_k, A_{k+1}. We construct an $n \times n$ matrix $[\![\pi]\!]$ via the following procedure.

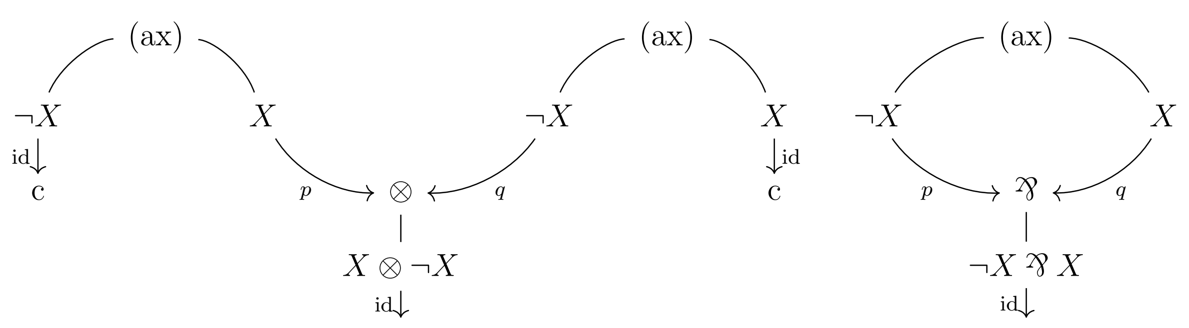
1. Label the left premise of every tensor link and every par link by p, label every right premise of every tensor link and every par link by q. Label the remaining edges by id.

2. For each $i, j \in \{1, ..., n\}$ there is a collection of maximal length persistent walks $\nu_1, ..., \nu_k$ of an unoriented atom of A_i and whose final edge is labelled A_j.

3. To each walk ν_t is a corresponding bounded linear operator o_t on l^2 given by the composite of the operators labelling the edges traversed in the walks ν_t where we take the adjoint of an operator if the corresponding edge is traversed in reverse direction.

4. We then define:
$$([\![\pi]\!])_{ji} := o_1 + \ldots + o_k \tag{16}$$

Example 1.2.7. Consider π of Example 1.2.5. We remove the cut-link to obtain a proof-*structure* π'. Label the left premise of each tensor and each par link by p and the right premise of each tensor and each par link by q (indeed these are the same p and q as in Section 1.1). Label the edges with conclusion c by the identity map id (this is the identity on the space ℓ^2).



(17)

Now, associated to each pair of conclusions is the collection of paths in π', here a *path* in π' means a finite sequence $(e_1, ..., e_n)$ of edges in π', where each e_i may be in either direction, and we only require that for all $i = 1, ..., n-1$ that e_i is not equal to e_{i+1} traversed in the opposite direction . We introduce some notation, associated to the pair $\neg X, X$ is the set of paths which we denote $\mathrm{Path}(\neg X, X)$, notice this set has one element, but $\mathrm{Path}(\neg X \parr X, \neg X \parr X)$ has infinitely many elements. Each path induces an operator $\ell^2 \longrightarrow \ell^2$ given by composing the labels on the edges in the path, where if an edge is traversed from the target to the source, we take the adjoint of the label. For example, the path

$$(\neg X, X, \neg X, X) \in \mathrm{Path}(\neg X, X) \tag{18}$$

has associated operator $\mathrm{id}\, q^* p\, \mathrm{id}^* = q^* p$.

Remark 1.2.8. Notice by Lemma 1.1.3 that $p^* q$ is the zero operator, this reflects the fact that although the edges labelled p, q are connected in the graph, the corresponding propositions $X, \neg X$ are "logically disconnected". Indeed, the formalisation of this concept of "logically disconnected" propositions is one of the fruits bared by this theory.

Ranging over all paths (needs to be rewritten) between all pairs of conclusions in π' defines a set of operators which can be organised into an incidence matrix as follows, we let r denote $pq^* + qp^*$ and assume index 1 corresponds to the conclusion $X \otimes \neg X$, index 2 to $\neg X \parr X$, index 3 to $\neg X$, and index 4 to X.

$$\begin{pmatrix} 0 & 0 & p & q \\ 0 & r & 0 & 0 \\ p^* & 0 & 0 & p^* q \\ q^* & 0 & q^* p & 0 \end{pmatrix} \tag{19}$$

Let $[\![\pi]\!]$ denote this matrix. Notice that the first two columns and first two rows are labelled by the formulas involved in the cut-link of π. Thus, we define σ to be the matrix which permutes the first two columns.

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

and consider $[\![\pi]\!]\sigma[\![\pi]\!]$, which is a matrix whose ij^{th} entry corresponds to the sum of operators corresponding to the paths in π' which traverse the cut once, where the start of the path is the conclusion in π' with label corresponding to column j, and whose end point is the conclusion with label corresponding to row i. In our current example, this is:

$$[\![\pi]\!]\sigma[\![\pi]\!] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & rp & rq \\ 0 & p^* r & 0 & 0 \\ 0 & q^* r & 0 & 0 \end{pmatrix} \tag{21}$$

Multiplying by $\sigma[\![\pi]\!]$ yields:

$$[\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p^* rp & q^* rp \\ 0 & 0 & p^* rq & q^* rq \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \tag{22}$$

What happens if we perform the same process to π after we have performed cut-elimination? Under this process, π corresponds to the proof consisting of a single axiom link:

$$\begin{array}{ccc} & (\mathrm{ax}) & \\ \neg X & & X \\ \downarrow & & \downarrow \\ \mathrm{c} & & \mathrm{c} \end{array} \tag{23}$$

which corresponds to the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{24}$$

which appears as a minor in (22). The general theory will show that this is not a coincidence.

Proposition 1.2.9. *Consider a walk ν in a proof net π which does not traverse the same edge twice in a row. Say ν begins at a conclusion A of π and ends at a conclusion B of π and consider the corresponding operator o_ν as defined in* (3) *of Definition 1.2.6. We have that $o_\nu \neq 0$ if and only if ν is persistent (necessarily of maximal length), see Definition 1.2.2.*

Proof. We can view o_ν as a word w written over the alphabet $\{p, q, p^*, q^*, \mathrm{id}, \mathrm{id}^*\}$. By Corollary 1.1.5 we have that o_ν is persistent if and only if w does not admit q^*p nor p^*q as a subword, and this happens if and only if the walk ν admits a pair of consecutive edges (e, e') such that e and e' are the premises of a common tensor or par link. This is the case if and only if w is not persistent. $\square$

Definition 1.2.10. Let π be a proof net and ζ the corresponding proof structure given by removing all edges corresponding to cut-links in π. Then if $A_1, ..., A_n$ are the conclusions of ζ and $m < n$ is such that $A_1, ..., A_m$ are conclusions of ζ but *not* of π, then we let σ_m denote the $(2m+n) \times (2m+n)$ matrix whose top left $2m \times 2m$ minor is given by the following.

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \tag{25}$$

The rest of the entries are 0.

Remark 1.2.11. Let π be a proof structure admitting a conclusion A and let π' be a proof structure admitting a conclusion $\neg A$. Let ζ denote the proof structure given by placing π and π' side by side and constructing a cut link with one premise A of π and the other premise $\neg A$ of π'.

$$\begin{tikzcd} \pi \arrow[d, dash, dashed] & & \pi' \arrow[d, dash, dashed] \\ A \arrow[dr, bend right] & & \neg A \arrow[dl, bend left] \\ & (\text{cut}) & \end{tikzcd} \tag{26}$$

Let X be some unoriented atom of A and denote by ν the maximal persistent walk in π of X. Corresponding to X is a copy of X amongst the set of unoriented atoms of $\neg A$. Let μ denote the maximal persistent walk in π' of this copy of X. Finally, let e, e' respevtively denote the left and right premise of the cut link in ζ displayed in (26). Associated to ν is formula B which is a conclusion of π and labels the final edge d of ν. Amongst the set of unoriented atoms of B is a copy of X. We now consider the maximal persistent walk θ of this copy of X associated to B in the proof net ζ.

In fact, θ is easy to describe, it is the following concatination, where $\cdot$ means concatination of walks.

$$\theta = \nu \cdot e \cdot e' \cdot \mu \tag{27}$$

Lemma 1.2.12. *Let ζ be a proof structure and let A_i, A_j be (possibly equal) conclusions of ζ. Let $\theta_1, ..., \theta_e$ denote the maximal length persistent walks in ζ corresponding to atomic axioms of A_i and whose final edge is labelled A_j, where each of $\theta_1, ..., \theta_e$ traverses a cut link in ζ exactly one time. Let $o_1, ..., o_e$ denote the induced operators as described in* (3) *of Definition 1.2.6. Then the ji^{th} entry of $[\![\zeta]\!]\sigma[\![\zeta]\!]$ is given by the sum $o_1 + \ldots + o_e$.*

Proof. Let $A_1, ..., A_n$ be the conclusions of the proof structure ζ' given by removing all the cut links from ζ, moreover, assume that $A_1, ..., A_n$ are ordered so that $A_1, ..., A_m$ are premises to cut links in ζ and for all odd $k < m$ assume A_k, A_{k+1} are premise to a common cut link l_k. For each $i, j \in \{1, ..., n\}$ and each k, let $\nu_1^k, ..., \nu_l^k$ be the collection of maximal length persistent walks in ζ' corresponding to the unoriented atoms of A_i such that the final edge in each $\nu_1^k, ..., \nu_l^k$ is labelled A_k, and let $\mu_1^{k+1}, ..., \mu_{l'}^{k+1}$ be the collection of maximal length persistent walks in ζ' corresponding to the unoriented atoms of A_{k+1} such that the final edge in each $\mu_1, ..., \mu_l$ is in A_j. Then by (27) we have that the following is the formal sum of maximal length persistent paths in ζ (note: not ζ') corresponding to unoriented atoms of A_i such that the final edge is labelled A_j, and where the displayed cut link (26) is traversed exactly one time.

$$\sum_{p=1}^{l} \sum_{q=1}^{l'} \nu_p^k \cdot e \cdot e' \cdot \mu_q^{k+1} \tag{28}$$

Summing over all k, we see that the formal sum of all maximal length persistent paths in ζ corresponding to unoriented atoms of A_i such that the final edge is labelled A_j, and where some cut link of ζ is traversed exactly one time is given as follows.

$$\sum_{\substack{k<m \\ k \text{ odd}}} \sum_{p=1}^{l} \sum_{q=1}^{l'} \nu_p^k \cdot e \cdot e' \cdot \mu_q^{k+1} \tag{29}$$

We now show that the operator corresponding to (29) is equal to the ji^{th} entry of $[\![\zeta]\!]\sigma[\![\zeta]\!]$. This is a calculation.

$$\begin{aligned} ([\![\zeta]\!]\sigma[\![\zeta]\!])_{ji} &= \Big(\sum_{k<m \text{ odd}} (o_{\mu_1}^{k+1} + \ldots + o_{\mu_{l'}}^{k+1})(o_{\nu_1}^{k} + \ldots + o_{\nu_l}^{k}) \Big) && (30) \\ &= \sum_{\substack{k<m \\ k \text{ odd}}} \sum_{p=1}^{l} \sum_{q=1}^{l'} o_{\mu_q}^{k+1} \circ o_{\nu_p}^{k} && (31) \end{aligned}$$

This is exactly the operator corresponding to (29). $\square$

Lemma 1.2.12 generalises easily.

Corollary 1.2.13. *Let ζ be a proof structure and let A_i, A_j be (possibly equal) conclusions of ζ. Let $\theta_1, ..., \theta_e$ denote the maximal length persistent walks in ζ corresponding to the atomic axioms of A_i and whose final edge is labelled A_j, where each of $\theta_1, ..., \theta_e$ traverses a cut link in ζ exactly m times. Let $o_1, ..., o_e$ denote the induced operators as described in* (3) *of Definition 1.2.6. Then the ji^{th} entry of $[\![\zeta]\!](\sigma[\![\zeta]\!])^n$ is given by the sum $o_1 + \ldots + o_e$.*

Corollary 1.2.14. *If π is a proof net and σ_m is as defined in Definition 1.2.10 then there exists an integer $n > 0$ such that $[\![\pi]\!](\sigma_m[\![\pi]\!])^n = 0$.*

Proof. Follows from Corollary 1.2.13 along with the fact that maximal length persistent walks are finite, as established in the proof of Corollary 1.2.4. $\square$

Definition 1.2.15. We define

$$\mathrm{Ex}([\![\pi]\!]) = [\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!] + \ldots \tag{32}$$

which by Corollary 1.2.14 is a well defined matrix.

Remark 1.2.16. Let π be a proof structure with conclusions $A_1, ..., A_n$ It follows from Corollary 1.2.13 that the ji^{th} entry of $\mathrm{Ex}([\![\pi]\!])$ is the sum of operators corresponding under (3) of Definition 1.2.6 to maximal length persistent walks corresponding to an unoriented atom of A_i and whose final edge is labelled A_j.

Corollary 1.2.17 (Geometry of Interaction One)**.** *Let π be a proof net and ζ the cut-free proof equivalent under cut elimination to π. Then the matrix $[\![\zeta]\!]$ exists as a minor in $Ex([\![\pi]\!])$ and any entry in $\mathrm{Ex}([\![\pi]\!])$ which is not in the minor corresponding to $[\![\zeta]\!]$ is equal to 0.*

Proof. In general, if $\gamma : \xi \longrightarrow_{(\mathrm{cut})} \xi'$ is a reduction then there is an induced mapping from maximal length persistent paths in ξ' to those in ξ given in the obvious way. incomplete but obvious how to finish it. □